Journal of Mathematical Systems, Estimation, and Control
Vol. 5, No. 4, 1995, pp. 1–22

Regions of Attraction of Closed-Loop Linear Systems with Saturated Linear Feedback*

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Abstract

In this article we address the problem of determining the region of attraction (**RA**) of a closed-loop single-input linear system with a saturated stabilizing linear feedback. It is shown that the shape of the region of attraction depends strongly on the number n_u of eigenvalues with positive real part of the open-loop system. In particular, under certain conditions on the control function, if the open-loop system has eigenvalues with strictly non-zero real part, the corresponding **RA** is homeomorphic to the cylinder $\mathbb{R}^{n-n_u} \times B^{n_u}$.

Key words: saturated linear feedback, non-linear dynamical systems, regions of attraction

AMS Subject Classifications: 93C10, 93D20

0 Basic Notation

$B^n(r)$	n-dimensional open ball of radius r.
S^n	n-dimensional unitary sphere.
n_s, n_u	number of eigenvalues of the matrix A with negative and positive real part, respectively.
$W^s(\gamma), W^u(\gamma)$	stable and unstable manifolds of the invariant set γ.
$\partial S, cl(S)$	boundary and clousure of the set S.
$\Omega(0)$	region of attraction of the origin.
$\sigma(A)$	the set of eigenvalues (spectrum) of the matrix A.
$\mathbb{C}^+, \mathbb{C}^-, \mathbb{C}^0$	complex numbers with positive, negative and zero real part.

*Received February 28, 1994; received in final form July 6, 1994. Summary appeared in Volume 5, Number 4, 1995.

1 Introduction

Consider the linear controllable system

$$\dot{x} = Ax + bu \tag{1.1}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}$. From the controllability property of the pair (A, b), there exists a linear feedback

$$u(x) = k^T x, \qquad k \in \mathbb{R}^n \tag{1.2}$$

which stabilizes system (1.1) (i.e., $\sigma(A + bk^T) \subset \mathbb{C}^-$). Let u_0 be a positive real number. If the input u is restricted to take values in the interval $[-u_0, u_0]$, a saturated feedback and a closed-loop non-linear vector field are obtained.

Definition 1.1 Let $u(x)$ be a state feedback defined as in (1.2). The *saturated linear feedback* $u_{\text{sat}}(x)$ is given by

$$u_{\text{sat}}(x) = \begin{cases} -u_0 & \text{if } u(x) \leq -u_0 \\ u(x) & \text{if } -u_0 < u(x) < u_0 \\ u_0 & \text{if } u(x) \geq u_0 \end{cases} \tag{1.3}$$

The *saturated vector field*, $Ax + bu_{\text{sat}}(x)$, will be denoted $f_{\text{sat}}(x)$.

Earlier studies on the saturated linear feedback problem have focused on the derivation of sufficient conditions for the global asymptotic stability of (1.1)-(1.3) when A is a marginally stable matrix (see for instance [9,10]). In a recent paper [1], for the two-dimensional case, we used qualitative methods to topologically characterize the region of attraction (**RA**) of (1.1)-(1.3) and its bifurcations. Along this methodological line, in this work we study the characterization of the **RA** for single-input n-dimensional hyperbolic ($\sigma(A) \cap \mathbb{C}^0 = \emptyset$) controllable linear systems (1.1) with a saturated linear feedback (1.3).

Specifically, we prove that, for a system whose open-loop eigenvalues have non-positive real part, the **RA** is unbounded. For completely unstable plants ($\sigma(A) \subset \mathbb{C}^+$), it is proved that the **RA** is bounded and homeomorphic to the n-dimensional ball. For stable open-loop systems, it is proved that all trajectories eventually tend towards some compact set of zero volume. For the case of systems whose eigenvalues have positive and negative real parts, it is found that a feedback which only relocates the eigenvalues with positive real part, makes the **RA** homeomorphic to the product of the **RA** s associated to the stable and stabilized parts. Consequently, the **RA** of the closed-loop system is homeomorphic to the cylinder $\mathbb{R}^{n-n_u} \times B^{n_u}$. For $n_u = 1$, and keeping fixed the relocated eigenvalues, the cylindric structure of the **RA** is retained under small changes in the locations of the open-loop stable eigenvalues. To estimate the **RA** we prove that $\partial\Omega(0) = \cup_i W^s(\gamma_j)$ when the critical elements γ_j in $\partial\Omega(0)$ are connected to the origin. Three aplication examples are presented to illustrate how the theoretical results improve the understanding of the problem and assist control design procedures.

2 The Saturated Closed-Loop System

The feedback $u_{\text{sat}}(x)$ induces a partition of $\mathbb{R}^n$ into three regions (S^+, S^- and S^0):

$$\begin{aligned} S^{+(-)} &= \{x \in \mathbb{R}^n : u_{\text{sat}}(x) = +(-)u_0\}; \\ S^0 &= \{x \in \mathbb{R}^n : |u(x)| < u_0\}. \end{aligned}$$

S^0 is an open set, and $S^+ \cup S^0 \cup S^- = \mathbb{R}^n$. $S^\pm = S^+ \cup S^-$ and S^0 are referred to as saturation and non-saturation regions, respectively. Note that the boundaries of the saturation regions S^+ and S^- are the $(n-1)$-dimensional hyperplanes $k^T x = \pm u_0$. On each of the regions S^+, S^0 and S^-, system (1.1)-(1.3) is linear. On $\mathbb{R}^n$, the system is piecewise linear and continuous.

Since $\sigma(A + bk^T) \subset \mathbb{C}^-$, the origin is a locally asymptotically stable (possibly, non-unique) equilibrium point of (1.1)-(1.3), and it is the only equilibrium point in S^0. Our main problem is to estimate the region of attraction $\Omega(0)$ of the origin. In the next proposition, it is proved that $\Omega(0)$ contains points in the interior of the saturation regions (S^+ and S^-).

Proposition 2.1 $\Omega(0) \cap [\mathbb{R}^n \backslash cl(S^0)] \neq \emptyset$.

Proof: Assume that $\Omega(0) \subset \text{cl}(S^0)$. Then, $\partial\Omega(0) \subset \text{cl}(S^0)$ and dist(0, $\partial\Omega(0)) > 0$. The invariance of $\partial\Omega(0)$ implies the existence of at least one trajectory $\gamma(t)$ contained in $\partial\Omega(0)$. Because in $\text{cl}(S^0)$ the saturated linear feedback coincides with the linear closed-loop feedback, γ satisfies: $\dot{\gamma}(t) = (A + bk^T)\gamma(t)$ for all $t \in \mathbb{R}$. The stability of $A + bk^T$ implies that $\gamma(t) \to 0$ when $t \to \infty$. This is a contradiction to the fact that $\text{dist}(0, \partial\Omega(0)) > 0$. Therefore, $\Omega(0)$ is not a subset of $\text{cl}(S^0)$ and the proposition is proved. ∎

In the next sections, it will be shown that the "size" of the intersection $\Omega(0) \cap [\mathbb{R}^n \backslash \text{cl}(S^0)]$ is, in general, rather large.

