A Complete Parameterization of All Rational Covariance Extensions

Yohei Kuroiwa*

1 Introduction
It is well known that the spectral density $\Phi$ of a purely nondeterministic, second-order, multivariate stationary stochastic process $\{y(t)\}$ with zero mean is given by the Fourier expansion

$$
\Phi(e^{i\theta}) = \sum_{k=-\infty}^{\infty} C_k e^{ik\theta}
$$
on the unit circle, where the covariance lags

$$
C_k = E\{y(t+k)y^T(t)\}, \quad k = 0, 1, 2, \ldots
$$
are the Fourier coefficients

$$
C_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\theta} \Phi(e^{i\theta}) d\theta.
$$

In the spectral estimation, for a given finite set of observations

$$
y_0, y_1, \ldots, y_N,
$$
the covariances are estimated by

$$
C_k = \frac{1}{N-k+1} \sum_{t=0}^{N-k} y(t+k)y^T(t).
$$

* e-mail: yohei@kth.se, Optimization and Systems Theory, Royal Institute of Technology, Stockholm, Sweden.
The basic problem in the spectral estimation is to estimate the spectral density $\Phi(e^{i\theta})$ by using the estimates of the covariances, and the most widely used model is rational functions [15, 24] and references therein. The estimated rational spectral density has a rational, stable and minimum-phase function $W(z)$ satisfying

$$\Phi(z) = W(z)W^*(z).$$

It is called modeling filter, which shapes a white noise into the stochastic process with the first $n+1$ covariances matching the estimated covariances. The problem to be considered in this paper is a rational covariance extension by ARMA modeling filter of degree $n$

$$W(z) = A(z)^{-1}\Sigma(z).$$

For a given sequence of covariances

$$C = [ C_0 \ C_1 \ \cdots \ C_n ],$$

which is positive in the sense that the Toeplitz matrix

$$T_C = \begin{bmatrix} C_0 & C_1 & \cdots & C_n \\ C_{-1} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ C_{-n} & \cdots & \cdots & C_0 \end{bmatrix}$$

is positive definite, and for each specified MA part of the modeling filter $\Sigma(z)$ of degree $n$, we seek a unique AR part of the modeling filter $A(z)$ of degree $n$ such that

$$W(z)W^*(z) = \hat{C}_0 + \sum_{k=1}^{\infty} \hat{C}_k(z^k + z^{-k});$$

$$\hat{C}_k = C_k \quad \text{for} \quad k = 0, 1, \ldots, n.$$

The system theoretic question is whether the MA part of the ARMA modeling filter provides a complete parameterization of all positive rational extensions of a covariance sequence. It is shown that, for each specified matricial $\Sigma(z)$, there exists an AR part of the modeling filter $W(z)$ [9, 10], and that, for each specified scalar $\Sigma(z)$, this AR part is unique [6, 5, 2]. It has been unanswered whether, for each specified matricial $\Sigma(z)$, there exists a unique AR part of the modeling filter for the solution to the rational covariance extension problem.

In this paper, we show the uniqueness of the AR part of the modeling filter for the given covariance data and each specified MA part of the modeling filter. We prove this uniqueness of the ARMA modeling filter for the solution to the rational covariance extension problem by applying degree theory to a nonlinear map which is homotopic to a nonlinear map determining the maximum entropy solution. In Section 2, we review basic facts about the covariance extension problem, the degree theory and the generalization of a result of the partial realization problem with fixed degree to the matricial setting. We state our main results in Section 3. The generalization of the presented theory to the other interpolation problems, where interpolation conditions are given by contour integral forms, is completed in [18].
2 Preliminaries on Rational Covariance Extension Problem

The symbols $R$ and $C$ denote real numbers and complex numbers. Denote by $R^{j \times k}$ $j \times k$ real matrices. In the rational covariance extension problem, it is assumed that the set of matrices (1) is given as a data, where $C_k \in R^{m \times m}$, $k = 0, \ldots, n$, and $C_0$ is Hermitian.

