Stability Analysis of Continuous-Time Macroeconometric Systems

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Abstract. There has been increasing interest in continuous-time macroeconomic models. This research investigates stability of the Bergstrom, Nowman, and Wymer continuous-time model of the U.K. when system parameters change. This particularly well-regarded continuous-time macroeconomic model is chosen to assure the empirical and potential policy relevance of the results. Stability analysis is important with this model for understanding the dynamic properties of the system and for determining which parameters are the most important to those dynamic properties. The main objective of this paper is to determine the boundaries of parameters at which instability occurs. Two types of boundaries are found: the transcritical bifurcation boundary and the Hopf bifurcation boundary, corresponding to two different ways that instability occurs when parameter values cross the bifurcation boundary.

The existence of the Hopf bifurcation boundary is particularly useful, since Hopf bifurcation may provide explanations for some cyclical phenomena in macroeconomy. Numerical algorithms are designed to locate the stability boundaries, which are displayed in three-dimensional diagrams. A notable and perhaps surprising fact is that both types of bifurcations can coexist with this well-regarded U.K. model—in the same neighborhood of the parameter space.

Keywords. stability, bifurcation, macroeconomic systems

1 Introduction

In recent years, there has been increasing interest in using continuous-time models to describe macroeconomic systems. Continuous-time econometrics has been very important for dynamic disequilibrium modeling. The specification of econometric models in continuous time rather than discrete time has several advantages, such as the characterization of the interaction between the variables during the unit observation period, more accurate representation of the partial adjustment processes in dynamic disequilibrium models, the independence of the unit of the observation period, and the capability of forecasting the continuous-time path of the variables. An informative discussion of the advantages is provided by Bergstrom (1996). Since the development of the first continuous-time macroeconometric model by Bergstrom and Wymer (1976), there has been a significant growth in the use of continuous-time econometric methods in macroeconomic modeling. Economy-wide continuous-time models have been developed for most of the leading industrial countries of the world (Bergstrom, Nowman, and Wymer 1992). The idea is to model a system by a set of differential equations. An important feature of the continuous-time models is that the estimator uses a discrete model that is satisfied by the observations generated by the differential equation system irrespective of the observation interval of the sample, so that the properties of the parameters of the differential equation system can be derived from the sampling properties of the discrete model. A recent survey was given by Bergstrom (1996).

Most research on continuous-time models focuses on estimation and model building for various economic systems. Continuous-time economic models have been built, for example, for the U.K. (Bergstrom and Wymer 1976; Knight and Wymer 1978), for the U.S. (Donaghy 1993), for The Netherlands (Nieuwenhuis 1994), and for Italy (Tullio 1972; Gandolfo and Padoan 1990). A complete list of different models is provided by...
Bergstrom (1996). With these models available, the next stage of research is naturally performance analysis. It is important to understand the structural properties of the continuous-time economic models. There are several papers dealing with stability of continuous-time models. Particularly, Bergstrom, Nowman, and Wymer (1992) and Donaghy (1993) examined the stability of the models for the U.K. and the U.S. economies, respectively. They noticed that for the estimated parameter values, these models are slightly unstable. Bergstrom, Nowman, and Wandasiewicz (1994) analyzed the effect of monetary and fiscal feedback controls on the stability of the U.K. model and found that simple fiscal policy feedbacks cannot stabilize the system. They further obtained a stabilizing controller, though the physical implementation of the controller is unclear, based on linear quadratic control theory. Nieuwenhuis and Schoonbeek (1997) investigated the relationship between the stability of continuous-time models and the structure of the matrices appearing in the models. Their results were obtained by analyzing the dominant diagonal structures of the matrices. Gandolfo (1992) considered sensitivity analysis of continuous-time models and their use in investigating bifurcations. Wymer (1997) suggested the study of singularities and bifurcations of continuous-time models. Barnett and colleagues (1996) explored, among other results, chaotic phenomena in economic systems.

This paper describes our recent effort in stability analysis of a continuous-time macroeconometric model of the U.K. as given by Bergstrom, Nowman, and Wymer (1992). An approach for finding the stability boundaries is provided.

The rest of the paper is organized as follows. Section 2 introduces the continuous-time macroeconometric model. Section 3 contains the linearized model, and uses the gradient method to find a set of parameter values under which the system is stable. Section 4 proposes a numerical algorithm for finding stability boundaries. Section 5 implements the algorithm for several special cases to locate explicit boundaries. Finally, a few conclusive remarks and further research directions are provided in the last section.

2 The Model

We consider Bergstrom, Nowman, and Wymer’s (1992) continuous-time macroeconometric model of the U.K. To introduce the model, two sets of variables are first defined.

2.1 Endogenous variables
The following endogenous variables are used in the differential equations that describe the model:

- \( C \) real private consumption
- \( E_n \) real non-oil exports
- \( F \) real current transfers abroad
- \( I \) volume of imports
- \( K \) amount of fixed capital
- \( K_a \) cumulative net real investment abroad (excluding changes in official reserve)
- \( L \) employment
- \( M \) money supply
- \( P \) real profits, interest, and dividends from abroad
- \( p \) price level
- \( Q \) real net output
- \( q \) exchange rate (price of sterling in foreign currency)
- \( r \) interest rate
- \( w \) wage rate
2.2 Exogenous variables
These terms define the exogenous variables of the model:

\( d_x \) dummy variable for exchange controls (\( d_x = 1 \) for 1974–1979, and \( d_x = 0 \) for 1980 onward)
\( E_o \) real oil exports
\( G_c \) real government consumption
\( p_f \) price level in leading foreign industrial countries
\( p_i \) price of imports (in foreign currency)
\( r_f \) foreign interest rate
\( T_1 \) total taxation policy variable defined by Bergstrom et al. (1992, p. 317)
\( T_2 \) indirect taxation policy variable defined by Bergstrom et al. (1992, p. 317)
\( t \) time
\( Y_f \) real income of leading foreign industrial countries

