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# Stabilizing Endogenous Fluctuations with Fiscal Policies: Global Analysis on Piecewise Continuous Dynamical Systems

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***Abstract.*** *This paper considers a one-sector overlapping-generations model with production. We observe that under standard assumptions on preferences and technology, the perfect-foresight equilibrium violates positivity constraints for large sets of initial conditions. The consideration of the positivity constraints of the consumer enables degenerate equilibrium, a perfect-foresight equilibrium that remains in the trivial steady state from a certain time, to be defined. To correct this unsatisfactory situation, we introduce a mechanism redistributing resources among generations by levying taxes.*

***Keywords.*** overlapping generations, perfect foresight, nonlinear dynamics, critical lines, piecewise differentiable dynamics

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## 1 Introduction

The aim of this study is to present the problem of definition of perfect-foresight equilibrium in an overlapping-generations model with production. The model we use is the one proposed in Reichlin 1986. The modeling gives rise to a two-dimensional dynamical system depending on a technological parameter.

Reichlin (1986) proved that there exists, in the neighborhood of the nontrivial-economy steady state, an attractive or repelling invariant closed curve. This result, based on the Neimark-Säcker theorem, is local: the invariant curve lies in the neighborhood of the fixed point, and the theorem states its existence only for parameters that lie in the neighborhood of the bifurcation value.

We provided, in Augeraud and Augier 1999, a global analysis of the same model. We gave mathematical proofs of the variation of the initial conditions basin leading to reasonable long-term economic equilibrium under variations of the technological parameter. The resulting phenomenon corresponds to the *blue sky catastrophe* (Medio 1992). The basin of attraction in an economy is studied mainly when there are multiple steady states (Soliman 1997).

Global analysis shows that perfect-foresight equilibrium is degenerate; it means that variables are null after a certain finite time.

After having defined the model (Section 2), we shall then explain in the modeling which hypothesis is responsible for the unsatisfactory long-term behavior of the economy (Section 3). Then to avoid this problem, we will propose some modifications in the modeling, based on government fiscal policy. The systems we obtained are piecewise continuous dynamical systems. The mathematical tools we use (Section 4) were developed by Mira (1964) and are called critical lines. These curves are useful tools for studying noninvertible systems or nondifferentiable two-dimensional systems. Details of the study of the dynamics are to be found in Section 5.

## 2 The Economic Model

We assume that the size of the population is constant. The model is a two-period overlapping-generations economy with production (Diamond 1965; Reichlin 1986).

Consumers work during the first period of their lives and consume during the second. The representative agent born at time $t$ supplies a quantity of labor $l_t$, receives wage income, saves it when young, and consumes $c_{t+1}$ when old.

We assume that the utility function of the representative agent is equal to $u(c_{t+1}) - v(l_t)$.[1] The following standard hypothesis on utility can be stated:

**Hypothesis 1.** *$u$ and $v$ are defined from $\mathbb{R}^+$ to $\mathbb{R}$, smooth on $\mathbb{R}^{+*}$.*

- *$u$ is strictly increasing and concave over $\mathbb{R}^{+*}$.*
- *$v$ is strictly increasing and convex over $\mathbb{R}^{+*}$.*

Under the perfect-foresight hypothesis, the agent's decision problem is the following:

$$\begin{cases} \max_{c_{t+1}, l_t} u(c_{t+1}) - v(l_t) \\ \qquad\qquad \text{subject to} \quad \begin{cases} s_t = w_t l_t \\ c_{t+1} = s_t R_{t+1} \\ c_{t+1} \geq 0,\ l_t \geq 0 \end{cases} \end{cases} \tag{2.1}$$

where $w_t$ is the real wage and $R_{t+1}$ the real rate of return[2] on savings.

Under assumption (2.1) and with $w_t > 0$ and $R_{t+1} > 0$, the decision problem has a unique solution satisfying $c_{t+1} > 0$ and $l_t > 0$. Futhermore, consumption demand and labor supply are the unique solution of the system

$$\begin{cases} c_{t+1} u'(c_{t+1}) = l_t v'(l_t) \\ c_{t+1} = R_{t+1} w_t l_t \end{cases} \tag{2.2}$$

Let us write[3] $U(x) = xu'(x)$ and $V(x) = xv'(x)$.

[1]This includes the following assumptions: first, utility is separable in labor and good, then disutility of labor is directly taken into account.
[2]$R_{t+1} = 1 + r_{t+1}$, where $r_{t+1}$ is the interest rate.
[3]These notations are the ones used by Reichlin (1986).

As in Reichlin's paper the following hypothesis is stated:

**Hypothesis 2.** $U' > 0$ *and* $\lim_{x \to 0} U(x) = 0$, $\lim_{x \to +\infty} U(x) = +\infty$.

This hypothesis enables us to consider the function $h(x) = U^{-1} \circ V(x)$ and to rewrite system (2.2) as

$$\begin{cases} c_{t+1} = h(l_t) \\ c_{t+1} = R_{t+1} w_t l_t \end{cases} \tag{2.3}$$

We can take the following constant relative risk of aversion (CRRA) utility function as an example:

$$U(c_{t+1}, l_t) = \frac{c_{t+1}^{1-\alpha}}{1-\alpha} - \frac{l_t^{\gamma}}{\gamma} \tag{2.4}$$

with $0 < \alpha < 1$ and $\gamma > 1$. Hypotheses (1) and (2) are thus verified. Let $\delta$ be $\frac{\gamma}{1-\alpha}$.

We write $Y_t$, the quantity produced at time $t$. The technology is described by a Leontief production function. The two production inputs $K_t$ (stock of capital) and $L_t$ (labor) are used in fixed proportions in the following way:

$$Y_t = \min\left(\frac{L_t}{a_0}, \frac{K_t}{a_1}\right) \tag{2.5}$$

We assume that capital depreciates, so that the profit function of producers can be rewritten as $Y_t - R_t K_t - w_t L_t$.

A consequence of profit maximization with the Leontief technology is

$$Y_t = \frac{L_t}{a_0} = \frac{K_t}{a_1}$$

The market-clearing assumptions are given by Hypothesis (3).