Given the finite sequence of the real matrices (1), which is positive in the sense that the Toeplitz matrix $T_C$ is positive definite, consider the class of infinite extensions

$$C_{n+1}, C_{n+2}, C_{n+3}, \ldots$$

of (1) such that an $m \times m$ matrix-valued function $F(z)$ defined by

$$F(z) := \frac{1}{2}C_0 + C_1 z^{-1} + C_2 z^{-2} + \ldots$$

is strictly positive real, i.e., it is analytic in the outside of the unit disc $\bar{D}^c := \{z \in C : |z| \geq 1\}$ and the Hermitian real part of $F(z)$ satisfies

$$F(z) + F^r(z) > 0$$

on the unit circle $T := \{\zeta \in C : |\zeta| = 1\}$. In the case that the Toeplitz matrix (2) is positive semidefinite, then, the function is called positive real if the Hermitian part of $F(z)$ satisfies

$$F(z) + F^r(z) \geq 0.$$  

The positive realness implies that $C_{-k} = C_k^T$. We use the notations $A > 0$ and $A \geq 0$ to denote that the matrix $A$ is positive definite and the matrix $A$ is positive semidefinite. Denote by $F^r(z)$ the paraconjugate Hermitian of $F(z)$, i.e., $F^r(z) := F(\overline{z}^{-1})^T$. It is well known that this covariance extension problem is equivalent to the matrix-valued Carathéodory-Fejér interpolation problem, and that a complete parameterization of all solutions to this interpolation problem is given by a linear fractional transformation [13, 17, 1] and reference therein.

For applications to the spectral estimation [15, 24], it is common that the spectral density is estimated by rational models, and that a bound of degree to the rational models is required [14]. Therefore, it is desirable to incorporate a degree constraint on a parameterization of all positive rational extensions of a covariance sequence. It is well known that there exists a minimal degree solution called maximum entropy solution, which maximizes the entropy rate of the estimated spectral density [4]. The parameterization by the linear fractional transformation cannot explicitly deal with the degree constraint since the choice of free parameters to yield minimal degree solutions is highly non-trivial except for the maximum entropy solution. On the other hand, if the strictly positive realness constraint is removed, a parameterization of all rational solutions to the scalar interpolation problem, of which the degree is bounded by the number of the interpolation conditions, is given in [10, 16], and to a multivariable interpolation problem in [12].
We present a generalization of this parameterization to the multivariable interpolation problem. It turns out that there exist two parameterizations of all rational functions interpolating the partial covariance sequence. Each parameterization corresponds to left and right coprime factorizations of the interpolants, respectively. The coprime factors are given by two sets of orthogonal matrix polynomials of the first and second kind [7, 25]. Let \( \Phi(z) \) be strictly positive real functions, satisfying

\[
C_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i k \theta} \Phi(e^{i \theta}) d\theta, \quad k = 0, 1, \ldots, n.
\]

Then, for each partial covariance sequence \((C_0, C_1, \ldots, C_n)\), there exists left orthogonal matrix polynomials of the first kind \(P_{\ell,k}(z)\), \(k = 0, 1, \ldots, n\), satisfying

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} P_{\ell,j}(e^{i \theta}) \Phi(e^{i \theta}) P_{\ell,k}(e^{i \theta}) d\theta = \delta_{jk} I.
\]

For each \(P_{\ell,k}(z)\), \(k = 0, 1, \ldots, n\), we can obtain left orthogonal matrix polynomials of the second kind \(Q_{\ell,k}(z)\), \(k = 0, 1, \ldots, n\), via

\[
Q_{\ell,k}(z) = \left[ P_{\ell,k}(z)(C_0 + 2C_1 z^{-1} + \cdots + 2C_k z^{-k}) \right]_0^k,
\]

where \([ \cdot ]_0^k\) denotes the extraction of the \(k\)-th order polynomial part. There also exist right orthogonal matrix polynomials of the first kind \(P_{r,k}(z)\), \(k = 0, 1, \ldots, n\), satisfying

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} P_{r,j}(e^{i \theta}) \Phi(e^{i \theta}) P_{r,k}(e^{i \theta}) d\theta = \delta_{jk} I
\]

For each \(P_{r,k}(z)\), \(k = 0, 1, \ldots, n\), we can obtain right orthogonal matrix polynomials of the second kind \(Q_{r,k}(z)\), \(k = 0, 1, \ldots, n\), via

\[
Q_{r,k}(z) = \left[ (C_0 + 2C_1 z^{-1} + \cdots + 2C_k z^{-k}) P_{r,k}(z) \right]_0^k.
\]

It is well known that these orthogonal polynomials satisfy the Szegö canonical recurrence relations [19], and that there exists a strictly positive real interpolant of minimal degree given by

\[
F(z) = \frac{1}{2} P_{\ell,n}^{-1}(z) Q_{\ell,n}(z) = \frac{1}{2} Q_{r,n}(z) P_{r,n}^{-1}(z).
\]

This interpolant is so-called maximum entropy solution [4]. Note that the McMillan degree of the maximum entropy solution is \(mn\) as seen from the state space realization of the rational function [3]. We state a generalization of the result in [10, 16]. We adopt the procedure in [10] to derive this generalization.