2.3 The Model
The dynamic behavior of the U.K. economy is described by the following 14 differential equations.

\[
D^2 \log C = \gamma_1 (\lambda_1 + \lambda_2 - D \log C) \\
+ \gamma_2 \log \left[ \frac{\beta_1 e^{-[\beta_3 (r - D \log p) + \beta_3 D \log p]} (Q + P)}{T_1 C} \right] \\
(1)
\]

\[
D^2 \log L = \gamma_3 (\lambda_2 - D \log L) + \gamma_4 \log \left[ \frac{\beta_4 e^{-\lambda_2 \{Q - \beta_5 K - \beta_6 \} - 1/\beta_6}}{L} \right] \\
(2)
\]

\[
D^2 \log K = \gamma_5 (\lambda_1 + \lambda_2 - D \log K) + \gamma_6 \log \left[ \frac{\beta_5 (Q/K)^{1+\beta_6}}{r - \beta_7 D \log p + \beta_8} \right] \\
(3)
\]

\[
D^2 \log Q = \gamma_7 (\lambda_1 + \lambda_2 - D \log Q) \\
+ \gamma_8 \log \left[ \frac{(1 - \beta_3 (q/p) \beta_3)}{Q} \right] \\
(4)
\]

\[
D^2 \log p = \gamma_9 (D \log (w/p) - \lambda_1) \\
+ \gamma_{10} \log \left[ \frac{\beta_{11} \beta_4 T_2 w e^{-\lambda_2 \{1 - \beta_3 (Q/K) \beta_3 \} - (1+\beta_5) + \beta_8}}{p} \right] \\
(5)
\]

\[
D^2 \log w = \gamma_{11} (\lambda_1 - D \log (w/p)) + \gamma_{12} D \log (p/qp) \\
+ \gamma_{13} \log \left[ \frac{\beta_{11} e^{-\lambda_1 \{Q - \beta_5 K - \beta_6 \} - 1/\beta_6}}{\beta_{12} e^{\lambda_2 t}} \right] \\
(6)
\]

\[
D^2 r = -\gamma_1 D r + \gamma_{15} \left[ \beta_1 + r_f - \beta_6 D \log q + \beta_{15} \frac{p(Q + P)}{M} - r \right] \\
(7)
\]
\[ D^2 \log I = \gamma_1 (\lambda_1 + \lambda_2 - D \log (pi/qp)) + \gamma_7 \log \left[ \frac{\beta_1 (qp/pi)^{\beta_0} (C + G + DK + E_n + E_0)}{(pi/qp)} \right] \] (8)

\[ D^2 \log E_n = \gamma_9 (\lambda_1 + \lambda_2 - D \log E_n) + \gamma_9 \log \left[ \frac{\beta_1 \gamma_{15} (p/qp)^{\beta_{17}}}{E_n} \right] \] (9)

\[ D^2 F = -\gamma_{20} DF + \gamma_{21} (Q + P - F) \] (10)

\[ D^2 P = -\gamma_{22} DP + \gamma_{23} \{ \beta_{20} + \beta_{21} (r_f - D \log p_f) \} K_a - P \] (11)

\[ D^2 K_a = -\gamma_{24} DK_a + \gamma_{25} \{ \beta_{22} + \beta_{23} (r_f - r) - \beta_{24} D \log q - \beta_{25} d_3 (Q + P) - K_a \} \] (12)

\[ D^2 \log M = \gamma_{26} (\lambda_3 - D \log M) + \gamma_{27} \log \left[ \frac{\beta_{26} e^{\lambda_3 t}}{M} \right] + \gamma_{29} D \log \left[ \frac{E_n + E_0 + P - F}{(pi/qp)} \right] + \gamma_{29} \log \left[ \frac{E_n + E_0 + P - F - DK_a}{(pi/qp)} \right] \] (13)

\[ D^2 \log q = \gamma_{30} D \log (p_f/qp) + \gamma_{31} \log \left[ \frac{\beta_{30} p_f}{qp} \right] + \gamma_{32} D \log \left[ \frac{E_n + E_0 + P - F}{(pi/qp)} \right] + \gamma_{33} \log \left[ \frac{E_n + E_0 + P - F - DK_a}{(pi/qp)} \right] \] (14)

where \( D \) is the differential operator, \( D x = dx/dt, D^2 x = d^2 x/dt^2 \), \( \beta_i, i = 1, 2, \ldots, 27, \gamma_j, j = 1, 2, \ldots, 33 \), and \( \lambda_k, k = 1, 2, 3 \), are structural parameters that can be estimated from historical data. A set of their estimates using quarterly data from 1974–1984 are given by Bergstrom, Nowman, and Wymer (1992, Table 2). These equations are formulated based on economic theory. The exact interpretations of these 14 equations are omitted here because they are not needed in this paper and can be found in Bergstrom, Nowman, and Wymer’s work (1992).

Equations (1)–(14) are nonlinear. To study the steady-state behavior, it was assumed by Bergstrom, Nowman, and Wymer (1992) that the exogenous variables satisfy the following conditions:

\[ d_x = 0 \]
\[ E_n = 0 \]
\[ G_e = g^* (Q + P) \]
\[ p_f = p_{f1} e^{\lambda_1 t} \]
\[ p_i = p_{i1} e^{\lambda_2 t} \]
\[ r_f = r_{f1} \]
\[ T_1 = T_{11} \]
\[ T_2 = T_{22} \]
\[ Y_f = Y_{f1} e^{(\lambda_1 + \lambda_2)/\beta_3 t} \]

where \( g^*, p_{f1}, p_{i1}, r_{f1}, T_{11}, T_{22}, Y_{f1} \), and \( \lambda_3 \) are constants.

