Optimal Control of a Class of Time-Varying Hyperbolic Distributed Parameter Systems

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Abstract: A class of hyperbolic distributed parameter time-varying system on the interval $[0, \infty)$ is considered. The corresponding linear-quadratic (LQ) optimal control problem is studied by using the infinite-dimensional state-space formulation and the well-known Riccati equation approach.

Keywords: Time-varying infinite-dimensional systems, evolution systems, LQ-Control, Riccati equation.

1 Introduction

Many unit operations in chemical plants include transport processes that can best be described by partial differential equations (PDEs): see [2] and [3]. When diffusive transport is negligible and convective transport is dominant, processes can be described by first-order hyperbolic PDEs. The class of such processes includes plug flow reactors.

In [2], the linear quadratic control problem was studied for a time-varying distributed parameter model of a plug flow reactor. The main motivation was the fact that time-varying rates of reaction arise from loss of catalyst activity which is an important issue in catalytic reactors. A simple exponential decay model form was adopted. Here our objective is to extend the previous work to a more general class of time-varying rates.

The contributions of this paper can be summarized as follows. In section 2, we describe the class of hyperbolic time-varying systems that we are interested in. In designing an LQ-controller, some useful results on the dynamical properties of the model are established in Section 3. The optimal control design problem is the subject of Section 3. An LQ-control

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feedback is computed by using the corresponding operator Riccati differential equation, whose
solution can be obtained via a related matrix Riccati partial differential equation.

2 Model Description

Consider a linear hyperbolic first-order partial differential equations time-varying system in
one spatial dimension:

$$\frac{\partial x}{\partial t} = -\frac{\partial x}{\partial z} + M(z, t)x(z, t) + B_0(z, t)u(z, t)$$

(1)

$$y(z, t) = C_0(z, t)x(z, t),$$

(2)

with boundary condition

$$x(0, t) = x_B, \quad \forall t \geq 0,$$

and initial condition

$$x(z, 0) = x_0(z), \quad \forall z \in [0, 1],$$

where $$x(\cdot, t) = [x_1(\cdot, t), \ldots, x_n(\cdot, t)]^T \in H := L^2(0, 1)^n$$ denotes the vector of state variables, $$z \in [0, 1] \in \mathbb{R}$$ and $$t \in [0, \infty)$$ denote position and time, respectively, $$u(\cdot, t) = [u_1(\cdot, t), \ldots, u_m(\cdot, t)]^T \in U := L^2(0, 1)^m$$ denotes the input variable, $$y(\cdot, t) = [y_1(\cdot, t), \ldots, y_p(\cdot, t)]^T \in Y := L^2(0, 1)^p$$ denotes the output variable. $$M, B_0$$ and $$C_0$$ are matrices whose entries are functions in $$L^\infty([0, 1] \times [0, \infty))$$. $$x_B$$ is a constant column vector and $$x_0(z) \in H$$. Without loss of generality assume that $$x_B = 0$$.

The previous system can be obtained from linearization around a desired equilibrium profile
of the nonlinear time-varying first-order PDE system:

$$\frac{\partial x}{\partial t} = -\frac{\partial x}{\partial z} + f(t, x, u)$$

$$y(t) = h(x(t)).$$

The equivalent state-space description of the model (1)-(2) is given by the following linear
abstract differential equation on the Hilbert space $$H$$:

$$\begin{cases} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ x(0) &= x_0 \in H \\ y(t) &= C(t)x(t). \end{cases}$$

(3)

Here $$\{A(t)\}_{t \geq 0}$$ is the family of linear operators defined on their domains:

$$D(A(t)) = \{x \in H : x \text{ is a.c.}, \frac{dx}{dz} \in H \text{ and } x(0) = 0\}$$

(4)

(where a.c. means that the function $$x$$ is absolutely continuous) by

$$A(t) = -\frac{d}{dz} + M(z, t) \cdot I,$$

(5)

and the family input and the output operators $$\{B(t)\}_{t \geq 0}$$ and $$\{C(t)\}_{t \geq 0}$$ are given by

$$B(t) := B_0(z, t) \cdot I$$ and $$C(t) := C_0(z, t) \cdot I,$$

(6)

respectively, where $$I$$ is the identity operator.
3 Trajectory and Exponential Stability

In this section, we are interested in the trajectory and the exponential stability of the linear model described in the previous section. The following lemma, whose proof can be made by using the perturbation theorem (see [7, Theorem 2.3, p.133]), is useful in order to prove the existence and uniqueness of the trajectory of the linear model (3).

Lemma 1. Consider the family of operators \( \{A(t)\}_{t \geq 0} \) given by (4)-(5). Then \( \{A(t)\}_{t \geq 0} \) is a stable family of infinitesimal generators.