**Theorem 1.** Any rational \(m \times m\) functions of McMillan degree at most \(mn\), of which power series expansion begins with

\[
\frac{1}{2} C_0 + C_1 z^{-1} + \cdots + C_n z^{-n},
\]

\((9)\)
admits the left coprime factorization $F(z) = A_r^{-1}(z)B_r(z)$ where $A_r(z)$ and $B_r(z)$ are $m \times m$ matrix polynomials of degree at most $n$. The left coprime factors are parameterized by

$$
A_r(z) = 2(P_{r,n}(z) + \Gamma_{r,1}P_{r,n-1}(z) + \cdots + \Gamma_{r,n}P_{r,0}(z))
$$

$$
B_r(z) = Q_{r,n}(z) + \Gamma_{r,1}Q_{r,n-1}(z) + \cdots + \Gamma_{r,n}Q_{r,0}(z),
$$

where $\Gamma_{r,k} \in \mathbb{R}^{m \times m}$, $k = 1, \cdots, n$, are free parameters. Similarly, any rational functions of McMillan degree at most $mn$, of which power series expansion that begins with (9) admits the right coprime factorization $F(z) = B_r(z)A_r^{-1}$ where $A_r(z)$ and $B_r(z)$ are $\ell \times \ell$ matrix polynomials of degree at most $n$. The right coprime factors are parameterized by

$$
A_r(z) = 2(P_{r,n}(z) + P_{r,n-1}(z)\Gamma_{r,1} + \cdots + P_{r,0}(z)\Gamma_{r,n})
$$

$$
B_r(z) = Q_{r,n}(z) + Q_{r,n-1}(z)\Gamma_{r,1} + \cdots + Q_{r,0}(z)\Gamma_{r,n},
$$

where $\Gamma_{r,k} \in \mathbb{R}^{m \times m}$, $k = 1, \cdots, n$, are free parameters.

**Proof.** Let us introduce a reciprocal of the $m \times m$ matrix polynomial $A(z)$, defined by $\hat{A}(z) = z^k A^*(z)$. Then, the $m \times m$ matrix polynomials $A_r(z)$ and $B_r(z)$ are related through

$$
B_r(z) = \left[(C_0 + 2C_{-1}z + \cdots + 2C_{-n}z^n)\hat{A}_r(z)\right]^n.
$$

The above represents a nonsingular transformation between $m \times m$ matrix polynomials of degree at most $n$. The two sets of matrix polynomials, $z^{n-k}P_k(z)$, $k = 0, \ldots, n$, and $z^{n-k}Q_k(z)$, $k = 0, \ldots, n$, form a basis for this space and they are related through

$$
z^{n-k}\hat{Q}_{k,l}(z) = \left[(C_0 + 2C_{-1}z + \cdots + 2C_{-n}z^n)z^{n-k}\hat{P}_{k,l}(z)\right]^n.
$$

Thus, $F^*(z)$ of which the power series expansion that begins with $\frac{1}{2}C_0 + C_{-1}z + \cdots + C_{-n}z^n$ is parameterized via

$$
F^*(z) = \frac{1}{2}(\hat{Q}_{l,n}(z) + z\hat{Q}_{l,n-1}(z)\Gamma_{l,1} + \cdots + z^n\hat{Q}_{l,0}(z)\Gamma_{l,n})
$$

$$
(\hat{P}_{l,n}(z) + z\hat{P}_{l,n-1}(z)\Gamma_{l,1} + \cdots + z^n\hat{P}_{l,0}(z)\Gamma_{l,n})^{-1}.
$$

The derivation of the right coprime factor representation of $F(z)$ is similar.