Under the assumption of the exogenous variables, it can be shown that \( C(t), \ldots, q(t) \) in Equations (1)–(14) changes at constant rates in equilibrium. In what follows, we study the behavior of the system of differential
equilibria of the system of Equations (1)–(14) corresponds to zero values of $y_1(t), y_2(t), \ldots, y_{14}(t)$ be defined as follows:

\[
y_1(t) = \log\left(\frac{C(t)}{C^* e^{(\lambda_1 + \lambda_2)t}}\right)
\]

\[
y_2(t) = \log\left(\frac{L(t)}{L^* e^{\lambda_1 t}}\right)
\]

\[
y_3(t) = \log\left(\frac{K(t)}{K^* e^{(\lambda_1 + \lambda_2)t}}\right)
\]

\[
y_4(t) = \log\left(\frac{Q(t)}{Q^* e^{(\lambda_1 + \lambda_2)t}}\right)
\]

\[
y_5(t) = \log\left(\frac{\beta(t)}{\beta^* e^{(\lambda_1 - \lambda_2)t}}\right)
\]

\[
y_6(t) = \log\left(\frac{w(t)}{w^* e^{(\lambda_3 - \lambda_2)t}}\right)
\]

\[
y_7(t) = r(t) - r^*
\]

\[
y_8(t) = \log\left(\frac{I(t)}{I^* e^{(\lambda_1 + \lambda_2)t}}\right)
\]

\[
y_9(t) = \log\left(\frac{E_n(t)}{E_n^* e^{(\lambda_1 + \lambda_2)t}}\right)
\]

\[
y_{10}(t) = \log\left(\frac{F(t)}{F^* e^{(\lambda_1 + \lambda_2)t}}\right)
\]

\[
y_{11}(t) = \log\left(\frac{P(t)}{P^* e^{(\lambda_1 + \lambda_2)t}}\right)
\]

\[
y_{12}(t) = \log\left(\frac{K_n(t)}{K_n^* e^{(\lambda_1 + \lambda_2)t}}\right)
\]

\[
y_{13}(t) = \log\left(\frac{M(t)}{M^* e^{\lambda_3 t}}\right)
\]

\[
y_{14}(t) = \log\left(\frac{q(t)}{q^* e^{(\lambda_1 + \lambda_2 + \beta_3 - \lambda_2)t}}\right)
\]

where $C^*, L^*, K^*, Q^*, P^*, w^*, r^*, I^*, E_n^*, F^*, P^*, K_n^*, M^*$, and $q^*$ are functions of the vector $(\beta, \gamma, \lambda)$ of 63 parameters in Equations (1)–(14) and the additional parameters $g^*$, $p_1^*$, $p_2^*$, $r_1^*$, $r_2^*$, $T_1^*$, $T_2^*$, $Y_f^*$, and $\lambda_4$. Then an equilibrium of the system of Equations (1)–(14) corresponds to zero values of $y_i(t) = 0$, $i = 1, 2, \ldots, 14$. The set of equations satisfied by $y_i(t)$, $i = 1, 2, \ldots, 14$ can be obtained from Equations (1)–(14):

\[
D^2 y_1 = -y_1 D y_1 + y_2 \{\log(Q^* e^{\lambda_1 t} + P^* e^{\lambda_2 t}) - \log(Q^* + P^*) - \beta_2 y_2 + (\beta_2 - \beta_3) D y_5 - y_1\} \tag{15}
\]

\[
D^2 y_2 = -y_3 D y_2 + y_4 \left\{\frac{1}{\beta_6} \log \left[\frac{(Q^*)^{-\beta_6} - (K^*)^{-\beta_6}}{(Q^*)^{-\beta_6} e^{-\beta_3 y_1} - (K^*)^{-\beta_6} e^{-\beta_3 y_5}}\right] - y_2\right\} \tag{16}
\]

\[
D^2 y_3 = -y_5 D y_3 + y_6 \left\{(1 + \beta_6)(y_4 - y_3) + \log(r^* - \beta_7 (\lambda_3 - \lambda_1 - \lambda_2) + \beta_8)\right.
\]

\[
- \log(y_7 + r^* - \beta_7 (D y_5 + \lambda_3 - \lambda_1 - \lambda_2) + \beta_{10})\} \tag{17}
\]

William A. Barnett and Yijun He 173
\[ D^2 y_i = -\gamma_i D y_i + \gamma_8 \left\{ \log \left[ \frac{1 - \beta_6 (q^* P^* / p^*)^{\beta_6}}{1 - \beta_6 (q^* P^*/p^*)^{\beta_6}} \right] + \log \left( C^* e^{\gamma_i} + g^* (Q^* e^{\gamma_i} + P^* e^{\gamma_{11}}) + K^* e^{\gamma_i} (D y_3 + \lambda_1 + \lambda_2) + E_n e^{\gamma_i} \right) \right\} - \log \left( C^* + g^* (Q^* + P^*) + K^* (\lambda_1 + \lambda_2) + E_n \right) - y_i \] (18)

\[ D^2 y_5 = \gamma_9 (D y_6 - D y_5) + \gamma_{10} \left\{ y_6 - y_5 - \frac{1 + \beta_6}{\beta_6} \log \left[ \frac{Q^*}{K^*} \right] \right\} + \frac{1 + \beta_6}{\beta_6} \log \left[ \frac{Q^*}{K^*} \right] \] (19)

\[ D^2 y_6 = \gamma_{11} (D y_5 - D y_6) - \gamma_{12} (D y_5 + D y_{11}) + \gamma_{13} \left\{ \frac{1}{\beta_6} \log ((Q^*)^{-\beta_6} - \beta_5 (K^*)^{-\beta_5}) \right\} - \frac{1}{\beta_6} \log ((Q^*)^{-\beta_6} - \beta_5 (K^*)^{-\beta_5}) \] (20)

