NECESSARY AND SUFFICIENT LYAPUNOV-LIKE CONDITIONS FOR ROBUST NONLINEAR STABILIZATION

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Abstract
In this work, we propose a methodology for the expression of necessary and sufficient Lyapunov-like conditions for the existence of stabilizing feedback laws. The methodology is an extension of the well-known Control Lyapunov Function (CLF) method and can be applied to very general nonlinear time-varying systems with disturbance and control inputs, including both finite- and infinite-dimensional systems.

Keywords: Control Lyapunov Function, stabilization, time-varying systems, nonlinear control.

1. Introduction

Feedback stabilization of nonlinear systems is a fundamentally important problem in control theory and practice. The purpose of this paper is to look at this problem from a control Lyapunov function point of view, but for a wide class of nonlinear time-varying systems. We aim to develop a methodology that not only results in necessary and sufficient conditions for robust feedback stabilization, but provides novel tools for the design of robust nonlinear controllers. To add to the generality of this framework, we will address partial stability with respect to output variables, instead of state variables. We first consider finite-dimensional nonlinear systems, and then show that the same methodology can be adapted to infinite-dimensional systems described by retarded functional differential equations. Specifically, we begin with finite-dimensional control systems in the general form:

\begin{equation}
\dot{x}(t) = f(t, d(t), x(t), u(t)), \quad Y(t) = H(t, x(t))
\end{equation}

where the vector fields \( f : \mathbb{R}^+ \times D \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n \), \( H : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^k \) are continuous with \( f(t, d, 0, 0) = 0 \), \( H(t, 0) = 0 \) for all \((t, d) \in \mathbb{R}^+ \times D\). And we ask the following question of feedback stabilizability: Under what conditions there exists a continuous feedback of the form:

\begin{equation}
u = k(t, x)
\end{equation}

such that the closed-loop system (1.1) with (1.2) is (uniformly) Robustly Globally Asymptotically Output Stable? See Section 2.1 for a precise definition.

The above-mentioned problem has been studied by several authors in past literature for a subclass of nonlinear control systems (1.1). For instance, in his pioneering work [1] Artstein studied the above existence problem for affine autonomous control systems without disturbances, \( U \subseteq \mathbb{R}^m \) being a closed convex set and output \( Y \) being identically the state of the system, i.e., \( H(t, x) = x \) (see also [24]). Sontag [21] extended the results by presenting an explicit formula of the feedback stabilizer for affine autonomous control systems without disturbances, \( U = \mathbb{R}^m \) and output \( Y \) being identically the state of the system. Sontag’s formula was exploited recently in [8] for the uniform
stabilization of time-varying systems. Freeman and Kokotovic in [3] extended the idea of the CLF in order to study affine control systems with disturbances, $U \subseteq \mathbb{R}^m$ being a closed convex set and output $Y$ being identically the state of the system, i.e., $H(t, x) = x$; they introduced the concept of the Robust Control Lyapunov Function (RCLF). In [9] the authors showed that the “small-control” property is not needed for non-uniform in time robust global stabilization of the state $(H(t, x) = x)$ of control systems affine in the control with $U = \mathbb{R}^m$. The result was extended in [12] for the general case of output stability. In all the above approaches the stabilizing feedback is constructed using a partition of unity methodology or Michael’s Theorem (e.g., [3, 21] when simple continuity of the feedback suffices). Control Lyapunov Functions have also been used for the design of discontinuous feedback laws (see for instance [2]), the design of static output feedback stabilizers (see [12, 25]), as well as for the design of adaptive nonlinear controllers (see [17, 20]).

However, so far the method of Lyapunov design of stabilizing feedback laws is more frequently applied to finite-dimensional systems of the form (1.1). In order to be able to extend the applicability of the method to infinite-dimensional systems of the form $\dot{x} = f(t, d, x, u)$ where the state $x$ belongs to an infinite-dimensional normed linear space $X$, one has to deal with Control Lyapunov Functionals $V : \mathbb{R}^+ \times X \rightarrow \mathbb{R}^+$, which present one (or many) of the following complications:

(i) In contrast to CLF in the finite-dimensional case, usually Control Lyapunov Functionals are simply locally Lipschitz mappings of the state (and, not necessarily, continuously differentiable);

(ii) Even if the mapping $f$ is affine in $u$, the (appropriate) derivative of the Control Lyapunov Functional $\dot{V}(t, d, x, u)$ is not necessarily affine in $u$;

(iii) The existing feedback construction methodology based on partition of unity arguments (see, e.g., [1, 24]) does not work because the state space $X$ is infinite-dimensional;

(iv) The feedback construction methodology based on Michael’s Theorem (see, e.g., [3]) does not work either because simple continuity of the feedback does not suffice or because the hypotheses of Michael’s Theorem cannot be verified.

