Results of ISS Type for Hysteretic Lur’e Systems: a Differential Inclusions Approach*

B. Jayawardhana†, H. Logemann‡, and E.P. Ryan§

1 Introduction

The paper comprises a study of absolute stability, input-to-state stability, and boundedness properties of a feedback interconnection of a finite-dimensional, linear, m-input, m-output system \((A, B, C)\) and a set-valued nonlinearity \(\Phi\). With reference to Figure 1, we assume that \(D\) is a set-valued map in which input or disturbance signals are embedded. The analytical framework is of sufficient generality to encompass feedback systems with hysteresis operators (that is, a causal rate-independent operator) in the feedback loop. To illustrate this, let \(F\) be a causal operator from \(\text{dom}(F) \subset L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^m)\) to \(L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^m)\), where \(\mathbb{R}_+ := [0, \infty)\), and consider the

*Based on research supported by the UK Engineering & Physical Sciences Research Council (Grant Ref: GR/S94582/01).
†Department of Discrete Technology and Production Automation, University of Groningen, 9747 AG Groningen, The Netherlands. Email: B.Jayawardhana@rug.nl
‡Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, United Kingdom. Email: hl@maths.bath.ac.uk
§Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, United Kingdom. Email: epr@maths.bath.ac.uk
feedback system (structurally of Lur’ë type), with input \( d \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^m) \), given by the functional differential equation

\[
\dot{x}(t) = Ax(t) + B(d(t) - (F(Cx))(t)).
\]

(1)

Assume that \( F \) can be embedded in a set-valued map \( \Phi \) in the sense that

\[
y \in \text{dom}(F) \implies (F(y))(t) \in \Phi(y(t)) \quad \text{for a.a. } t \in \mathbb{R}_+.
\]

If the input \( d \) is such that \( d(t) \in D(t) \) for almost all \( t \), then any solution of (1) is \textit{a fortiori} a solution of the feedback interconnection in Figure 1. In this sense, properties of solutions of the feedback interconnection are inherited by solutions of (1). Under particular regularity assumptions on \( D \) and \( \Phi \), generalized sector conditions on \( \Phi \), and positive-real conditions related to the linear component \((A, B, C)\), we establish input-to-state stability (in the sense of [10], but extended to differential inclusions) and boundedness properties of solutions of the system in Figure 1.

2 Set-valued nonlinearities and differential inclusions

A set-valued map \( y \mapsto \Phi(y) \subset \mathbb{R}^m \), with non-empty values and defined on \( \mathbb{R}^m \), is said to be \textit{upper semicontinuous} at \( y \in \mathbb{R}^m \) if, for every open set \( U \) containing \( \Phi(y) \), there exists an open neighbourhood \( Y \) of \( y \) such that \( \Phi(Y) := \bigcup_{z \in Y} \Phi(z) \subset U \); the map \( \Phi \) is said to be \textit{upper semicontinuous} if it is upper semicontinuous at every \( y \in \mathbb{R}^m \). The set of upper semicontinuous compact-convex-valued maps

\[
\Phi : \mathbb{R}^m \to \{ S \subset \mathbb{R}^m \mid S \text{ non-empty, compact and convex} \}
\]

is denoted by \( U \). Let \( D : \mathbb{R}_+ \to \{ S \subset \mathbb{R}^m \mid S \neq \emptyset \} \) be a set-valued map. The map \( D \) is said to be \textit{measurable} if the preimage \( D^{-1}(U) := \{ t \in \mathbb{R}_+ \mid D(t) \cap U \neq \emptyset \} \) of every open set \( U \subset \mathbb{R}^m \) is Lebesgue measurable: \( D \) is said to be \textit{locally essentially bounded} if \( D \) is measurable and the function \( t \mapsto |D(t)| := \sup\{ \|\xi\| \mid \xi \in D(t) \} \) is in \( L^\infty_{\text{loc}}(\mathbb{R}_+) \).

The set of all locally essentially bounded set-valued maps \( \mathbb{R}_+ \to \{ S \subset \mathbb{R}^m \mid S \neq \emptyset \} \) is denoted by \( B \). For \( D \in B, I \subset \mathbb{R}_+ \) an interval and \( 1 \leq p \leq \infty \), the \( L^p \)-norm of the restriction of the function \( t \mapsto |D(t)| \) to the interval \( I \) is denoted by \( \|D\|_{L^p(I)} \).