By observing that the domain of the family \( \{A(t)\}_{t \geq 0} \) is independent of \( t \), the following theorem can be proved by using [7, Theorem 4.6, p.143].

Theorem 3.1. Consider the family of operators \( \{A(t)\}_{t \geq 0} \) given by (4)-(5). Then, there exists a unique evolution system \( U_A(\cdot, \cdot) : \{(t, s) \in \mathbb{R}^2 : t \geq s\} \rightarrow \mathcal{L}(H) \) such that

\[
\frac{\partial}{\partial t} U_A(t, s)x = A(t)U_A(t, s)x, \ \forall x \in D(A(t)), \ 0 \leq s \leq t.
\]

Moreover, there are constants \( M \geq 1 \) and \( w \) such that

\[
\|U_A(t, s)\| \leq Me^{w(t-s)}, \ 0 \leq s \leq t.
\]

Now we are in a position to state the following stability theorem.

Theorem 3.2. Consider the family of operators \( \{A(t)\}_{t \geq 0} \) given by (4)-(5). Then \( \{A(t)\}_{t \geq 0} \) generates an exponentially stable evolution system.

Proof: This result can be proved by two ways. The first one is based on the corresponding Lyapunov equation and it suffices to prove that the latter has a nonnegative solution. The second one is based on [7, Theorem 8.1, p.173].

4 Control Design

This section is devoted to the linear-quadratic optimal problem (see e.g. [4] and reference therein) in order to design a LQ-state feedback controller for system (3). For any system given by (3) let us consider the LQ-control problem: for any initial state \( x_0 \in H \), we want to minimize the cost functional

\[
J(u) = \int_0^{\infty} \{|C(s)x(s)|^2 + |u(s)|^2\}ds \tag{7}
\]

overall controls \( u \in L^2(0, \infty; U) \) subject to the differential equation constraint (3).

The solution of this problem can be obtained by finding the minimal nonnegative self-adjoint bounded operator \( Q(t) \) which solves the following operator Riccati equation

\[
\dot{Q} + A^*Q + QA - QBB^*Q + C^*C = 0, \text{ in } [0, \infty] \tag{8}
\]

Note that the fact that \( A(t) \) generates an exponentially stable evolution system (see Theorem 3.2) then \( (A(t), B(t)) \) is \( C(t) \)-stabilizable. An immediate consequence of Theorems 3.1 and 3.2 is the following result: see [4, Theorem 5.2, p. 507].
**Theorem 4.1.** The operator Riccati equation (8) has a unique nonnegative bounded solution $Q$. Moreover, the optimal control $u^*$ is given by

$$u^*(t) = -B^*(t)Q(t)x^*(t),$$

and the optimal cost criterion is given by

$$J(u^*) = \langle Q(0)x_0, x_0 \rangle.$$

It turns out that the solution of corresponding operator Riccati differential equation is based on the solution of a matrix Riccati partial differential equation.

**Lemma 2.** Let us consider the matrix Riccati partial differential equation:

$$\frac{\partial \Psi}{\partial t} = -\frac{\partial \Psi}{\partial z} + M^* \Psi + \Psi M - C^*C + \Psi BB^*\Psi, \quad \Psi(t, 1) = 0, \quad t \in [0, \infty). \quad (9)$$

The latter has a unique nonnegative bounded solution on $[0, 1] \times [0, \infty)$.

**Proof:** By using the method of characteristics, the matrix Riccati partial differential equation becomes the following matrix Riccati differential equation along the characteristics

$$\frac{d \Psi}{d r} = M^* \Psi + \Psi M + Q - \Psi S \Psi, \quad \Psi(1) = 0. \quad (10)$$

Then by using [1, Corollary 6.7.36], it can be shown that equation (10) has a nonnegative solution.

Now we are in a position to state the following theorem.

**Theorem 4.2.** Let us consider the time-varying linear model (3)-(6). Let $\Psi(t, z) = \Psi^*(t, z) \geq 0$ be the nonnegative solution of the matrix partial differential equation (9). Then $Q(t) := \Psi(t, z)I$ is the unique nonnegative solution of the operator Riccati differential equation (8).

**Proof:** By assuming that the solution of the operator Riccati equation (8) has the form $Q(t) = \Psi(t, z)I$, it can be shown by straightforward calculation that if $\Psi$ is a nonnegative solution of the matrix Riccati partial differential equation (9) then $Q$ is a nonnegative solution of equation (8).

**5 Conclusion**

In this paper, the linear quadratic optimal control problem has been studied for a class of hyperbolic time-varying PDEs. To design the LQ-controller, some useful dynamical properties of the model have been established. An LQ-control feedback has been computed by using an operator Riccati differential equation, whose solution can be obtained via a related matrix Riccati partial differential equation.
Bibliography