\[ D^2 y_7 = -\gamma_{14} D y_5 + \gamma_5 \left\{ \beta_{15} \frac{P^* e^{\gamma_i} (Q^* e^{\gamma_{11}} + P^* e^{\gamma_{11}})}{M^* e^{\gamma_{11}}} - \beta_{15} \frac{P^* (Q^* + P^*)}{M^*} - \beta_{14} D y_{14} - y_5 \right\} \] (21)

\[ D^2 y_8 = -\gamma_{18} D y_9 - \gamma_{19} \left\{ \beta_{16} (y_5 + y_{14}) + y_9 \right\} \] (23)

\[ D^2 y_{10} = -(\gamma_{20} + 2(\lambda_1 + \lambda_2)) D y_{10} - (D y_{10})^2 + \gamma_{21} \beta_{19} \left\{ \frac{Q^* e^{\gamma_{11}} + P^* e^{\gamma_{11}}}{F^* e^{\gamma_{10}}} - \frac{Q^* + P^*}{F^*} \right\} \] (24)

\[ D^2 y_{11} = -\gamma_{22} + 2(\lambda_1 + \lambda_2) D y_{11} - (D y_{11})^2 \] (25)

\[ D^2 y_{12} = -(\gamma_{24} + 2(\lambda_1 + \lambda_2)) D y_{12} - (D y_{12})^2 + \gamma_{25} \left\{ \beta_{22} + \beta_{23} (r_{y}^* - r^* - \gamma) \right\} \] (26)

\[ D^2 y_{13} = -\gamma_{26} D y_{13} - \gamma_{27} y_5 + \gamma_9 \left\{ \frac{E_n^t e^{\gamma_{11}} D y_{10} + P^* e^{\gamma_{11}} D y_{11} - F^* e^{\gamma_{10}} D y_{10}}{E_n^t e^{\gamma_{11}} + P^* e^{\gamma_{11}} - F^* e^{\gamma_{10}}} \right\} + y_5 + D y_{14} - D y_{13} \] (27)
\[
D^2y_{14} = -\gamma_{y_0}(Dy_5 + Dy_{14}) - \gamma_{y_1}(y_5 + y_{14}) \\
+ \gamma_{y_3} \left\{ E_n^y e^{iy_0} Dy_9 + P^y e^{yi_3} D y_{14} - F^y e^{iy_0} D y_{10} + Dy_5 + Dy_{14} - Dy_8 \right\} \\
+ \gamma_{y_3} \left\{ \log[E_n^y e^{iy_0} + P^y e^{yi_3} - F^y e^{iy_0} - K_n^y e^{yi_3}(Dy_{12} + \lambda_1 + \lambda_2)] \right\} \\
- \log[E_n^y + P^y - F^y - K_n^y(\lambda_1 + \lambda_2)] + y_5 + y_{14} - y_8 \right\}
\] (28)

Equations (15)–(28) form an autonomous system with 0 being an equilibrium for any parameter values of \( \beta_i, \gamma_j, \lambda_k \). System (15)–(28) might have other equilibria. However, as a first step, we focus on the properties of the trajectories of the system of Equations (15)–(28) near the equilibrium 0.

3 Linearization of Macroeconometric Equations

Consider an ordinary differential equation
\[
Dx(t) = f(x(t))
\] (29)
where \( x \in \mathbb{R}^n \) is the state vector and the mapping \( f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n \) is continuously differentiable (with respect to each argument). Suppose that \( x^* \in \mathbb{R}^n \) is a constant vector satisfying
\[
f(x^*) = 0
\]
Then \( x^* \) is an equilibrium of the system. Let \( \tilde{A} \) be the Jacobian matrix of \( f(x) \) evaluated at \( x^* \):
\[
\tilde{A} = \frac{\partial f(x)}{\partial x} |_{x=x^*}
\]
Then the following linear system
\[
Dy = \tilde{A}y
\] (30)
is called the linearized system of Equation (29) around the equilibrium \( x^* \). The advantage of linearization is that the (stability) behavior of trajectories of the nonlinear system of Equation (29) in a close neighborhood of the equilibrium \( x^* \) can be studied through that of its linearization of Equation (30). Briefly, if all eigenvalues of \( \tilde{A} \) have negative real parts, then Equation (29) is stable in the neighborhood of \( x^* \), meaning that every trajectory approaches \( x^* \) as \( t \to \infty \) when the initial state \( x(0) \) is sufficiently close to \( x^* \). If at least one of the eigenvalues of \( \tilde{A} \) has a positive real part, then Equation (29) is unstable in the neighborhood of \( x^* \). In this case, there exists an initial state \( x(0) \) (arbitrarily close to \( x^* \)) for which \( x(t) \) does not approach \( x^* \) as \( t \to \infty \). If all eigenvalues of \( \tilde{A} \) have nonpositive real parts and at least one has a zero real part, the stability of Equation (29) usually cannot be determined from the matrix \( \tilde{A} \). One needs to analyze higher-order terms in order to determine the stability of the system. In most cases, one needs to examine the system behavior along a certain manifold to determine the stability (Khalil 1992).

Since the concept of stability adopted here is concerned with a close neighborhood of an equilibrium only, it is referred to as local stability. In this paper, we only consider local stability, particularly the local stability around the equilibrium \( x^* = 0 \).