**Hypothesis 3.** $K_t = s_{t-1}$ *and* $L_t = l_t$.

A consequence is that capital stock is the only asset in the model, and labor demand and supply are equal.

## 3 The Problem

We assume that the perfect-foresight hypothesis holds. Benhabib and Laroque (1988) give the definition of an intertemporal perfect-foresight equilibrium. Perfect-foresight expectations are such that an agent's expectation should be the actual future sequence predicted by the model (Kehoe and Levine 1985).

**Property 3.1.** *In Reichlin's model, the perfect-foresight equilibrium is associated with a sequence* $(s_t)_t$ *satisfying*

$$s_{t+1} = \frac{1}{a_1} s_t - h\left(\frac{a_0}{a_1} s_{t-1}\right) \tag{3.1}$$

*if* $s_{t+1} > 0$. *If there exists* $t_1$ *such that* $s_{t+1} \leq 0$, *then* $s_t = 0$ *for all* $t \geq t_1$.

*Proof.* The definition of intertemporal equilibrium gives $Y_{t+1} = c_{t+1} + s_{t+1}$. Equation (2.5) and market-clearing conditions give $Y_{t+1} = \frac{1}{a_1} s_t$ and $L_t = \frac{a_0}{a_1} s_{t-1}$. The resolution of the consumer program, in case constraints are not saturated, enables one to prove the property.

If $s_{t+1}$ is negative, we have to come back to expectation formation and take into account the positivity constraints of the consumer program.

The presentation of Böhm and Wenzelburger 1999 enables one to express the dynamics according to expectations $(\widehat{R_{t+1}})_t$. The economic law is the solution of consumer and firm programs and the market's constraints.

**Lemma 3.1.** *The economic law, for $\widehat{R_{t+1}} > 0$, is given by*

$$\begin{cases} s_t = \frac{h(\frac{a_0}{a_1}s_{t-1})}{\widehat{R_{t+1}}} \\ R_t = \frac{1}{a_1} - \frac{h(\frac{a_0}{a_1}s_{t-1})}{\widehat{R_{t+1}}s_{t-1}} \end{cases} \tag{3.2}$$

*Proof.* The resolution of the consumer program yields $\widehat{R_{t+1}}s_t = h(l_t)$. As the producer's maximization and market's constraints give $l_t = \frac{a_0}{a_1}s_{t-1}$, we obtain the first equation of system (3.2).

To get the second equation, one must use the condition of null profit, written as $Y_t = R_tK_t + w_tL_t$. As the technology is Leontief's, the result is

$$R_t = \frac{1}{a_1} - \frac{a_0}{a_1}w_t.$$

The following condition $(\widehat{R_{t+1}}w_tl_t = h(l_t))$, obtained according to the consumer program, enables one to prove Lemma 3.1.

**Lemma 3.2.** *When the perfect-foresight hypothesis holds, expectation is given by*

$$\widehat{R_{t+1}} = \frac{h(\frac{a_0}{a_1}s_{t-1})}{\frac{1}{a_1}s_{t-1} - h(\frac{a_0}{a_1}s_{t-2})}$$

*Proof.* Economic law (3.2) gives

$$\widehat{R_{t+1}} = \frac{h(\frac{a_0}{a_1}s_{t-1})}{s_t}$$

So $\hat{R}_t = \frac{h(\frac{a_0}{a_1}s_{t-2})}{s_{t-1}}$. Then the second equation of system (3.2) gives $R_t$. As $\hat{R}_t = R_t$ under the perfect-expectation hypothesis, $\hat{R}_{t+1}$ is as given in Lemma 3.2.

**Remark 3.1.** Note here that the expectation function is characterized by a memory of order 2 with variables $s_{t-1}$ and $s_{t-2}$. This order $= 2$ memory can give rise to an indetermination problem (Hahn 1966; Laitner 1982).

**Lemma 3.3.** *If there exists $t_1$ such that $s_{t_1} = 0$, then for all $t > t_1$, $s_t = 0$.*

*Proof.* If the expression of $s_t$ given by Equation (6) is negative, it means, according to Lemma 3.2, that $\widehat{R_{t+1}} < 0$. As this has no economic meaning as $R = 1 + r$, the agent then expects $\widehat{R_{t+1}} = 0$. Resolution of the consumer program gives $s_t = 0$. Leontief's technology then gives $L_{t+1} = K_{t+1} = Y_{t+1} = 0$. Then prices $R_{t+1}$ and $w_t$ are null. Agents who are young at time $t$ have then saved nothing, $c_{t+1} = 0$, so $s_{t+1} = 0$. An immediate recurrence enables one to end the proof.

**Definition 3.1.** *We call a degenerate equilibrium a sequence $(s_t)_t$ such that*

- *$(s_t)_t$ is associated with a perfect-foresight equilibrium.*
- *there exists $t_1$ such that for all $t \geq t_1 s_t = 0$.*

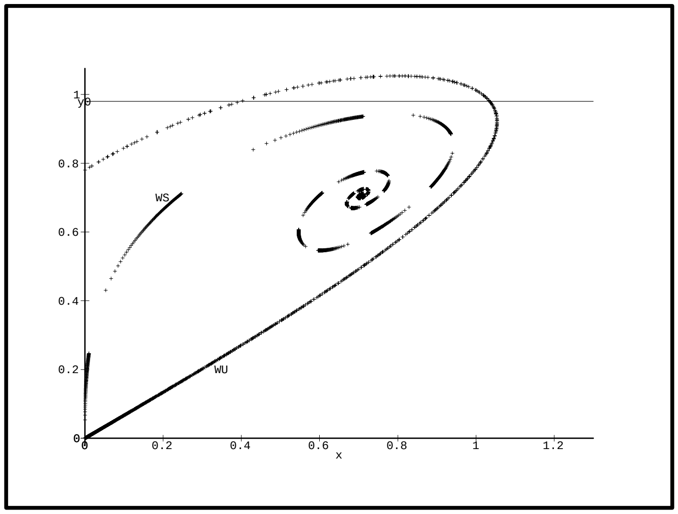



**Figure 1**
Invariant manifolds of the trivial fixed point for $h(x) = x^2$, $a = 1.5$.

### *3.1 Review of local dynamical properties of the system*

Reichlin (1986) has studied local properties of system (3.1). He proved that for $a_1 < 1$, the system has two steady states. When utility is the CRRA function given by Equation (2.4), we have