Particularly, all of the above complications are encountered when control systems described by Retarded Functional Differential Equations (RFDEs) are studied, i.e., systems of the form

$$
\dot{x}(t) = f(t, d(t), T_r(t)x, u(t)), \quad Y(t) = H(t, T_r(t)x)
$$

$$
x(t) \in \mathbb{R}^n, Y(t) \in Y, d(t) \in D, u(t) \in U
$$

(1.3)

where $r > 0$ is a constant, $f : \mathbb{R}^+ \times D \times C^0([-r, 0]; \mathbb{R}^n) \times U \rightarrow \mathbb{R}^n$, $H : \mathbb{R}^+ \times C^0([-r, 0]; \mathbb{R}^n) \rightarrow Y$ satisfy $f(t, d, 0, 0) = 0$, $H(t, 0) = 0$ for all $(t, d) \in \mathbb{R}^+ \times D$, $D \subseteq \mathbb{R}^l$ is a non-empty compact set, $U \subseteq \mathbb{R}^m$ is a closed convex set with $0 \in U$, $Y$ is a normed linear space and $T_r(t)x = x(t + \theta); \theta \in [-r, 0]$.

It should be emphasized that by allowing the output to take values in an abstract normed linear spaces we are in a position to consider:

- outputs with no delays, e.g. $Y(t) = h(t, x(t))$ with $Y = \mathbb{R}^k$,

- outputs with discrete or distributed delay, e.g. $Y(t) = h(x(t), x(t - \tau))$ or $Y(t) = \int_{t-\tau}^t h(t, \theta, x(\theta))d\theta$ with $Y = \mathbb{R}^k$,

- functional outputs with memory, e.g. $Y(t) = h(t, \theta, x(t + \theta)); \theta \in [-r, 0]$ or the identity output $Y(t) = T_r(t)x = x(t + \theta); \theta \in [-r, 0]$ with $Y = C^0([-r, 0]; \mathbb{R}^k)$.

Control Lyapunov functions and functionals for control systems described by RFDEs have been used in [5, 6, 7, 19]. As the second contribution of the present work, we show how all complications mentioned above for infinite-dimensional systems can be solved, and consequently we obtain Lyapunov-like necessary and sufficient conditions for systems of the form (1.3). Since the methodology that we describe in the present work allows the construction of locally Lipschitz stabilizing feedback laws, it is expected that it can be used for general infinite-dimensional control systems.

All proofs of the results of the present work are provided in [16].
2. Finite-Dimensional Control Systems

In this section, we consider control systems of the form (1.1) under the following hypotheses:

(H1) The vector fields $f: \mathbb{R}^n \times D \times \mathbb{R}^n \times U \to \mathbb{R}^n$, $H: \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^k$ are continuous and for every bounded interval $I \subset \mathbb{R}^+$ and every compact set $S \subset \mathbb{R}^n \times U$ there exists $L \geq 0$ such that $|f(t, d, x, u) - f(t, d, y, v)| \leq L|x - y| + L|u - v|$ for all $(t, d) \in I \times D$, $(x, u) \in S$, $(y, v) \in S$ (i.e., the mapping $\mathbb{R}^+ \times D \times \mathbb{R}^n \times U \to (t, d, x, u) \to f(t, d, x, u) \in \mathbb{R}^n$ is locally Lipschitz with respect to $(x, u)$),
(H2) the set \( D \subset \mathbb{R}^I \) is compact and \( U \subset \mathbb{R}^n \) is a closed convex set,

(H3) \( f(t,d,0,0) = 0, H(t,0) = 0 \) for all \((t,d) \in \mathbb{R}^+ \times D\).

In order to present the main results on finite-dimensional systems of the form (1.1) we need to present in detail the basic steps of the method. The methodology consists of the following steps:

2.I. Notions of Output Stability
2.II. Lyapunov-like criteria for Output stability
2.III. Definition of the Output Robust Control Lyapunov Function
2.IV. Converse Lyapunov theorems for output stability

2.I. Notions of Output Stability

We first analyze the output stability notions used in the present work. Consider the system

\[
\begin{align*}
\dot{x}(t) &= f(t,d(t),x(t)) \quad , \quad Y(t) = H(t,x(t)) \\
x(t) &\in \mathbb{R}^n , \quad d(t) \in D , \quad Y(t) \in \mathbb{R}^k
\end{align*}
\]

(2.1)

where the vector fields \( f : \mathbb{R}^+ \times D \times \mathbb{R}^n \rightarrow \mathbb{R}^n \), \( H : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^k \) are continuous and \( D \subset \mathbb{R}^I \) is compact. We assume that for every \((t_0,x_0,d) \in \mathbb{R}^+ \times \mathbb{R}^n \times M_D\) there exist \( h \in (0,\infty] \) and a unique absolutely continuous mapping \( x : [t_0,t_0+h) \rightarrow \mathbb{R}^n \) with \( x(t_0) = x_0 \) and \( \dot{x}(t) = f(t,d(t),x(t)) \) a.e. for \( t \in [t_0,t_0+h) \). Moreover, we assume that \( f(t,d,0) = 0, H(t,0) = 0 \) for all \((t,d) \in \mathbb{R}^+ \times D\). The solution \( x : [t_0,t_0+h) \rightarrow \mathbb{R}^n \) of (2.1) at time \( t \geq t_0 \) with initial condition \( x(t_0) = x_0 \) corresponding to input \( d \in M_D \) will be denoted by \( x(t,t_0,x_0;d) \).

