Small Time Local Controllability and Stabilizability of Spacecraft Attitude Dynamics using Control Moment Gyroscopes

A. A. Paranjape*, and S. P. Bhat†

1 Introduction

Momentum exchange devices such as reaction wheels and control moment gyroscopes (CMGs) form an important class of torque actuators for spacecraft attitude control. A single gimbal CMG comprises of a rapidly spinning rotor mounted on a gimbal. The orientation of the rotor can be changed by applying torque to the gimbal, and the reaction torque thus generated serves to control the spacecraft attitude. The magnitude of torque produced by the CMG is a function of the constant rotor speed and gimbal rotation rate, while the direction is tangential to the circle along which the CMG rotor is constrained to rotate. Clearly, a single CMG cannot produce torques in all directions. In order to obtain torques along three independent directions, as well as for redundancy, CMGs are used in arrays consisting of multiple CMGs. Unfortunately, every CMG array possesses singular configurations

*Graduate Student, Department of Aerospace and Ocean Engineering, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061; adityap@iitbombay.org.
†Associate Professor, Department of Aerospace Engineering, Indian Institute of Technology Bombay, Mumbai 400076, India; bhat@aero.iitb.ac.in.
[1] at which the mapping from gimbal angle rates to output torque becomes singular. Consequently, for each singular configuration, there exists a singular direction along which the CMG array is unable to produce torque while in that configuration.

A considerable amount of research related to CMGs has focused on steering algorithms to maneuver CMG arrays to produce desired torque profiles while avoiding singular configurations. Although the presence of singular configurations constrains the ability of a CMG array to produce arbitrary torque profiles, it is not evident that singular configurations affect the ability of the array to achieve steering or stabilization. Given the significant amount of attention that the problem of singular configurations has received in the attitude control literature [2, 3, 4, 5, 6, 7, 8], it is of interest to know exactly which system theoretic properties of a spacecraft attitude control system are affected by the presence of singular CMG configurations.

The purpose of this paper is to investigate whether the presence of singular configurations poses an obstruction to the small time local controllability (STLC) and stabilizability of the spacecraft attitude dynamics at rest equilibria. Our work builds upon the results of [9], in which it was shown that the combined spacecraft-CMG system can be steered between any two states possessing the same total angular momentum inspite of the presence of singular CMG configurations.

We identify a special class of singular configurations called critically singular configurations at which the magnitude of the total angular momentum of the CMG array has a critical value. In Section 4, we show that spacecraft attitude dynamics are STLC at equilibria in which the CMG array is not in a critically singular configuration. Sufficient conditions for STLC [10] are violated at equilibria in which the CMG array is in a critically singular configuration such that no CMG has its gimbal axis along the total angular momentum vector. However, the attitude dynamics are STLC if the critically singular configuration is such that there are at least two CMGs whose torques are not colinear, and whose gimbal axes coincide with the total angular momentum of the CMG array. We also show that the attitude dynamics are not STLC at any equilibrium if the spacecraft carries only one CMG on board.

In Section 5, we show that equilibria in which the configuration of the CMG array is not critically singular are stabilizable using smooth static state feedback. On the other hand, equilibria in which the magnitude of the total angular momentum of the CMG array is either a local maximum or a nonzero local minimum cannot be stabilized using continuous feedback. As a consequence, it follows that, if the spacecraft carries only one CMG, then the attitude dynamics are not stabilizable through continuous feedback at any equilibrium.

We begin by introducing the necessary terminology, notation and description of the dynamics in sections 2 and 3.

## 2 Preliminaries

The set SO(3) of $3 \times 3$ special orthogonal matrices is a three-dimensional Lie group. The Lie algebra so(3) of SO(3) is the set of $3 \times 3$ real skew-symmetric matrices with the matrix commutator as the bracket operation. We denote by $\times$
the usual cross product on \( \mathbb{R}^3 \). Define \( S : \mathbb{R}^3 \to \text{so}(3) \) by

\[
S(a) = \begin{bmatrix}
0 & -a_3 & a_2 \\
a_3 & 0 & -a_1 \\
-a_2 & a_1 & 0
\end{bmatrix}, \quad a \in \mathbb{R}^3.
\]

For every \( a \in \mathbb{R}^3 \), \( S(a) \) is simply the matrix representation in the standard basis of the linear map \( b \mapsto (a \times b) \) on \( \mathbb{R}^3 \). We denote the tangent space to \( \text{SO}(3) \) at \( R \in \text{SO}(3) \) by \( T_R\text{SO}(3) \). For every \( R \in \text{SO}(3) \), \( T_R\text{SO}(3) = \{RG : G \in \text{so}(3)\} = \{RS(g) : g \in \mathbb{R}^3\} \).

We denote the Euclidean norm on \( \mathbb{R}^3 \) by \( \| \cdot \| \) and the two-dimensional unit sphere \( \{x \in \mathbb{R}^3 : \|x\| = 1\} \) by \( S^2 \).