The feedback system shown in Figure 1 corresponds to the initial-value problem

\[
\dot{x}(t) - Ax(t) = B(D(t) - \Phi(Cx(t))), \quad x(0) = x^0 \in \mathbb{R}^n, \quad D \in B,
\]

(2)

where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{m \times n} \) and \( \Phi \in U \). By a solution of (2) we mean an absolutely continuous function \( x : [0, \omega) \to \mathbb{R}^n, 0 < \omega \leq \infty \), such that \( x(0) = x^0 \) and the differential inclusion in (2) is satisfied almost everywhere on \([0, \omega)\); a solution is \textit{maximal} if it has no proper right extension that is also a solution; a solution is \textit{global} if it exists on \([0, \infty)\). We record the following existence result (a consequence of, for example, [3, Corollary 5.2]).

**Lemma 1.** Let \( \Phi \in U \). For each \( x^0 \in \mathbb{R}^n \) and each \( D \in B \), the initial-value problem (2) has a solution. Moreover, every solution can be extended to a maximal solution \( x : [0, \omega) \to \mathbb{R}^n \) and, if \( x \) is bounded, then \( x \) is global.
3 Input-to-state stability: the main results

In the context of the differential inclusion (2), the transfer-function matrix of the linear system given by \((A, B, C)\) is denoted by \(G\), i.e., \(G(s) = C(sI - A)^{-1}B\).

We assemble the following hypotheses which will be variously invoked in the theory developed below. Recall that \(K\) is the set of all functions \(\varphi : \mathbb{R}_+ \to \mathbb{R}_+\) that are continuous and strictly-increasing with \(\varphi(0) = 0\); \(\mathcal{K}_\infty \subset K\) is the set of all unbounded functions of class \(K\); \(\mathcal{K}\) is the set of all functions \(\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+\) such that \(\beta(\cdot, t) \in K\) for each \(t \in \mathbb{R}_+\) and, for each \(r \in \mathbb{R}_+, \beta(r, t) \downarrow 0\) as \(t \to \infty\).

\((H1)\) There exist numbers \(a < b\) and \(\delta > 0\) such that
\[
\langle ay - v, by - v \rangle \leq 0 \quad \forall v \in \Phi(y), \forall y \in \mathbb{R}^m,
\]
\(G(I + aG)^{-1} \in H_\infty\) and \((I + bG)(I + aG)^{-1} - \delta I\) is positive real.

\((H2)\) \(\Phi(0) = \{0\}\) and there exist numbers \(a > 0\), \(\delta \in (0, 1)\) and \(\theta \geq 0\) such that
\[
\frac{a}{2}\|y\|^2 \leq \langle y, v \rangle \quad \forall v \in \Phi(y), \forall y \in \mathbb{R}^m,
\]
\[
\|v - a\delta y\| \leq \langle y, v - a\delta y \rangle \quad \forall v \in \Phi(y), \forall y \in \mathbb{R}^m \text{ with } \|y\| \geq \theta
\]
and \(G(I + \delta aG)^{-1}\) is positive real.

\((H3)\) There exist \(\varphi \in \mathcal{K}_\infty\) and numbers \(b > 0\) and \(\delta \in [0, 1)\) such that
\[
\max \left\{ \varphi(\|y\|)\|y\|, \frac{\|y\|^2}{b} \right\} \leq \langle y, v \rangle \quad \forall v \in \Phi(y), \forall y \in \mathbb{R}^m
\]
and \((\delta/b)I + G\) is positive real.

\((H4)\) \(\Phi(0) = \{0\}\) and there exist \(\varphi \in \mathcal{K}_\infty\) and a number \(\theta \geq 0\) such that
\[
\varphi(\|y\|)\|y\| \leq \langle y, v \rangle \quad \forall v \in \Phi(y), \forall y \in \mathbb{R}^m,
\]
\[
\|v\| \leq \langle y, v \rangle \quad \forall v \in \Phi(y), \forall y \in \mathbb{R}^m \text{ with } \|y\| \geq \theta
\]
and \(G\) is positive real.

