Convergence in Trace Norm of LQ-Optimal Actuator Locations *

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1 Introduction

Many systems of interest are modelled by partial differential equations. Examples include control of interior noise, particularly in automobiles and in aircraft; and control of structural vibrations and heating processes. In most situations, there is freedom on where to place the actuators. Performance depends strongly on actuator location; see for instance [14].

Actuators should be located at positions that optimize performance. The actuator location problem has been considered by many researchers; see for instance [7, 12, 13]. Here we are interested in linear-quadratic regulators. Letting Π indicate the optimal solution to the regulation problem, and \( x_0 \) the initial condition, the most natural measures of performance are

\[
\max_{\|x_0\|=1} \langle x_0, \Pi x_0 \rangle = \|\Pi\|
\]

which provides a bound on the response to the worst initial condition and

\[
\text{trace}(\Pi \Xi)
\]

which is the expected response if the initial condition is random with zero mean and covariance \( \Xi \). In this paper the issues associated with using approximations in calculating the best location where the initial condition is random are considered. Hence the cost to be minimized is the trace.

In practice, the equations for the optimal control cannot be solved and the control is calculated using an approximation. Conditions for strong convergence for

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the approximating Riccati operator $\Pi_n$, with fixed actuator locations, to converge to the exact operator $\Pi$ are known; see for instance, [1, 9, 10, 11]. However, determining the the optimal actuator location for the optimal control introduces an additional layer of numerical calculation. Strong convergence of $\Pi_n$ is not sufficient to obtain correct results: The optimal cost and corresponding actuator locations of the approximating sequence may not converge. The sequence of Riccati operators needs to converge in a suitable norm to the exact operator. If, in addition to standard approximation assumptions, the input operator $B$ and output operator $C$ are compact operators, the Riccati operators converge uniformly [15]. For analytic semigroups, with additional compactness assumptions, uniform convergence of the Riccati operators can be shown in a general setting that applies to unbounded input operators $B$ and provides convergence estimates [11, Thm. 4.1.4.1,4.5.1.1]. For general semigroups, conditions for convergence in the Hilbert-Schmidt norm of $\Pi_n$ to $\Pi$ are given in [6], but the approximation space needs to lie in the domain of the semigroup generator. This assumption is not satisfied by many finite-element type approximation schemes. Convergence in trace of $\Pi_n$ to $\Pi$ for hereditary differential systems with finite-dimensional input and output operators is shown in [9, Thm. 6.9, Cor. 7.1].

In this paper it is shown that if the input and output spaces are both finite-dimensional, the optimal cost, the trace of the Riccati operator, is continuous with respect to the actuator location. Conditions under which the approximating optimal performance converges to the optimal performance, along with a corresponding sequence of actuator locations are established. The results are illustrated with an example.

## 2 Well-Posedness of Optimal Actuator Problem

Consider systems described by

$$\frac{dz}{dt} = Az(t) + Bu(t), \quad z(0) = z_0$$

(1)

on a Hilbert space $\mathcal{H}$ where $A$ with domain $D(A)$ generates a strongly continuous semigroup $S(t)$ on $\mathcal{H}$ and $B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$ where $\mathcal{U}$ is a Hilbert space.

The linear-quadratic (LQ) controller design objective is to find a control $u(t)$ so that the cost functional

$$J(u, z_0) = \int_0^\infty \langle Cz(t), Cz(t) \rangle + \langle u(t), Ru(t) \rangle dt$$

(2)

is minimized where $R \in \mathcal{L}(\mathcal{U}, \mathcal{U})$ is a self-adjoint positive definite operator weighting the control, $C \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$ (with Hilbert space $\mathcal{Y}$) weights the state, and $z(t)$ is determined by (1).

**Definition 2.1.** The pair $(A, B)$ is stabilizable if there exists $K \in \mathcal{L}(\mathcal{U}, \mathcal{H})$ such that $A - BK$ generates an exponentially stable semigroup.
Definition 2.2. The pair \((C, A)\) is detectable if there exists \(F \in \mathcal{L}(\mathcal{Y}, \mathcal{H})\) such that \(A - FC\) generates an exponentially stable semigroup.