2.1 Equilibrium points

As a point of departure to study the topological configuration of the **RA**, in this section we obtain the set of equilibrium points of the saturated system. Consider system (1.1)-(1-3) with the matrix A invertible. In addition to the origin, the equilibrium points

$$e^{\pm} = \mp A^{-1} b u_0 \tag{2.1}$$

of the open-loop systems, and

$$\dot{x} = Ax \pm b u_0 \tag{2.2}$$

are candidates for equilibrium points of the saturated system (1.1)-(1.3).

Lemma 2.2 *Let the pair (A, b) be controllable, A invertible and $u(x)$ a linear feedback that stabilizes (1.1). Then, e^+ and e^- are equilibrium points of the system*

$$\dot{x} = Ax + b u_{\text{sat}}(x) = f_{\text{sat}}(x) \tag{2.3}$$

if and only if n_u is odd.

Proof: Consider the continuous function $E : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$, given by

$$E(x,s) = Ax + (1-s) b u_{\text{sat}}(x).$$

For any $\alpha > \|A^{-1}\| \; \|b\| u_0$ we have that $E(x,s) \neq 0$ for all $(x,s) \in \partial B^n(\alpha) \times [0,1]$. Consequently, $f_{\text{sat}}(x)$ and Ax are homotopically equivalent vector fields. This implies (see [5]) that the total degree of both vector fields coincide; where the total degree, $T \deg(f(x))$, is defined by

$$T\deg(f(x)) = \sum_{\bar{x}_i \in E} \text{index}(\bar{x}_i)$$

where $E = \{x \in \mathbb{R}^n : f(x) = 0\}$. Then, we have

$$T\deg(f_{\text{sat}}(x)) = T\deg(Ax) = \text{sgn}(\det(A)) = (-1)^{n_s}. \tag{2.4}$$

Since the origin is an asymptotically stable equilibrium point of f_{sat} (in fact, the pair (A, b) is controllable), index$(0) = (-1)^n$. Thus,

$$T\deg(f_{\text{sat}}(x)) = (-1)^n + \sum_{\bar{x}_i \in E^*} \text{index}(\bar{x}_i) \tag{2.5}$$

where $E^* = E \backslash \{0\}$. Because the origin is the unique equilibrium point of $f(x)$ in S^0, we have that $E^* \subset S^+ \cup S^-$. Therefore, $E^* \subseteq \{e^+, e^-\}$, with e^+ and e^- being candidate equilibrium points of (1.1)-(1.3). Since $Df_{\text{sat}}(e^{\pm}) = A$, if e^+ and e^- are equilibrium points of (1.1)-(1.3), then index$(e^+) =$ index$(e^-) = (-1)^{n_s}$. Finally, from (2.4) and (2.5) we have that e^+ and e^- are not equilibrium points of (1.1)-(1.3) only if $(-1)^n + 2(-1)^{n_s} = (-1)^{n_s}$ $[(-1)^n = (-1)^{n_s}]$ or equivalently only if n_u is odd [even]. In other words, e^+ and e^- are equilibrium points of (1.3) if and only if n_u is odd.

∎

The following theorem is an immediate consequence of the last lemma.

Theorem 2.3 *Let the pair (A, b) be controllable, A invertible and $u(x)$ a linear feedback which stablizes (1.1). Then, if n_u is odd, (2.3) has three equilibrium points: one attractor and two saddle points of n_u-type when $n_u \neq n$, or one attractor and two repulsors when $n_u = n$. If n is even, (2.3) has only one equilibrium point which is an attractor.*

The last case to be considered is when the matrix A is not invertible. That is, A has at least one zero eigenvalue.

Theorem 2.4 *Let (A, b) be a controllable pair with $det(A) = 0$, and u be a linear feedback which stabilizes (1.1). Then, (2.3) has only one equilibrium point which is an attractor.*

Proof: Suppose x^0 is a solution of $Ax^0 = \pm bu_0$. By hypothesis

$$\begin{aligned} \mathbb{R}^n &= \text{span}\{b, Ab, \ldots, A^{n-1}b\} \\ &= \text{span}\{Ax^0, A^2x^0, \ldots, A^nx^0\} \subset \text{Im}(A). \end{aligned}$$

This implies that the matrix A is invertible, which is a contradiction.

∎

Remark 2.5 The structure of the equilibrium points for the m-input ($m \geq 2$) case is a rather complex problem. In particular, it involves the existence of non-differentiable (or non-standard) bifurcations where one equilibrium point disappears because it collapses with the saturation boundary (border collision bifurcation). For instance, if A is stable, while the origin is the unique equilibrium point of the saturated system (2.3), a two-input saturated system may have either one, five, or nine equilibrium points. The independence of the total degree of the linear system on the saturation of the control implies that the sum of the indices of such equilibrium points is invariant and equal to $(-1)^{n_s}$. This is the result which allows us to count and clasify the equilibrium points of (2.3) (see [3]).

3 Regions of Attraction

In this paper the main result is the characterization of the topological configuration of the region of attraction of the saturated system (2.3). Because our approach is based on (smooth) dynamical systems theory, instead of function $f_{\text{sat}}(x)$ we will consider a smooth function $f^{\epsilon}_{\text{sat}}(x)$ which coincides with $f_{\text{sat}}(x)$ in $\mathbb{R}^n \backslash W_\epsilon$ where

$$W_\epsilon = \{x: \; -\epsilon \leq k^T x + u_0 \leq \epsilon \;\text{ or }\; -\epsilon \leq k^T x - u_0 \leq \epsilon\}.$$

Observe that W_ϵ satisfies $W_{\epsilon_1} \subset W_{\epsilon_2}$ if $\epsilon_1 < \epsilon_2$ and has the following property: for any compact set D , $\lim\limits_{\epsilon \to 0} \text{Vol}(D \cap W_\epsilon) = 0$.

Given $\epsilon > 0$, define the C^∞ function $\varphi^\epsilon(r)$ as an approximation of the saturation function

$$\varphi(r) = \begin{cases} u_0 & \text{if } r \geq u_0 \\ r & \text{if } -u_0 < r < u_0 \\ -u_0 & \text{if } r \leq -u_0 \end{cases} \tag{3.1}$$

such that $\varphi^\epsilon(r) \equiv \varphi(r)$ for all $r \in (-\infty, -u_0 - \epsilon) \cup (-u_0 + \epsilon, u_0 - \epsilon) \cup (u_0 + \epsilon, \infty)$ and $0 \leq (d/dr)\varphi^\epsilon \leq 1$. Then, the function $f^\epsilon_{\mathrm{sat}}(x) = Ax + b\varphi^\epsilon(k^T x)$ is a C^∞ function which coincides with $f_{\mathrm{sat}}(x)$ in $\mathbb{R}^n \backslash W_\epsilon$.

To prove the next proposition, assume that $\partial\Omega(0)$ is an embedded submanifold of $\mathbb{R}^n$ (smooth manifold). In fact, the smoothness hypothesis could be weakened.

