Some Results on Practical Stabilizability of Discrete-Time Switched Systems

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Abstract

In this paper, we report some recent development on practical stabilizability of discrete-time switched systems. We first introduce some practical stabilizability notions for discrete-time switched systems. Then we propose some sufficient conditions for the ϵ-practical asymptotic stabilizability of such systems. Furthermore, we focus on a class of discrete-time switched systems — namely, switched systems with constant increments, and present an approach to estimating the minimum bound for practical stabilizability. Since such class of systems are usually derived by discretizing continuous-time switched systems with integrator subsystems, we also explore the relationship between the minimum bound and the sampling period.

Extended Abstract

Recently, in our papers [7, 8, 9, 10, 11, 12], we have noted that, under appropriate switching laws, switched systems whose subsystems have different or no equilibria may still exhibit interesting behaviors similar to those of conventional stable or asymptotically stable systems near an equilibrium. Such behaviors are defined as practical stabilizability (local behavior) and practical asymptotic stabilizability (behavior in a larger region) in these papers. They are natural extensions of the traditional concepts of practical stability [2, 3], which are concerned with bringing the system trajectories to be within given bounds.

The results reported in [7, 8, 9, 10, 11, 12] are mainly concerned with practical stabilizability of continuous-time switched systems. For a survey of practical stabilizability and its relationship to the conventional stabilizability of continuous-time switched systems, the reader is referred to our recent paper [11] for more information. Up to now, there have only been very few results reported in the literature which are related to the boundedness or practical stability of discrete-time switched systems (see, e.g., [1, 4, 5, 6]).

In this paper, we will formally propose some notions of practical stability and stabilizability for discrete-time switched systems and present some sufficient conditions for the ϵ-practical asymptotic stabilizability of such systems. Based on the sufficient conditions, we will then focus on a special class of discrete-time system, i.e., switched systems with constant increments, and present an approach to estimating the minimum bound for practical stabilizability. Since such class of systems are usually derived by discretizing continuous-time switched systems with integrator subsystems, we also explore the relationship between the minimum bound and the sampling period. A longer version of this paper can be found in [13].

In the sequel, we use ||·|| to denote the 2-norm, $B(x, r)$ to denote the open ball $\{y \in \mathbb{R}^n : \|y-x\| < r\}$ and $B[x, r]$ the closed ball. We use $S_r$ to denote the $r$-sphere around the origin, i.e., $S_r = \{x \in \mathbb{R}^n : \|x\| = r\}$. By a domain $D$ around the origin, we mean an open connected subset of $\mathbb{R}^n$ containing the origin. $\text{Int}(A)$ denotes the interior of a set $A \subset \mathbb{R}^n$.

A. Discrete-Time Switched Systems and Practical Stabilizability Notions

In this paper, we consider discrete-time switched systems which consist of discrete-time subsystems

$$x(k+1) = f_i(x(k)), \quad i \in I \triangleq \{1, 2, \ldots, M\}. \tag{1}$$

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In (1), every \( f_i : \mathbb{R}^n \to \mathbb{R}^n \) is locally Lipschitz continuous. The active subsystem at each time instant is orchestrated by a switching law, which will be formally defined below. Given an initial state \( x(0) \) at discrete time instant 0, the switching law generates a switching sequence \( \sigma = ((0, i_0), (1, i_1), \cdots, (k, i_k), \cdots) \) \((i_k \in I) \) which indicates that subsystem \( i_k \) is active at discrete time instant \( k \). Unlike continuous-time switched systems, for a discrete-time switched system we do not need to be concerned with the Zenoness problem (i.e., infinite number of switchings in finite time intervals), since such a system can only switch its dynamics at discrete time instant \( k \)'s. Discrete-time switched systems are often derived from the discretization of continuous-time switched systems with a sampling period \( T \). In such cases, the state \( x(k) \) corresponds to the continuous-time state at time \( kT \). It can be readily seen that such a discretized system cannot switch infinitely fast.

**Definition 1 (Switching Law)** For a discrete-time switched system (1), a switching law \( S \) is defined as a mapping \( \mathbb{R}^n \to \Sigma_{[0, \infty)} \) which specifies a switching sequence \( \sigma \in \Sigma_{[0, \infty)} \) for any initial state \( x(0) \). Here \( \Sigma_{[0, \infty)} \triangleq \{ \text{switching sequence } \sigma \text{ over } 0 \leq k < \infty \} \).

**Remark 1** \( S \) over \( 0 \leq k < \infty \) as defined above is often determined by some rules or algorithms, which describe how to generate a switching sequence for a given \( x(0) \), rather than mathematical formulae.