Theorem 2.3. [3, Thm 6.2.4, 6.2.7] If (1) with cost (2) is stabilizable and detectable, then the cost has a minimum for every \(z_0 \in \mathcal{H}\). Furthermore, there exists a self-adjoint non-negative operator \(\Pi \in \mathcal{L}(H, H)\) such that

\[
\min_{u \in L^2(0, \infty; \mathcal{U})} J(u, z_0) = \langle z_0, \Pi z_0 \rangle.
\]

The operator \(\Pi\) is the unique non-negative solution to the operator equation

\[
\langle Az_1, \Pi z_2 \rangle + \langle \Pi z_1, Az_2 \rangle + \langle Cz_1, Cz_2 \rangle - \langle B^* \Pi z_1, R^{-1} B^* \Pi z_1 \rangle = 0 \tag{3}
\]

for all \(z_1, z_2 \in D(A)\). Defining \(K = R^{-1} B^* \Pi\), the corresponding optimal control is \(u = -Kz(t)\) and \(A - BK\) generates an exponentially stable semigroup.

Consider now the situation where there are \(m\) actuators with locations that can be varied over some compact set \(\Omega \subset \mathbb{R}^q\). Parametrize the actuator locations by \(r\) and indicate the corresponding input operator by \(B(r)\). Note that \(r\) is a vector of length \(m\) with components in \(\Omega\) so that \(r\) varies over a space denoted by \(\Omega^m\). For each \(r\) we have an optimal control problem (2) which we indicate by \(J^r(u, z_0)\). If the initial condition is random, with zero mean and variance \(\Xi\) then the expected cost is

\[
\text{trace} (\Xi^\frac{1}{2} \Pi (r) \Xi^\frac{1}{2}),
\]

or since \(\Pi\) is self-adjoint and non-negative, \(\| \Xi^\frac{1}{2} \Pi (r) \Xi^\frac{1}{2} \|_1\) where \(\| \cdot \|_1\) indicates the nuclear norm. Assume for simplicity of exposition that the variance is unity. The performance for a particular \(r\) is \(\mu(r) = \| \Pi(r) \|_1\) and the optimal performance

\[
\hat{\mu} = \inf_{r \in \Omega^m} \| \Pi(r) \|_1.
\]

The operator \(\Pi\) does not always have a finite nuclear norm and it may not even be a compact operator. However, if the input and output spaces, \(\mathcal{U}\) and \(\mathcal{Y}\) are both finite-dimensional then \(\Pi\) is a nuclear operator [4, Thm. 3.3]. It will now be shown that finite-dimensionality of \(\mathcal{U}\) and \(\mathcal{Y}\) also implies that the optimal cost is continuous with respect to the actuator location.

Theorem 2.4. Let \(B(r) \in \mathcal{L}(\mathcal{U}, \mathcal{H}), r \in \Omega^m\), be a family of input operators such that for any \(r_0 \in \Omega^m, u \in \mathcal{U}\),

\[
\lim_{r \to r_0} B(r)u = B(r_0)u.
\]

Assume that \((A, B(r))\) are all stabilizable and that \((A, C)\) is detectable where \(C \in \mathcal{L}(\mathcal{H}, \mathcal{Y})\). If \(\mathcal{U}\) and \(\mathcal{Y}\) are finite-dimensional, then the corresponding Riccati operators \(\Pi(r)\) are continuous functions of \(r\) in the nuclear norm:

\[
\lim_{r \to r_0} \| \Pi(r) - \Pi(r_0) \|_1 = 0.
\]

Proof: Consider \((A, B(r_0))\) at some arbitrary point \(r_0 \in \Omega\). Choose some \(K\) so that \(A - B(r_0)K\) generates an exponentially stable semigroup with bound \(Me^{-\alpha t},\)
where $M \geq 1$, $\alpha > 0$. Since $\mathcal{U}$ is finite-dimensional, $\|B(r) - B(r_0)\| \to 0$ as $r \to r_0$. Let $\delta$ be such that $A - B(r)K$ generates an exponentially stable semigroup with bound $Me^{-\frac{\alpha}{2}t}$ for all $\|B(r) - B(r_0)\| < \delta$. There is $\epsilon > 0$ such that for all $|r - r_0| < \epsilon$, $\|B(r) - B(r_0)\| < \delta$. We thus have a sequence of systems $(A, B(r))$ with $\|T_r(t)\| \leq Me^{-\frac{\alpha}{2}t}$ where $T_r(t)$ indicates the semigroup generated by $A - B(r)K$. For any $z_0 \in \mathcal{H}$,