Proposition 3.1 *Let A be a matrix with $\sigma(A) \subset \mathbb{C}^- \cup \mathbb{C}^0$. Then, if $\partial\Omega(0)$ is a smooth manifold, the region of attraction $\Omega(0)$ of (2.3) is unbounded.*

Proof: Suppose $\Omega(0)$ is bounded. $\sigma(A) \subset \mathbb{C}^- \cup \mathbb{C}^0$ implies that $\mathrm{div} f^\epsilon_{\mathrm{sat}}(x) \leq 0$ for all $x \in S^\pm \backslash W_\epsilon$, and $\sigma(A + bk^T) \subset \mathbb{C}^-$ implies that there exist $\beta > 0$ such that $\mathrm{div} f^\epsilon_{\mathrm{sat}}(x) = -\beta$ for all $x \in S^0 \backslash W_\epsilon$. On the other hand, it can be easily proved the existence of a number h such that $Re\lambda \leq h$ for all $\lambda \in \sigma(A + rbk^T)$ where $0 \leq r \leq 1$. This implies that $\mathrm{div} f^\epsilon_{\mathrm{sat}}(x) < nh$ for all $x \in W_\epsilon$. Then, the boundedness of $\Omega(0)$, there exist ϵ^* and $\gamma > 0$ such that

$$\int_{\Omega(0)} \mathrm{div}\ f^\epsilon_{\mathrm{sat}}(x) dV_n \leq nh \mathrm{Vol}(\Omega(0) \cap W_\epsilon) - \beta \mathrm{Vol}(S^0 \cap \Omega(0) \backslash W_\epsilon) < -\gamma$$

for all $\epsilon < \epsilon^*$. From the invariance of $\partial\Omega(0)$ follows that $\langle f_{\mathrm{sat}}(x), \nu(x) \rangle = 0$ for all $x \in \partial\Omega(0)$, where $\nu(x)$ is the outward unit normal vector field on the boundary $\partial\Omega(0)$. This, together with the condition $\lim \mathrm{Vol}(D \cap W_\epsilon) = 0$, implies the existence of a positive number L such that

$$\left| \int_{\partial\Omega(0)} \langle f^\epsilon_{\mathrm{sat}}(x), \nu(x) \rangle dV_{n-1} \right| = \left| \int_{\partial\Omega(0) \cap W_\epsilon} \langle f^\epsilon_{\mathrm{sat}}(x), \nu(x) \rangle dV_{n-1} \right| \leq \epsilon L.$$

For ϵ small enough this is a contradiction to the divergence theorem (see [14]). Therefore, $\Omega(0)$ is not bounded.

∎

Remark 3.2 When n ($n \neq 5$) is odd, the proof of Proposition 3.1 can be simplified: if $\Omega(0)$ is bounded, $\partial\Omega(0)$ is homeomorphic to the S^{n-1} sphere. Since $n-1$ is even, due to Theorem A.4 (appendix), $\partial\Omega(0)$ has at least one equilibrium point. This is a contradiction to Lemma 2.2.

In what follows, the approximation of $f_{\mathrm{sat}}(x)$ by the smooth vector field $f^\epsilon_{\mathrm{sat}}(x)$ will be obviated. Next we prove three fundamental results.

Theorem 3.3 *Let A be a matrix with all its eigenvalues in the open right half-plane. Then, $\Omega(0)$ is bounded and homeomorphic to B^n. Moreover, for $n \neq 5$, if $\partial\Omega(0)$ is a smooth manifold, then it is homeomorphic to S^{n-1}.*

Proof: Since $\sigma(A) \subset \mathbb{C}^+$, there exists a positive-definite symmetric matrix P that satisfies $PA + A^T P = I$. Introduce the Lyapunov function $V = x^T P x$, and obtain

$$\begin{aligned} \dot{V} &= \|x\|^2 + (b^T P x + x^T P\, b) u_{\text{sat}}(x) \\ &\geq \|x\|^2 - 2u_0 \|P\|\, \|b\|\, \|x\|. \end{aligned} \tag{3.2}$$

Then $\dot{V} > 0$ whenever $\|x\| > 2u_0\|P\|\,\|b\|$. Consequently, $\Omega(0)$ is bounded. The theorem follows from Proposition A.1 and A.2 (appendix).

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Theorem 3.4 *Let A be a matrix with all its eigenvalues having negative real part. Then, the solutions of (2.3) are bounded and there is a compact invariant set of zero volume which contains the ω-limit sets $\omega(x)$ of all points $x \in \mathbb{R}^n$. If $\partial\Omega(0)$ is a smooth manifold, $\Omega(0)$ is not bounded.*

Proof: First, we prove that the solutions of (2.3) are bounded. Let P be a positive-definite symmetric matrix that satisfies $PA + A^T P = -I$. If we define the Lyapunov function $V = x^T P x$, we obtain

$$\begin{aligned} \dot{V} &= -\|x\|^2 + (b^T P x + x^T P\, b) u_{\text{sat}}(x) \\ &\leq -\|x\|^2 + 2u_0 \|P\|\, \|b\|\, \|x\|. \end{aligned} \tag{3.3}$$

$\dot{V} < 0$ if $\|x\| > 2u_0\|P\|\,\|b\| =: r$. A Lyapunov argument implies that the solutions of (2.3) are bounded (ultimately contained in $V_r = \{x \in \mathbb{R}^n : V(x) \leq \sup_{y \in B^n(r)} V(y)\}$).

Let $\alpha > 0$ be a constant such that $\mathbb{R}e(\lambda) < -\alpha$ for all $\lambda \in \sigma(A) \cup \sigma(A + bk^T)$. From the well known Abel-Jacobi-Liouville formula, we have

$$\det\ e^{At} = e^{(tr\ A)t} \leq e^{-\alpha t}; \quad \det\ e^{A+bk^T} = e^{tr(A+bk^T)t} \leq e^{-\alpha t}.$$

Let c be a positive number such that $B_r \subset D = \{x \in \mathbb{R}^n : V(x) \leq c\}$. Hence, the flow maps the compact set D into the compact set $\phi_t D$ with $\text{Vol}(\phi_t D) = (\det \phi_t)\ \text{Vol}(D) \leq \text{Vol}(D)\ \exp(-\alpha t)$. Because all trajectories cross inwards the boundary of D, $\phi_{t_1}(D) \subset \phi_{t_2}(D)$ if $t_1 > t_2$. Then, every trajectory is ultimately contained in $D^* = \cap_{t>0} \phi_t(D)$. This implies the existence of a zero-volume set D^*, contained in $B^n(r)$, towards which all trajectories tend (see [15]). Finaly, unboundedness of $\Omega(0)$ follows from Proposition 3.1.

∎

In general, global boundedness of trajectories does not imply global asymptotic stability. The following simulation example shows that typical non-linear phenomena may appear.

Example 3.5 Consider the one-input three-dimensional system:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= -6x_1 - 11x_2 - 6x_3 + u, \quad |u| \leq 1. \end{aligned}$$

The open-loop system is stable with eigenvalues $\{-1, -2, -3\}$. The linear feedback:

$$u(x) = (6 + \lambda^3)x_1 + (11 - 3\lambda^2)x_2 + (6 + 3\lambda)x_3$$

relocates all the closed-loop eigenvalues at $\lambda < 0$. According to Theorem 3.4 the saturated vector field $f_{\mathrm{sat}}(x)$ has all its trajectories bounded, and the origin as the unique equilibrium point (Lemma 2.2). If the origin is not a global attractor, there are trajectories that converge to a closed orbit.