Definition 2.1: We say that (2.1) is Robustly Forward Complete (RFC) if for every \( T \geq 0 \), \( r \geq 0 \) it holds that:

\[
\sup\{\|x(t_0 + h,t_0,x_0;d)\| : \|x_0\| \leq r, t_0 \in [0,T], d(\cdot) \in M_D\} < +\infty
\]

(2.2)

Clearly, the notion of robust forward completeness implies the standard notion of forward completeness, which simply requires that for every initial condition the solution of the system exists for all times greater than the initial time, or equivalently, the solutions of the system do not present finite escape time. Conversely, an extension of Proposition 5.1 in [18] to the time-varying case shows that every forward complete system (2.1) whose dynamics are locally Lipschitz with respect to \((t,x)\), uniformly in \( d \in D\), is RFC. All output stability notions used in the present work will assume RFC.

We continue with the notion of (non-uniform in time) Robust Global Asymptotic Output Stability (RGAOS) as a generalization of the notion of Robust Output Stability (see [10,11]). Let us denote by \( Y(t) = H(t,x(t_0,x_0;d)) \) the value of the output for the unique solution of (2.1) at time \( t \) that corresponds to input \( d \in M_D \) with initial condition \( x(t_0) = x_0 \).

Definition 2.2: Consider system (2.1) and suppose that (2.1) is RFC. We say that system (2.1) is (non-uniformly in time) Robustly Globally Asymptotically Output Stable (RGAOS) if it satisfies the following properties:

P1(Output Stability) For every \( \varepsilon > 0 \), \( T \geq 0 \), it holds that

\[
\sup\{\|Y(t)\| : t \geq t_0, \|x_0\| \leq \varepsilon, t_0 \in [0,T], d(\cdot) \in M_D\} < +\infty
\]

and there exists a \( \delta : = \delta(\varepsilon,T) > 0 \) such that:

\[
\|x_0\| \leq \delta, t_0 \in [0,T] \Rightarrow Y(t) \leq \varepsilon, \forall t \geq t_0, \forall d(\cdot) \in M_D
\]

P2(Uniform Output Attractivity on compact sets of initial data) For every \( \varepsilon > 0 \), \( T \geq 0 \) and \( R \geq 0 \), there exists a \( \tau : = \tau(\varepsilon,T,R) \geq 0 \) such that:
\[ |x_0| \leq R, t_0 \in [0, T] \Rightarrow |y(t)| \leq \varepsilon, \forall t \geq t_0 + \tau, \forall d(\cdot) \in M_D \]

The notion of Uniform Robust Global Asymptotic Output Stability was given in [22,23] and is a special case of (non-uniform in time) RGAOS.

**Definition 2.3:** Consider system (2.1) and suppose that (2.1) is RFC. We say that system (2.1) is **Uniformly Robustly Globally Asymptotically Output Stable (URGAOS)** if it satisfies the following properties:

**P1 (Uniform Output Stability)** For every \( \varepsilon > 0 \), it holds that
\[
\sup_{t \geq t_0} \left| y(t) \right| \leq \varepsilon, \quad \forall t \geq t_0, \quad \forall d(\cdot) \in M_D
\]
and there exists a \( \delta = \delta(\varepsilon) > 0 \) such that:
\[
|x_0| \leq \delta, t_0 \geq 0 \Rightarrow |y(t)| \leq \varepsilon, \quad \forall t \geq t_0, \quad \forall d(\cdot) \in M_D
\]

**P2 (Uniform Output Attractivity on compact sets of initial states)** For every \( \varepsilon > 0 \) and \( R \geq 0 \), there exists a \( \tau = \tau(\varepsilon, R) \geq 0 \), such that:
\[
|x_0| \leq R, t_0 \geq 0 \Rightarrow |y(t)| \leq \varepsilon, \quad \forall t \geq t_0 + \tau, \quad \forall d(\cdot) \in M_D
\]

Obviously, for the case \( H(t, x) = x \) the notions of RGAOS and URGAOS coincide with the notions of **non-uniform in time Robust Global Asymptotic Stability (RGAS)** as given in [9] and **Uniform Robust Global Asymptotic Stability (URGAS)** as given in [18], respectively. Also note that if there exists \( a \in K_\infty \) with \( |x| \leq a(H(t, x)) \) for all \((t, x) \in \mathbb{R}^+ \times \mathbb{R}^n\), then (U)RGAOS implies (U)RGAS.

### 2.II. Lyapunov-like Criteria for Output Stability

For a locally bounded function \( V: \mathbb{R} \to \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R} \), we define
\[
V^0(t, x, v) := \limsup_{h \to 0^+, v \to 0} \frac{V(t + h, x + hv) - V(t, x)}{h} \tag{2.3}
\]

The reader should notice that the function \((t, x, v) \to V^0(t, x, v)\) may take values in the extended real number set \( \mathbb{R}^\ast = [-\infty, +\infty] \). However, for locally Lipschitz functions \( V: \mathbb{R}^\ast \times \mathbb{R}^n \rightarrow \mathbb{R} \), the function \((t, x, v) \to V^0(t, x, v)\) is locally bounded. It should be clear that for locally Lipschitz functions \( V: \mathbb{R}^\ast \times \mathbb{R}^n \rightarrow \mathbb{R} \) it holds that:
\[
V^0(t, x, v) = \limsup_{h \to 0^+} \frac{V(t + h, x + hv) - V(t, x)}{h} \tag{2.4}
\]

The main reason for introducing the above Dini derivative is the following lemma.