- $(0, 0)$ is a saddle with eigenvalues 0 and $\frac{1}{a_1}$.
- $\left(\left(\frac{1}{a_1} - 1\right)^{\frac{1}{\delta-1}} \left(\frac{a_1}{a_0}\right)^{\frac{\delta}{\delta-1}}, \left(\frac{1}{a_1} - 1\right)^{\frac{1}{\delta-1}} \left(\frac{a_1}{a_0}\right)^{\frac{\delta}{\delta-1}}\right)$, which is an attractive focus for $\left(\frac{1}{a_1}, \delta\right) \in \left\{\left(\frac{1}{a_1}, \delta\right), \left(\frac{1}{a_1}\right)^2 - 4\delta\left(\frac{1}{a_1} - 1\right) < 0 \text{ and } \delta\left(\frac{1}{a_1} - 1\right) < 1\right\}$ and a repelling focus for $\left(\frac{1}{a_1}, \delta\right) \in \left\{\left(\frac{1}{a_1}, \delta\right), \left(\frac{1}{a_1}\right)^2 - 4\delta\left(\frac{1}{a_1} - 1\right) < 0 \text{ and } \delta\left(\frac{1}{a_1} - 1\right) > 1\right\}$.

Reichlin then considered bifurcation according to technological parameter $a_1$. He proved, by applying the *Neimark-saker* bifurcation theorem (Hale and Kocak 1991) that there exists locally an invariant attractive closed curve. The parameter for which the bifurcation occurs is $a_1 = \frac{\delta}{\delta+1}$.

### *3.2 Review of global dynamical properties of the system*

It has been proved (Augeraud and Augier 1999) that there exists a value $a_1^*$ of parameter $a_1$ such that for all $a_1 < a_1^*$, there exists a value $s_{\min}(a) \in \mathbb{R}^+$ such that for all $s_0 > s_{\min}(a)$, for all $s_1 \in \mathbb{R}^+$, $s_t$ is a degenerate equilibrium.

Figure 1 illustrates this situation for a peculiar value of $a_1$.

## 4 Mathematical Techniques

Critical lines enable us to give a geometrical characterization of the attractors. In this section, we describe the tools we use in a general framework. These tools were first introduced in Gumowski and Mira 1964.

### *4.1 Setting of the general framework*

Let $F_a$ be a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by

$$F_a\colon\ (x_n, y_n) \in \mathbb{R}^2 \to (x_{n+1}, y_{n+1}) \in \mathbb{R}^2$$

Note that the notation $F_a$ indicates that the mapping depends on a parameter $a$. If we were not interested in this dependence, the mapping would only be written $F$.

**Hypothesis 4.** *$F$ is proper (that is, the inverse image of a compact is compact). Under this hypothesis, each point in $\mathbb{R}^2$ has a finite number of preimages.*

The systems on which we would apply the critical-lines method have the following form:

- Type 1:

$$Fa \begin{cases} x_{n+1} = f(x_n, y_n) \\ y_{n+1} = g(x_n, y_n) \end{cases}$$

  where $f$ and $g$ are $\mathcal{C}^k$ functions ($k \geq 1$).

- Type 2: A piecewise dynamical system for which each part is defined with systems of type 1.

### *4.2 Definition of critical lines*

#### 4.2.1 Properties using the topological nature of the system

**Definition 4.1.** *1. Let $Z_i$ be the set of points having $i$ preimages of order 1.*

2. *Let us say that $X_0$ satisfies the property $\mathcal{P}$ iff there exists a neighborhood $\mathcal{U}$ of $X_0$ and a neighborhood $\mathcal{V}$ of $F(X_0)$ such that $F$ is a bijection from $\mathcal{U}$ to $\mathcal{V}$.*
3. *We call the set of critical points the curve $LC_{-1}$ defined by*

$$LC_{-1} = \{X_0 \text{ that do not satisfy the property } \mathcal{P}\}$$

4. *Let $LC$ be*

$$F(LC_{-1}) = LC$$

**Remark 4.1.** Points in $\overline{Z_i} \cap \overline{Z_j} (i \neq j)$ belong to $LC$. The following example[4] shows that $LC$ may intersect $Z_i$'s interior. Let us consider $f$ such that

$$f\colon\ x \to x(x^2 - 1)^2 \tag{4.1}$$

The system that is defined by $f$ has a set $Z_1$ that is defined by

$$Z_1 = ]-\infty, -\sqrt{\frac{1}{5}}\left(\frac{16}{25}\right)[\ \cup\ ]\sqrt{\frac{1}{5}}\left(\frac{16}{25}\right), \infty[ \text{ and a set } Z_3 = ]-\sqrt{\frac{1}{5}}\left(\frac{16}{25}\right), \sqrt{\frac{1}{5}}\left(\frac{16}{25}\right)[.$$

The critical lines are as follows:

$$LC_{-1} = \{-1\} \cup \left\{-\sqrt{\frac{1}{5}}\right\} \cup \left\{\sqrt{\frac{1}{5}}\right\} \cup \{1\}$$

and

$$LC = \left\{-\sqrt{\frac{1}{5}}\left(\frac{16}{25}\right)\right\} \cup \{0\} \cup \left\{\sqrt{\frac{1}{5}}\left(\frac{16}{25}\right)\right\}.$$

[4]The following example is defined on $\mathbb{R}$ for simplicity reasons. If we consider $F_a$ for which $f$ is the one given in Equation (4.1) and $g$ is such that $g(x, y) = y$, we can show that the result we have remains true in $\mathbb{R}^2$.