With numerical simulations, the following dynamic structure was found:

a) For $-29 < \lambda < 0$: the origin is a global attractor; that is $\Omega(0) = \mathbb{R}^3$.

b) For $\lambda \cong -29.1$: a non-hyperbolic closed orbit appears, and $\Omega(0)$ is homeomorphic to $\mathbb{R} \times B^2$.

c) For $\lambda < -29.2$: there exist two limit cycles, one stable and one saddle-type. $\Omega(0)$ is homeomorphic to $\mathbb{R} \times B^2$.

Theorem 3.6 *Let A be a matrix with n_u eigenvalues with positive real part and n_s eigenvalues with negative real part, $n_u + n_s = n$. Then, for the saturated system (2.3), there exist two positive numbers r_1, r_2; and a linear coordinate transformation T such that* $\mathrm{cl}(\Omega(0))$ *is contained in the cylinder $T^{-1}(\mathbb{R}^{n_s} \times B^{n_u}(r_1))$, and given any initial point x^0, there is a time $t^*(x^0)$ such that the trajectory $x(t)$ is contained in the cylinder $T^{-1}(B^{n_s}(r_2) \times \mathbb{R}^{n_u})$ for all $t > t^*$. In particular, $\omega(x^0) \subset T^{-1}(B^{n_s}(r_2) \times \mathbb{R}^{n_u})$.*

Proof: There is a linear change of coordinates T which transforms system (1.1) into the following system

$$\dot{x} = \begin{bmatrix} A^+ & 0 \\ 0 & A^- \end{bmatrix} x + \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \end{bmatrix} u, \tag{3.4}$$

where A^- is a $(n_s \times n_s)$ stable matrix, A^+ is an $(n_u \times n_u)$ unstable matrix, and $x = (x_1, x_2)$. Let P_1 and P_2 be positive-definite symmetric matrices that satisfy $P_1 A^+ + A^{+T} P_1 = I_1$ and $P_2 A^- + A^{-T} P_2 = -I_2$. If we set $V(x) = x_1^T P_1 x_1$, (3.2) implies the existence of a positive number r_1 such that, for any $x^0 = (x_1^0, x_2^0)$ with $\|x_1^0\| \geq r_1$, the solution $\phi_t(x^0)$ diverges. In other words, any bounded solution must satisfy $\|x_1^0\| < r_1$. It follows that $T[\Omega(0)] \subset \mathbb{R}^{n_s} \times B^{n_u}(r_1)$. If $V(x) = x_2^T P_2 x_2$, by (3.3) there exist $r_2 > 0$ and, for any $x^0 = (x_1^0, x_2^0)$ with $\|x_1^0\| < r_2$, a positive time $t^*(x^0) > 0$, such that the trajectory $\phi_t(x^0) = (x_1(t), x_2(t))$ satisfies $\|x_2(t)\| < r_2$ for all $t > t^*$. Then, $\omega(x^0) \subset T^{-1}(B^{n_s}(r_2) \times \mathbb{R}^{n_u})$.

∎

Corollary 3.7 *Let A be as in Theorem 3.6. Then the ω-limit set of any trajectory on $\partial\Omega(0)$ is non-empty and contained in the bounded set $T^{-1}(B^{n_s}(r_2) \times B^{n_u}(r_1))$.*

Proof: Let $x^0 \in \partial\Omega(0)$. From Theorem 3.6, $\partial\Omega(0) \subset T^{-1}(\mathbb{R}^{n_s} \times B^{n_u}(r_1))$ and $\omega(x^0) \subset T^{-1}(B^{n_s}(r_2) \times \mathbb{R}^{n_u})$. Hence $\omega(x^0) \subset T^{-1}(B^{n_s}(r_2) \times B^{n_u}(r_1))$.

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Remark 3.8 Note that theorems 3.3 and 3.6 and the proof of boundedness of trajectories of open-loop stable systems (Theorem 3.4) are valid for any bounded multi-input non-linear control law.

3.1 Regions of attraction for unstable saturated systems

We shall prove that the shape of $\Omega(0)$ depends strongly on n_u and in particular on the number of eigenvalues that have been "relocated" by feedback. To fix ideas we present the following definition. Let us express the spectrum of A in (1.1) as the union of two disjoint symmetric sets: $\sigma(A) = \sigma_1(A) \cup \sigma_2(A), \sigma_1(A) \cap \sigma_2(A) = \emptyset$, where $\sigma_1(A) = \{\lambda_1, \ldots, \lambda_m\}$ and $\sigma_2(A) = \{\lambda_{m+1}, \ldots, \lambda_n\}$. Consider a change of coordinates $T = (T_1, T_2)$ which transforms the controllable system (1.1) into the system

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + b_1 u \\ \dot{x}_2 &= A_2 x_2 + b_2 u \end{aligned} \tag{3.5}$$

in such a way that

$$\sigma(A_1) = \{\lambda_1, \ldots, \lambda_m\}$$

and

$$\sigma(A_2) = \{\lambda_{m+1}, \ldots, \lambda_n\}.$$

Definition 3.9 *From the controllability of the pair (A, b), for any symmetric set of m complex numbers, Δ, there exists a vector k_1 such that $\sigma(A_1 + b_1 k_1^T) = \Delta$. If $\sigma(A_1) \cap \sigma(A_1 + b_1 k_1^T) = \emptyset$ we will say that the feedback function $u(x) = k_1^T T_1 x$* relocates *the eigenvalues $\{\lambda_1, \ldots, \lambda_m\}$ of (1.1).*

It is easy to prove that the dynamic behavior of the open-loop system on the invariant subspace associated to the non-relocated eigenvalues, is not affected by saturation. To see this, observe that the saturated system (2.3) is transformed, by the change of coordinates T, into the system

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + b_1 \varphi(k_1^T x_1) \\ \dot{x}_2 &= A_2 x_2 + b_2 \varphi(k_1^T x_1) \end{aligned}$$

where φ is the saturation function (3.1). Then, the dynamical behavior on the invariant subspace asociated to the not relocated eigenvalues, $\{x_1 = 0\}$, is given by $\dot{x}_2 = A_2 x_2$, which is precisely the open-loop dynamics of the non-relocated part of the system.

By a "minimum-energy-control" (**MEC**) feedback, we mean a control feedback which only relocates the unstable eigenvalues (see [13]). Formally, a **MEC** feedback for system (1.1), is a function $u(x) = u_1(x_1)$ where $u_1(x_1)$ is a stabilizing control for the unstable part of (3.4). The **MEC** feedback is used when control is expensive (see [8]). Here we prove that, for any **MEC** feedback, $\Omega(0)$ is homeomorphic to $\mathbb{R}^{n_s} \times B^{n_u}$ when $0 < n_u < n$, $n_u + n_s = n$.

Theorem 3.10 *Let A be a matrix with n_u eigenvalues with positive real part and n_s eigenvalues with negative real part $n_u + n_s = n$. Assuming that $u(x)$ is a* **MEC** *feedback, $\Omega(0)$ is homeomorphic to $\mathbb{R}^{n_s} \times B^{n_u}$.*

Proof: Let T be the change of coordinates which transforms systems (1.1) into (3.4). Let $u^1(x) = k_1^T x_1$ be a **MEC** feedback. Then, $A^+ + \bar{b}_1 k_1^T$ is asymptotically stable. Consider the saturation of $u^1(x), u^1_{\text{sat}}(x_1)$, defined by (1.3) and let $\Omega_1(0)$ be the **RA** of the corresponding closed-loop system. From Theorem 3.4, it follows that $\Omega_1(0)$ is homeomorphic to B^{n_u}.