Given two smooth vector fields \( f \) and \( g \) on a smooth manifold \( \mathcal{N} \), we denote their Lie bracket by \( [f, g] \). If \( \mathcal{N} \) is an embedded submanifold of a manifold \( \mathcal{M} \), and \( \hat{f} \) and \( \hat{g} \) are \( C^\infty \) extensions to \( \mathcal{M} \) of the vector fields \( f \) and \( g \), respectively, then \( [f, g] \) is the restriction of \( [\hat{f}, \hat{g}] \) to \( \mathcal{N} \). In particular, if \( \mathcal{M} = \mathbb{R}^n \) for some \( n \), then, for every \( x \in \mathcal{N} \subseteq \mathcal{M} \), the canonical identification between \( T_x\mathcal{M} \) and \( \mathbb{R}^n \) yields

\[
[f, g](x) = \frac{d}{dh} \bigg|_{h=0} [\hat{g}(x + hf(x)) - \hat{f}(x + hg(x))].
\]

In the sequel, we will find it convenient to apply the formula (1) to vector fields defined on \( \text{SO}(3) \), which can be viewed as an embedded submanifold of \( \mathbb{R}^n \) for \( n = 9 \). An alternative approach to computing Lie brackets on \( \text{SO}(3) \) is described in [11, 12].

### 3 Attitude Dynamics

We describe the attitude of a rigid body using a matrix \( R \in \text{SO}(3) \) such that the multiplication of the body components of a vector by \( R \) gives the components of that vector with respect to a reference inertial frame. The attitude kinematics of the rigid body are then described by the equation

\[
\dot{R}(t) = R(t)S(\omega(t)),
\]

where \( \omega(t) \in \mathbb{R}^3 \) denotes the instantaneous body-frame components of the angular velocity of the spacecraft relative to the reference inertial frame.

Next, we consider the dynamics of a rigid spacecraft that is equipped with an array of \( m > 0 \) SGCMGs. The instantaneous body components of the total angular momentum vector of the spacecraft with respect to an inertial observer are given by

\[
H(t) = J\omega(t) + \nu(\theta(t)),
\]

where \( J \in \mathbb{R}^{3 \times 3} \) is the symmetric moment-of-inertia matrix of the spacecraft about the body-fixed frame, and \( \nu : \mathbb{R}^m \to \mathbb{R}^3 \) gives the body components of the spin angular momentum of the CMG array as a function of the vector of gimbal angles

\[
\theta = [\theta_1 \cdots \theta_m]^T \in \mathbb{R}^m.
\]
For every $\theta \in \mathbb{R}^m$, the vector $\nu(\theta)$ can be written as $\nu(\theta) = \nu_1(\theta_1) + \nu_2(\theta_2) + \ldots + \nu_m(\theta_m)$, where $\nu_i : \mathbb{R} \to \mathbb{R}^3$ gives the body components of the spin angular momentum of the $i$th CMG as a function of the $i$th gimbal angle $\theta_i \in \mathbb{R}$. For each $i = 1, \ldots, m$, we denote by $\nu'_i : \mathbb{R} \to \mathbb{R}^3$ and $\nu''_i : \mathbb{R} \to \mathbb{R}^3$ the first and second derivatives, respectively, of $\nu_i$ with respect to $\theta_i$. Since the spin angular momentum of each CMG has a constant magnitude, it follows that $\nu'_i(\theta_i)$ is orthogonal to $\nu_i(\theta_i)$ while $\nu''_i(\theta_i)$ equals $-\nu_i(\theta_i)$ for every $i = 1, \ldots, m$, and every $\theta_i \in \mathbb{R}$. We do not require all CMGs to be identical. In particular, we do not assume that the angular momenta of all the CMGs have the same magnitude.

A **singular configuration** of the CMG array corresponds to a $\theta \in \mathbb{R}^m$ such that the span of the vectors $\{\nu'_1(\theta_1), \ldots, \nu'_m(\theta_m)\}$ has dimension less than three. We denote the set of singular configurations by $\mathcal{S} \subset \mathbb{R}^m$. Given $v \in \mathbb{R}^2$, we let $\mathcal{S}_v = \{\theta \in \mathcal{S} : v^T \nu'_i(\theta) = 0, i = 1, \ldots, m\}$. It is clear that, for every $\theta \in \mathcal{S}$, there exists $v \in \mathcal{S}_v$ such that $v^T \nu'_i(\theta) = 0$ for all $i \in \{1, \ldots, m\}$. Thus $\mathcal{S} = \cup_{v \in \mathcal{S}_v} \mathcal{S}_v$. If $v \in \mathcal{S}_v$ and $\theta \in \mathcal{S}$ are such that $\theta \notin \mathcal{S}_v$, then $v$ is a singular direction corresponding to the singular configuration $\theta$. A critically singular configuration of the CMG array is a singular configuration that is a critical point of the smooth function $\theta \mapsto \|\nu(\theta)\|^2$. It is easy to verify that the set $\mathcal{C}$ of critically singular configurations is given by $\mathcal{C} = \{\theta \in \mathcal{S} : v^T \nu'_i(\theta) = 0, i = 1, \ldots, m\}$. Thus, if $\theta \in \mathbb{R}^m$ satisfies $\|\nu(\theta)\| \neq 0$, then $\theta \in \mathcal{C}$ if and only if $\theta \in \mathcal{S}_v$, for $v = \|\nu(\theta)\|^{-1} \nu(\theta) \in \mathbb{R}^2$. On the other hand, if $\theta \notin \mathcal{C}$ satisfies $\|\nu(\theta)\| = 0$, then $\theta \in \mathcal{S}$.