In many problems such as the continuous-time macroeconomic system of Equations (15)–(28), the function \( f(x) \), and consequently the coefficient matrix \( \tilde{A} \) of the corresponding linearized system (30), depend on some parameters. In this case, write Equation (30) in the following form:
\[
Dy = \tilde{A} (\theta) y
\] (31)
where \( \theta \in \Theta \) is the vector of parameters taking values in the parameter space \( \Theta \). Since \( \theta \) may change eigenvalues of \( \tilde{A} (\theta) \), the stability of Equation (30) might depend on \( \theta \).

To determine stability boundaries of the continuous-time macroeconometric system shown by Equations (15)–(28), we consider its linearization. The parameter \( \theta \) is chosen as those terms that were estimated from real data:
\[
\theta = [\beta_1, \ldots, \beta_{27}, \gamma_1, \ldots, \gamma_{53}, \lambda_1, \lambda_2, \lambda_3]'
\]

William A. Barnett and Yijun He

175
So $\theta \in \mathbb{R}^{63}$ is a 63-dimensional column vector. The feasible region $\Theta$ is specified by the bounds of the parameters (see Table 2 in the work of Bergstrom, Nowman, and Wymer [1992]). It is a bounded region. The linearized system of Equations (15)–(28) is

$$D^2 y_1 = -\gamma_1 D y_1 + \gamma_2 \left\{ \frac{Q^* y_1 + P^* y_1}{Q^* + P^*} - \beta_2 y_1 + (\beta_2 - \beta_3) D y_5 - y_1 \right\}$$  (32)

$$D^2 y_2 = -\gamma_3 D y_2 + \gamma_4 \left\{ \frac{(Q^*)^{-\beta_4} y_4 - \beta_5 (K^*)^{-\beta_6} y_6}{(Q^*)^{-\beta_4} - \beta_5 (K^*)^{-\beta_6}} - y_2 \right\}$$  (33)

$$D^2 y_3 = -\gamma_5 D y_3 + \gamma_6 \left\{ (1 + \beta_6)(y_4 - y_3) - \frac{y_2 - \beta_7 D y_5}{r^* - \beta_7(\lambda_3 - \lambda_1 - \lambda_2) + \beta_8} \right\}$$  (34)

$$D^2 y_4 = -\lambda_7 D y_4 + \gamma_8 \left\{ -y_4 - \frac{\beta_9 (Q^* P^*)^{\beta_9}}{1 - \beta_5 (Q^* P^*)^{\beta_6}} \beta_{10} (y_5 + y_1) + \frac{C^* y_1 + g^*(Q^* y_4 + P^* y_1) + K^* D y_5 + K^*(\lambda_1 + \lambda_2)y_5 + E_n y_0}{C^* + g^*(Q^* + P^*) + K^*(\lambda_1 + \lambda_2) + E_n} \right\}$$  (35)

$$D^2 y_5 = y_0(D y_6 - D y_5) + \gamma_{10} \left\{ (1 + \beta_6) \frac{\beta_6 (Q^*/K^*)^{\beta_6}}{1 - \beta_5 (Q^*/K^*)^{\beta_6}} (y_1 - y_5) + y_2 - y_5 \right\}$$  (36)

$$D^2 y_6 = y_{11}(D y_5 - D y_6) - y_{12}(D y_5 + D y_{14}) + \gamma_{13} \frac{(Q^*)^{-\beta_4} y_4 - \beta_5 (K^*)^{-\beta_6} y_3}{(Q^*)^{-\beta_4} - \beta_5 (K^*)^{-\beta_6}}$$  (37)

$$D^2 y_7 = -\gamma_{14} D y_7 + \gamma_{15} \left\{ -\beta_{14} D y_{14} - y_7 \right\}$$  (38)

$$D^2 y_8 = y_{16}(D y_5 + D y_{14} - D y_8) + \gamma_{17} \left\{ (1 + \beta_{16})(y_5 + y_{14}) - y_8 \right\}$$  (39)

$$D^2 y_9 = -\gamma_{18} D y_9 - \gamma_{19} \left\{ \beta_{18}(y_5 + y_{14}) + y_9 \right\}$$  (40)

$$D^2 y_{10} = -[y_{20} + 2(\lambda_1 + \lambda_2)] D y_{10} + \frac{\gamma_{20} \beta_{19}}{P^*} [Q^*(y_1 - y_{10}) + P^*(y_{11} - y_{10})]$$  (41)

$$D^2 y_{11} = -[y_{22} + 2(\lambda_1 + \lambda_2)] D y_{11} + \gamma_{23} \left\{ \beta_{20} + \beta_{23} (r^* - \lambda_4) \right\} \frac{K_a}{P^*} (y_{12} - y_{11})$$  (42)

$$D^2 y_{12} = -[y_{24} + 2(\lambda_1 + \lambda_2)] D y_{12} + \gamma_{25} \left\{ -\beta_{24} \frac{Q^* + P^*}{K_a} D y_{14} - \beta_{23} \frac{Q^* + P^*}{K_a} y_7 \right\}$$  (43)
\[ D^2y_{13} = -\gamma_{26}y_{13} - \gamma_{27}y_{13} + \gamma_{28} \left( \frac{E^s_n y_9 + P^s y_{11} - F^s y_{10} + D y_5 + y_{14} - y_6}{E^s_n + P^s - F^s} \right) \]
\[ + \gamma_{29} \left( \frac{E^s_n y_9 + P^s y_{11} - F^s y_{10} - K^s_n(\lambda_1 + \lambda_2)y_{12} - K^s_n D y_{12} + y_5 + y_{14} - y_6}{E^s_n + P^s - F^s - K^s_n(\lambda_1 + \lambda_2)} \right) \]
\[ D^2y_{14} = -\gamma_{30}(D y_5 + D y_{14}) - \gamma_{31}(y_5 + y_{14}) + \gamma_{32} \left( \frac{E^s_n y_9 + P^s y_{11} - F^s y_{10} + D y_5 + y_{14} - y_6}{E^s_n + P^s - F^s} \right) \]
\[ + \gamma_{33} \left( \frac{E^s_n y_9 + P^s y_{11} - F^s y_{10} - K^s_n(\lambda_1 + \lambda_2)y_{12} - K^s_n D y_{12} + y_5 + y_{14} - y_6}{E^s_n + P^s - F^s - K^s_n(\lambda_1 + \lambda_2)} \right) \]