After closing the loop, system (3.4) is transformed into the following triangular non-linear system

$$\begin{aligned} \dot{x}_1 &= A^+ x_1 + \bar{b}_1 u^1_{\text{sat}}(x_1) = f_1(x_1) \\ \dot{x}_2 &= A^- x_2 + \bar{b}_2 u^1_{\text{sat}}(x_1) = f_2(x_1, x_2). \end{aligned} \tag{3.6}$$

From the stability of matrices A^- and $A^+ + \bar{b}_1 k_1^T$ follows that (3.6) is locally asymptotically stable. Moreover, the following conditions are fulfilled: i) The set $\{x : x_1 = 0\}$ is an invariant manifold; ii) system (3.6), restricted to $x_1 = 0$, is globally asymptotically stable (coincides with $\dot{x}_2 = A^- x_2$); iii) the manifold $x_1 = 0$ attracts the set $\Omega_1(0) \times \mathbb{R}^{n_s}$, and iv) for any point in $\Omega_1(0) \times \mathbb{R}^{n_s}$ the corresponding solution of (3.6) is bounded in the future (consequence of Theorem 3.6). Then, (3.6) satisfies the non-local stability conditions for triangular non-linear system given in [11]. This implies $\Omega(0) = \Omega_1(0) \times \mathbb{R}^{n_s}$ and the theorem is proved (see Figure 1).

∎

Dependency of the control function on just n_u variables,

$$u_{\mathrm{sat}}(x) = u^1_{\mathrm{sat}}(x_1),$$

implies that system (2.3) (with a **MEC** feedback) has, at its stability boundary, n_s invariant direction which are parallel to the saturation hyperplanes S^+ and S^-. Specifically, when $n_u = 1$, the stability boundary consist of two $(n-1)$-dimensional hyperplanes parallel to S^+ and S^- which do not depend on the gain k_1.

Corollary 3.11 *Let A be a matrix with $n_u = 1$, $n_s = n-1$ and $u(x)$ be a* **MEC** *feedback. Then,*

$$\Omega_{\mathrm{MEC}}(0) = T^{-1}\{x \in \mathbb{R}^n \,:\, |x_1| < |b_1|u_0/\lambda^+\}$$

where λ^+ is the positive eigenvalue of A, and $T : \mathbb{R}^n \to \mathbb{R}^n$ is the change of coordinates which transforms (1.1) into (3.4). Furthermore, $\Omega_{\mathrm{MEC}}(0)$ is independent of the specific **MEC** *feedback chosen.*

Remark 3.12 For $n > 1$ the corollary is not valid. In other words, $\Omega_{\mathrm{MEC}}(0)$ is independent of the feedback gains, if and only if $n_u = 1$. For $n_u = 2$, a simple example

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -6x_1 + 5x_2 + u; \end{aligned}$$

with $u = \varphi((-4\epsilon + 6)x_1 + (-4\epsilon^2 - 5)x_2)$, where φ is the saturation function (3.1), shows that $\partial\Omega_\epsilon(0)$ is a limit cycle that converges to zero when the gains converges to infinity ($\epsilon \to \infty$).

At this point, the results obtained topologically characterize the shape of the **RA** of (2.3). In what follows we shall address the problem of estimating the **RA** of (2.3). In principle, the estimation of the **RA**s is part of the problem of the characterization of the global behavior of non-linear systems, and is far from being solved. In [4] the estimation of the **RA** has been addresed analizing the behavior of the system at the boundary. The main result in this direction has been stated in the appendix (Theorem A.3). Next we prove that the conditions of Theorem A.3 are satisfied. We require the following definition and observations from dynamical systems.

Definition 3.13 *Two compact invariant sets P_1 and P_2 are connected (heteroclinic) if there exists a trajectory whose $\alpha-$ and ω-limit sets are equal to P_1 and P_2.*

It is well known [17] that a connection between saddle points can be broken by small perturbations, while a connection between an attractor or a repulsor and any other hyperbolic equilibrium point is structurally stable. In our case, system (2.3) with a **MEC** feedback has the following properties: i) if n_u is odd, e^+ and e^- are connected with the origin; ii) if n_u is even, there exists at least one invariant set (using simulations, a limit cycle was always found) which is connected with the origin; Assuming structural stability of our system, a small change in the locations of the negative eigenvalues of the system (3.6) does not break the connections of the invariant sets with the origin.

The next lemma is a consequence of the λ-lemma [4]. Its proof follows along similar lines of a part of the proof of theorem 3.7 of [4]. We include the proof for the sake of completness.

Lemma 3.14 *Let γ be a hyperbolic critical element of system (2.3). If γ is connected with the origin, then $W^s(\gamma) \subset \partial\Omega(0)$.*

Proof: If γ is connected with the origin, then $W^u(\gamma) \cap \Omega(0) \neq \emptyset$. Let $D \subseteq W^u(\gamma) \cap \Omega(0)$ be an m-disk, $m = \dim(W^u(\gamma))$. Let $y \in W^s(\gamma)$ be arbitrary. For any $\epsilon > 0$, let N be an m-disk transversal to $W^s(\gamma)$ at γ, contained in an ϵ-neighborhood of y. By the λ-lemma, there exist $t > 0$ such that $\phi_t(N)$ is so close to D that $\phi_t(N)$ contains a point $p \in \Omega(0)$. Thus $\phi_{-t}(p) \in N$. Since $\Omega(0)$ is invariant, this shows that $N \cap \Omega(0) \neq \emptyset$. Letting $\epsilon \to 0$, one proves that $y \in \mathrm{cl}(\Omega(0))$. Thus $W^s(\gamma) \subset \mathrm{cl}(\Omega(0))$. Since, $W^s(\gamma)$ and $\Omega(0)$ are disjoint, it follows that $W^s(\gamma) \subset \partial\Omega(0)$.

∎

Theorem 3.15 *If all critical elements of (2.3), γ_j $(j = 1, 2, \ldots)$ on $\partial\Omega(0)$ are hyperbolic and connected with the origin, then*

$$\partial\Omega(0) = \bigcup_j W^s(\gamma_j).$$

Proof: The result follows from Theorem A.3. Observe that condition iii) follows from Corollary 3.7, and that, due to Lemma 3.14, condition ii) can be replaced by the connectivity condition.

∎

For the general case, we already know that the boundary of the **RA** is contained in a cylinder. On the other hand, we know that for the **MEC** case, the **RA** is a cylinder. An interesting problem arises: what happens when the **MEC** case is perturbed by gain deviations. Specifically, for $n_u = 1$, we will prove that $\Omega(0) \subset \Omega_{\mathrm{MEC}}(0)$ and $\Omega(0)$ retains the cylindric structure. For three dimensional systems with small gain deviations, numerical simulations exhibited boundaries as cylinders with radius tending to zero [16] (see figure 2). We conjecture that both results are valid for the n-dimensional case.