**Lemma 2.4:** Let \( V: \mathbb{R}^\ast \times \mathbb{R}^n \rightarrow \mathbb{R} \) be a locally bounded function and let \( x: [t_0, t_{\text{max}}] \to \mathbb{R}^n \) be a solution of (2.1) with initial condition \( x(t_0) = x_0 \in \mathbb{R}^n \) corresponding to certain \( d \in M_D \), where \( t_{\text{max}} \in (t_0, +\infty) \) is the maximal existence time of the solution. Then it holds that
\[
\limsup_{h \to 0^+} \frac{|V(t + h, x(t + h) - V(t, x(t))|}{h} \leq V^0(t, x; D^x(t)), \text{ a.e. on } [t_0, t_{\text{max}}] \tag{2.5}
\]
where \( D^x(t) = \lim_{h \to 0^+} \frac{1}{h} (x(t + h) - x(t)) \).

Having introduced an appropriate derivative for Lyapunov functions, we are now in a position to give Lyapunov-like criteria for RGAOS and URGAOS.
Proposition 2.5: Consider system (2.1) and the following statements:

(Q1) There exist a locally Lipschitz function $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$, functions $a_1, a_2 \in K_\infty$, $\beta, \mu \in K_+$, a function $q \in \mathcal{E}$ and a $C^0$ positive definite function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$a_1(\|\mu(t)x, H(t,x)\|) \leq V(t,x) \leq a_2(\beta(t)|x|), \quad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$$

(2.6)

and such that the following inequality holds for all $(t,x,d) \in \mathbb{R}^+ \times \mathbb{R}^n \times D$:

$$V^0(t,x; f(t,d,x), q(t) \leq -\rho(V(t,x))$$

(2.7)

(Q2) Hypothesis (Q1) holds with $\beta(t) = 1$, $q(t) = 0$.

(Q3) The mapping $\mathbb{R}^+ \times D \times \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz with respect to $x \in \mathbb{R}^n$.

If hypotheses (Q1), (Q3) hold then system (2.1) is RGAOS. If hypothesis (Q2) holds, then system (2.1) is URGAS.

2.11. Definition of the Output Robust Control Lyapunov Function

We next give the definition of the Output Robust Control Lyapunov Function for system (1.1). The definition is in the same spirit with the definition of the notion of Robust Control Lyapunov Function given in [3] for continuous-time finite-dimensional control systems. The small-control property in the following definition constitutes a time-varying version of the small-control property for the autonomous case [1,3,21].

Definition 2.6: We say that (1.1) admits an Output Robust Control Lyapunov Function (ORCLF) if there exists a locally Lipschitz function $\Psi : \mathbb{R}^+ \times \mathbb{R}^n \times U \to \mathbb{R} \cup \{\infty\}$ with $\Psi(t,0,0) = 0$ for all $t \geq 0$ such that for each $u \in U$ the mapping $(t,x) \to \Psi(t,x,u)$ is upper semi-continuous, a function $q \in \mathcal{E}$ and a $C^0$ positive definite function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ such that the following inequality holds:

$$\inf_{u \in U} \Psi(t,x,u) \leq q(t), \quad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$$

(2.8)

Moreover, for every finite set $\{u_1, u_2, \ldots, u_p\} \subset U$ and for every $\lambda_i \in [0,1]$ $(i = 1, \ldots, p)$ with $\sum_{i=1}^p \lambda_i = 1$, it holds that:

$$\sup_{d \in D} \left\{ t, x \in [0,1] \right\} \left\{ f(t,d,x, \sum_{i=1}^p \lambda_i u_i) \right\} \leq -\rho(V(t,x)) + \max_{i=1,...,p} \left\{ \Psi(t,x,u_i) \right\}, \quad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$$

(2.9)

If in addition to the above there exist $\gamma \in K_\infty$, $\gamma \in K_+$ such that for every $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$ there exists $u \in U$ with $|u| \leq a(\gamma(t)|x|)$ such that

$$\Psi(t,x,u) \leq q(t)$$

(2.10)

then we say that $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ satisfies the “small-control” property.

For the case $H(t,x) = x$ we simply call $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ a State Robust Control Lyapunov Function (SRCLF).
2.IV. Converse Lyapunov theorems for output stability

In this section we are going to exploit the converse Lyapunov theorem for RGAOS presented in [11].

2.V. Main Results

We are now ready to state our main results for the finite-dimensional case (1.1).

Theorem 2.8: Consider system (1.1) under hypotheses (H1-3). The following statements are equivalent:

(a) There exists a $C^\infty$ function $k : \mathbb{R}^+ \times \mathbb{R}^n \to U$ with $k(t,0) = 0$ for all $t \geq 0$, in such a way that the closed-loop system (1.1) with $u = k(t,x)$ is RGAOS.

(b) There exists a $C^0$ function $k : \mathbb{R}^+ \times \mathbb{R}^n \to U$ with $f(t,d,x,k(t,x))$ being locally Lipschitz with respect to $x$ and $f(t,d,0,k(t,0)) = 0$ for all $(t,d) \in \mathbb{R}^+ \times D$, such that the closed-loop system (1.1) with $u = k(t,x)$ is RGAOS.

(c) System (1.1) admits an ORCLF, which satisfies the small control property with $q(t) = 0$.

(d) System (1.1) admits an ORCLF.