As seen from (10), the choice $\Gamma_{\ell,k} = 0$ yields the maximum entropy solution. Except for this maximum entropy solution, it is highly non-trivial to describe the set of $\Gamma_{\ell,k}$, for which the rational functions $F(z) = A_r^{-1}(z)B_r(z)$ are strictly positive real. The parameterization (10) implies that strictly positive real rational functions of minimal degree are represented as

$$
F(z) := A^{-1}(z)B(z).
$$

(11)
\(A(z)\) and \(B(z)\) are \(m \times m\) matricial pseudopolynomials of degree \(n\), defined by

\[
A(z) := AG \\
B(z) := BG \\
A := \begin{bmatrix} A_0 & A_1 & \cdots & A_n \end{bmatrix} \\
B := \begin{bmatrix} B_0 & B_1 & \cdots & B_n \end{bmatrix} \\
G := \begin{bmatrix} I & z^{-1}I & \cdots & z^{-n}I \end{bmatrix}^T
\]

where \(A_k \in R^{m \times m}\) and \(B_k \in R^{m \times m}\), \(k = 0, \ldots, n\). The strictly positive realness of 
\(F(z)\) requires that \(A(z)\) and \(B(z)\) are outer, i.e., \(A^{-1}(z)\) and \(B^{-1}(z)\) are analytic in \(D^c\). We denote by \(\mathcal{A}\) the set of \(m \times m\) outer matricial pseudopolynomials of degree at most \(n\). It is non-convex and connected. We use a notation \(A \in \mathcal{A}\) to denote that the \(m \times m\) matricial pseudopolynomials \(A(z)\) of degree at most \(n\) are outer.

In view of (4),

\[
\Phi(z) := F(z) + F^*(z)
\]

is a rational spectral density of the \(m\)-dimensional nondeterministic stationary stochastic process. It is well known that the spectral density has a unique minimum-phase spectral factor, i.e., a rational and minimum-phase transfer function \(W(z)\) analytic in \(D^c\) such that

\[
W(z)W^*(z) = F(z) + F^*(z).
\]

It is clear that \(W(z)\) has the form

\[
W(z) = A^{-1}(z)\Sigma(z)
\]

for an \(m \times m\) outer matricial pseudopolynomial \(\Sigma(z)\) of degree \(n\), defined by

\[
\Sigma(z) := \Sigma_0 + \Sigma_1 z^{-1} + \cdots + \Sigma_n z^{-n} = \Sigma G
\]

where \(\Sigma := \begin{bmatrix} \Sigma_0 & \Sigma_1 & \cdots & \Sigma_n \end{bmatrix} \in R^{m \times m(n+1)}\). The \(\Sigma(z)\) is a minimum-phase spectral factor of a matricial pseudopolynomial \(D(z)\) [21], defined by

\[
D(z) := A(z)B^*(z) + B(z)A^*(z).
\]

The maximum entropy solution (8) has the modeling filter \(z^n P^{-1}_{\ell, n}(z)\) of AR type, and thus, the corresponding spectral density is given by

\[
\Phi(z) = A(z)^{-1}A^{-*}(z),
\]

which has no finite zeros. It is well known that this maximum entropy solution [4], which maximizes the entropy rate of the spectral density

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \log \det \Phi(e^{i\theta}) d\theta,
\]
is computed by the Levinson algorithm [23]. In [8, 2], it is shown that this maximum entropy solution is given at a stationary condition

$$\frac{\partial J}{\partial A^T} = 0$$  \hspace{1cm} (13)\

of a non-convex optimization problem

$$\min_{A \in \mathcal{A}} J(A),$$

where

$$J(A) = \text{tr} A^T A^T - \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \det A^* (e^{i\theta}) d\theta.$$\

The stationary condition (13) yields a linear equation and determines the unique maximum entropy solution, while the optimization problem is non-convex since the domain $\mathcal{A}$ is non-convex. The approach employed in this paper is degree theory instead of optimization. We consider a family of nonlinear maps, which is homotopic to a nonlinear map $G(A)$, defined by

$$G(A) := A^T - \frac{1}{2\pi} \int_{-\pi}^{\pi} A^{-*}G^* d\theta. \hspace{1cm} (14)$$

The stationary condition (13), at which the maximum entropy solution is determined, is given by $G(A) = 0$. Before presenting the main results to be proved by degree theory, we present a brief review of some basic facts from the degree theory [20].