The next result shows that for $n_u = 1$ the largest **RA** is obtained when $u(x)$ is a **MEC** feedback.

Theorem 3.16 *Let A be a matrix with $n_u = 1$, $n_s = n-1$, and let $u(x)$ be any linear (or non-linear) feedback with $|u(x)| \leq u_0$. Then the* **RA**, *$\Omega(0)$, related to the feedback $u(x)$, satisfies the relation*

$$\Omega(0) \subset \Omega_{\text{MEC}}(0).$$

Proof: The unstable part of (3.4) can be written as follows

$$\dot{x} = \lambda^+ x_1 + b_1 u. \tag{3.7}$$

Integration of (3.7), for $u = u(x)$, yields

$$e^{-\lambda^+ t} x_1(t) = x_1^0 + \int_0^t e^{-\lambda^+ s} b_1 u(x(s)) ds.$$

Since $|u(x)| \leq u_0$,

$$|x_1(t)| \geq e^{\lambda^+ t}(|x_1^0| - |b_1|u_0/\lambda^+) + |b_1|u_0/\lambda^+.$$

When $|x_1^0| \geq |b_1|u_0/\lambda^+$, the trajectory satisfies $|x_1(t)| \geq |b_1|u_0/\lambda^+$. Then $\Omega(0) \subset \{x \in \mathbb{R}^n : |x_1| < |b_1|u_0/\lambda^+\} = \Omega_{\text{MEC}}(0)$.

∎

Theorem 3.17 *Let A be a matrix with $n_u = 1$, and $n_s = n-1$. If there is a connection between e^+, e^- and the origin, then, $\Omega(0)$ is homeomorphic to $\mathbb{R}^{n-1} \times B^1$. In particular, $\partial\Omega(0) = W^s(e^+) \cup W^s(e^-)$.*

Proof: From Theorem 3.15 follows that

$$e^+, e^- \in \partial\Omega(0)$$

and

$$W^s(e^+), \quad W^s(e^-) \subset \partial\Omega(0).$$

Observe that $Df_{\text{sat}}(e^\pm) = A$ implies that $W^s(e^\pm)$ locally coincides with the $(n-1)$ dimensional stable linear manifolds $E^s(e^\pm)$ of the open-loop system (2.1). Then $E^s(e^+)$ and $E^s(e^-)$ can be divided in two parts: one ($E^s_w(e^+)$ and $E^s_w(e^-)$) which does not intersect the saturation hyperplanes, S^+ and S^-; and a second part ($E^s_i(e^+)$ and $E^s_i(e^-)$) which intersects the hyperplanes. In particular, $E^s_w(e^+)(E^s_w(e^-))$ is contained in $W^s(e^+)(W^s(e^-))$.

Let y be a point in $E^s_i(e^+)$. Hence, there exists $t^* > 0$ such that $\phi_{t^*}(y) \in S^+$. Because $\phi_t(y)$ cannot stay in S for all $t < t^*$, there is a $t^{**} < t^*$ for which $\phi_{t^{**}}(y) \in S^-$(if $\phi_{t^{**}}(y) \in S^+$ the e^--origin connection is broken because $\dim(E^s_i(e^+)) = n-1$). The saddle properties of e^- imply that $\phi_t(y) \in S^-$ for all $t < t^{**}$. In S^-, $E^s(e^-)$ is a negative attractor, which means that $\phi_t(y) \to E^s(e^-)$ when $t \to -\infty$. It follows that $W^s(e^\pm)$ is unbounded in any direction with $\dim(W^s(e^\pm)) = n-1$. This implies that the only critical elements of (2.3) in $\partial\Omega(0)$ are e^+, e^-. From Theorem 3.15 follows that $\partial\Omega(0) = W^s(e^+) \cup W^u(e^-)$ and the theorem is proved.

■

Remark 3.18 An interesting question is to analize what happens when the connection between the invariant elements in $\partial\Omega(0)$ and the origin are broken. In this case, a topological bifurcation occurs. For two-dimensional systems and $n_u = 1$ (see [1]), as the gains are increased (moving the negative eigenvalues to the left), there is a critical value where the connection with the origin is broken and a connection between e^+ and e^- is created (see [1]). After that value, the $e^- - e^+$ connection with the origin is broken and $\Omega(0)$ remains bounded. The parts $W^u(e^+)$ and $W^u(e^-)$ which, for a **MEC** feedback converge to the origin, now diverge (see [1]). It must be pointed out that the last two of the above cases were not detected by simulations in [13].

4 Examples

In control design for linear systems with saturated input, a fundamental problem is to find a feedback such that the origin becomes an attractor for a suitable set D of initial conditions [7]. The assumption that the set D must be contained in a non-saturation region S^0 is a major disadvantage of most existing design techniques. Such limitation is aggravated as the gain of the controller increases or when the system is open-loop unstable. In principle, a control design technique should benefit from *a priori* information on the geometric structure of the region of attraction $\Omega(0)$. In particular, it can be proved (using Theorem 2.2 of [12]) that, given any compact subset D of $\Omega_{\mathrm{MEC}}(0)$, there is an ϵ-neighborhood of u_{MEC} for which D is also contained in the corresponding **RA**, $\Omega(0)$. Therefore, one can design first a **MEC** feedback in such a way that $D \subset \Omega(0)$. After that, the stable eigenvalues can be relocated if care is taken to keep $D \subset \Omega(0)$. In this section, the mentioned points are illustrated by three application examples whose study is based on the theoretical results obtained in the preceding sections.

Example 4.1 The following single-input two-dimensional system is the linearization around an unstable equilibrium point of a chemical reactor [2]:

$$\begin{aligned} \dot{x}_1 &= -2x_1 - 0.03125x_2 \\ \dot{x}_2 &= 200x_1 + 4.25x_2 + u \quad |u| \leq 10.0 \end{aligned} \tag{4.1}$$

where x_1 and x_2 are reactor concentration and temperature, and u is the coolant temperature. The open-loop eigenvalues are (0.75, -3.0), so $n_u = 1$ and $\det(A) \neq 0$. The linear feedback

$$u(x) = [(-2.25 - 2k_1 + k_2)x_1 - (0.0703 + 0.03125k_1)x_2]/0.03125 \tag{4.2}$$

stabilizes (4.1) at the origin. Suppose the assigned closed-loop eigenvalues are real and equal to $\lambda < 0$. Then we have $k_1 = -2\lambda$ and $k_2 = \lambda^2$.

If $\lambda = -3.0$, (4.2) is a **MEC** feedback. From Theorem 3.10, the **RA** of the saturated vector field $Ax + bu_{\mathrm{sat}}(x)$ is homeomorphic to the cylinder $\mathbb{R} \times B^1$. $\Omega_{\mathrm{MEC}}(0)$ can be obtained using Corollary 3.11.

For non-**MEC** saturated feedbacks, the **RA** was investigated with numerical simulations guided by Theorem 3.17. As a first step, we looked for possible connections between the origin and the saddle-type equilibrium points $e^+ = (-0.1388, 8.889)$ and $e^- = (0.1388, -8.889)$. (From Lemma 2.2, e^+ and e^- are equilibrium points of the saturated vector field $Ax + bu_{\mathrm{sat}}(x)$). Since $\partial\Omega(0)$ is one-dimensional, $e^{\pm}$ are the only critical elements on $\partial\Omega(0)$.