or in matrix form

\[ \dot{x} = A(\theta)x \]

where

\[ x = [y_1 \ y_2 \ y_3 \ y_4 \ y_5 \ y_6 \ y_{13} \ y_{14}]^T \in \mathbb{R}^{28} \]

and \( A(\theta) \in \mathbb{R}^{28 \times 28} \) is the coefficient matrix. For the set of estimated values of \( \{\beta_i\} \), \( \{y_j\} \), and \( \{\lambda_k\} \) given in Table 2 of Bergstrom, Nowman, and Wymer’s 1992 work, all the eigenvalues of \( A(\theta) \) are stable (having negative real parts) except three:

\[ s_1 = 0.0053 \]
\[ s_2 = 0.0090 + 0.0453i \]
\[ s_3 = 0.0090 - 0.0453i \]

where \( i = \sqrt{-1} \) is the imaginary unit. However, the real parts of the unstable eigenvalues are (relatively) so small that it is unclear whether they are caused by errors in estimation or by the structural properties of the system itself.

Note that the system described by Equations (15)–(28), or the linearized system of Equations (32)–(45), operates in a locally unstable region. We are interested in locating the stable region and the boundary. Our approach is to first find a stable subregion of \( \Theta \), and then expand the subregion to find its boundary. To this end, we next find a parameter vector \( \theta^* \in \Theta \) such that Equation (46) is stable. From this \( \theta^* \) we will find the stable region of \( \theta \) and the stability boundaries. We use the gradient method to find a \( \theta^* \) such that all eigenvalues of \( A(\theta^*) \) have strictly negative real parts.

To find such a \( \theta^* \), we consider the following problem of minimizing the maximum real part of eigenvalues of the matrix \( A(\theta) \):

\[ \min_{\theta \in \Theta} R_{\text{max}}(A(\theta)) \]

where

\[ R_{\text{max}}(A(\theta)) = \max_i \{\text{real}(\lambda_i): \lambda_1, \lambda_2, \ldots, \lambda_{28} \text{ are eigenvalues of } A(\theta)\} \]

Since the dimension of \( A \) is 28, which is relatively high, it is infeasible to have a closed-form expression for \( R_{\text{max}}(A(\theta)) \). We use the gradient method to solve the minimization problem of Equation (47).

Consider the following recursive algorithm. Let \( \theta_0 \) be the estimated set of parameter values given in Table 2 of Bergstrom, Nowman, and Wymer’s work (1992). At step \( n, n \geq 0 \), with \( \theta_n \), let

\[ \theta_{n+1} = \pi_{\Theta} \left[ \theta_n - a_n \frac{\partial R_{\text{max}}(A(\theta))}{\partial \theta} |_{\theta = \theta_n} \right] \]
unstable eigenvalues are created. Two types of stability boundaries may occur, according to the way eigenvalues are continuous functions of entries of $A$.

After several iterations (20 iterations in this case), the algorithm arrived at the following $\theta^*$:

$$\theta^* = [0.9400, 0.2256, 2.3894, 0.2030, 0.2603, 0.1930, 0.1829, 0.0183, 0.2470, -0.2997, 1.0000, 23.5000, -0.0100, 0.1260, 0.0082, 13.5460, 0.4562, 1.0002, 0.0097, 0.0049, 0.2812, -0.1000, 44.9050, 0.1431, 0.0004, 71.421, 0.8213, 3.9998, 0.8973, 0.8698, 0.8697, 0.1064, 0.0010, 3.9901, 0.3652, 1.0818, 0.0081, 3.9588, 0.6626, 0.1172, 0.8452, 0.0421, 1.4280, 0.3001, 3.9969, 3.6512, 3.9995, 4.0000, 3.9995, 3.9410, 0.5681, 0.0040, 0.7684, 0.0427, 0.1183, 0.0708, 2.3187, 0.1659, 0.0017, 0.0000, 0.0100, 0.0100, 0.0067]$$

The corresponding $R_{\max}(A(\theta^*)) = -0.0039$. Therefore, all eigenvalues of $A(\theta^*)$ have strictly negative real parts and the system of Equation (46) is stable at $\theta^*$. Starting from this stable point, in the next section we will find the stable region of the parameter space and the stability boundaries.

### 4 Determination of Stability Boundaries

The goal of this section is to find stability boundaries. Since the linearized system of Equation (46) only determines the local stability of Equations (15)–(28), we are dealing with local stabilities, as opposed to global stabilities.

The system of Equations (15)–(28) can be written as

$$Dx = A(\theta)x + F(x; \theta)$$

(48)

where $F(x; \theta) = O(x^2)$ includes terms of higher orders.

On one hand, we have seen in the previous section that $A(\theta)$ has three eigenvalues with strictly positive real parts for the set of parameter values given in Table 2 of Bergstrom, Nowman, and Wymer's work (1992). On the other hand, all eigenvalues of Equation (46) have strictly negative real parts for the set of parameter values given in Table 2 of Bergstrom, Nowman, and Wymer's work (1992). Since eigenvalues are continuous functions of entries of $A(\theta)$, there must exist at least one eigenvalue of $A(\theta)$ with a zero real part on the stability boundary. Two types of stability boundaries may occur, according to the way unstable eigenvalues are created.

#### 4.1 Transcritical bifurcation boundary

A transcritical bifurcation occurs when the system has an equilibrium with a geometrically simple zero eigenvalue at the bifurcation point and additional transversality conditions are satisfied (see, for example, Gandolfo's (1996) work for the exact conditions).