Suppose that $U, V \subset \mathbb{R}^n$ are open connected subsets and that $F : U \rightarrow V$ is a continuously differentiable function on $U$. It is frequently of considerable interest to know in advance the number of solutions of the nonlinear equation

$$y = F(x)$$
in some specified set. Denote by $\partial U$ the boundary of the set $U$. Suppose that, for a given

$$y \notin F(\partial U),$$

the Jacobian matrix of $F$, denoted by $\text{Jac}_x(F)$, at all $x$ is nonsingular for all

$$x \in U_s := \{x \in U|y = F(x)\}.$$\

Then, the degree of $F$ with respect to $y$ is defined by

$$\deg_y(F) := \sum_{U_s} \text{sign} \det \text{Jac}_x(F).$$
One of the important properties of the degree of map is the homotopy invariance of the degree. Let $H$ be a jointly continuous map from $U \times [0,1] \rightarrow V$ such that $H(x,0) = G(x)$ and $H(x,1) = F(x)$. Suppose that $y$ satisfies $H(x,\lambda) \neq y$ for all $(x,\lambda) \in \partial U \times [0,1]$. Then,

$$\deg_y(F) = \deg_y(G)$$

holds. We only consider the degree with respect to zero. Thus, we simply denote $\deg_y(F)$ as $\deg(F)$.

### 3 A Complete Parameterization of All Rational Covariance Extensions

We state the main result of this paper.

**Theorem 2.** Suppose that the Toeplitz matrix $T_C$ is positive definite. Then, for the given partial covariance sequence $(C_0, C_1, \ldots, C_n)$ and each outer $m \times m$ matricial pseudopolynomials $\Sigma(z)$ of degree $n$, there exists a unique outer $m \times m$ matricial pseudopolynomials $A(z)$ of degree $n$ such that $W(z) = A^{-1}(z)\Sigma(z)$ is the minimum spectral factor of the spectral density $\Phi(z)$ satisfying

$$\Phi(z) = \hat{C}_0 + \sum_{k=1}^{\infty} \hat{C}_k (z^k + z^{-k});$$

$$\hat{C}_k = C_k \text{ for } k = 0, 1, \ldots, n.$$ 

The unique outer $m \times m$ matricial pseudopolynomial $A(z)$ is determined by solving a nonlinear equation $F(A) = 0$, where the nonlinear map $F(A) : A \rightarrow \mathbb{R}^{m \times m(n+1)}$ is defined by

$$F(A) := AT_C - \frac{1}{2\pi} \int_{-\pi}^{\pi} \Sigma(e^{i\theta})\Sigma^*(e^{i\theta})A^{-*}(e^{i\theta})G^* d\theta.$$ 

(15)

Moreover, a map $C \times \Sigma \rightarrow A$ is smooth, and, for the fixed $C$, the map $\Sigma \rightarrow A$ is diffeomorphic.

**Proof.** First, we shall show that the solution to $F(A) = 0$ cannot be on the boundary of $A$. Denote by $\mathcal{A}$ the closure of $A$ and by $\partial A$ the boundary of $A$. The set $\partial A$ consists of $m \times m$ matricial pseudopolynomials of which some of the roots are on $T$. The condition that $A(z)$ has roots on $T$ is equivalent to a condition of the positive realness of $F(z) = A^{-1}(z)B(z)$ [22, 9].

**Lemma 3.** A rational function is positive real if and only if

$$\det[A(z) + B(z)]$$

is positive for all $z$.这就说明了正实性的条件。
is analytic in $D^c$, and $D(z)$ defined by (12) is nonnegative definite on $T$.

This lemma implies that $\det[A(z) + B(z)]$ has no roots in $D^c$. However, roots on the boundary are possible [11]. Assume that the $A(z) + B(z)$ has a root $e^{j\alpha} \in T$ with geometric multiplicity one. Then, there exists a vector $u \in C^n$ such that

$$u^* [A(e^{j\alpha}) + B(e^{j\alpha})] = 0.$$ 

From

$$\frac{1}{2} [A(z)B^*(z) + B(z)A^*(z)] = |A(z) + B(z)|^2 - |A(z) - B(z)|^2$$

$$= |\Sigma(z)|^2$$

$$\geq 0,$$

it follows that $A(z) - B(z)$ shares the same root as $A(z) + B(z)$. Therefore, it is a root of $A(z)$ and $B(z)$ as well. And it is also a root of the $\Sigma(z)$. This contradicts the assumption that the $\Sigma(z)$ is outer. This contradiction implies that the interpolant, which is determined by solving the nonlinear equation $F(A) = 0$, is strictly positive real. Thus, the solution $A(z)$ has no roots on $T$. This asserts that

$$0 \notin F(\partial A),$$

and that the degree of $F$ with respect to zero can be computed.