Assuming that no external torques act on the spacecraft, the equation representing the attitude dynamics of the spacecraft are given by the Euler’s equation $H(t) + \omega(t) \times H(t) = 0$. Substituting from (3) in Euler’s equation yields

$$J\ddot{\omega}(t) = -\omega(t) \times (J\omega(t) + \nu(\theta(t))) - \sum_{i=1}^{m} \nu'_i(\theta_i(t))u_i(t), \quad (4)$$

$$\dot{\theta}_i(t) = u_i(t), \quad i = 1, 2, \ldots, m, \quad (5)$$

where $\dot{\theta}_i$ is the gimbal rate of the $i$th CMG. In deriving (4), we have followed [2, 3, 4, 5, 6] and assumed that the moments of inertia of the CMG gimbals are negligible in comparison with those of the spacecraft, so that the matrix $J$ does not depend on the gimbal angles. Equations (2), (4) and (5) represent a control system on the $(6 + m)$-dimensional manifold $SO(3) \times \mathbb{R}^3 \times \mathbb{R}^m$ with gimbal rates $u_i, i = 1, \ldots, m$, as inputs.

The terms in (4) involving the gimbal rates represent the control torque experienced by the spacecraft due the gimbal motions of the CMG array. Note that this control torque is contained in span$\{\nu'_1(\theta_1), \ldots, \nu'_m(\theta_m)\}$ at every instant. Hence, at those instants at which the CMG array passes through a singular configuration, the control torque that can be generated is constrained to lie in a linear subspace of dimension less than three, and the system is momentarily underactuated.

The inertial components of the total angular momentum of the spacecraft give rise to a function $P : SO(3) \times \mathbb{R}^3 \times \mathbb{R}^m \to \mathbb{R}^3$ given by $P(R, \omega, \theta) = R(J\omega + \nu(\theta))$. Since the inertial components of the total angular momentum are constant along the motion of the spacecraft, for a given initial inertial angular momentum $\mu \in \mathbb{R}^3$, the dynamics given by (2), (4) and (5) evolve on the angular momentum level set.
\( M_\mu \overset{\text{def}}{=} P^{-1}(\mu) \subseteq \text{SO}(3) \times \mathbb{R}^3 \times \mathbb{R}^m \). It follows that the control system defined by (2), (4) and (5) is not controllable on the manifold \( \text{SO}(3) \times \mathbb{R}^3 \times \mathbb{R}^m \).

To investigate controllability and stabilizability properties of the system (2), (4) and (5) on an angular momentum level set, consider \( \mu \in \mathbb{R}^3 \). It is easy to verify that the map \( \phi_\mu : \text{SO}(3) \times \mathbb{R}^m \rightarrow \text{SO}(3) \times \mathbb{R}^3 \times \mathbb{R}^m \) given by \( \phi_\mu(R, \theta) = (R, J^{-1}(R^T \mu - \nu(\theta)), \theta) \) is a diffeomorphism between \( \text{SO}(3) \times \mathbb{R}^m \) and \( M_\mu \). Hence the angular momentum level set is diffeomorphic to the \((3+m)\)-dimensional manifold \( N \overset{\text{def}}{=} \text{SO}(3) \times \mathbb{R}^m \). It follows that states in \( M_\mu \) include all possible combinations of spacecraft attitudes and CMG configurations, including singular configurations.

On \( M_\mu \), the attitude kinematics (2) reduce to
\[
\dot{R}(t) = R(t)S(J^{-1}(R^T(t)\mu - \nu(\theta(t)))).
\] (6)

Equations (5) and (6) describe the combined dynamics of the spacecraft and the CMG array on the angular momentum level set \( M_\mu \), and define a control system of the form
\[
\dot{y}(t) = f_\mu(y(t)) + g_1(y(t))u_1(t) + \cdots + g_m(y(t))u_m(t),
\] (7)
on the manifold \( N \), where \( y = (R, \theta) \in N \) represents the spacecraft attitude and the CMG configuration. The drift vector field \( f_\mu \) and the control vector fields \( g_1, g_2, \ldots, g_m \), are analytic vector fields on \( N \) given by
\[
f_\mu(R, \theta) = (RS (J^{-1}(R^T \mu - \nu(\theta))), 0),
\] (8)
\[
g_i(R, \theta) = (0, e_i),
\] (9)
where, for each \( i = 1, 2, \ldots, m \), \( e_i \in \mathbb{R}^m \) is the vector whose \( i \)th element is 1, the rest being zero. In order to apply standard controllability results, we assume that the vector of gimbal rates \( u = [u_1 \cdots u_m]^T \) is a piecewise continuous function of time that has finite right and left limits at every instant of discontinuity, and that takes values in a connected set \( \Omega \subseteq \mathbb{R}^m \) containing 0 in its interior.

For each \( \mu \in \mathbb{R}^3 \), the set of equilibria of (7) is given by \( \mathcal{E}_\mu = \{(R, \theta) \in N : R^T \mu - \nu(\theta) = 0\} \).