$$
\langle \Pi(r)z_0, z_0 \rangle \leq J(-Kz(t), z_0)
= \int_0^\infty \|CT_r(t)z_0\|^2 + \|R^{1/2}KT_r(t)z_0\|^2 dt
\leq c\|z_0\|^2
$$

for some constant $c > 0$. This implies that $\|\Pi(r)\| \leq c$. This, and Datko's Theorem then implies that the semigroups $S_r(t)$ generated by $A - B(r_0)K(r)$ where $K(r) = R^{-1}B^*(r)\Pi(r)$ are bounded by $M_3e^{-\beta t}$ for some $M_3 \geq 1$, $\beta > 0$. (See the proof of Theorem 2.1 in [10] for details.) Thus, any sequence $(A, B(r_i))$ with $r_i \to r_0$ is exponentially stabilizable by $K(r_i)$ with a uniform decay rate. Hence for all $z \in \mathcal{H}$ [8, Thm. 5.3]:

$$
\lim_{r \to r_0} \|\Pi(r)z - \Pi(r_0)z\| = 0.
$$

Since $\mathcal{U}$ is finite-dimensional this implies not only strong, but also uniform, convergence of the optimal feedback operators $K(r)$ to $K_0 = R^{-1}B^*(r_0)\Pi(r_0)$. Furthermore, letting $S_0(t)$ indicate the semigroup generated by $A - B(r_0)K(r_0)$, $\|S_0(t)\| \leq M_3e^{-\beta t}$ and $S_r(t)$ converges strongly to $S_0(t)$, uniformly on bounded intervals of time. Furthermore, since $B(r)K_r$ converges uniformly to $B(r_0)K_0$, $S^*_r(t)$ also converge strongly, uniformly on bounded intervals of time to $S^*_0(t)$.

Define the operators from $\mathcal{H}$ to $L_2(0, \infty; \mathcal{Y} + \mathcal{U})$,

$$
C_0(t) = \left[ \begin{array}{c} C \\ R^{1/2}K_0 \end{array} \right] S_0(t), \quad C_r(t) = \left[ \begin{array}{c} C \\ R^{1/2}K_r \end{array} \right] S_r(t).
$$

Since $\mathcal{Y} + \mathcal{U}$ is finite-dimensional, $C_0$ and $C_r$ are Hilbert-Schmidt operators [5, Thm. 4]. It will be shown that $C_r$ converges to $C_0$ in Hilbert-Schmidt norm. Since $\Pi(r_0) = C^*_0C_0$ and similarly $\Pi(r) = C^*_rC_r$, this will imply that $\Pi_r$ converges to $\Pi$ in nuclear norm [17, Thm. 7.8].

Letting $\{\phi_j\}_{j=1}^\infty$ be an orthonormal basis for $\mathcal{H}$, and $\{e_i\}_{i=1}^M$ an orthonormal basis for $\mathcal{Y} + \mathcal{U}$,

$$
\|C_0 - C_r\|_{HS} = \sum_{j=1}^\infty \int_0^\infty \|C_0\phi_j(t) - C_r\phi_j(t)\|^2_{\mathcal{Y} + \mathcal{U}} dt
\leq \sum_{j=1}^\infty \int_0^\infty \sum_{i=1}^M |\langle C_0\phi_j(t) - C_r\phi_j(t), e_i \rangle_{\mathcal{Y} + \mathcal{U}}|^2 dt
\leq \sum_{j=1}^\infty \sum_{i=1}^M \|\phi_j(t) - c_0(t)\|^2_{\mathcal{H}} dt
\leq \sum_{j=1}^\infty \sum_{i=1}^M \|\phi_j(t) - c_r(t)\|^2_{\mathcal{H}} dt
$$
where \( c_{ri}(t) = S_r(t)^* \left[ C^* K_r R^{1/2} \right] e_i \) and \( c_{0i} \) is defined similarly. By Levi’s Theorem,