Remark 2. (a) \((H1)\) is a set-valued version of the familiar multivariable sector condition. A routine calculation shows that (3) holds if and only if
\[
\left\| v - \frac{a + b}{2} y \right\| \leq \frac{b - a}{2}\|y\| \quad \forall v \in \Phi(y), \forall y \in \mathbb{R}^m.
\]
(b) If \(m = 1\) (the single-input, single-output case), then the combined frequency-domain assumptions in \((H1)\) admit a graphical characterization in terms of the Nyquist diagram of \(G\) (see, e.g., [5, pp. 268]).

(c) Conditions (4) and (7) can be viewed as the limits of (3) and (6), respectively, as \(b \to \infty\).

(d) A sufficient condition for (6) to hold is the “nonlinear” sector condition
\[
\langle \varphi(y)\|y\|^{-1} y - v, by - v \rangle \leq 0 \quad \forall v \in \Phi(y), \forall y \in \mathbb{R}^m,
\]
which is (3) with the term $ay$ replaced by $\varphi(y)\|y\|^{-1}y$ (which should be interpreted as taking the value 0 for $y = 0$). It is easy to construct counterexamples which show that (9) is not necessary for (6) to hold.

(e) If $m = 1$ and (4) holds, then (5) is trivially satisfied for any $\theta \geq 1$ and any $\delta \in [0,1)$. Similarly, if $m = 1$ and (7) holds, then (8) is satisfied for every $\theta \geq 1$.

(f) If (6) holds for some $\varphi \in K_\infty$ and for some $b > 0$, then $\Phi(0) = \{0\}$ and, furthermore, (8) is satisfied for any $\theta > 0$ satisfying $\varphi(\theta) \geq b$.

**Definition 3.** System (2) is said to be input-to-state stable with bias $c \geq 0$ if every maximal solution of (2) is global, and there exist $\beta_1 \in KL$ and $\beta_2 \in K_\infty$ such that, for all $x_0 \in \mathbb{R}^n$ and all $D \in B$, every global solution $x$ satisfies

$$\|x(t)\| \leq \max \{\beta_1(\|x_0\|, t), \beta_2(\|D\|_{L^\infty([0,t])} + c)\} \quad \forall t \in \mathbb{R}_+.$$  

System (2) is input-to-state stable if it is input-to-state stable with bias 0.

System (2) has the converging-input-converging-state property if, for all $x_0 \in \mathbb{R}^n$ and all $D \in B$ with $\|D\|_{L^\infty([t,\infty))} \to 0$ as $t \to \infty$, every maximal solution $x$ of (2) is global and satisfies $x(t) \to 0$ as $t \to \infty$. The following lemma shows in particular that if system (2) is input-to-state stable, then it has the converging-input-converging-state property.

**Lemma 4.** Assume that system (2) is input-to-state stable with bias $c \geq 0$ and let $\beta_1$ and $\beta_2$ be as in Definition 3. Then, for all $x_0 \in \mathbb{R}^n$ and all $D \in B$, every global solution $x$ of (2) satisfies

$$\limsup_{t \to \infty} \|x(t)\| \leq \limsup_{t \to \infty} \beta_2(\|D\|_{L^\infty([t,2t])} + c).$$  

We now arrive at the main results on input-to-state stability (proofs of which can be found in [4]).

**Theorem 5.** Let the linear system $(A, B, C)$ be stabilizable and detectable. Assume that (H1) holds. Then, every maximal solution of (2) is global and there exist positive constants $c_1$, $c_2$ and $\varepsilon$ such that, for all $x_0 \in \mathbb{R}^n$ and $D \in B$, every global solution $x$ satisfies

$$\|x(t)\| \leq c_1 e^{-\varepsilon t}\|x_0\| + c_2 \|D\|_{L^\infty([0,t])} \quad \forall t \in \mathbb{R}_+.$$  

In particular, system (2) is input-to-state stable.

**Theorem 6.** Let the linear system $(A, B, C)$ be minimal. Assume that at least one of hypotheses (H2), (H3) or (H4) holds. Then system (2) is input-to-state stable.

In [1] it has been proved, for single-valued $\Phi$ and $D$, that, if (H4) holds, then (2) is input-to-state stable. Therefore, Theorem 6 can be considered as a
generalization of the main result in [1].