## 4 Small Time Local Controllability

In this section, we analyze the STLC property of (7) at its equilibrium points. Our first two results show that equilibrium points in which the CMG array is not in a critically singular configuration possess the STLC property. For both these results, STLC is proved using the sufficient condition for STLC given in Theorem 7.3 from [10]. For applying the results of [10], recall that a bracket involving the vector fields \( f_\mu, g_1, \ldots, g_m \) is \textit{bad} if \( f_\mu \) appears in the bracket an odd number of times while every input vector field \( g_i, i \in \{1, \ldots, m\} \), appears an even number of times. Also, given a vector \( l = [l_0, l_1, \ldots, l_m]^T \) of nonnegative integer weights associated with the vector fields \( f_\mu, g_1, \ldots, g_m \), the \( l \)-degree of a bracket involving the vector fields \( f_\mu, g_1, \ldots, g_m \) is the sum of the weights of all vector fields appearing in the bracket counting multiplicities.
Before stating the results, we list the expressions for a few brackets involving the drift and control vector fields of (7). These brackets were computed by treating $\mathcal{N}$ as an embedded submanifold of $\mathbb{R}^{3 \times 3} \times \mathbb{R}^m$ and using (1). Given $\mu \in \mathbb{R}^3$, $x = (R, \theta) \in \mathcal{N}$ and $i, j \in \{1, \ldots, m\}$ such that $i \neq j$, we have

$$[g_i, g_j](x) = 0, \quad [f_\mu, g_i](x) = (RS(J^{-1} \nu'_i(\theta_i)), 0), \quad (10)$$

$$[g_j, [f_\mu, g_i]](x) = 0, \quad [g_i, [f_\mu, g_i]](x) = (-RS(J^{-1} \nu_i(\theta_i)), 0), \quad (11)$$

$$[[f_\mu, g_i], [f_\mu, g_j]](x) = (RS(J^{-1} \nu'_i(\theta_i) \times J^{-1} \nu'_j(\theta_j)), 0). \quad (12)$$

If, in addition, $x \in \mathcal{E}_\mu$, then

$$[f_\mu, [f_\mu, g_i]](x) = (RS(J^{-1}(J^{-1} \nu'_i(\theta_i) \times R^T \mu)), 0). \quad (13)$$

**Theorem 4.1.** Let $\mu \in \mathbb{R}^3$ and suppose $p = (R, \theta) \in \mathcal{E}_\mu$ is such that $\theta \notin \mathcal{S}$. Then, the spacecraft attitude dynamics described by (7)-(9) are STLC at $p$.

**Proof.** By our assumption on $p$, there exist distinct $i, j, k \in \{1, 2, \ldots, m\}$ such that $\nu'_i(\theta_i)$, $\nu'_j(\theta_j)$ and $\nu'_k(\theta_k)$ are linearly independent. It follows from equation (10) that $[f_\mu, g_i](p)$, $[f_\mu, g_j](p)$, and $[f_\mu, g_k](p)$ are linearly independent. Choose the weights $l_0 = 1$ and $l_n = 3$ for $n = 1, 2, \ldots, m$. The three good brackets $[f_\mu, g_i]$, $[f_\mu, g_j]$ and $[f_\mu, g_k]$ have an $l$-degree of 4 for our choice of weights. Bad brackets with the smallest $l$-degree that are non-zero at $p$ are all of the form $[g_n, [f_\mu, g_n]]$ for $n \in \{1, 2, \ldots, m\}$, and have $l$-degree 7. It follows that all bad brackets can be neutralized at $p$ by good brackets of a smaller degree. The result now follows from Theorem 7.3 in [10].

Our next result asserts that the STLC property holds at equilibrium points in which the CMG array is in a singular, but not critically singular, configuration. The proof of this result depends on the following lemma.

**Lemma 4.2.** Suppose $\theta \in \mathbb{R}^m$ and $v \in S^2$ are such that $\theta \in \mathcal{S}_v \setminus \mathcal{C}$. Then, there exist $i, j \in \{1, 2, \ldots, m\}$ such that $v^T(J^{-1} \nu'_i(\theta_i) \times \nu(\theta)) \neq 0$ and $\nu'_i(\theta_i) \times \nu'_j(\theta_j) \neq 0$.

**Proof.** To arrive at a contradiction, suppose that $\nu'_i(\theta_i) \times \nu'_j(\theta_j) = 0$ for all $i, j \in \{1, 2, \ldots, m\}$. It follows that $\nu_i(\theta_i)^T \nu'_j(\theta_j) = 0$ for all $i, j \in \{1, 2, \ldots, m\}$. Hence, $\nu(\theta)^T \nu'_j(\theta_j) = 0$ for all $j \in \{1, 2, \ldots, m\}$. Therefore, $\theta \in \mathcal{C}$, which is a contradiction. The contradiction implies that, for every $k \in \{1, 2, \ldots, m\}$, there exists an $n \in \{1, 2, \ldots, m\}$ such that $\nu'_k(\theta_k) \times \nu'_n(\theta_n) \neq 0$. In particular, the vectors $\{\nu'_1(\theta_1), \ldots, \nu'_m(\theta_m)\}$ span the two-dimensional subspace orthogonal to the vector $v$.

Since $\theta \in \mathcal{S}_v \setminus \mathcal{C}$, it follows that $\nu(\theta) \neq 0$ and $v \times \nu(\theta) \neq 0$. To arrive at a contradiction, suppose that $v^T(J^{-1} \nu'_i(\theta_i) \times \nu(\theta)) = 0$ for all $i \in \{1, 2, \ldots, m\}$. Using basic properties of the scalar triple product and noting that $J = J^T$, it follows that $\nu'_i(\theta_i)^T(J^{-1}(v \times \nu(\theta))) = 0$ for all $i \in \{1, 2, \ldots, m\}$. Thus the vector $J^{-1}(v \times \nu(\theta))$ is
orthogonal to every vector in the two-dimensional subspace orthogonal to \( v \). Hence there exists \( \gamma \in \mathbb{R} \setminus \{0\} \) such that \( v \times \nu(\theta) = \gamma Jv \). This implies that \( v^T Jv = 0 \), which contradicts the fact that \( J \) is positive definite. The contradiction implies that there exists \( i \in \{1, \ldots, m\} \) such that \( v^T (J^{-1} \nu'_i(\theta_1) \times \nu(\theta)) \neq 0 \). As shown in the previous paragraph, there exists \( j \) such that \( \nu'_i(\theta_1) \times \nu'_j(\theta_2) \neq 0 \). \( \square \)

**Theorem 4.3.** Let \( \mu \in \mathbb{R}^3 \) and suppose \( p = (R, \theta) \in \mathcal{E}_\mu \) is such that \( \theta \in \mathcal{S} \setminus \mathcal{C} \). Then, the spacecraft attitude dynamics described by (7)-(9) are STLC at \( p \).