\[
\| C_0 - C_r \|_{HS} = \int_0^\infty \sum_{j=1}^\infty \sum_{i=1}^M |\langle \phi_j(t), c_{0i}(t) - c_{ri}(t) \rangle_H|^2 dt \\
= \sum_{i=1}^M \int_0^\infty \sum_{j=1}^\infty |\langle \phi_j(t), c_{0i}(t) - c_{ri}(t) \rangle_H|^2 dt \\
= \sum_{i=1}^M \int_0^\infty \| c_{0i}(t) - c_{ri}(t) \|^2_H dt.
\]

Now, \( \| c_{0i}(t) - c_{ri}(t) \|_H = \| S_0(t)^* \left[ C^* K_0^* R^{1/2} \right] e_i - S_r(t)^* \left[ C^* K_r^* R^{1/2} \right] e_i \| \leq \| (S_0(t)^* - S_r^*(t)) \left[ C^* K_0^* R^{1/2} \right] e_i \| + \| S_r(t)^* [K_r^* - K_0^*] R^{1/2} e_i \|. \)

Since \( S_r^*(t) \) converge strongly on bounded intervals to \( S_0^*(t) \) and \( K_r^* \) converges uniformly to \( K_0^* \), \( \| c_{0i}(t) - c_{ri}(t) \|_H \) converges to 0 on bounded intervals of time. Uniform exponential stability of \( S_r^*(t) \) then implies that \( \lim_{r \to r_0} \int_0^\infty \| c_{0i}(t) - c_{ri}(t) \|_H dt = 0 \)
and so \( \lim_{r \to r_0} \| C_0 - C_r \|_{HS} = 0 \) which implies \( \lim_{r \to r_0} \| \Pi_0 - \Pi(r) \|_1 = 0 \).

The following result now follows immediately from the compactness of \( \Omega \).

**Corollary 2.5.** There exists an optimal actuator location \( \hat{r} \) such that

\[
\| \Pi(\hat{r}) \|_1 = \inf_{r \in \Omega^n} \| \Pi(r) \|_1 = \hat{\mu}.
\]

### 3 Calculation of Optimal Locations

For most problems on an infinite-dimensional Hilbert space \( H \), the operator equation (3) cannot be solved and the the control is calculated using a finite-dimensional approximation. Let \( H_n \) be a finite-dimensional subspace of \( H \) and \( P_n \) be the orthogonal projection of \( H \) onto \( H_n \). The space \( H_n \) is equipped with the norm inherited from \( H \). Consider a sequence of operators \( A_n \in \mathcal{L}(H_n, H_n), \quad B_n \in \mathcal{L}(U, H_n) \). This leads to a sequence of approximations

\[
\frac{dz}{dt} = A_n z(t) + B_n u(t), \quad z(0) = P_n z_0
\]
with cost functional

\[
J(u, z_0) = \int_0^\infty \langle C_n z(t), C z(t) \rangle + \langle u(t), Ru(t) \rangle dt
\]  

(5)

where \( C_n = C|_{\mathcal{H}_n} \). If \((A_n, B_n)\) is stabilizable and \((A_n, C_n)\) is detectable, then the cost functional has the minimum cost \( \langle P_n z_0, \Pi_n P_n z_0 \rangle \) where \( \Pi_n \) is the unique non-negative solution to the algebraic Riccati equation

\[
A_n^* \Pi_n + \Pi_n A_n - \Pi_n B_n R^{-1} B_n^* \Pi_n + C_n^* C_n = 0
\]  

(6)

on the finite-dimensional space \( \mathcal{H}_n \).

The feedback control \( K_n = R^{-1} B_n^* \Pi_n \), is used to control the original system (1). Assumptions that guarantee that \( \Pi_n \) converges to \( \Pi \) in some sense are required in order for this approach to be valid. The following set of assumptions is standard.