Next let us introduce some practical stability notions for discrete-time switched systems. They are extensions of practical stability notions for continuous-time switched systems (see [11]). We mainly discuss the case of the origin and let the initial time instant be 0. To study the practical stability and stabilizability of switched systems, we do not require that \( f_i(0) = 0 \) for any \( i \in I \) (i.e., the origin does not need to be a common equilibrium of the subsystems).

**Definition 2** Assume that a switching law \( S \) is specified for the discrete-time switched system (1). Given an \( \epsilon > 0 \), the system is

- \( \epsilon \)-practically stable (about the origin) under \( S \) if there is a \( \delta > 0 \) such that
  \[
  x(0) \in B(0, \delta) \Rightarrow x(k) \in B(0, \epsilon), \ \forall k \geq 0.
  \]

- \( \epsilon \)-practically asymptotically stable on a given domain \( D \) around the origin under \( S \) if it is \( \epsilon \)-practically stable under \( S \), and for every \( x(0) \in D \), there is a \( K = K(x(0)) \geq 0 \) such that
  \[
  x(k) \in B(0, \epsilon), \ \forall k \geq K.
  \]

**Remark 2** One difference between \( \epsilon \)-practical stability and conventional stability is that, in Definition 2, \( \epsilon \) is a prespecified constant which cannot be varied. Therefore, \( \epsilon \)-practical stability is concerned with the boundedness of the state trajectory in the given \( \epsilon \)-neighborhood around the origin.

**Definition 3** Given an \( \epsilon > 0 \), the discrete-time switched system (1) is

- \( \epsilon \)-practically stabilizable (about the origin) if there is a switching law \( S \) such that the system is \( \epsilon \)-practically stable under \( S \).

- \( \epsilon \)-practically asymptotically stabilizable on a given domain \( D \) around the origin if there is a switching law \( S \) such that the system is \( \epsilon \)-practically asymptotically stable on \( D \) under \( S \).

- globally \( \epsilon \)-practically asymptotically stabilizable if it is \( \epsilon \)-practically asymptotically stabilizable on \( D = \mathbb{R}^n \).

**Remark 3** If system (1) is \( \epsilon \)-practically stabilizable, we say \( \epsilon \) is a bound for practical stabilizability for the system. In general, not every positive number can serve as a bound for practical stabilizability for a given discrete-time switched system. Given a switched system, if \( \epsilon \) is too small, it can be true that there does not exist a switching law to make the system \( \epsilon \)-practically stable (one such example will be given in the Section C.1 when we study switched systems with constant increments). Also note that given \( 0 < \epsilon_1 \leq \epsilon_2 \), if the system is \( \epsilon_1 \)-practically stabilizable then it must be \( \epsilon_2 \)-practically stabilizable. Hence, a more interesting problem is how to find the minimum (or infimum) bound \( \epsilon \) for practical stabilizability. If \( \epsilon \) is the minimum bound and a switching law \( S \) is determined to \( \epsilon \)-practically stabilize the system, \( S \) can then be used to make the system \( \epsilon_1 \)-practically stable for any \( \epsilon_1 > \epsilon \).
In view of Remark 3, one interesting topic to explore is how to answer the following question.

**Question 1** Given a discrete-time switched system (1) which is $\epsilon$-practically stabilizable for some $\epsilon > 0$, how to find the minimum (or infimum) bound for practical stabilizability?

**B. Sufficient Conditions for $\epsilon$-Practical Asymptotic Stabilizability**

Methods using energy functions similar to the conventional Lyapunov function methods can be applied to determine the practical stabilizability of the discrete-time switched system (1). In the sequel, we call a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ an *energy function* if it is positive definite, i.e., $V(0) = 0$ and $V(x) > 0, \forall x \neq 0$.

**Theorem 1** Assume an $\epsilon > 0$ is given. Also assume that an energy function $V(x)$ is given for the discrete-time switched system (1). If there exist $\rho_1 > \rho_2 > 0$ such that the following conditions are satisfied

(a). The sets $\Omega_{\rho_1} = \{x \in \mathbb{R}^n : V(x) \leq \rho_1\}$ and $\Omega_{\rho_2} = \{x \in \mathbb{R}^n : V(x) \leq \rho_2\}$ are bounded.

(b). $\Omega_{\rho_2} \subseteq B(0, \epsilon) \subseteq \Omega_{\rho_1}$.

(c). For any $x \in \text{Int}(\Omega_{\rho_1} - \Omega_{\rho_2})$,

$$\min_{i \in I} \{V(f(x)) - V(x)\} < 0.$$  \hspace{1cm} (4)

(d). For any $x \in \Omega_{\rho_2}$, $f_i(x) \in B(0, \epsilon), \forall i \in I$.

then the discrete-time switched system (1) is $\epsilon$-practically asymptotically stabilizable on the domain $\text{Int}(\Omega_{\rho_1})$.