**Proof.** Since \( \theta \in \mathcal{S} \), there exists \( v \in S^2 \) such that \( v^T \nu'_i(\theta_1) = 0 \) for every \( i \in \{1, 2, \ldots, m\} \). Since \( \theta \notin \mathcal{C} \), we may assume that \( v \) satisfies \( v \times \nu(\theta) \neq 0 \). We choose weights \( l_0 = 1 \) and \( l_i = 3 \) for \( i = 1, 2, \ldots, m \). It follows from Lemma 4.2 that there exist distinct \( i, j \in \{1, 2, \ldots, m\} \) such that \( \nu'_i(\theta_1) \times \nu'_j(\theta_2) \neq 0 \), and \( v^T [J^{-1} \nu'_i(\theta_1) \times \nu(\theta)] \neq 0 \). Because \( v^T \nu'_i(\theta_1) = v^T \nu'_j(\theta_2) = 0 \), it follows that \( \nu'_i(\theta_1), \nu'_j(\theta_2) \) and \( J^{-1} \nu'_i(\theta_1) \times \nu(\theta) \) are linearly independent. It follows from (10) and (13) in the appendix that the good brackets \([f_\mu, g_1](p), [f_\mu, g_2](p)\) and \([f_\mu, [f_\mu, g_1]](p)\) having \( l \)-degrees 4, 4, and 5, respectively, are linearly independent. Bad brackets with the smallest \( l \)-degree that are non-zero at \( p \) are of the form \([g_i, [f_\mu, g_i]]\) and have an \( l \)-degree 7. Thus, all bad brackets can be neutralized by good brackets with smaller \( l \)-degree. The result now follows from Theorem 7.3 of [10]. \( \square \)

Next, we consider the STLC property at equilibria in which the CMG array is in a critically singular configuration. Our next result states that an equilibrium point in which the CMG array is in a critically singular configuration possesses the STLC property provided there are at least two CMGs which have their gimbal axes along the singular direction of the array, and which produce linearly independent torques in the plane orthogonal to the singular direction.

**Theorem 4.4.** Let \( \mu \in \mathbb{R}^3 \). Let \( p = (R, \theta) \in \mathcal{E}_\mu \) be such that \( \theta \in \mathcal{C} \). Suppose that \( v \in S^2 \) is such that \( \theta \in \mathcal{S}_v \) and \( \nu_i(\theta_1)^T v = 0 \) for \( i = 1, 2, \ldots, k \), where \( 2 \leq k < m \). Further, suppose there exist distinct \( i, j \in \{1, 2, \ldots, k\} \) such that \( \nu'_i(\theta_1) \) and \( \nu'_j(\theta_2) \) are linearly independent. Then, the spacecraft attitude dynamics described by (7)-(9) are STLC at \( p \).

**Proof.** Let \( l_0 = 1 \), \( l_i = 3 \) for \( i \in \{1, \ldots, k\} \), and \( l_i = 5 \) for \( i \in \{k + 1, \ldots, m\} \). Without loss of generality, we may assume that \( \nu'_i(\theta_1) \) and \( \nu'_j(\theta_2) \) are linearly independent. It then follows that the vectors \( \nu'_i(\theta_1) \) and \( \nu'_j(\theta_2) \) span the two-dimensional subspace orthogonal to the vector \( v \). Because \( \nu_i(\theta_1)^T v = 0 \) for all \( i \in \{1, 2, \ldots, k\} \), it follows that \( \nu_1(\theta_1), \ldots, \nu_k(\theta_k) \in \text{span} \{\nu'_i(\theta_1), \nu'_j(\theta_2)\} \). It follows from (10) and (11) that the bad brackets \([g_1, [f_\mu, g_1]], \ldots, [g_k, [f_\mu, g_k]]\) having \( l \)-degree 7 can be neutralized by the good brackets \([f_\mu, g_1]\) and \([f_\mu, g_2]\), whose \( l \)-degree is 4. The good bracket \([f_\mu, g_1], [f_\mu, g_2]\) has \( l \)-degree 8. Moreover, it can be verified from equations (10) and (12) that \([f, g_1], [f, g_2]\) and \([[f_\mu, g_1], [f_\mu, g_2]]\) are all linearly independent and hence neutralize the bad brackets \([g_i, [f_\mu, g_i]]\) for \( i \in \{k + 1, \ldots, m\} \) whose \( l \)-
degree is 11. All other bad brackets which are non-zero at \( p \) must have at least 3 \( f_\mu \)'s and at least 2 \( g_i \)'s for some \( i \in \{1, 2, \ldots, m\} \), and hence have an \( l \)-degree of at least \( 3 + 6 = 9 \). The result now follows from Theorem 7.3 of [10]. \( \square \)

Our next result asserts that the sufficient condition for STLC given in [10] (which we refer to as Sussmann’s condition) does not hold at an equilibrium in which the CMG array is in a critically singular configuration such that none of the angular momenta of the individual CMGs lies in the plane orthogonal to the singular direction or, equivalently, none of the CMG gimbal axes coincide with the singular direction of the CMG array. This result depends on the following lemma.