Theorem 2.9: Consider system (1.1) under hypotheses (H1-3). If system (1.1) admits an ORCLF, which satisfies the small-control property and inequalities (2.6), (2.10) with $\beta(t) = 1$, $q(t) = 0$, then there exists a continuous mapping $k : \mathbb{R}^+ \times \mathbb{R}^n \to U$ with $k(t,0) = 0$ for all $t \geq 0$, which is $C^\infty$ on the set $\mathbb{R}^+ \times (\mathbb{R}^n \setminus \{0\})$, such that

i) for all $(t_0,x_0,d) \in \mathbb{R}^+ \times \mathbb{R}^n \times M_D$ the solution $x(t)$ of the closed-loop system (1.1) with $u = k(t,x)$, i.e. the solution of

$$\dot{x}(t) = f(t,d(t),x(t),k(t,x(t)))$$

(2.11)

with initial condition $x(t_0) = x_0 \in \mathbb{R}^n$, corresponding to input $d \in M_D$ is unique,

ii) system (2.11) is URGAOS.

Moreover, if the ORCLF $V$ and the function $\Psi$ involved in property (ii) of Definition 2.6 are time independent then the continuous mapping $k$ is time invariant. Finally, if in addition there exist functions $\eta \in K^+, \varphi \in C^\infty(A;U)$ where $v \in \{1,2,\ldots\}$, $A = \cup_{t \in \mathbb{R}^+} \{x \in \mathbb{R}^n \mid x < 4\eta(t)\}$ with $\varphi(t,0) = 0$ for all $t \geq 0$, such that

$$\Psi(t,x,\varphi(t,x)) \leq 0, \text{ for all } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n \text{ with } |x| \leq 2\eta(t)$$

(2.12)

then the continuous mapping $k$ is of class $C^\infty(\mathbb{R}^+ \times \mathbb{R}^n;U)$.

3. Extensions to Systems Described by Retarded Functional Differential Equations

In this section we extend the methodology presented in Section 2, to infinite-dimensional systems described by Retarded Functional Differential Equations (RFDEs). Particularly, we consider control systems of the form (1.3) under the following hypotheses:

(S1) The mapping $(x,u,d) \to f(t,d,x,u)$ is continuous for each fixed $t \geq 0$ and such that for every bounded $I \subseteq \mathbb{R}^+$ and for every bounded $S \subseteq C^0([-r,0];\mathbb{R}^n) \times U$, there exists a constant $L \geq 0$ such that:
\[
(x(0) - y(0))' \left( f(t, d, x, u) - f(t, y, y, u) \right) \leq L \max_{\tau \in [-r, 0]} \| x(t) - y(t) \|^2 = L \| x - y \|^2
\]

\forall t \in I, \forall (x, u, y, u) \in S \times S, \forall d \in D

Hypothesis (S1) is equivalent to the existence of a continuous non-decreasing function \( L : \mathbb{R}^+ \to \mathbb{R}^+ \), with the following property:

\[
(x(0) - y(0))' \left( f(t, d, x, u) - f(t, d, y, u) \right) \leq L(t + \| x \| + \| y \| + \| u \|) \| x - y \|^2
\]

\forall (t, x, y, d, u) \in \mathbb{R}^+ \times C^0([-r, 0]; \mathbb{R}^n) \times \mathbb{R}^+ \times \mathbb{R}^d \times U

(3.1)

(S2) For every bounded \( \Omega \subset \mathbb{R}^+ \times \mathbb{R}^d \times C^0([-r, 0]; \mathbb{R}^n) \times U \) the image set \( f(\Omega) \subset \mathbb{R}^n \) is bounded.

(S3) There exists a countable set \( A \subset \mathbb{R}^+ \), which is either finite or \( A = \{ t_k ; k = 1, \ldots, \infty \} \) with \( t_{k+1} > t_k > 0 \) for all \( k = 1, 2, \ldots \) and \( \lim_{k \to \infty} t_k = +\infty \), such that mapping \( (t, x, u, d) \in (\mathbb{R}^+ \setminus A) \times C^0([-r, 0]; \mathbb{R}^n) \times U \times D \to f(t, d, x, u) \) is continuous. Moreover, for each fixed \( (t_0, x, u, d) \in \mathbb{R}^+ \times C^0([-r, 0]; \mathbb{R}^n) \times U \times D \), we have

\[
\lim_{t \to t_0^+} f(t, d, x, u) = f(t_0, d, x, u).
\]

(S4) For every \( \epsilon > 0 \), \( t \in \mathbb{R}^+ \), there exists \( \delta = \delta(\epsilon, t) > 0 \) such that

\[
\sup_{t \in \mathbb{R}^+} \left\{ \left| f(t, d, x, u) - f(t, d, x, v) \right| : t, d \in D, u \in U, \| e - t \| + \| x \| + \| u \| < \delta \right\} < \epsilon.
\]

(S5) The mapping \( u \to f(t, d, x, u) \) is Lipschitz on bounded sets, in the sense that for every bounded \( I \subset \mathbb{R}^+ \) and every bounded \( S \subset C^0([-r, 0]; \mathbb{R}^n) \times U \), there exists a constant \( L_U \geq 0 \) such that

\[
\left| f(t, d, x, u) - f(t, d, x, v) \right| \leq L_U \| u - v \|, \forall t \in I, \forall (x, u, x, v) \in S \times S, \forall d \in D
\]

Hypothesis (S5) is equivalent to the existence of a continuous, non-decreasing function \( L_U : \mathbb{R}^+ \to \mathbb{R}^+ \), with the following property:

\[
\left| f(t, d, x, u) - f(t, d, x, v) \right| \leq L_U \left( t + \| x \| + \| y \| \right) \| u - v \|
\]

\forall (t, x, d, u, v) \in \mathbb{R}^+ \times C^0([-r, 0]; \mathbb{R}^n) \times D \times U \times U

(3.2)

(S6): The set \( D \subset \mathbb{R}^I \) is compact and \( U \subset \mathbb{R}^m \) is a closed convex set.