In the following corollaries (to Theorems 5 and 6, respectively), we will consider not only nonlinearities satisfying at least one of the conditions (3), (4), (6) and (7) for all arguments \( y \in \mathbb{R}^m \), but also nonlinearities \( \Phi \in \mathcal{U} \) with the property that there exists a set-valued map \( \tilde{\Phi} \in \mathcal{U} \) satisfying at least one of the conditions (3), (4), (6) and (7) and a compact set \( K \subset \mathbb{R}^m \) such that

\[
y \in \mathbb{R}^m \setminus K \implies \Phi(y) \subset \tilde{\Phi}(y).
\]

(10)

In particular, single-input, single-output hysteretic elements can be subsumed by this set-valued formulation provided that the “characteristic diagram” of the hysteresis is contained in the graph of some \( \Phi \in \mathcal{U} \).

**Corollary 7.** Let the linear system \( (A, B, C) \) be stabilizable and detectable. Let \( \Phi \in \mathcal{U} \) be such that there exist a set-valued map \( \tilde{\Phi} \in \mathcal{U} \) and a compact set \( K \subset \mathbb{R}^m \) such that (10) holds. Assume that (H1) holds with \( \Phi \) replaced by \( \tilde{\Phi} \). Then, every maximal solution of (2) is global and there exist positive constants \( c_1, c_2 \) and \( \varepsilon \) such that, for all \( x^0 \in \mathbb{R}^n \) and \( D \in \mathbb{B} \), every global solution \( x \) satisfies

\[
\|x(t)\| \leq c_1 e^{-\varepsilon t}\|x^0\| + c_2 (\|D\|_{L^\infty[0,t]} + E) \quad \forall t \in \mathbb{R}_+,
\]

where

\[
E := \sup_{y \in K} \sup_{v \in \Phi(y)} \inf_{v \in \tilde{\Phi}(y)} \|v - \tilde{v}\|.
\]

(11)

In particular, system (2) is input-to-state stable with bias \( E \).

**Corollary 8.** Let the linear system \( (A, B, C) \) be minimal and let \( \Phi \in \mathcal{U} \) be such that there exist a set-valued map \( \tilde{\Phi} \in \mathcal{U} \) and a compact set \( K \subset \mathbb{R}^m \) such that (10) holds. Assume that at least one of the hypotheses (H2), (H3) or (H4) holds with \( \Phi \) replaced by \( \tilde{\Phi} \). Then system (2) is input-to-state stable with bias \( E \) given by (11).

### 4 Hysteretic feedback systems

We return to the feedback interconnection of Figure 1, but now in a single-input, single-output setting and with a hysteretic operator \( F \) in the feedback path. An operator \( F : C(\mathbb{R}_+) \to C(\mathbb{R}_+) \) is a hysteresis operator if it is causal and rate independent. Here rate independence means that \( F(y \circ \zeta) = (Fy) \circ \zeta \) for every \( y \in C(\mathbb{R}_+) \) and every time transformation \( \zeta \), where \( \zeta : \mathbb{R}_+ \to \mathbb{R}_+ \) is said to be a time transformation if it is continuous, non-decreasing and surjective. Conditions on \( F \) which ensure well-posedness of the feedback interconnection (existence and uniqueness of solutions of the associated initial-value problem) are expounded in, for example, [8] and [9]. The so-called Preisach operators are among the most general and most important hysteretic operators: in particular, they can model complex hysteresis effects such as nested loops in input-output characteristics. Therefore, and for clarity of presentation, we focus on the class of Preisach operators.

A basic building block for these operators is the backlash operator. A discussion
of the backlash operator (also called play operator) can be found in a number of references, see for example [2], [6] and [7]. Let $\sigma \in \mathbb{R}_+$ and introduce the function $b_\sigma : \mathbb{R}^2 \to \mathbb{R}$ given by

$$b_\sigma(v_1,v_2) := \max \{v_1 - \sigma, \min\{v_1 + \sigma, v_2\}\} = \begin{cases} v_1 - \sigma, & \text{if } v_1 < v_1 - \sigma \\ v_2, & \text{if } v_2 \in [v_1 - \sigma, v_1 + \sigma] \\ v_1 + \sigma, & \text{if } v_2 > v_1 + \sigma. \end{cases}$$