(A1) Let \( S_n(t) \) indicate the semigroup generated by \( A_n \). For each \( z \in \mathcal{H} \), we have

\[
(i) \quad \| S_n(t) P_n z - S(t) z \| \to 0,
\]

\[
(ii) \quad \| S_n^*(t) P_n z - S^*(t) z \| \to 0
\]

uniformly in \( t \) on bounded intervals.

(A2) (i) For each \( u \in \mathcal{U}, \| B_n u - B u \| \to 0 \), and for each \( z \in \mathcal{H} \), \( \| B_n^* P_n z - B^* z \| \to 0 \),

(ii) For each \( z \in \mathcal{H} \), \( \| C_n P_n z - C z \| \to 0 \), and for each \( y \in \mathcal{Y} \), \( \| C_n y - C^* y \| \to 0 \).

(A3) (i) The family of pairs \((A_n, B_n)\) is uniformly exponentially stabilizable, that is, there exists a uniformly bounded sequence of operators \( K_n \in \mathcal{L}(\mathcal{H}_n, \mathcal{U}) \) such that

\[
\left\| e^{(A_n - B_n K_n) t} P_n z \right\| \leq M_1 e^{-\omega_1 t} |z|
\]

for some positive constants \( M_1 \geq 1 \) and \( \omega_1 \).

(ii) The family of pairs \((A_n, C_n)\) is uniformly exponentially detectable, that is, there exists a uniformly bounded sequence of operators \( F_n \in \mathcal{L}(\mathcal{Y}, \mathcal{H}_n) \) such that

\[
\left\| e^{(A_n - F_n C_n) t} P_n \right\| \leq M_2 e^{-\omega_2 t}, \quad t \geq 0,
\]

for some positive constants \( M_2 \geq 1 \) and \( \omega_2 \).

Assumptions (A1)-(A3) imply strong convergence of the Riccati operator \( \Pi_n \) [10, Thm. 2.1, Cor. 2.2] and also that there exists constants \( M_2 \geq 1, \alpha_2 > 0 \), independent of \( n \), such that

\[
\left\| e^{(A_n - B_n R^{-1} B_n^* \Pi_n) t} \right\| \leq M_2 e^{-\alpha_2 t}.
\]

In order to use approximations to find the optimal actuator location, the approximate Riccati operators should converge in nuclear norm.
Theorem 3.1. Assume that \((A, B)\) is stabilizable and \((A, C)\) is detectable, and that \(\mathcal{U}\) and \(\mathcal{Y}\) are finite-dimensional. Let \((A_n, B_n, C_n)\) be a sequence of approximations to \((A, B, C)\) such that assumptions (A1)-(A3) are satisfied. Then

\[ \lim_{n \to \infty} \| \Pi_n P_n - \Pi \|_1 = 0. \]

Proof: Since assumptions (A1)-(A3) are satisfied, the feedback operators \(K_n = R - B_n^* \Pi_n\) are such that the semigroups \(S_n(t)\) generated by \(A_n - B_n K_n\) are uniformly exponentially stable and also \(\| \Pi_n \| \leq c\) for some constant \(c\) [10, Thm. 2.1, Cor. 2.2]. Since \(\mathcal{U}\) is finite-dimensional we have not only strong, but also uniform, convergence of the optimal feedback operators \(K_n = R - B_n^* \Pi_n\) to \(K = R - B^* \Pi\).

Letting \(S(t)\) indicate the semigroup generated by \(A - BK\), \(S_n(t)\) converge strongly to \(S(t)\), uniformly on bounded intervals of time. Furthermore, since \(B_n K_n\) converges uniformly to \(BK\), \(S_n^*(t)\) also converge strongly, uniformly on bounded intervals of time to \(S^*(t)\).

Define the operators from \(\mathcal{H}\) to \(L_2(0, \infty; \mathcal{Y} + \mathcal{U})\),

\[ C(t) = \left[ \begin{array}{c} C \\ \frac{-1}{2} K \end{array} \right] S(t), \quad C_n(t) = \left[ \begin{array}{c} C_n \\ \frac{-1}{2} K_n \end{array} \right] S_n(t). \]

The remainder of the proof is identical to that of Theorem 2.4: \(C_n\) converges to \(C\) in Hilbert-Schmidt norm and hence \(\Pi_n\) converges to \(\Pi\) in nuclear norm.