**Lemma 4.5.** Let \( \mu \in \mathbb{R}^3 \), and suppose \( p = (R, \theta) \in \mathcal{E}_\mu \) is such that \( \theta \in \mathcal{C} \). Then \( \text{ad}_{f_\mu}^n g_i(p) \in \text{span} \{ [f_\mu, g_1](p), \ldots, [f_\mu, g_m](p) \} \) for all \( n \geq 1 \) and \( i \in \{1, \ldots, m\} \).

**Proof.** Choose \( i \in \{1, \ldots, m\} \). First, assume \( \mu \neq 0 \). Then it follows that \( \nu(\theta) \neq 0 \) and \( \theta \in \mathcal{S}_\nu \) for \( v = \| \nu(\theta) \|^{-1} \nu(\theta) \). We assert that, for every \( n \geq 1 \), there exist functions \( h_{n;i} : \mathcal{N} \to \mathbb{R}^3 \) and \( c_{n;i} : \mathcal{N} \to \mathbb{R}^3 \) satisfying \( \nu(\theta)^T h_n(p) = 0 \) and \( c_{n;i}(p) = 0 \) such that, for every \( x = (R, \hat{\theta}) \in \mathcal{N} \),

\[
\text{ad}_{f_\mu}^n g_i(x) = (\hat{R} S(J^{-1} h_{n;i}(x)), 0) + (\hat{R} S(c_{n;i}(x)), 0). \tag{14}
\]

We prove this claim by induction on \( n \). It follows from (11) that the claim is true for \( n = 1 \) with \( h_{1;i}(x) = \nu_i(x) \) and \( c_{1;i}(x) = 0 \) for all \( x \in \mathcal{N} \). Next, suppose the claim is true for some integer \( k \). Consider \( x = (R, \hat{\theta}) \in \mathcal{N} \), and denote \( \hat{\omega}(x) = J^{-1}(\hat{R}^T \mu - \nu(\hat{\theta})) \). An algebraic computation using (1) yields

\[
\text{ad}_{f_\mu}^{k+1} g_i(x) = (\hat{R} S(J^{-1} h_{k+1;i}(x)) + \hat{R} S(c_{k+1;i}(x)), 0), \tag{15}
\]

where \( h_{k+1;i}(x) \overset{\text{def}}{=} J^{-1} h_{k;i}(x) \times \hat{R}^T \mu \) and \( c_{k+1;i}(x) \overset{\text{def}}{=} \hat{\omega}(x) \times (J^{-1} h_{k;i}(x) + c_{k;i}(x)) + J^{-1} (c_{k;i}(x) \times R^T \mu) - \psi(x) \) with

\[
\psi(x) = \lim_{s \to 0} \frac{1}{s} \left[ J^{-1}(h_{k;i}(x + sf_\mu) - h_{k;i}(x)) + (c_{k;i}(x + sf_\mu) - c_{k;i}(x)) \right].
\]

Since \( f_\mu(p) = 0 \), it follows that \( \psi(p) = 0 \), and hence, by the induction hypothesis, \( c_{k+1;i}(p) = 0 \). Furthermore, \( h_{k+1;i}(p) \) equals \( J^{-1} h_{k;i}(p) \times \nu(\theta) \) and satisfies \( \nu(\theta)^T h_{k+1;i}(p) = 0 \). Induction now implies that (14) holds for all \( n \geq 1 \). The statement now follows from the expressions for the vector fields \( [f_\mu, g_1], \ldots, [f_\mu, g_m] \), in (10) by evaluating (14) at \( p \) and noting that \( \nu(\theta)^T h_{n;i}(p) = 0 \), \( \nu(\theta) \neq 0 \) and \( \theta \in \mathcal{C} \) imply that \( h_{n;i}(p) \in \text{span} \{ \nu_1(\theta_1), \ldots, \nu_m(\theta_m) \} \).