### 4.2.2 Properties based on the differentiability of the system

**Property 4.1.**

$$LC_{-1} \subseteq J_0$$

where

$$J_0 = \{(x, y), |Jac(F)(x, y)| = 0\}$$

*The proof of this property is based on the local inversion theorem. Note that the previous relation can be a strict inclusion; by analogy with what can be done in dimension 1, we have to eliminate from $J_0$ all points corresponding to inflection points.*

*When $F$ is of type 2, we have the following property.*

**Property 4.2.**

$$LC_{-1} \subseteq J_0 \cup ND$$

*where*

$$ND = \{(x, y), F(x, y) \textit{ is not differentiable}\}$$

### 4.2.3 Properties using the algebraic nature of the system

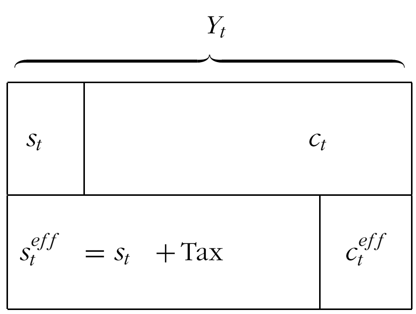
**Remark 4.2.** Points in $LC$ have preimages whose multiplicity is strictly superior to 1.

## 5 Redistribution of Resources

We have showed that if positivity constraints are not satisfied at time $t$, the quantities chosen by consumers after time $t$ are trivial. The economic interpretation of this result is clear: a young agent's savings are insufficient to keep the economy on a feasible growth path. It follows that the government levies a tax on the older generation and uses the proceeds to make a transfer payment to the younger generation (Azariadis 1993). The consumption (saving) at time $t$ will be written $c_t^{eff}$ ($s_t^{eff}$), where *eff* stands for "effective."

We assume that the government policy is continuous according to the set of economic variables. Such a hypothesis eliminates peculiar cases in which the fiscal policy is implemented through lump-sum tax schemes or taxation on one state variable. An example of such a case is given by a proportional tax policy on consumption.

We shall study several tax schemes: a tax scheme in a function of $s_{t-1}$ and another tax scheme depending on $s_{t-2}$.




The government distributes wealth as it would have been distributed in a competitive market. The fiscal policy appears as an alternative adjustment mechanism.

In the models we present here, the government levies taxes at different times $t$. This action has to be isolated, because of problems of time inconsistency (Kydland and Prescott 1977). Indeed, individual consumers are not able to predict government action.

### 5.1 Tax producing a system depending on $s_{t-1}$ only

The government levies a tax $T(s_{t-1}, s_{t-2})$ at time $t$, such that

$$as_{t-1} - h(s_{t-2}) + T(s_{t-1}, s_{t-2}) = \bar{w}s_{t-1}$$

where $\bar{w}$ is given a priori.

**Remark 5.1.** Tax $T$ could be interpreted as a way of creating minimum wages $\bar{w}$. We assume that the government applies this fiscal policy when

$$as_{t-1} - h(s_{t-2}) < s_{t-1}\bar{w}$$

This tax policy's conditions imply that the economic dynamical system is defined by piecewise continuous functions.

**Remark 5.2.** The government has to take into account the following wealth constraint:

$$T(s_{t-1}, s_{t-2}) < h(s_{t-2})$$

This inequality means that the government cannot levy a tax higher than the older generation's income. It implies that $\bar{w}$ must be inferior to $a$.

### 5.2 The dynamics

Let $x_t = s_t$ and $y_t = s_{t-1}$. The dynamics $(S)$ is given by map $F_{a,\bar{w}}$ from $\mathbb{R}^2_+$ to $\mathbb{R}^2_+$ such that

$$F_{a,\bar{w}}\colon\ (x_{t-1}, y_{t-1}) \longrightarrow \begin{cases} \text{If } \frac{ax_{t-1}-h(y_{t-1})}{x_{t-1}} \geq \bar{w}, & \begin{cases} x_t = ax_{t-1} - h(y_{t-1}) \\ y_t = x_{t-1} \end{cases} \\ \text{If } \frac{ax_{t-1}h(y_{t-1})}{x_{t-1}} < \bar{w}, & \begin{cases} x_t = \bar{w}x_{t-1} \\ y_t = x_{t-1} \end{cases} \end{cases} \tag{5.1}$$

Map $F_{a,\bar{w}}$ is continuous and piecewise differentiable. Let

$$\begin{array}{rclcrcl} F_1\colon \mathbb{R}^2_+ & \to & \mathbb{R}^2 & \quad\text{and}\quad & F_2\colon \mathbb{R}^2_+ & \to & \mathbb{R}^2_+ \\ \begin{pmatrix} x \\ y \end{pmatrix} & \mapsto & \begin{pmatrix} ax - h(y) \\ x \end{pmatrix} & & \begin{pmatrix} x \\ y \end{pmatrix} & \mapsto & \begin{pmatrix} \bar{w}x \\ x \end{pmatrix} \end{array}$$

Let $(S_1)$ $((S_2))$ be the dynamical system defined by $F_1$ $(F_2)$.

#### 5.2.1 Description of critical lines

We can consider the following sets:

1. Set $ND$ of points where $F_{a,\bar{w}}$ is not differentiable:

$$ND = \{(x, y), (a - \bar{w})x - h(y) = 0\}$$

*Notations*: $ND$ partitions $\mathbb{R}^2_+$ into $\mathcal{S}_1$ $(\mathcal{S}_2)$, defined by

$$\mathcal{S}_1 = \left\{(x, y), \frac{ax - h(y)}{x} > \bar{w}\right\} \left(\mathcal{S}_2 = \left\{(x, y), \frac{ax - h(y)}{x} < \bar{w}\right\}\right)$$

on which $(\mathcal{S}_1)((\mathcal{S}_2))$ is defined. $ND$ is the border between $\mathcal{S}_1$ and $\mathcal{S}_2$.