When $\det(A(\theta)) = 0$, $A(\theta)$ has at least one zero eigenvalue. So the first condition we are going to use to find the transcritical bifurcation boundary is

$$\det(A(\theta)) = 0$$

(49)

Note that $A(\theta)$ in the linearized system (46) is a sparse matrix. Analytical forms of stability boundaries can be obtained for most parameters. To demonstrate the feasibility of this approach, we consider finding the stability boundaries for $\beta_2$ and $\beta_5$.

**Theorem 1.** The stability boundary for $\beta_2$ and $\beta_5$ is determined by

$$1.36\beta_2\beta_5 + 21.78\beta_5 - 2.05\beta_2 - 10.05 = 0$$

(50)

**Proof.** Denote

$$A(\theta) = [a_{ij}]$$

We know from Equations (32)–(45) that only the following entries of $A(\theta)$ are functions of $\beta_2$ and $\beta_5$. All other
entries do not depend on $\beta_2$ and $\beta_5$. 

$$a_{2,10} = \gamma_2 (\beta_2 - \beta_3), \quad a_{2,13} = -\gamma_2 \beta_2$$

$$a_{4,5} = -\gamma_4 \frac{(K^*)^{-\beta_5} \beta_5}{(Q^*)^{-\beta_5} - (K^*)^{-\beta_5} \beta_5}, \quad a_{4,7} = \gamma_4 \frac{(Q^*)^{-\beta_5}}{(Q^*)^{-\beta_5} - (K^*)^{-\beta_5} \beta_5}$$

$$a_{10,5} = -\gamma_{10} (1 + \beta_0) \frac{\beta_5(Q^*/K^*)^{\beta_5}}{1 - \beta_5(Q^*/K^*)^{\beta_5}}, \quad a_{10,7} = \gamma_{10} (1 + \beta_0) \frac{\beta_5(Q^*/K^*)^{\beta_5}}{1 - \beta_5(Q^*/K^*)^{\beta_5}}$$

$$a_{12,5} = -\gamma_{13} \frac{(K^*)^{-\beta_5} \beta_5}{(Q^*)^{-\beta_5} - (K^*)^{-\beta_5} \beta_5}, \quad a_{12,7} = \gamma_{13} \frac{(Q^*)^{-\beta_5}}{(Q^*)^{-\beta_5} - (K^*)^{-\beta_5} \beta_5}$$

Setting parameter values at $\theta^*$ except for $\beta_2$ and $\beta_5$, we obtain from direct calculation that

$$\det(A) = -4.63 \times 10^{-15} - 0.005 \times 10^{-15} - 2.05 \times 10^{-16} \beta_2 + 2.178 \times 10^{-15} \beta_5 + 1.36 \times 10^{-16} \beta_2 \beta_5$$

$$0.48 - 0.32 \beta_5$$

Hence, Equation (50) immediately follows from setting $\det(A(\theta)) = 0$. ■

The boundary of Equation (50) is illustrated as the dashed line in Figure 1.

### 4.2 Hopf bifurcation boundary

A Hopf bifurcation occurs at points where the system has a pair of purely imaginary but nonzero eigenvalues, and additional transversality conditions are satisfied, according to the Hopf theorem (Guckenheimer and Holmes 1983).

Consider the case of $\det(A(\theta)) \neq 0$ but where $A(\theta)$ has at least one pair of purely imaginary eigenvalues (with zero real parts and nonzero imaginary parts).
Figure 2
Trajectories when \((\beta_2, \beta_5)\) is crossing the transcritical bifurcation boundary.
Figure 3
Phase portrait of \( (x_1, x_{10}, x_{27}) \) when \((\beta_2, \beta_5)\) is crossing the transcritical bifurcation boundary.
Figure 4
Trajectories when \((\beta_2, \beta_5)\) is crossing the Hopf bifurcation boundary.

(a) \((\beta_2, \beta_5)\) in stable region

(b) \((\beta_2, \beta_5)\) on the boundary

(c) \((\beta_2, \beta_5)\) in unstable region after crossing the boundary
Figure 5
Phase portrait of \((x_1, x_{10}, x_{27})\) when \((\beta_2, \beta_5)\) is crossing the Hopf bifurcation boundary.
Figure 6
Stability boundary for \( \beta_2, \beta_3, \) and \( \beta_{15}. \)

To find Hopf bifurcation boundaries, let \( p(s) = \text{det}(sI - A(\theta)) \) be the characteristic polynomial of \( A(\theta) \), and express it as

\[
p(s) = c_0 + c_1 s + c_2 s^2 + c_3 s^3 + \cdots + c_{n-1} s^{n-1} + s^n
\]

where \( n = 28 \) for Equation (46). Construct the following \((n-1) \times (n-1)\) matrix:

\[
S = \begin{bmatrix}
c_0 & c_2 & \cdots & c_{n-2} & 1 & 0 & 0 & \cdots & 0 \\
0 & c_0 & c_2 & \cdots & c_{n-2} & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & c_0 & c_2 & c_4 & \cdots & 1 \\
c_1 & c_3 & \cdots & c_{n-1} & 0 & 0 & \cdots & 0 \\
0 & c_1 & c_3 & \cdots & c_{n-1} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & c_1 & c_3 & \cdots & c_{n-1} \\
\end{bmatrix}
\]

\( n-2 \) rows

\( n-2 \) rows

Let \( S_0 \) be obtained by deleting rows 1 and \( n/2 \) and columns 1 and 2, and let \( S_1 \) be obtained by deleting rows 1 and \( n/2 \) and columns 1 and 3. Then the following theorem of Guckenheimer, Myers, and Sturmfels (1997) gives a condition for \( A(\theta) \) to have exactly one pair of purely imaginary eigenvalues.