**Proof:** Detailed proof can be found in the longer version of this paper [13]. In the proof, we explicitly construct a switching law under which the switched system is $\epsilon$-practically asymptotically stabilizable on $\text{Int}(\Omega_{\rho_1})$.

**Remark 4** If in Theorem 1, $V(x)$ is radially unbounded and condition (c) can be further strengthened to be $\min_{i \in I} \{V(f(x)) - V(x)\} < 0, \forall x \in (\mathbb{R}^n - \Omega_{\rho_2})$, then the system is globally $\epsilon$-practically asymptotically stabilizable.

**C. Practical Stabilizability of Switched Systems with Constant Increments**

In this section, we will focus on a special class of discrete-time switched systems — namely, switched systems with constant increments, and derive some necessary and sufficient conditions for their practical stabilizability. Moreover, we will explore the answer to Question 1 for such systems by presenting an approach to estimating the minimum (or infimum) bound for practical stabilizability.

In the sequel, we consider discrete-time switched systems with constant increments whose subsystems are of the form

$$x(k + 1) = x(k) + a_i, \quad a_i \in \mathbb{R}^n, i \in I.$$  \hspace{1cm} (5)

**Remark 5** The above discrete-time switched system (5) may be obtained by discretizing a continuous-time switched system with integrator subsystems

$$\dot{x}(t) = \dot{a}_i, \quad \dot{a}_i \in \mathbb{R}^n, i \in I$$  \hspace{1cm} (6)

with a sampling period $T$. The resultant discretized system is

$$\dot{x}((k + 1)T) = \dot{x}(kT) + \dot{a}_iT, \quad i \in I.$$  \hspace{1cm} (7)

If we define the discrete-time system state $x(k) \triangleq \dot{x}(kT)$ and denote $a_i \triangleq \dot{a}_iT$, we obtain the system (5).

The following lemma provides a necessary condition for the switched system (5) to be $\epsilon$-practically asymptotically stabilizable for some $\epsilon > 0$. 

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Lemma 1 If the discrete-time switched system with constant increments (5) is $\epsilon$-practically asymptotically stabilizable on some domain $D$ around the origin for some $\epsilon > 0$, then we must have

$$0 \in \text{Int}(C)$$

where $C$ is the convex hull of the set $\{a_i : i \in I\}$, i.e., $C = \text{conv}(\{a_i : i \in I\}) = \{\sum_{i=1}^{M} \lambda_i a_i : \lambda_i \geq 0, i \in I \text{ and } \sum_{i=1}^{M} \lambda_i = 1\}$. \hfill \square

Proof: Detailed proof can be found in the longer version of this paper [13].

In fact, as will be seen later, the condition in Lemma 1 is also sufficient for the system (5) to be $\epsilon$-practically asymptotically stabilizable for some $\epsilon > 0$. We will explicitly derive such an $\epsilon$ in Section C.1.

C.1. Estimation of Minimum Bound for Practical Stabilizability

For the switched system (5), if $a_i \neq 0$, $\forall i \in I$, then we find that for any $\epsilon < \min_{i \in I} \|a_i\|$, the system cannot be $\epsilon$-practically stabilized. To see this, we only need to notice that if $x(k) = 0$, then $x(k + 1)$ must be equal to one of the $a_i$'s and hence is outside $B(0, \epsilon)$. In view of this, we conclude that not every $\epsilon > 0$ can serve as a bound for practical stabilizability for the system (5). Hence a natural question to ask is: If (8) is satisfied for a switched system (5), does there exist some $\epsilon > 0$ such that the system is $\epsilon$-practically asymptotically stabilizable? The answer to this question is yes. In the following, we will explicitly derive one such $\epsilon$ value (i.e., a bound for practical stabilizability).

Instead of finding any bound for practical stabilizability, we will try to address Question 1 by estimating the minimum bound for practical stabilizability. Our approach to estimating such a bound is based on Theorem 1. We also point out that, for a switched system (5) satisfying (8), once we found a bound for practical stabilizability, the system must be globally $\epsilon$-practically asymptotically stabilizable.

In the following, we adopt the energy function $V(x) = x^T P x$ where $P = P^T \in \mathbb{R}^{n \times n}$ is a real positive definite matrix. According to Theorem 1, if we can determine a bounded set in $\mathbb{R}^n$ such that $\min_{i \in I} \{V(x + a_i) - V(x)\} < 0$ for any $x$ outside of it, then Theorem 1 can be applied (as long as we choose an $\Omega_{\rho_2}$ which contains this bounded set and then choose an $\epsilon > 0$ such that $B(0, \epsilon)$ satisfies condition (d) of Theorem 1).