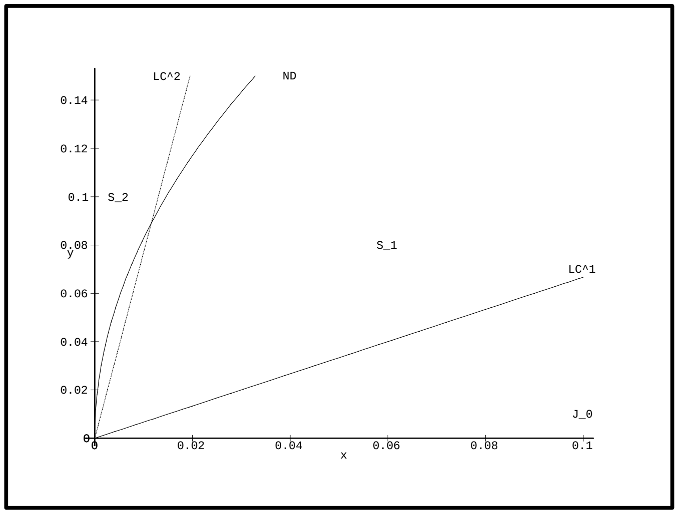



**Figure 2**
Critical lines for $h(x) = 2x^2$, $a = 1.5$ and $w = 0.13$.

2. Set $J_0$ where the jacobian of $F_{a,\bar{w}}$ is defined and vanishes.

$$J_0 = \{(x, y) \in \mathbb{R}_+^* \times \mathbb{R}_+^*, h'(y) = 0\} \cup \mathcal{S}_2$$

   As utility is CRRA, $y = 0$ is the unique solution of $h'(y) = 0$.

Let $LC^2$ be the image of $ND$ and $LC^1$ the image of $J_0$. Let $LC$ be $LC^1 \cup LC^2$. Calculation gives $LC^1$ and $LC^2$:

$$\begin{aligned} LC^2 &= \{(x, y), x = \bar{w}y\} \\ LC^1 &= \{(x, y), x = ay\} \end{aligned}$$

Figure 2 presents these curves.

**Remark 5.3.** The image of $\mathcal{S}_2$ by $F_2$ is line $LC^2$.

**Remark 5.4.** Let $M$ be a point of $ND$ and $\mathcal{U}$ be a neighborhood of $M$. Let $\mathcal{V}$ be the image of $\mathcal{U}$ by $F$. Then $F\backslash_{\mathcal{U}}$: $\mathcal{U} \to \mathcal{V}$ is not a bijection.

We can use the following property.

**Property 5.1.** *Each half-line given by $y = \alpha x$ (with $\alpha > 0$ and $x \geq 0$) intersects ND at two points: the trivial point and one belonging to $\mathbb{R}^+ \times \mathbb{R}^+$.*

*Proof.* Let $\varphi$: $x \to (a - \bar{w})x - h(\alpha x)$. The solutions of equation $\varphi(x) = 0$ give the abscissa of intersection points between $ND$ and the half-line. As utility is given by a CRRA function, we have $h(0) = 0$ and $\lim_{x\to\infty} h(x) = \infty$.

$$\varphi'(x) = (a - \bar{w}) - \alpha h'(\alpha x)$$

As $h'$ is strictly increasing from $[0, \infty[$ to $[0, \infty[$ and unbounded, there exists a unique $v_0$ such that $\varphi'(v_0) = 0$. The graph of $\varphi$ is the following:

$$
\begin{array}{c|ccccc}
x & 0 & & v_0 & & \infty \\
\hline
\varphi(x) & & \nearrow & & \searrow & \\
 & 0 & & & & -\infty
\end{array}
$$

So there exists a unique positive $v_1$ solution of $\varphi(x) = 0$.

**Remark 5.5.** The line $\bar{w}y = x$ intersects $ND$ at $(0, 0)$ and another point in $\mathbb{R}^+ \times \mathbb{R}^+$ that we call $a_0 = (x^*, y^*)$.

The study of the order-1 images of $\mathcal{S}_1$ and $\mathcal{S}_2$ gives the following property.

**Property 5.2.** *$Z_0 = \{(x, y), \bar{w}y > x, x \in \mathbb{R}^+\}$ has no preimage.*

*Proof.* Consider the images by $F_1$ ($F_2$) of $\mathcal{S}_1$ ($\mathcal{S}_2$).

- $F_2$ maps $\mathcal{S}_2$ on the line $\bar{w}y = x$. Points belonging to $Z_0$ have no preimages by $F_2$.
- Moreover, let $(x, y) \in \mathcal{S}_1$. By definition of $\mathcal{S}_1$, we have $ax > \bar{w}x + h(y)$. The image $(x_1, y_1)$ of $(x, y)$ satisfies $(x_1 > \bar{w}x, y_1 = x)$. So $x_1 > \bar{w}y_1$. This means that $(x_1, y_1) \notin Z_0$.

**Remark 5.6.** If there exist attractors, they do not belong to $Z_0$.

**5.2.2 Remarks on parameters** Remark 5.2 says that parameters belong to set $\{(a, \bar{w}), (a, \bar{w}) \in ]1, \infty] \times [0, a]\}$.

**Property 5.3.** *The steady state $P$ of $(S_1)$ belongs to the domain of definition of $(S_1)$ if and only if $\bar{w} < 1$.*

*Proof.* It is very easy to prove this property. We simply have to note that $P$ belongs to $y = x, x \in \mathbb{R}^+$ and to find conditions of intersection of this half-line and $ND$.

**Property 5.4.** *If $\bar{w} > 1$, for all $(x_0, y_0) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, trajectories are unbounded.*

*Proof.*

- If $x_t \in \mathcal{S}_1$, then $ax_t - f(y_t) > \bar{w}x_t$. But $x_{t+1} = ax_t - f(y_t)$. So $x_{t+1} > \bar{w}x_t$.
- If $x_t \in \mathcal{S}_2$, $x_{t+1} = \bar{w}x_t$.

So for all $t \geq 0$, we have $x_{t+1} \geq \bar{w}x_t$. Then $(x_t)_t$ is strictly increasing and goes to infinity.

**Remark 5.7.** The study of the case in which $\bar{w} \geq 1$ shows that the economy has a growth rate equal to $\bar{w}$ from a given date $t_0$. The tax then becomes predictable, thus invalidating the model we have chosen.

The previous remark leads us to consider parameter $\bar{w} < 1$. We must restrict the domain of the parameter $a$ more, using the following hypothesis.

**Hypothesis 5.** *We assume there exists $t$ such that $a_t$ (image of order $t$ of $a_0$ by $F_{a,\bar{w}}$) does not belong to $\mathcal{S}_2$.*

This hypothesis is observed numerically. So the domain of the parameters that we shall take into account is

$$\{(a, \bar{w}), (a, \bar{w}) \in ]a', a_3[ \times [0, 1[\}$$

**5.2.3 Study in the phase plane** The methodology that we are going to use to describe the dynamics is based on the study of the critical lines.