Next, assume \( \mu = 0 \) and \( \theta \in \mathcal{C} \). The assertion of the lemma clearly holds for \( n = 1 \), while (13) implies that \( \text{ad}_{f_\mu}^{k+1} g_i(p) = 0 \). Since \( f_\mu(p) = 0 \), an easy computation shows that \( \text{ad}_{f_\mu}^{k+1} g_i(p) = 0 \) whenever \( \text{ad}_{f_\mu}^{k} g_i(p) = 0 \). Induction now implies that \( \text{ad}_{f_\mu}^{n} g_i(p) = 0 \) for all \( n \geq 2 \). Clearly, the assertion of the lemma holds for all \( n \geq 2 \). This completes the proof. \( \square \)
Further, suppose that at least one critically singular. Since we assign arbitrary weights to \( f_{\mu}, g_1, \ldots, g_m \), respectively. The lowest \( l \)-degree bad bracket that is non-zero at \( p \) is \([g_k, [f_{\mu}, g_k]]\) and has \( l \)-degree \( 2l_k + l_0 \), where \( l_k = \min\{l_1, \ldots, l_m\} \). Since \([g_1, g_1] = 0\) for all \( i, j \in \{1, 2, \ldots, m\} \), a nonzero good bracket must have at least one \( f_{\mu} \). Our choice of \( k \) implies that a nonzero good bracket of a smaller \( l \)-degree than \([g_k, [f_{\mu}, g_k]]\) will have at most one \( g_i \) for \( i \in \{1, 2, \ldots, m\} \). Hence all good brackets of a lower \( l \)-degree than the bracket \([g_k, [f_{\mu}, g_k]]\) are of the form \( \text{ad}_p^{f_{\mu}, g_i}(p) \), where \((n - 1)l_0 + l_i < 2l_k \). Lemma 4.5 implies that span\{\text{ad}_p^{f_{\mu}, g_i}(p) | i = 1, 2, \ldots, m, n = 0, 1, 2, \ldots \} = \text{span}\{[f_{\mu}, g_1](p), \ldots, [f_{\mu}, g_m](p)\}. \) On the other hand, since \( \nu_i(\theta_k) \notin \text{span}\{\nu_i'(\theta_1), \ldots, \nu_m'(\theta_m)\} \), it follows that \([g_k, [f_{\mu}, g_k]](p)\) is not contained in \( \text{span}\{[f_{\mu}, g_1](p), \ldots, [f_{\mu}, g_m](p)\} \). Hence, it follows that no linear combination of brackets of the form \( \text{ad}_p^{f_{\mu}, g_i} \) of a smaller \( l \)-degree can neutralize the bad bracket \([g_k, [f_{\mu}, g_k]]\) at \( p \). Because our choice of weights was arbitrary, it follows that Sussmann’s condition is violated at \( p \). □

Finally, we prove that when a spacecraft carries a single CMG, the attitude dynamics are not STLC at any equilibrium point. The proof of this negative result is based on Proposition 6.3 of [13].

**Theorem 4.7.** Suppose \( q = 1 \). Let \( \mu \in \mathbb{R}^3 \) and let \( p = (R, \theta) \in \mathcal{E}_\mu \). Then, the spacecraft attitude dynamics described by (7)-(9) are not STLC at \( p \).

**Proof.** Note that in the case of a single CMG, every CMG configuration is critically singular. Since \( \nu_1(\theta_1) = 0 \), Lemma 4.5 and (11) imply that \([g_1, [f_{\mu}, g_1]](p) \notin \text{span}\{\text{ad}_p^{f_{\mu}, g_1}(p) | k = 1, 2, \ldots \} \). Since \( f_{\mu}(p) = 0 \), the expressions for \([g_1, [f_{\mu}, g_1]] \) and \( g_1 \) further imply that \([g_1, [f_{\mu}, g_1]](p) \) is not contained in \( \text{span}\{f_{\mu}(p), g_1(p), \text{ad}_p^{f_{\mu}, g_1}(p) | k = 1, 2, \ldots \} \). Since the system (7) is analytic, Proposition 6.3 of [13] implies that the dynamics are not STLC at \( p \). □

## 5 Stabilizability

In this section, we analyze the feedback stabilizability of equilibria of (7). Our first result below asserts that equilibrium points of (7) in which the CMG array is not in a critically singular configuration is locally asymptotically stabilizable using smooth feedback. The proof is based on showing that the Jacobian linearization of the dynamics at every such equilibrium is controllable.
Theorem 5.1. Let $\mu \in \mathbb{R}^3$ and suppose $p = (R, \theta) \in E_\mu$ is such that $\theta \notin C$. Then $p$ is locally asymptotically stabilizable by smooth static state feedback.

Proof. As in the proofs of theorems 4.1 and 4.3, it can be shown that the dimension of the subspace spanned by $\{ad_{f_\mu}^ng_i(p)\mid n \geq 0, i = 1, 2, \ldots, m\}$ is $m + 3$. This implies that the linearization of the system at $p$ is controllable [14], and hence stabilizable. It now follows that (7) is locally asymptotically stabilizable at $p$ by smooth static state feedback [15, Prop. 1].

Our next result shows that equilibrium points at which the magnitude of the total angular momentum of the CMG array either has a local maximum or a nonzero local minimum, cannot be stabilized using continuous feedback. Note that, in both cases, the CMG array is in a critically singular configuration. The proof is based on showing that Brockett’s necessary condition [15, 16] for continuous stabilizability is violated.

Theorem 5.2. Let $\mu \in \mathbb{R}^3$, and suppose $p = (R, \theta) \in E_\mu$ is such that either

i) $\theta$ is a local maximizer of the function $\beta \mapsto \|\nu(\beta)\|^2$ or,

ii) $\theta$ is a local minimizer of the function $\beta \mapsto \|\nu(\beta)\|^2$ and $\|\nu(\theta)\| \neq 0$.

Then $p$ is not locally asymptotically stabilizable through continuous feedback.