The stable and unstable invariant directions of these equilibrium points are $v_s = (0.03125, 1.0)$, and $v_u = (-0.01136, 1.0)$.

To test connection between e^+ and the origin, initial conditions in the direction $\{e^+ + \alpha v_u\}$ were considered. If at least one trajectory converges to the origin, then it is connected with e^+. From a symmetry argument, the same is true for e^-. It is easy to see that $e^{\pm}$ are hyperbolic. Then, according to Theorem 3.15 if both e^+ and e^- are connected with the origin, then $e^+, e^- \in \partial\Omega(0)$ and $W^s(e^+) \cup W^s(e^-) = \partial\Omega(0)$. For the saturation of (4.2), both connections were found for $-4.1399 < \lambda < 0$. For $\lambda \cong -4.1399$, the system presents a topological bifurcation: the connection between the equilibrum points e^+, e^- and the origin breaks and a heteroclinic connection γ apears. In this case $\partial\Omega(0) = \gamma$, so that $\Omega(0)$ is bounded. For $\lambda < -4.1399$ there are no connections; $e^+, e^- \notin \Omega(0)$, and $\partial\Omega(0)$ is an unstable limit cycle.

Example 4.2 Consider the one-input bounded control for a linearized model of an inverted pendulum on a cart [8]. Specifically, we are interested in describing the geometry of the **RA**. The process dynamics is given by the following four-dimensional linear model:

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -F/M & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -g/L & 0 & g/L & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1/M \\ 0 \\ 0 \end{pmatrix} u \tag{4.3}$$

where M, F, g, L are (positive) physical constants. The input $u(t)$ is the force on the cart. The state variables are $d, \dot{d}, d + L\phi$, and $\dot{d} + L\dot{\phi}$, where d is the cart displacement referred to the origin, and ϕ is the pendulum angle with respect to the vertical.

The spectrum of matrix A, and its corresponding eigenvectors are:

$$\begin{array}{rclrcl}
\lambda_1 &=& 0 & v_1 &=& (1,0,1,0) \\
\lambda_2 &=& -F/M & v_2 &=& (-M/F, 1, -MFg/(M^2g \\
 & & & & & -F^2L), M^2g/ \\
 & & & & & (M^2g - F^2L)) \\
\lambda_{3,4} &=& \pm g/L & v_{3,4} &=& (0,0,1,\pm g/L)
\end{array}$$

The coordinate transformation $z = Tx = (v_1, v_2, v_3, v_4)x$, carries system (4.3) into a system of the form (3.5) where A_1 is a 2×2 stable diagonal matrix with eigenvalues $\{-g/L, -F/M\}$, and A_2 is a 2×2 unstable diagonal matrix with eigenvalues $\{0, g/L\}$. Suppose a **MEC** feedback $u = u(x_2)$ is applied. By Theorem 2.4, system (4.3)-(1.3) has only the origin as equilibrium point . If $\Omega_2(0)$ is the asymptotic region of attraction for the unstable part of (3.5), then, the asymptotic region of attraction of (4.3)-(1.3) is $\Omega(0) = \mathbb{R}^2 \times \Omega_2(0)$.

Although in this work we have not addressed the problem of linear systems with positive and zero real part eigenvalues, for two dimensional systems, it is possible to see that the zero eigenvalue behaves as an unstable one [1]. In this case, $\partial\Omega_2(0)$ is an unstable limit cycle (see figure 3). In original coordinates, the cart-pendulum region of attraction turns out to be the product of the plane generated by two eigenvectors associated to the stable eigenvalues and the interior of the limit cycle contained in the plane generated by the other eigenvectors.

Example 4.3 Consider the linearized model for the depth control of a submarine [7]:

$$\begin{array}{rcl}
\dot{x}_1 &=& x_2 \\
\dot{x}_2 &=& x_3 \\
\dot{x}_3 &=& -0.005x_3 + 0.005u \quad |u| \leq 0.005
\end{array} \tag{4.4}$$

where x_1 designates depth. The open-loop eigenvalues are (0, 0, -0.005), and therefore the system is not hyperbolic. The linear feedback:

$$\begin{array}{rcl}
u(x) &=& \left[1.117 \times 10^{-5}\lambda x_1 - (1.117 \times 10^{-5} - 5.2 \times 10^{-3}\lambda)x_2 + \right. \\
 & & \left. (\lambda - 5.2 \times 10^{-3})x_3\right] / 0.005
\end{array} \tag{4.5}$$

relocates the closed-loop eigenvalues at $(\lambda, -2.6 \times 10^{-3} \pm 2.1 \times 10^{-3}i)$. The last two eigenvalues have the same assignation as in [7]. For $\lambda = -0.005$, (4.5) is a **MEC** feedback. Then, $\Omega(0)$ is homeomorphic to $\mathbb{R} \times \Omega_2(0)$, where $\Omega_2(0)$ is the **RA** for a two-dimensional pure integrator. It is well known (see[1]) that $\Omega_2(0) = \mathbb{R}^2$, thus $\Omega(0) = \mathbb{R}^3$. That is, under the action of a saturated **MEC** feedback, system (4.4) is globally asymptotically stable. For "small" deviations of the **MEC** design, the saturated system remains globally asymptotically stable (GAS). For "large" deviations, this affirmation is not true anymore.

With numerical simulations, the following dynamical structure was found:

a) For $-0.076 < \lambda < 0$: the saturated system is GAS.

b) For $\lambda \leq -0.076$: the closed-loop saturated system is unstable in-the-large. There are trajectories that escape to infinity. Here $\Omega(0)$ is homeomorphic to $\mathbb{R} \times B^2$.

Statement a) is consistent with the value $\lambda = -0.0039$ that Gutman and Hagander [7] derived by solving an optimization problem, and which is near the λ value of the **MEC** case.

5 Conclusions

The problem of determining the region of attraction (**RA**) for linear systems subjected to saturated linear feedbacks has been studied. It was found that the number of eigenvalues with positive real part of the open-loop system, n_u, determines the shape of the **RA**. Specifically, the following results were obtained: (i) If $\sigma(A) \subset \mathbb{C}^+$, the **RA**, $\Omega(0)$, is bounded and homeomorphic to B^n; (ii) If $\sigma(A) \subset \mathbb{C}^-$, the system trajectories are bounded and converge to a compact invariant set of zero volume; (iii) If $\sigma(A) \cap \mathbb{C}^0 = \emptyset$, for a **MEC** feedback, the **RA** is homeomorphic to the cylinder $\mathbb{R}^{n-n_u} \times B^{n_u}$; (iv) If $\sigma(A) \cap \mathbb{C}^0 = \emptyset$ and $n_u = 1$, when the **MEC** case is perturbed by gain deviations the result in (iii) is still valid. In all cases, the nature of the equilibrium points was established. To estimate the **RA** it was proved that $\partial\Omega(0) = \bigcup_i W^s(\gamma_j)$ when the critical elements γ_j in $\partial\Omega(0)$ are connected to the origin. Various aplication examples were presented to illustrate how the proposed *a priori* geometric characterization of the **RA** is used.