Let $C_{pm}(\mathbb{R}_+)$ denote the space of continuous piecewise monotone functions defined on $\mathbb{R}_+$. For all $\sigma \in \mathbb{R}_+$ and $\xi \in \mathbb{R}$, define the operator $B_{\sigma,\xi} : C_{pm}(\mathbb{R}_+) \to C(\mathbb{R}_+)$ by

$$B_{\sigma,\xi}(y)(t) = \begin{cases} b_\sigma(y(0),\xi) & \text{for } t = 0, \\ b_\sigma(y(t),(B_{\sigma,\xi}(u))(t_i)) & \text{for } t_i < t \leq t_{i+1}, i = 0, 1, 2, \ldots, \end{cases}$$

where $0 = t_0 < t_1 < t_2 < \ldots$, $\lim_{n \to \infty} t_n = \infty$ and $u$ is monotone on each interval $[t_i, t_{i+1}]$. We remark that $\xi$ plays the role of an “initial state”. It is not difficult to show that the definition is independent of the choice of the partition $(t_i)$. Figure 2 illustrates how $B_{\sigma,\xi}$ acts. It is well-known that $B_{\sigma,\xi}$ extends to a

![Figure 2. Backlash hysteresis](image)

Lipschitz continuous hysteresis operator on $C(\mathbb{R}_+)$ (with Lipschitz constant $L = 1$), the so-called backlash operator, which we shall denote by the same symbol $B_{\sigma,\xi}$.

Let $\xi : \mathbb{R}_+ \to \mathbb{R}$ be a compactly supported and globally Lipschitz function with Lipschitz constant 1. Let $\mu$ be a regular signed Borel measure on $\mathbb{R}_+$. Denoting Lebesgue measure on $\mathbb{R}$ by $L$, let $w : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ be a locally $(\mu_L \otimes \mu)$-integrable function and let $w_0 \in \mathbb{R}$. The operator $P_\xi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ defined by

$$(P_\xi(y))(t) = \int_0^\infty \int_0^\infty (B_{\sigma,\xi}(y))(t) w(s,\sigma)\mu_L(ds)\mu(d\sigma) + w_0$$

$$\forall u \in C(\mathbb{R}_+), \forall t \in \mathbb{R}_+, \quad (12)$$

is called a Preisach operator, cf. [2, p. 55]. It is well-known that $P_\xi$ is a hysteresis operator (this follows from the fact that $B_{\sigma,\xi}(u)$ is a hysteresis operator for every $\sigma \geq 0$). Under the assumption that the measure $\mu$ is finite and $w$ is essentially bounded,
the operator \( P_\xi \) is Lipschitz continuous with Lipschitz constant \( L = |\mu|(\mathbb{R}_+)||w||_\infty \) (see [7]) in the sense that

\[
\sup_{t \in \mathbb{R}_+} |P_\xi(y_1)(t) - P_\xi(y_2)(t)| \leq L \sup_{t \in \mathbb{R}_+} |y_1(t) - y_2(t)| \quad \forall y_1, y_2 \in C(\mathbb{R}_+).
\]

This property ensures the well-posedness of the feedback interconnection.

Setting \( w(\cdot, \cdot) = 1 \) and \( w_0 = 0 \) in (12), we obtain the Prandtl operator \( P_\xi : C(\mathbb{R}_+) \to C(\mathbb{R}_+) \) defined by

\[
P_\xi(y)(t) = \int_0^\infty (B_{\sigma, \xi(\sigma)}(y))(t) \mu(d\sigma) \quad \forall u \in C(\mathbb{R}_+), \quad \forall t \in \mathbb{R}_+.
\]

(13)

For \( \xi(\cdot) = 0 \) and \( \mu \) given by \( \mu(E) = \int_E \chi_{[0,5]}(\sigma) \, d\sigma \) (where \( \chi_{[0,5]} \) denotes the indicator function of the interval \([0, 5]\)), the Prandtl operator (13) is illustrated in Figure 3. The next proposition identifies conditions under which the Preisach operator (12)

satisfies a generalized sector bound. For simplicity, we assume that the measure \( \mu \) and the function \( w \) are non-negative (an important case in applications), although the proposition can be extended to signed measures \( \mu \) and sign-indefinite functions \( w \).