Proof. First, suppose $\theta$ is a local maximizer of the function $\beta \mapsto \|\nu(\beta)\|^2$. There exists an open neighbourhood $V$ of $p$ such that $\nu(\theta)^T(R^T\mu) > 0$ and $\|\nu(\theta)\| \geq \|\nu(\theta)\|$ for all $(\bar{R}, \bar{\theta}) \in V$. To arrive at a contradiction, assume that $p$ is stabilizable through continuous feedback. Brockett’s necessary condition for continuous stabilizability [15, Thm. 3], [16, Thm. 1] when applied to (7) implies that the image of $V$ under $h : (\bar{R}, \bar{\theta}) \mapsto \bar{R}^T\mu - \nu(\bar{\theta})$ contains an open neighbourhood of 0. Hence there exist $(\bar{R}, \bar{\theta}) \in V$ and $\epsilon > 0$ such that $\bar{R}^T\mu - \nu(\bar{\theta}) = -\epsilon v(\bar{\theta})$. Therefore, $\nu(\bar{\theta}) = \bar{R}^T\mu + \nu(\bar{\theta})$, which gives $\|\nu(\bar{\theta})\|^2 = \|\bar{R}^T\mu\|^2 + \epsilon^2 \|\nu(\bar{\theta})\|^2 + 2\epsilon(\bar{R}^T\mu)^T\nu(\bar{\theta})$. However, $\|\bar{R}^T\mu\|^2 = \|\mu\|^2 = \|\nu(\bar{\theta})\|^2$ and $(\bar{R}^T\mu)^T\nu(\bar{\theta}) > 0$. Therefore, it follows that $\|\nu(\bar{\theta})\|^2 > (1 + \epsilon^2)\|\nu(\bar{\theta})\|^2$, which contradicts our assumption that $\bar{\theta}$ is a global maximizer of $\|\nu(\cdot)\|$ on $V$. The contradiction implies that $p$ is not stabilizable through continuous feedback.

Next, suppose $\theta$ is a local minimizer of the function $\beta \mapsto \|\nu(\beta)\|^2$ and $\|\nu(\theta)\| \neq 0$. There exists an open neighbourhood $V$ of $p$ such that $\nu(\theta)^T\nu(\theta) > 0$ and $\|\nu(\theta)\| \geq \|\nu(\theta)\|$ for all $(\bar{R}, \bar{\theta}) \in V$. To arrive at a contradiction, assume that $p$ is stabilizable through continuous feedback. Brockett’s necessary condition for continuous stabilizability [15, Thm. 3], [16, Thm. 1] when applied to (7) implies that the image of $V$ under the map $h : (\bar{R}, \bar{\theta}) \mapsto \bar{R}^T\mu - \nu(\bar{\theta})$ contains an open neighbourhood of 0. Hence, there exist $(\bar{R}, \bar{\theta}) \in V$ and an $\epsilon \in (0, 1)$ such that $\bar{R}^T\mu - \nu(\bar{\theta}) = \epsilon v(\bar{\theta})$. Therefore, $\bar{R}^T\mu = \nu(\bar{\theta}) + \epsilon v(\bar{\theta})$, which gives $\|\bar{R}^T\mu\|^2 = \|\nu(\bar{\theta})\|^2 + \epsilon^2 \|\nu(\bar{\theta})\|^2 + 2\epsilon(\nu(\bar{\theta})^T(\nu(\bar{\theta})$. However, $\|\bar{R}^T\mu\|^2 = \|\mu\|^2 = \|\nu(\theta)\|^2$ and $\nu(\bar{\theta})^T\nu(\bar{\theta}) > 0$. Therefore, it follows that $\|\nu(\bar{\theta})\|^2 < (1 + \epsilon^2)\|\nu(\bar{\theta})\|^2$, which contradicts our assumption that $\bar{\theta}$ is a local minimizer of $\|\nu(\cdot)\|$ on $V$. The con-
tradiction implies that \( p \) is not stabilizable through continuous feedback. \( \square \)

As a corollary of the above result, we show that no equilibrium of a spacecraft actuated by a single CMG is stabilizable through continuous feedback.

**Corollary 5.3.** Suppose \( m = 1 \) and let \( \mu \in \mathbb{R}^3 \). Suppose \( p = (R, \theta) \in \mathcal{E}_\mu \). Then \( p \) is not locally asymptotically stabilizable through continuous feedback.

**Proof.** The result follows from Theorem 5.2 by noting that, for a single CMG, the function \( \beta \mapsto \|\nu(\beta)\|^2 \) is a constant, positive-valued function, and hence every configuration is a local maximizer of the function \( \beta \mapsto \|\nu(\beta)\|^2 \). \( \square \)

### 6 Conclusions

The purpose of this paper was to investigate in what manner the presence of singular CMG configuration affects local controllability and stabilizability of the attitude dynamics of a spacecraft actuated by a CMG array. Our main achievement has been to identify a class of singular CMG configurations called critically singular configurations, at which local controllability and stabilizability may be problematic. While equilibria in which the configuration of the CMG array is possibly singular, but not critically singular, are stabilizable and possess the STLC property, for equilibria in which the configuration of the CMG array is critically singular, standard tools such as Sussman’s sufficient condition for STLC and Brockett’s necessary condition for stabilizability yield definite conclusions only under additional assumptions on the CMG configuration. We have shown, for instance, that the attitude dynamics are not stabilizable at equilibrium points where the CMG array is in a critically singular configuration in which the angular momentum magnitude of the CMG array either has a local maximum or a nonzero local minimum. Similarly, we have shown that Sussmann’s sufficient condition for STLC is violated at equilibrium points where the CMG array is in a critically singular configuration such that no CMG gimbal axis is along the total angular momentum vector. On the other hand, equilibria in which there are at least two CMGs whose gimbal axes are along the total angular momentum and whose torques are linearly independent possess the STLC property. In summary, the STLC and stabilizability of equilibria involving critically singular configurations remain largely open, and appear to be challenging analysis questions.

### References