**Lemma 5.1.** *There exists a set globally invariant whose dimension is at least 1.*

*Proof.* We are going to prove this lemma by building this set. Let us consider $a_0 = (x^*, y^*)$ (which is the nontrivial intersection of *ND* and its image by $F_{a,\bar{w}}$). Such an intersection exists and is unique, according to Property 5.1.

- Let $a_i$ be the point such that $a_i = F^i(a_0)$ and $\widetilde{a_i a_{i+1}}$ is the image of order $i$ of the part of $LC^2$ between $a_0$ and $a_1$. As $\bar{w} < 1$, $a_1$ belongs to $\mathcal{S}_1$.
- We map $F_1$ until there exists $t$ such that $a_t \notin \mathcal{S}_1$ (such a $t$ exists, according to Hypothesis 5). Let $b_0$ be the first intersection point between $\widetilde{a_{t-1} a_t}$ and *ND*. Let $b_1$ be the image of $b_0$ by $F$. Only two cases can occur:
  - $a_{t+1} \in \{(x, y), \bar{w}y = x, x > x^*\}$.
  - $a_{t+1} \in \{(x, y), \bar{w}y = x, < x < x^*\}$.

**Remark 5.8.** $a_{t+1}$ cannot belong to $\{(x, y), \bar{w}y = x, 0 < x < \bar{w}x^*\}$, which would mean that $a_t$ belongs to $\mathcal{S}_2 \cap Z_0$.

- Consider the case in which $a_{t+1} \in \{(x, y), \bar{w}y = x, x > x^*\}$. The two following situations can occur:
  - $b_1 \in [a_1, a_{t+1}]$. There exists $n \geq 0$ such that $F_2^n(a_{t+1}) \in \widetilde{a_0 a_1}$. Indeed, as $\bar{w} < 1$ the abscissa of $F_2^n(a_{t+1})$ decreases with $n$. Furthermore, if one considers three points of abscissas strictly decreasing, the abscissas of their images are in the same order. So curve $\mathcal{C}_1$ defined by $\cup_{i=0\ldots t}\widetilde{a_i a_{i+1}} \cup \widetilde{a_{t+1} a_0}$ is globally invariant.
  - $a_{t+1} \in [a_1, b_1]$. As previously, there exists $n \geq 0$ such that $F_2^n(a_{t+1}) \in \widetilde{a_0 a_1}$. So curve $\mathcal{C}'_1$ defined by $\cup_{i=0\ldots t}\widetilde{a_i a_{i+1}} \cup [a_{t+1} b_1] \cup \widetilde{a_{t+1} a_0}$ is globally invariant.
- Consider the case in which $a_{t+1} \in \{(x, y), \bar{w}_y = x, \bar{w}x^* < x < x^*\}$.
  - $b_1 \in [a_1, a_{t+1}]$ or $b_1 \in [a_{t+1}, a_0]$. Curve $\mathcal{C}_2$, defined by $\cup_{i=1\ldots t-1}\widetilde{a_i a_{i+1}} \cup \widetilde{a_t a_{t+1}} \cup [a_{t+1}, a_0]$, is globally invariant.
  - $b_1 > a_0$. Curve $\mathcal{C}'_2$, defined by $\cup_{i=1\ldots t-1}\widetilde{a_i a_{i+1}} \cup \widetilde{a_t a_{t+1}} \cup [a_{t+1}, b_1]$, is globally invariant.

We shall call $\mathcal{C}$ the constructed curve.

**Remark 5.9.** We should now consider the conditional character of the intervention of the government. We should reject, and consider as unacceptable, the situations for which the government has to intervene over two consecutive periods, because then the tax becomes predictable by the agents. (This remark in parallel to Remark 5.7.)

**Remark 5.10.** $\mathcal{C} = \cup_{n=0}^{t} F^n(\widetilde{a_0 a_1})$.

**Property 5.5.** *$\mathcal{C}$ is an attracting set.*

*Proof.* We have to show that there exists an open set $U$ of $\mathcal{C}$ such that for all $M \in U$, $\lim_{n\to\infty} F^n(M) \to \mathcal{C}$. Let $V$ be a neigborhood of $\widetilde{a_t a_{t+1}}$. Let us consider $V_0 = \cup_{n=0\ldots t} F^{-n}(V)\mathbb{R}^+ \times \mathbb{R}^+ \cup F(\widetilde{a_t a_{t+1}})^o$ (where $^o$ is the interior of the considered set). Let $y \in V_0$. Then there exists $n$ such that $F^n(y) \in \mathcal{S}_2 \backslash Z_0$. So there exists $m \geq n$ such that $F^n(y) \in \mathcal{C}$.

**5.2.4 Dynamics on the attractor** We will not take into account dynamics on the curve $\mathcal{C}$. De Vilder (1995) shows with an example of the same nature that various behaviors can be envisaged according to the nature of $\mathcal{C}$:

- If $\mathcal{C}$ is homeomorphic to a circle (case where $\mathcal{C}$ is $\mathcal{C}_1$), then according to the value of the number of rotations, the dynamics can have a periodic or dense orbit (Guckenheimer and Holmes 1983).
- Otherwise, the dynamics of $\mathcal{C}$ may be chaotic.

### *5.3 Tax producing a system depending on $s_{t_2}$ only*

We assume that the government levies a tax of amount $T(s_{t-1}, s_{t-2})$, such that

$$as_{t-1} - h(s_{t-2}) + T(s_{t-1}, s_{t-2}) = Ts_{t-2}$$

where $T$ is given a priori. As previously, we assume for continuity reasons that the government levies the tax when

$$as_{t-1} - h(s_{t-2}) < Ts_{t-2}$$

We have to take into account the wealth constraint:

$$T(s_{t-1}, s_{t-2}) < h(s_{t-2})$$

This constraint can be written as

$$Ts_{t-2} < as_{t-1}$$

The dynamical system in which one is interested is given by the following map $F$:

$$F_{a,T}\colon\ (s_{t-1}, L_{t-1}) \longrightarrow \begin{cases} \text{If } as_{t-1} - h(L_{t-1}) \geq TL_{t-1},\ (S_1)F_1\colon & \begin{cases} s_t = as_{t-1} - h(L_{t-1}) \\ L_t = s_{t-1} \end{cases} \\ \text{If } as_{t-1} - h(L_{t-1}) < TL_{t-1},\ (S_2)F_2\colon & \begin{cases} s_t = TL_{t-1} \\ L_t = s_{t-1} \end{cases} \end{cases}$$

Map $F$ is continuous, piecewise differentiable.