(S7) The mapping \( H(t, x) \) is Lipschitz on bounded sets, in the sense that for every bounded \( I \subset \mathbb{R}^+ \) and for every bounded \( S \subset C^0([-r, 0]; \mathbb{R}^n) \), there exists a constant \( L_H \geq 0 \) such that:

\[
\left\| H(t, x) - H(t, y) \right\| \leq L_H \| x - y \|, \forall t \in I, \forall (x, y) \in S \times S
\]

Following the methodology described in Section 2, we next analyze in detail the following steps of the method:

3.I. Notions of Output Stability
3.II. Lyapunov-like criteria for Output stability
3.III. Definition of the Output Robust Control Lyapunov Functional
3.IV. Converse Lyapunov theorems for output stability

3.I. Notions of Output Stability

We consider uncertain dynamical systems described by RFDEs of the form:

\[
\dot{x}(t) = f(t, d(t), T_r(t)x), \quad y(t) = H(t, T_r(t)x)
\]

\( x(t) \in \mathbb{R}^n, y(t) \in Y, d(t) \in D \)

(3.3)

where \( r > 0 \) is a constant, \( f : \mathbb{R}^+ \times D \times C^0([-r, 0]; \mathbb{R}^n) \to \mathbb{R}^n \), \( H : \mathbb{R}^+ \times C^0([-r, 0]; \mathbb{R}^n) \to \mathbb{R}^d \) satisfy \( f(t, d, 0) = 0 \),
\[ H(t, 0) = 0 \quad \text{for all} \quad (t, d) \in \mathbb{R}^+ \times D, \quad D \subseteq \mathbb{R}^l \quad \text{is a non-empty compact set,} \quad Y \quad \text{is a normed linear space and} \quad T_r(t) = x(t + \theta) ; \quad \theta \in [-r, 0], \quad \text{under the following hypotheses:} \]

\textbf{(Q1)} The mapping \((x, d) \to f(t, d, x)\) is continuous for each fixed \(t \geq 0\) and such that for every bounded \(I \subseteq \mathbb{R}^+\) and for every bounded \(S \subset C^0([-r, 0]; \mathbb{R}^n)\), there exists a constant \(L \geq 0\) such that
\[
\left( x(0) - y(0) \right) \left( f(t, d, x) - f(t, d, y) \right) \leq L \|x - y\|^2 \\
\forall t \in I, \quad \forall (x, y) \in S \times S, \quad \forall \alpha \in D
\]

\textbf{(Q2)} For every bounded \(\Omega \subset \mathbb{R}^+ \times D \times C^0([-r, 0]; \mathbb{R}^n)\) the image set \(f(\Omega) \subset \mathbb{R}^n\) is bounded.

\textbf{(Q3)} There exists a countable set \(A \subset \mathbb{R}^+\), which is either finite or \(A = \{t_k ; k = 1, \ldots, \infty\}\) with \(t_{k+1} > t_k > 0\) for all \(k = 1, 2, \ldots\) and \(\lim t_k = +\infty\), such that mapping \((t, x, d) \in (\mathbb{R}^+ \setminus A) \times C^0([-r, 0]; \mathbb{R}^n) \times D \to f(t, d, x)\) is continuous. Moreover, for each fixed \((t_0, x, d) \in \mathbb{R}^+ \times C^0([-r, 0]; \mathbb{R}^n) \times D\), we have \(\lim \limits_{t \to t_0^+} f(t, d, x) = f(t_0, d, x)\).

\textbf{(Q4)} Hypothesis \((S7)\) holds for the output map.

For systems of the form \((3.3)\) under hypotheses \((Q1-4)\) we adopt the definitions of RGAOS and URGAOS given in [14] for a wide class of deterministic systems with disturbances (see also [4,13] for usual state stability notions).

3.II. Lyapunov-like criteria for Output stability

Let \(x \in C^0([-r, 0]; \mathbb{R}^n)\) and \(V : \mathbb{R}^+ \times C^0([-r, 0]; \mathbb{R}^+)) \to \mathbb{R}\) be a locally bounded functional. By \(E_h(x; \nu)\), where \(0 \leq h < r\) and \(\nu \in \mathbb{R}^n\) we denote the following operator:
\[
E_h(x; \nu) := \begin{cases} 
    x(0) + (\theta + h) \nu & \text{for} \quad -h < \theta \leq 0 \\
    x(\theta + h) & \text{for} \quad -r \leq \theta \leq -h 
\end{cases} 
\] (3.4)
and we define
\[
V^0(t, x; \nu) := \lim_{h \to 0^+} \sup_{\nu \in C^0([-r, 0]; \mathbb{R}^n)} \frac{V(t + h, E_h(x; \nu + \nu h) - V(t, x)}{h} (3.5)
\]

The following lemma presents some elementary properties of the generalized derivative given above. Notice that the function \((t, x, \nu) \to V^0(t, x; \nu)\) may take values in the extended real number set \(\mathbb{R}^* = [-\infty, +\infty]\).