To prove the uniqueness of the solution to $F(A) = 0$, first, we show that the determinant of the Jacobian of the map $F(A)$ is positive. To this end, compute the directional derivative

$$\delta F(A) = \delta AT_C + \frac{1}{2\pi} \int_{-\pi}^{\pi} \Sigma \Sigma^* A^{-*} G^* \delta A^T A^{-*} G^* d\theta$$

and show that, for all $\delta A \in R^{m \times m(n+1)}$, the trace of the symmetric part of (17)

$$\frac{1}{2} \text{tr} [\delta F(A) \delta A^T + \delta A \delta F^*(A)]$$

is positive. For economy of notation, we simply write $\int \Omega$ to denote integrals of the form $\frac{1}{2\pi} \int_{-\pi}^{\pi} \Omega(e^{i\theta}) d\theta$ in the rest of the proof. Let us define $\delta A := \delta AG$ and $\delta R := \delta AA^{-1}$. Then, (18) is

$$\text{tr} \delta AT_C \delta A^T + \frac{1}{2} \int \text{tr} [\Sigma^2 (A^{-*} \delta A^* A^{-*} \delta A^* + \delta AA^{-1} \delta AA^{-1})]$$

$$= \text{tr} \delta AT \delta A^T + \frac{1}{2} \int \text{tr} [\Sigma^2 (|\delta R - \delta R^*|^2 + |\delta R|^2 + |\delta R^*|^2)].$$

Thus, (18) is positive, i.e., the symmetric part of the Jacobian of $F(A)$ is positive definite, and its uniform lower bound is given by the minimum eigenvalue of the Toeplitz matrix. Thus, the minimum eigenvalue of the Jacobian is positive. Hence, the determinant of the Jacobian is also positive. According to the degree theory,
the number of the solutions to the nonlinear equation $F(A) = 0$ is the same as the degree of the map $F(A)$ with respect to zero. To show the uniqueness of the solution to the nonlinear equation $F(A) = 0$, we see that the degree of the map $F(A)$ with respect to zero is in fact one. It is shown that there is the unique maximum entropy solution, which is determined by solving the nonlinear equation $G(A) = 0$, defined by (14). Thus, $\deg G = 1$. Then, we construct a homotopy from $G(A)$ to $F(A)$. In fact, there is a jointly continuous map $H : A \times [0, 1] \rightarrow \mathbb{R}^{m \times m(n+1)}$ defined by

$$H(A, \lambda) := (1 - \lambda)G(A) + \lambda F(A), \quad \lambda \in [0, 1],$$

where $H(A, 0) = G(A)$ and $H(A, 1) = F(A)$ hold. Hence, the homotopy invariance of the degree of the maps assures that $\deg F = 1$. Hence, there is a unique solution to the nonlinear equation $F(A) = 0$. Note that the non-zero degree implies the existence of the solution.

Finally, we show that this unique solution $\Phi(z) = A^{-1}(z)\Sigma(z)\Sigma^*(z)A^{-*}(z)$ satisfies the interpolation conditions

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\theta} \Phi(e^{i\theta}) d\theta = C_k, \quad k = 0, 1, \ldots, n.$$

Denote the Fourier expansion of $\Phi(e^{i\theta})$ by

$$\Phi(e^{i\theta}) = \sum_{k=-\infty}^{\infty} F_k e^{ik\theta}.$$

The second term of $F(A)$ is

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \Sigma(e^{i\theta})\Sigma^*(e^{i\theta})A^{-*}(e^{i\theta})G^* d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} A(e^{i\theta})\Phi(e^{i\theta}) d\theta$$

$$= A \frac{1}{2\pi} \int_{-\pi}^{\pi} G(\Phi(e^{i\theta}))G^* d\theta = AT_F,$$

where $T_F$ is the Toeplitz matrix with components $F_k, k = 1, \ldots, n$, Thus, $F(A) = A(T_C - T_F)$ and each Fourier coefficient $F_k, k = 0, 1, \ldots, n$ is a function of $A$. It is clear that $C_k = F_k, k = 0, 1, \ldots, n$ yields $F(A) = 0$. It has already been shown that the existence and uniqueness of the solution to $F(A) = 0$. Hence, the corresponding unique $\Phi(z)$ satisfies the interpolation conditions.

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Bibliography