**5.3.1 Description of critical lines** As previously, we are interested in the two following sets:

1. Set $ND$ of points where $F_{a,T}$ is not differentiable.

$$ND = \{(x, y),\ ax - h(y) = Ty\}$$

2. Set $J_0$ where the jacobian of $F_{a,T}$ exists and vanishes.

$$J_0 = \{(x, y),\ y = 0\}$$

The images of these sets are called $LC^1$ and $LC^2$. They make up a partition of $\mathbb{R}^+ \times \mathbb{R}^+$ into three areas. $LC^1$ and $LC^2$ are the following sets:

$$\begin{aligned} LC^1 &= \left\{(x, y),\ ay = x + h\left(\frac{x}{T}\right)\right\} \\ LC^2 &= \{(x, y),\ ay = x\} \end{aligned}$$

If $(x, y) \in \{(x, y), ay > x + h(\frac{x}{T})\}$ , then $(x, y)$ has no preimage by $F$.
If $(x, y) \in \{(x, y), ay > x + h(\frac{x}{T}), y > \frac{x}{a}\}$ , then $(x, y)$ has more than one preimage.
If $(x, y) \in \{(x, y), y < \frac{x}{a}\}$ , then $(x, y)$ has exactly one preimage.

In this paragraph, we can apply a method similar to the one we used previously. We can work on the following example, assuming that $f$ is defined by $f(x) = 2x^2$. The critical lines of order 0, $LC^1$ and $LC^2$, make up a partition of $\mathbb{R}^+ \times \mathbb{R}^+$ into three areas ($Z_0$–$Z_3$–$Z_1$).

$$\begin{aligned} LC^1 &= \left\{(x, y), ay = x + 2\left(\frac{x}{T}\right)^2\right\} \\ LC^2 &= \{(x, y), ay = x\} \end{aligned}$$

If $(x, y) \in \{(x, y), ay > x + 2(\frac{x}{T})^2\}$ , then $(x, y)$ has no preimage by $F$.
If $(x, y) \in \{(x, y), ay > x + 2(\frac{x}{T})^2, y > \frac{x}{a}\}$ , then $(x, y)$ has three preimages.
If $(x, y) \in \{(x, y), y < \frac{x}{a}\}$ , then $(x, y)$ has exactly one preimage.

**Property 5.6.** *$LC^1$ cuts the curve ND into two points: the trivial point and a nontrivial point we call $a_0$.*

*Proof.* A point $(x, y)$ belonging to $LC^1$ satisfies $y = \frac{x+(\frac{x}{T})^2}{a}$. In replacing this formula in the equation of *ND* we have

$$x\left(a - \frac{T}{a}\right) - 2x^2\left(\frac{1}{a^2} + \frac{1}{aT}\right) - x^3\left(\frac{8}{a^2T^2}\right) - x^4\left(\frac{4}{a^2T^4}\right)$$

So the trivial point belongs to $LC^1$ and *ND*. Let $\varphi$ be the map such that
$\varphi\colon\ x \to (a - \frac{T}{a}) - 2x(\frac{1}{a^2} + \frac{1}{aT}) - x^2(\frac{8}{a^2T^2}) - x^3(\frac{4}{a^2T^4})$. The variation diagram of $\varphi$ is the following:

| $x$ | $0$ | | $\infty$ |
|---|---|---|---|
| $\varphi''(x)$ | | — | |
| $\varphi'(x)$ | $<0$ | $\searrow$ | $-\infty$ |
| $\varphi(x)$ | $>0$ | $\searrow$ | $-\infty$ |

So there is a unique solution to the equation $\varphi(x) = 0$.

**5.3.2 Study in the phase plane** As previously, we shall construct a curve $\mathcal{C}$. The algorithm is the same as the one described previously. The only difference lies in the fact that $\mathcal{C}$ is not necessarily an invariant curve. The following property is given in Barugola 1984.

**Property 5.7.** *$\mathcal{C}$ is the border of a region of the plane called $d'$ such that*

- *$d'$ is a compact set.*
- *$F_{a,T}(d') \subseteq d'$.*

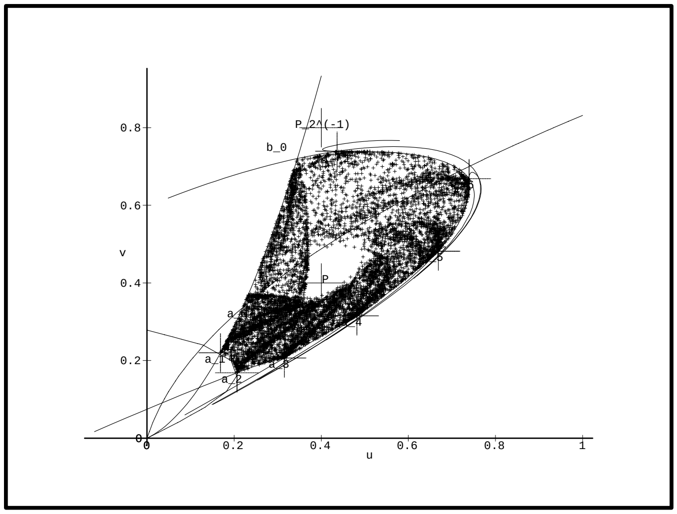



**Figure 3**
Absorbing area for $a = 1.8$ and $T = 0.5$.

- *for all neighborhoods $\mathcal{U}$ of $d'$, for all $M \in \mathcal{U} \backslash d'$ there exists $n > 0$ such that $F^n_{a,T}(M) \in d'$. This last property ensures that $d'$ is superabsorbing (Abraham, Gardini, and Mira 1997).*

Such a region is called an absorbing area (Abraham, Gardini, and Mira 1997). Figure 3, illustrating the previous construction, has been realized for $a = 1.8$, $T = 0.5$.