\textbf{Lemma 3.1:} Let \(V : \mathbb{R}^+ \times C^0([-r, 0]; \mathbb{R}^+)) \to \mathbb{R}\) be a locally bounded functional and let \(x \in C^0([t_0 - r, t_{\max}); \mathbb{R}^n)\) a solution of \((3.3)\) under hypotheses \((Q1-4)\) with initial condition \(x(t_0) = x_0 \in C^0([-r, 0]; \mathbb{R}^n)\), corresponding to certain \(d \in M_D\), where \(t_{\max} \in [t_0, +\infty)\) is the maximal existence time of the solution. Then it holds that
\[
\lim_{h \to 0^+} \sup_{x \in C^0([-r, 0]; \mathbb{R}^n)} h^{-1}(V(t + h, T_r(t + h)x) - V(t, T_r(t)x)) \leq V^0(t, T_r(t)x; D^* x(t)) , \text{ a.e. on } [t_0, t_{\max}] (3.6)
\]
where \(D^* x(t) = \lim_{h \to 0^+} h^{-1}(x(t + h) - x(t))\).

An important class of functionals is presented next.

\textbf{Definition 3.2:} We say that a continuous functional \(V : \mathbb{R}^+ \times C^0([-r, 0]; \mathbb{R}^+)) \to \mathbb{R}^+\), is “almost Lipschitz on bounded sets”, if there exist non-decreasing functions \(L_V : \mathbb{R}^+ \to \mathbb{R}^+, \quad P : \mathbb{R}^+ \to \mathbb{R}^+, \quad G : \mathbb{R}^+ \to [1, +\infty)\) such that for all \(R \geq 0\), the following properties hold:
For every $x, y \in \mathbb{R}^n$ with $\|x\|_\infty \leq R$, it holds that:

$$\|V(t, y) - V(t, x)\| \leq L_V(R) \|y - x\|_\infty, \ \forall t \in [0, R]$$

For every absolutely continuous function $x : [-r, 0] \to \mathbb{R}^n$ with $\|x\|_\infty \leq R$ and essentially bounded derivative, it holds that:

$$\|V(t + h, x) - V(t, x)\| \leq h P(R) \left(1 + \sup_{-r \leq \tau \leq 0} \|\dot{x}(\tau)\|\right), \ \forall t \in [0, R] \text{ and } 0 \leq h \leq \frac{1}{G \left(\sup_{-r \leq \tau \leq 0} \|\dot{x}(\tau)\|\right)}$$

The following proposition is based on the results obtained in [15] and provides Lyapunov-like criteria for RGAOS and URGAOS for (3.3).

**Proposition 3.3:** Consider system (3.3) under hypotheses (Q1-4). Suppose that there exist functions $a_1, a_2 \in K_{\infty}$, $\beta, \mu \in K^+$, $q \in \mathfrak{E}$, a positive definite continuous function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ and a mapping $V : \mathbb{R}^+ \times C^0([-r, 0]; \mathbb{R}^n) \to \mathbb{R}^+$, which is almost Lipschitz on bounded sets, such that the following inequalities hold for all $(t, x, d) \in \mathbb{R}^+ \times C^0([-r, 0]; \mathbb{R}^n) \times D:

$$\max\{a_1(\|H(t, x)\|), a_1(\|\dot{x}(\tau)\|)\} \leq V(t, x) \leq a_2(\beta(t)\|x\|)$$

$$V^0(t, x, f(t, d, x)) \leq -\rho(V(t, x)) + q(t)$$

Then system (3.3) is RGAOS. Moreover, if $\beta(t) = 1$ and $q(t) = 0$ then system (3.3) is URGAOS.

### 3.III. Definition of the Output Robust Control Lyapunov Functional

We next give the definition of the Output Robust Control Lyapunov Functional for system (1.3). The definition is in the same spirit with Definition 2.6 of the notion of ORCLF for finite-dimensional control systems.

**Definition 3.4:** We say that (1.3) admits an **Output Robust Control Lyapunov Functional (ORCLF)** if there exists an almost Lipschitz on bounded sets functional $\Psi : \mathbb{R}^+ \times \mathbb{R}^n \times U \to \mathbb{R} \cup \{+\infty\}$ with $\Psi(t, 0, 0) = 0$ for all $t \geq 0$ such that for each $u \in U$ the mapping $(t, \varphi) \to \Psi(t, \varphi, u)$ is upper semi-continuous, a function $q \in \mathfrak{E}$, a continuous mapping $\Phi : \mathbb{R}^+ \times C^0([-r, 0]; \mathbb{R}^n) \to \mathbb{R}^+$ being completely locally Lipschitz with respect to $x \in C^0([-r, 0]; \mathbb{R}^n)$ with $\Phi(t, 0) = 0$ for all $t \geq 0$ and a $C^0$ positive definite function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ such that the following inequality holds:

$$\inf_{u \in U} \Psi(t, \varphi, u) \leq q(t), \ \forall t \geq 0, \forall \varphi = (\varphi_1, \ldots, \varphi_p)^T \in \mathbb{R}^p$$