**Theorem 2.** The matrix \( A(\theta) \) has precisely one pair of pure imaginary eigenvalues if

\[
det(S) = 0, \quad \frac{\text{det}(S_0)}{\text{det}(S_1)} > 0
\]

If \( \text{det}(S) \neq 0 \) or if \( \frac{\text{det}(S_0)}{\text{det}(S_1)} < 0 \), \( A(\theta) \) has no pure imaginary eigenvalues. \( \blacksquare \)

If \( \text{det}(S) = 0 \) and \( \frac{\text{det}(S_0)}{\text{det}(S_1)} = 0 \), then \( A \) may have more than one pair of pure imaginary eigenvalues. Therefore, the condition for the Hopf bifurcation boundary is

\[
det(S) = 0, \quad \frac{\text{det}(S_0)}{\text{det}(S_1)} \geq 0 \quad (51)
\]
We will use Equation (51) to determine candidates of the Hopf bifurcation boundaries as well as segments of true boundaries.

In principle, the approach outlined in the proof of Theorem 1 can also be applied to find Hopf bifurcation boundaries. However, in most cases, the analytical formula of $\text{det}(S)$ is not available. The following numerical procedure could be used to find the boundaries.

### 4.3 Procedure (P1)

For the sake of simplicity, we only consider two parameters here, say $\mu_1$ and $\mu_2$.

1. For any fixed $\mu_1$, we treat $\mu_2$ as a function of $\mu_1$ and find the $\mu_2$ satisfying the condition of Equation (49); i.e., $b(\mu_2) = \text{det}(A(\theta)) = 0$. First find the number of zeros of $b(\mu_2)$. Starting with approximations of zeros, use the following gradient algorithm to find all zeros of $b(\mu_2)$

$$\theta_2(n + 1) = \pi_{\theta_2}(\theta_2(n) - a_n b(\mu_2)|_{\mu_2=\theta_2(n)})$$

(52)

where $\{a_n, n = 0, 1, 2, \ldots\}$ is a sequence of positive step sizes.

2. Repeat the same procedure to find all $\mu_2$ terms satisfying Equation (51).

3. Plot all the pairs of $(\theta_1, \theta_2)$.

4. Check all parts of the plot to find the segments representing the stability boundaries. Then, parts of the curves found in step 1 are boundaries of transcritical bifurcation boundaries, while parts of the curves found in step 2 are Hopf bifurcation boundaries.

### 5 Case Studies

In this section, the numerical procedure (P1) is used to find explicit stability boundaries for several sets of parameters. In order to be able to view the boundaries, we only consider two or three parameters. The procedure is applicable to any number of parameters.

#### 5.1 Case I: $\beta_2$ and $\beta_5$

We first find the stability boundaries for $\beta_2$ and $\beta_5$ for the system in Equation (46). Assume that other parameters operate at $\theta^*$. The result is illustrated in Figure 1, in which the dashed line is given by...
Figure 8
Stability boundary for $\gamma_6$, $\beta_{15}$, and $\beta_2$. 
$\det(A(\theta)) = 0$, and the solid line is the set of parameter pairs satisfying Equation (51). The shaded area shows the stable region. All other regions give the unstable system of Equation (46). It can also be seen from Figure 1 that the segment of the dashed line defining the stable region is the boundary of transcritical bifurcation boundaries, while the other segment of the same line is not a stability boundary at all. Similarly, the segment of the solid line defining the stable region is a Hopf bifurcation boundary. The other part of the solid line is not a stability boundary. The stability behavior of Equation (46) along the stability boundaries is unclear, and is a subject of ongoing research.

Of particular interest is the cross-point of the two stability boundaries, which is approximately $(\beta_2, \beta_3) = (1.785, 0.566)$. At this point, the coefficient matrix has three eigenvalues with zero real parts: $s_1 = 0.0000$, $s_2 = -0.0000 + 0.0336i$, and $s_3 = -0.0000 - 0.0336i$.

Figures 2–5 illustrate the trajectories of $x$ and the phase portraits of $(x_1, x_{10}, x_2)$ when the parameters cross the two boundaries.

5.2 Case II: $\beta_2$, $\beta_3$, and $\beta_{15}$

We now add the parameter $\beta_{15}$ into Case I. Using again procedure (P1), we find the surface of the stability boundary for $\beta_2$, $\beta_3$, and $\beta_{15}$ as shown in Figure 6.

5.3 Case III: $\gamma_8$ and $\beta_{15}$

In this case, we find stability boundaries for parameters $\gamma_8$ and $\beta_{15}$. Assume that other parameters operate at $\theta^*$. The result is illustrated in Figure 7, in which only Hopf bifurcation boundaries exist. The shaded area shows the stable region.

5.4 Case IV: $\gamma_8$, $\beta_{15}$, and $\beta_2$

In this case, we consider the three-dimensional stability boundary for $\gamma_8$, $\beta_{15}$, and $\beta_2$. Similar to Case III, only Hopf bifurcation boundaries exist for the three parameters. Figure 8 illustrates the boundary viewed from two different directions.

6 Conclusion

We proposed a procedure for determining the stability boundaries within the plausible range of parameter values for the Bergstrom, Nowman, and Wymer continuous-time macroeconometric model. A trajectory simulation of the linearized model for different settings of the parameter values shows that the behavior of the system is consistent with the prediction of stability boundaries.

This paper reports on the first results from an ongoing research project. Based upon our current results, we plan to explore further cases of system behavior when the parameters are set exactly on the stability boundaries. We also plan to investigate whether any of the parameter settings within the unstable region can support chaos. In short, the current results are only a first step, but are critical as motivation for the future research we now contemplate.

References


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