For the sequence of approximating problems \((A_n, B_n(r), C_n)\) define, analogously to \(J^r\) and \(\hat{\mu}, J^r_n(u, z_0), \mu_n(r)\) and \(\hat{\mu}_n\). Theorem 2.4 and Corollary 2.5 apply to these finite-dimensional problems and so the performance measure \(\mu_n(r)\) is continuous with respect to \(r\) and the optimal performance \(\hat{\mu}_n\) is well-defined. It will now be shown that the optimal cost converges as the approximation order increases as well as a corresponding sequence of optimal actuator locations.

Theorem 3.2. Assume a family of control systems \((A, B(r), C)\) with finite-dimensional input space \(\mathcal{U}\) and output space \(\mathcal{Y}\) such that

1. \((A, B(r))\) are stabilizable and \((A, C)\) are detectable,

2. for any \(r_0 \in \Omega, u \in \mathcal{U}\), \(\lim_{r \to r_0} B(r)u = B(r_0)u\).

Choose some approximation scheme such that assumptions (A1)-(A3) are satisfied for each \((A, B(r), C)\) with \(B_n(r) = P_n B(r), C_n = C|\tau_n\). Letting \(\hat{r}\) be an optimal actuator location for \((A, B(r), C)\) with optimal cost \(\hat{\mu}\) and defining similarly \(\hat{r}_n, \hat{\mu}_n\), it follows that

\[ \hat{\mu} = \lim_{n \to \infty} \hat{\mu}_n, \]

and there exists a subsequence \(\{\hat{r}_m\}\) of \(\{\hat{r}_n\}\) such that

\[ \hat{\mu} = \lim_{m \to \infty} \| \Pi(\hat{r}_m) \|_1. \]
Proof:

\[ \hat{\mu}_n = \inf_{r \in \Omega} \| \Pi_n(r) \|_1 \]
\[ \leq \| \Pi_n(\hat{r}) \|_1 \]
\[ \leq \| \Pi_n(\hat{r}) - \Pi(\hat{r}) \|_1 + \| \Pi(\hat{r}) \|_1 \]
\[ = \| \Pi_n(\hat{r}) - \Pi(\hat{r}) \|_1 + \hat{\mu}. \]

Since \( \| \Pi_n(\hat{r}) - \Pi(\hat{r}) \|_1 \to 0 \) (Thm. 3.1),
\[ \limsup \hat{\mu}_n \leq \hat{\mu}. \]

It remains only to show that
\[ \liminf \hat{\mu}_n \geq \hat{\mu}. \]

To this end, choose a subsequence \( \mu_m \to \liminf \hat{\mu}_n, \) with corresponding actuator locations \( \hat{r}_m. \) Since the sequence \( \{\hat{r}_m\} \) lies in a compact set it has a convergent subsequence, also denoted \( \{\hat{r}_m\}, \) with limit \( \hat{r}. \)

And so \( \| B_m(\hat{r}_m) - P_m B(\hat{r}) \| \) converges to zero. By assumption (A3), there is a uniformly bounded sequence \( K_m(\hat{r}) \in \mathcal{L}(\mathcal{H}, U) \) such that \( A_m - B_m(\hat{r}) K_m(\hat{r}) \) generate semigroups bounded by \( Me^{-\omega_1 t} \) for some \( M > 0, \) \( \omega_1 > 0. \) For some \( \epsilon < \omega_1/M, \) choose \( N \) large enough that \( \| B_m(\hat{r}_m) - B_m(\hat{r}) \| < \epsilon \) for \( m > N. \) Then for all \( m > N, A_m - B_m(\hat{r}_m) K_m(\hat{r}) \) generates an exponentially stable \( C_0 \)-semigroup with bound \( Me^{(-\omega_1+M)\epsilon t}. \) The assumptions of Theorem 3.1 are satisfied by the sequence \( (A_m, B_m(\hat{r}_m), C_m) \) and so \( \| \Pi_m(\hat{r}_m) - \Pi(\hat{r}) \|_1 \to 0. \) Thus,

\[ \liminf \hat{\mu}_n = \lim_{m \to \infty} \mu_m \]
\[ = \lim_{m \to \infty} \| \Pi_m(\hat{r}_m) \|_1 \]
\[ = \| \Pi(\hat{r}) \|_1 \]
\[ \geq \hat{\mu}. \]

Thus, \( \liminf \hat{\mu}_n \geq \hat{\mu} \) and so \( \hat{\mu}_n = \hat{\mu} \) as required.