Moreover, for every finite set $\{u_1, u_2, \ldots, u_N\} \subset U$ and for every $\lambda_i \in [0, 1]$ ($i = 1, \ldots, N$) with $\sum_{i=1}^N \lambda_i = 1$, it holds that:

$$\sup_{d \in D} V^0\left(t, x, f\left(t, d, x, \sum_{i=1}^N \lambda_i u_i\right)\right) \leq -\rho(V(t, x)) + \max_{i=1, \ldots, N} \left[\Psi(t, \Phi(t, x), u_i)\right], \ \forall (t, x) \in \mathbb{R}^+ \times C^0([-r, 0]; \mathbb{R}^n)$$

If in addition to the above there exist $a \in K_{\infty}$, $\gamma \in K^+$ such that for every $(t, \varphi) \in \mathbb{R}^+ \times \mathbb{R}^n$ there exists $u \in U$ with
\[ |p| \leq a(\gamma(t)|\phi|) \] such that

\[ \Psi(t, \phi, u) \leq q(t) \]  \hspace{1cm} (3.11)

then we say that \( V: \mathbb{R}^+ \times C^0([-r,0]; \mathbb{R}^n) \to \mathbb{R}^+ \) satisfies the “small-control” property.

For the case \( H(t,x) = x \in C^0([-r,0]; \mathbb{R}^n) \) we simply call \( V: \mathbb{R}^+ \times C^0([-r,0]; \mathbb{R}^n) \to \mathbb{R}^+ \) a State Robust Control Lyapunov Functional (SRCLF).

3.IV. Converse Lyapunov theorems for output stability

In this work we are going to exploit the converse Lyapunov theorems for RGAOS and URGAOS presented in [15].

3.V. Main Results

We are now in a position to state our main results for the infinite-dimensional case (1.3).

**Theorem 3.5:** Consider system (1.3) under hypotheses (S1-7). The following statements are equivalent:

(a) There exists a continuous mapping \( U: \mathbb{R}^+ \times C^0([-r,0]; \mathbb{R}^n) \to \mathbb{R}^+ \) \( \varphi(t,x) \to k(t,x) \in U \) being completely locally Lipschitz with respect to \( x \in C^0([-r,0]; \mathbb{R}^n) \) with \( k(t,0) = 0 \) for all \( t \geq 0 \), such that the closed-loop system (1.3) with \( u = k(t,T,T_x) \) is RGAOS.

(b) System (1.3) admits an ORCLF, which satisfies the small control property with \( q(t) = 0 \).

(c) System (1.3) admits an ORCLF.

**Theorem 3.6:** Consider system (1.3) under hypotheses (S1-7). The following statements are equivalent:

(a) System (1.3) admits an ORCLF, which satisfies the small-control property and inequalities (3.7), (3.11) with \( \beta(t) = 1, q(t) = 0 \). Moreover, there exist continuous mappings \( \eta \in K^+, A \varphi(t,\varphi) \to K(t,\varphi) \in U \) where \( A = \cup_{t \geq 0} \{t \times \{\varphi \in \mathbb{R}^n : |\varphi| \leq 4\eta(t)\} \) being locally Lipschitz with respect to \( \varphi \) with \( K(t,0) = 0 \) for all \( t \geq 0 \) and such that

\[ \Psi(t, \Phi(t,x), K(t, \Phi(t,x))) \leq 0, \text{ for all } (t,x) \in \mathbb{R}^+ \times C^0([-r,0]; \mathbb{R}^n) \text{ with } |\Phi(t,x)| \leq 2\eta(t) \]  \hspace{1cm} (3.12)

where \( \Phi = (\Phi_1, \ldots, \Phi_p) \) : \( \mathbb{R}^+ \times C^0([-r,0]; \mathbb{R}^n) \to \mathbb{R}^p \) and \( \Psi : \mathbb{R}^+ \times \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) are the mappings involved in property (ii) of Definition 3.4.

(b) There exists a continuous mapping \( \Phi : \mathbb{R}^+ \times C^0([-r,0]; \mathbb{R}^n) \to \mathbb{R}^p \) with \( k(t,0) = 0 \) for all \( t \geq 0 \), such that the closed-loop system (1.3) with \( u = k(t,T,T_x) \) is URGAOS.

**Remark 3.7:** From the proof of Theorem 3.6 it becomes apparent that if statement (a) of Theorem 3.6 is strengthened so that the ORCLF \( V \), the mappings \( \Phi = (\Phi_1, \ldots, \Phi_p) \) : \( \mathbb{R}^+ \times C^0([-r,0]; \mathbb{R}^n) \to \mathbb{R}^p, \Psi \) involved in property (ii) of Definition 3.4 and the mapping \( K : A \to U \) are time independent then the continuous mapping \( k \), whose existence is guaranteed by statement (b) of Theorem 3.6, is time invariant.

4. Conclusions

In the present work we have showed how the well-known “Control Lyapunov Function (CLF)” methodology can be generalized to a broader class of nonlinear time-varying systems with both disturbance and control inputs, which include infinite-dimensional control systems described by retarded functional differential equations (RFDE). Necessary and sufficient conditions for the existence of stabilizing feedback are developed for the non-affine uncertain finite-dimensional case (1.1). The case of uncertain control systems described by RFDEs of the form (1.3)
is studied. It is our belief that the present work can be used as a starting point for the discovery of necessary and sufficient Lyapunov-like conditions for the existence of stabilizing feedback for a wide class of infinite-dimensional control systems.

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References