We estimate such a bounded set by a $B[0, \tilde{r}]$. To find such a $\tilde{r}$, we consider $\alpha x$, $\alpha \geq 0$ for each $x$ on the unit sphere $S_1$. In the sequel in Theorem 2, we will show that for every $x \in S_1$, when $\alpha$ is large enough,

$$\min_{i \in I} \{V(\alpha x + a_i) - V(\alpha x)\} < 0$$

is always satisfied given (8). Hence for each $x \in S_1$, we can always find a minimum value $\alpha_{\min}(x)$ such that $\min_{i \in I} \{V(\alpha x + a_i) - V(\alpha x)\} < 0$, $\forall \alpha > \alpha_{\min}(x)$. Now by taking the maximum of all such $\alpha_{\min}(x)$ values over all $x \in S_1$, we can obtain $\tilde{r}$.

To formally carry out the aforementioned approach, we note that

$$V(\alpha x + a_i) - V(x) = (\alpha x + a_i)^T P(\alpha x + a_i) - \alpha^2 x^T P x = 2 \alpha x^T P a_i + a_i^T P a_i.$$  \hfill (10)

To continue our discussion, let us introduce the following lemma.

Lemma 2 If (8) holds, then for every $x \in S_1$, there exists an $i \in I$ such that

$$2 x^T P a_i < 0.$$  \hfill (11)

Proof: Detailed proof can be found in the longer version of this paper [13].

Equipped with Lemma 2, given an $x \in S_1$, we can always decompose the subsystem index set $I$ into $I_1^x$ and $I_2^x$. Here $I_1^x$ contains all the subsystem index $i$ such that $2 x^T P a_i < 0$ and must be nonempty. And $I_2^x$ contains all the subsystem index $i$ such that $2 x^T P a_i \geq 0$. Now we present the main results in this section.

Theorem 2 Consider a discrete-time switched systems with constant increments (5). If (8) holds, then for every $x \in S_1$, the minimum $\alpha \geq 0$ such that $\min_{i \in I} \{V(\alpha x + a_i) - V(\alpha x)\} < 0$, $\forall \alpha > \alpha$ is

$$\alpha_{\min}(x) = \min_{i \in I_1^x} \left( -\frac{a_i^T P a_i}{2 x^T P a_i} \right).$$  \hfill (12)

$\square$
Once we obtain $\alpha_{\min}(x)$ for every $x \in S_1$, we can then choose
\[
\hat{r} = \max_{x \in S_1} \alpha_{\min}(x).
\]

Consequently, we can choose the smallest set $\Omega_{\rho_2} = \{x \in \mathbb{R}^n : V(x) \leq \rho_2\}$ such that $B[0,\hat{r}] \subseteq \Omega_{\rho_2}$. Finally, we estimate the minimum bound for practical stabilizability to be
\[
\hat{\epsilon} = \max_{x \in \Omega_{\rho_2}} \|x\| + \max_{i \in I} \|a_i\|.
\]

(14)

With such an $\hat{\epsilon}$, Theorem 1 can be applied to establish the global $\epsilon$-practical asymptotic stabilizability of the switched system (5) for any $\epsilon > \hat{\epsilon}$.

C.2. The Effect of Discretization of Choice of Sampling Period

As mentioned in Remark 5, a switched system (5) is often obtained by discretizing a continuous-time switched system (6) with a sampling period $T$. Hence, the $a_i$'s in (5) are functions of $T$, i.e., $a_i = \hat{a}_iT$. In this case, the $\hat{r}$ and $\hat{\epsilon}$ obtained in (13) and (14) will also be functions of $T$. We denote them as $\hat{r}(T)$ and $\hat{\epsilon}(T)$. It can then be shown that $\hat{r}(T) = T\hat{r}(1)$ and
\[
\hat{\epsilon}(T) = T\hat{\epsilon}(1).
\]

(15)

(15) has important implications for the design of practical stabilizing switching laws for the continuous-time switched system (6). If $0 \in \text{Int}(\hat{C})$ where $\hat{C} = \text{conv}(\{\hat{a}_i : i \in I\})$, then given any $\epsilon > 0$, we can always design a periodic switching law with period $T$ under which the system (6) is globally $\epsilon$-practically asymptotically stabilizable.

In the paper, we will present more details on this topic.

D. A Computational Approach and Examples

In this section, we will present a numerical approach based on the results in Section C.1 to effectively estimate the minimum bound for practical stabilizability of switched system (5). Numerical examples will also be given in this paper.

It should be noted that the approach for estimating the minimum bound in Section C.1 can also be extended to discrete-time switched systems with affine subsystems, i.e., switched system with subsystems
\[
x(k+1) = A_ix(k) + b_i, \quad A_i \in \mathbb{R}^{n \times n}, b_i \in \mathbb{R}^n, i \in I.
\]

(16)

In the longer version of the paper [13], we also present some preliminary results on the estimation of the minimum bound for practical stabilizability for such class of discrete-time switched systems.

References