**Proposition 9.** Let \( P_\xi \) be the Preisach operator defined in (12). Assume that the measure \( \mu \) is non-negative, \( a_1 := \mu(\mathbb{R}_+) < \infty \) and \( a_2 := \int_0^\infty \sigma \mu(d\sigma) < \infty \). Furthermore, assume that

\[
b_1 := \inf_{(s, \sigma) \in \mathbb{R} \times \mathbb{R}_+} w(s, \sigma) \geq 0, \quad b_2 := \sup_{(s, \sigma) \in \mathbb{R} \times \mathbb{R}_+} w(s, \sigma) < \infty
\]

and set

\[
a_P := a_1 b_1, \quad b_P := a_1 b_2, \quad c_P := a_2 b_2 + |w_0|.
\]

(14)

Then, for all \( y \in C(\mathbb{R}_+) \) and all \( t \in \mathbb{R}_+ \),

\[
y(t) \geq 0 \quad \Rightarrow \quad a_P y(t) - c_P \leq (P_\xi(y))(t) \leq b_P y(t) + c_P,
\]

(15)
\[ y(t) \leq 0 \implies b_p y(t) - c_P \leq (P_\xi(y))(t) \leq a_p y(t) + c_P, \quad (16) \]

Furthermore, for every \( \eta > 0 \),
\[ |y(t)| \geq c_P / \eta \implies (a_p - \eta)y^2(t) \leq (P_\xi(y))(t)y(t) \leq (b_p + \eta)y^2(t). \]

Let \( P_\xi \) be a Preisach operator, defined as in (12), satisfying the hypotheses of Proposition 9. Let \( a_p, b_p \) and \( c_P \) be given by (14) and define \( \Phi, \tilde{\Phi} \in \mathcal{U} \) by
\[
\Phi(y) := \left\{ v \in \mathbb{R} \mid a_p y - c_P \leq v \leq b_p y + c_P \right\}, \quad y \geq 0
\]
\[
\Phi(y) := \left\{ v \in \mathbb{R} \mid b_p y - c_P \leq v \leq a_p y + c_P \right\}, \quad y < 0.
\]

\[ \tilde{\Phi}(y) := \left\{ v \in \mathbb{R} \mid (a_p - \eta)y^2 \leq vy \leq (b_p + \eta)y^2 \right\}, \]
where \( \eta > 0 \). In view of (15) and (16),
\[ y \in C(\mathbb{R}_+) \implies (P_\xi(y))(t) \in \Phi(y(t)) \quad \forall \ t \in \mathbb{R}_+.
\]

Moreover, writing \( K := [-c_P / \eta, \ c_P / \eta] \), we have
\[ \Phi(y) \subset \tilde{\Phi}(y) \quad \forall \ y \in \mathbb{R} \setminus K \quad \text{and} \quad E := \sup_{y \in \mathbb{R}} \sup_{v \in \Phi(y)} \inf_{\tilde{v} \in \tilde{\Phi}(y)} |v - \tilde{v}| = c_P.
\]

Let the linear system \( (A, B, C) \) (with transfer function \( G \)) be stabilizable and detectable. Write \( a := a_p - \eta, \ b := b_p + \eta \) and assume that \( G/(1 + aG) \in H^\infty \) and, for some \( \delta > 0 \), \( (1 + bG)/(1 + aG - \delta) \) is positive real. Then hypothesis (H1) holds with \( m = 1 \) and \( \tilde{\Phi} \) replacing \( \Phi \). We are now in a position to invoke Corollary 7 to conclude properties of solutions of the single-input, single-output, functional differential equation
\[
\dot{x}(t) = Ax(t) + B[d(t) - (P_\xi(Cx))(t)], \quad x(0) = x^0. \quad (17)
\]

We reiterate that, for each \( x^0 \in \mathbb{R}^n \) and \( d \in L^\infty_{\text{loc}}(\mathbb{R}_+) \), (17) has unique global solution. An application of Corollary 7 (with \( D(t) = \{d(t)\} \) for all \( t \in \mathbb{R}_+ \)) yields the existence of constants \( \varepsilon, c_1, c_2 > 0 \) such that, for every global solution \( x \),
\[
\|x(t)\| \leq c_1 e^{-\varepsilon t}\|x^0\| + c_2 \left( \|d\|_{L^\infty[0,t]} + c_P \right) \quad \forall \ t \in \mathbb{R}_+,
\]
showing in particular that (17) is input-to-state stable with bias \( c_P \). Furthermore, by Lemma 4,
\[ \lim_{t \to \infty} d(t) = 0 \implies \limsup_{t \to \infty} \|x(t)\| \leq c_2 c_P. \]
Bibliography