Appendix

Recall the following concepts and results from dynamical system theory (see for instance [17, 4]). Consider a differential equation

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n, \qquad (A.1)$$

that satisfies a sufficient condition which implies uniqueness and global existence of solutions (there are no finite escape times). Denote by ϕ_t the trajectory of (A.1) with $\phi_0(x) = x$. (A.1) generates a flow (dynamical system $\phi_t(x)$). A *critical element* of the vector field $f(x)$ is either a closed orbit or an equilibrium point. A set $S \subset \mathbb{R}^n$ is an *invariant set* of (A.1) if every trajectory of (A.1), with initial point in S, remains in S of all $t \in \mathbb{R}$. The boundary ∂S and the clousure $\bar{S}$ of an invariant set S are also invariant sets. A closed invariant set S is an *attractor* if there exists some invariant neighborhood V of S such that for all $x \in V$, $\phi_t(x) \to S$ as $t \to \infty$. An attractor contains the ω-limit sets of the neighborhood V. Suppose x_e is an asymptotically stable (AS) equilibrium point of system (A.1). In the global analysis of dynamical systems, one is interested in estimating the extent of the *region of attraction* (**RA**), $\Omega(x_e)$, of x_e.

$$\Omega(x_e) = \{x \in \mathbb{R}^n : \omega(x) = \{x_e\}\}. \qquad (A.2)$$

where $\omega(x)$ is the ω-limit set of x. The **RA** is also referred to as basin or stability region.

Proposition A.1 *If x_e is an asymptotically stable equilibrium point, then $\Omega(x_e)$ is an open, invariant set which is homeomorphic to $\mathbb{R}^n$.*

Proposition A.2 *If $\Omega(x_e)$ is not dense in $\mathbb{R}^n$, then the boundary of $\Omega(x_e)$, $\partial\Omega(x_e)$, is a closed invariant set of dimension $n-1$. If $\Omega(x_e)$ is bounded and $\partial\Omega(x_e)$ is a smooth manifold, then , for $n \neq 5$, $\partial\Omega(x_e)$ is homeomorphic to the $(n-1)$ dimensional sphere.*

Suppose that x_e is a *hyperbolic equilibrium point* of f (i.e.; the eigenvalues of the Jacobian matrix $J_x f$ at x_e, have non-zero real parts). The set of points in $\mathbb{R}^n$ that have x_e as ω-limit is called the *stable manifold* of $x_e : W^s(x_e)$, and the set of points that have x_e as α-limit is called the *unstable manifold* of $x_e : W^u(x_e)$. $W^s(x_e)$ and $W^u(x_e)$ are invariant sets under $\phi_t(x)$.

If A, B are injectively immersed manifolds in M, we say that they satisfy the transversality condition if either: i) at every point of the intersection $x \in A \cap B$, the tangent spaces of A and B span the tangent space of M at x,

$$T_x(A) + T_x(B) = T_x(M) \quad \text{for} \quad x \in A \cap B$$

or, ii) they not intersect at all.

To estimate the **RA** of a particular system, the following general result is important for our analysis.

Theorem A.3 [4] *Consider a dynamical system (A.1) that satisfies the following conditions.*

i) All critical elements on the stability boundary are hyperbolic.

ii) The stable and unstable manifolds of the critical elements on the stability boundary satisfy the transversality condition.

iii) Every trajectory on the stability boundary approaches one of the critical elements as $t \to \infty$.

Let x_i, $i = 1, 2, \ldots$, be the equilibrium points and γ_j, $j = 1, 2, \ldots$, be the closed orbits on the stability boundary $\partial\Omega(0)$ of an asymptotically stable equilibrium point. Then

$$\partial\Omega(0) = \bigcup_i W^s(x_i) \bigcup_j W^s(\gamma_j).$$

Finally, we have the following fundamental theorem.

Theorem A.4 *Let $f(x)$ be a vector field defined on a compact manifold M, which is diffeomorphic to a sphere S^{2k}. Then, there exists at least one point $x_e \in M$ such that $f(x_e) = 0$.*

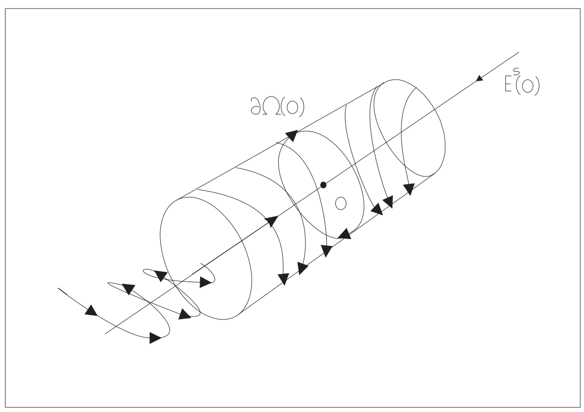

Figure 1: Asymptotic region of attraction $\Omega(0)$, for a 3-dimensional system with $n_u = 2$, when a **MEC** feedback is applied to the system. Here, $\partial\Omega(0)$ is diffeomorphic to $\mathbb{R} \times S^1$.

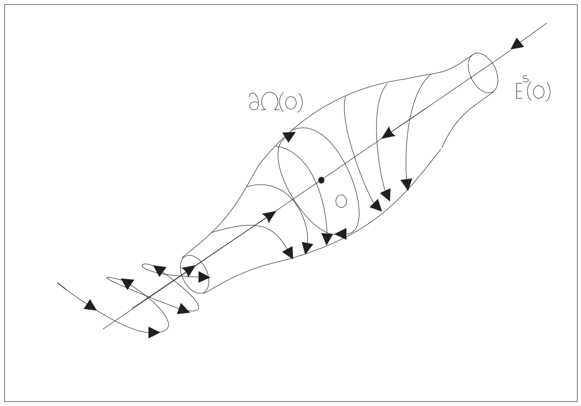

Figure 2: Asymptotic region of attraction $\Omega(0)$, for a 3-dimensional system with $n_u = 2$, when a small gain perturbation of the **MEC** feedback is applied to the system.

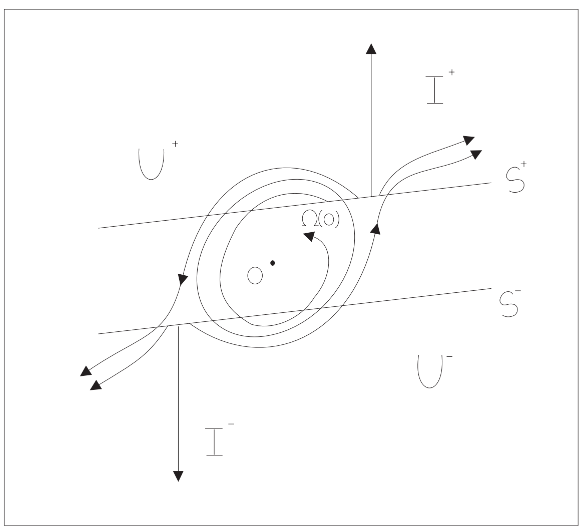

Figure 3: Geometry of the limit cycle appearing in the dynamics of the inverted pendulum on a cart.

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Communicated by Ruth Curtain