The previous example allows us to show an annular zone that looks[5] chaotic and in which a hole $W$ appears. It would be of interest to investigate the existence of an annular shape for the attractor.

**Use of critical lines for the determination of the value of bifurcation of the parameter for which the attractor is or is not annular.** One considers a variation of parameter $T$. One is interested in the analytical determination of the value of $T$ for which hole $W$ does not exist. The determination of this value of bifurcation is made by means of two theorems presented in Barugola and Cathala 1986.

The two theorems are as follows.
Let $P$ be a fixed point situated inside hole $W$:

**Theorem 5.1.** *If none of the preimages of $P$ (other than $P$) belong to $d'$, then the hole $W$ containing $P$ exists.*

**Theorem 5.2.** *If at least one antecedent of $P$ (other than $P$) belongs to $d'$, then the hole $W$ containing $P$ does not exist.*

The idea of the proof of these two theorems is as follows: If hole $W$ belongs to a zone $Z_n$ with $n \neq 0, 1$, $W$ admits multiple preimages; in particular, there are preimages $W^{-1}$ of $W$ not included in $W$. Let us assume that $W$ contains a repelling fixed point $P$. $W^{-1}$ contains a preimage $P^{-1}$ of the point $P$. Let us suppose that there is a zone $W$ and that $P^{-1}$ belongs to the annular chaotic zone. Then the image of $W^{-1}$ also belongs to

[5] The presence of chaos has not been proved analytically here, but there exist certain similarities with other situations that we know to be truly chaotic (Guckenheimer and Holmes 1983).

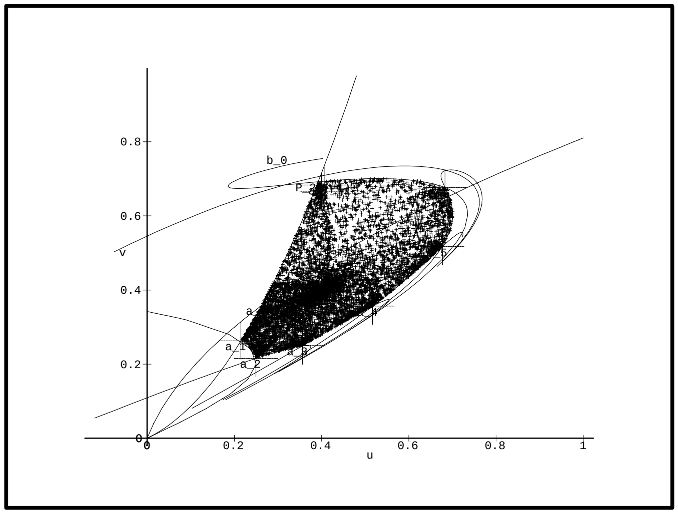



**Figure 4**
Absorbing area for $a = 1.8$ and $T = 0.6$.

the chaotic zone, which contradicts the existence of hole $W$. These two theorems allow us to determine the value of bifurcation for which the chaotic zone does not contain any repelling internal part.

Let us return to the previous example and let us calculate the value of bifurcation for which the chaotic attracteur is full. As the fixed point $P = (\frac{a-1}{2}, \frac{a-1}{2})$ belongs to $Z_3$, it has three images:

- itself: $P = (\frac{a-1}{2}, \frac{a-1}{2})$.
- another preimage by $F_1$: $P_1^{-1} = (\frac{a-1}{2}, -\frac{a-1}{2})$.
- a preimage by $F_2$: $P_2^{-1} = (\frac{a-1}{2}, \frac{a-1}{2T})$.

Only point $P_2^{-1}$ can belong to the absorbing domain. To calculate the corresponding value of $T$ in the bifurcation, it is sufficient to calculate for which value $P_2^{-1}$ crosses a critical line.

**5.3.3 Return to the modeling** In the study of the previous dynamical system, we did not check if the constraint function of the tax (tax lower than the wealth of the older people) was satisfied. We are now going to make this determination a posteriori and clarify the choice of the parameters that allow this constraint to be satisfied.

Let us recall that the constraint on wealth is set

$$CW = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+, Ty < ax\}$$

**Property 5.8.** *If $\frac{a^2}{T} - 1 > 0$, $CW$ intersects $LC^1$ into two points: the trivial point and a nontrivial one that we call $I$.*

*Proof.* The equation of $LC^1$ is $ay - x - h(\frac{x}{T})$. Let $(x, y)$ be the intersection point of $LC^1$ and $CW$. $x$ satisfies $x(\frac{a^2}{T} - 1) - h(\frac{x}{T}) = 0$. We then use the same reasoning as in Property 5.1 to end the proof.

If $b_0$ belongs to the part of $LC^1$ contained between $(0, 0)$ and $I$, then the constraint on wealth is satisfied. Otherwise, it is not, and the range of parameters does not allow the government to implement this tax.

### *5.4 Time inconsistency problem*

As the tax does not appear in the consumer program, agents cannot predict it. The character of nonpredictability should be verified a posteriori. We can observe that if $b_0$ is closed to *ND*, then the tax seems not to be predictable.

## 6 Conclusion

This study raises serious doubts about regarding the global-stability properties of the OLG model à la Reichlin. The perfect-foresight hypothesis is not accurate enough to keep the economy on a stable growth path. Further research on the analysis of global stability should examine alternative expectations mechanisms. This question goes beyond the scope of our work. Nevertheless, we can give an answer through the government's stabilizing fiscal policy. The redistributive policy among generations appears as an interesting stabilizing adjusment mechanism. Despite this stabilizing fiscal policy, the economic dynamics is characterized by weak stability properties.

A small change of parameters induces a large variation in the qualitative behavior of the dynamics. This property reflects perhaps the intrinsic instability generated by spatial and intertemporal interactions among agents. In this sense, mathematical tools are not an obstacle to the economic anaysis (McCallum 1983), but on the contrary they reveal the intrinsic instability of a society based on competitive markets.

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