Since \( \hat{\mu} = \lim \hat{\mu}_n = \liminf \hat{\mu}_n, \) (7) implies that

\[ \hat{\mu} = \| \Pi(\hat{r}) \|_1 \]
\[ = \lim_{m \to \infty} \| \Pi(\hat{r}_m) \|_1. \]

where the latter equality follows from continuity of performance with respect to actuator location (Theorem 2.4). Thus, as was to be shown, a sequence of approximating actuator locations yield performance arbitrarily close to optimal.

Example: Weakly Damped Beam
Consider a simply supported Euler-Bernoulli beam and let \( w(r, t) \) denote the deflection of the beam from its rigid body
motion at time $t$ and position $r$. The deflection is controlled by applying a force $u(t)$ centered on the point $r$ with width $w$. Letting $\Delta = 0.001$ indicate the width of the actuator, and $r$ its location,

$$b(r) = \begin{cases} 
  1/\Delta, & |r| < \frac{\Delta}{2} \\
  0, & |r| \geq \frac{\Delta}{2}.
\end{cases}$$

If we normalize the variables and include viscous damping with parameter $c_d$, we obtain the partial differential equation

$$\frac{\partial^2 w}{\partial t^2} + c_d \frac{\partial w}{\partial t} + \frac{\partial^4 w}{\partial x^4} = b(r)u(t), \quad t \geq 0, 0 < x < 1,$$

with boundary conditions

$$w(0, t) = 0, \quad w''(0, t) = 0, \quad w(1, t) = 0, \quad w''(1, t) = 0. \quad (8)$$

Define the state-space $H = H^2_0(0, 1) \times L^2(0, 1)$ with state $z(t) = (w(\cdot, t), \frac{\partial}{\partial t} w(\cdot, t))$. A state-space formulation of the above partial differential equation problem is

$$\frac{d}{dt} x(t) = Ax(t) + Bu(t),$$

where

$$A = \begin{bmatrix} 
  0 & I \\
  -\frac{d^4}{dx^4} & -c_d I
\end{bmatrix}, \quad B = \begin{bmatrix} 
  0 \\
  b(r)
\end{bmatrix},$$

with domain

$$D(A) = \{ (\phi, \psi) \in H^2_0(0, 1) \times H^2_0(0, 1) \text{ with } \phi'' \in H^2_0(0, 1) \}. $$

Let $\phi_i(x)$ indicate the eigenfunctions of $\frac{\partial^4 w}{\partial x^4}$ with boundary conditions (8). Defining $X_n$ to be the span of $\phi_i, i = 1..n$, we choose $H_n = X_n \times X_n$. This type of approximation satisfies assumptions (A1)-(A3) and the sequence of solutions $\Pi_n$ to the corresponding finite-dimensional ARE’s converge strongly to the exact solution $\Pi$.

Since there is only one control, choose control weight $R = 1$. If we choose state weight $C = I$, the trace of $\Pi_n$ does not even converge for a fixed actuator location, as shown in Figure 1. This indicates that $\Pi$ is not a nuclear operator. Suppose we try instead to reduce the deflection at the midpoint so the weight $Cz = w(0.5)$ where $w$ is the first component of the state $z$. The input and output spaces are both one-dimensional. We obtain the sequence of optimal actuator locations and performance shown in Figures 2-3. As predicted by the theory, the optimal locations and performance converge.
Figure 1. \( \text{Trace} \parallel \Pi_n \parallel \) Viscously damped beam, \( C = I \), actuator at \( x = 0.5 \)

Figure 2. Optimal actuator locations \( \hat{r}_n \) for viscously damped beam, \( C = \text{deflection at } x = 0.5 \)

Figure 3. Optimal performance Trace \( \Pi_n \) for viscously damped beam, \( C = \text{deflection at } x = 0.5 \)
Bibliography


