ABSTRACT

A procedure based on the use of the describing function is presented to identify the parameters of a nonlinear system incorporating cubic stiffness elements. The procedure uses frequency response data at resonance, obtained from different forcing levels, to identify both the linear and nonlinear parameters of the system. Under noisy conditions an effective variance method of solution is used which shows less sensitivity to noise than the classical weighted least squares method. Furthermore, the method gives encouraging results when used as a means for locating cubic elements situated within the system. Simulated experiments and results are included to illustrate the feasibility of the method under noisy measurement conditions.

List of Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_k$</td>
<td>nonlinear parameter, $(k = 1, \ldots, n+1)$</td>
</tr>
<tr>
<td>$c_k$</td>
<td>damping coefficient, $(k = 1, \ldots, n+1)$</td>
</tr>
<tr>
<td>$F$</td>
<td>amplitude of applied force</td>
</tr>
<tr>
<td>$f(t)$</td>
<td>applied force</td>
</tr>
<tr>
<td>$i$</td>
<td>mode number</td>
</tr>
<tr>
<td>$j$</td>
<td>force level</td>
</tr>
<tr>
<td>$k_k$</td>
<td>stiffness coefficient, (subscript $k = 1, \ldots, n+1$)</td>
</tr>
<tr>
<td>$m, c, k, b$</td>
<td>mass, damping, stiffness and nonlinear parameters of a single degree of freedom system, equation 1</td>
</tr>
<tr>
<td>$m_k$</td>
<td>mass, $(k = 1, \ldots, n)$</td>
</tr>
<tr>
<td>$n$</td>
<td>number of degrees of freedom</td>
</tr>
<tr>
<td>${p}$</td>
<td>vector of parameters</td>
</tr>
<tr>
<td>$[S_p]$</td>
<td>covariance of residuals</td>
</tr>
<tr>
<td>$s$</td>
<td>number of force levels used experimentally</td>
</tr>
<tr>
<td>$[T]$</td>
<td>matrix relating $p$ and $y$</td>
</tr>
<tr>
<td>$X_{i,k}$</td>
<td>resonant amplitude for the $i$th mode measured at the $k$th coordinate</td>
</tr>
<tr>
<td>$x$</td>
<td>displacement of single degree of freedom system</td>
</tr>
<tr>
<td>$x_k$</td>
<td>displacement of the $k$th coordinate, $(k = 1, \ldots, n)$</td>
</tr>
<tr>
<td>${y}$</td>
<td>vector of frequencies squared</td>
</tr>
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**Greek**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta_{i,k}$</td>
<td>modal relationship</td>
</tr>
<tr>
<td>${\varepsilon}$</td>
<td>residual error in frequency</td>
</tr>
<tr>
<td>$\mu_k$</td>
<td>mass ratio</td>
</tr>
<tr>
<td>$\phi$</td>
<td>phase difference between displacements and force</td>
</tr>
<tr>
<td>$\omega_{i,k}$</td>
<td>backbone frequency</td>
</tr>
<tr>
<td>$\omega_{i,k}^{B(i,k)}$</td>
<td>backbone frequency</td>
</tr>
</tbody>
</table>

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Several techniques for identifying nonlinear systems from experimental data have been developed in recent years. Some detect nonlinearity while others are capable of characterizing the nonlinearity and/or identifying parameters. Among the parameter identification methods, there exists a number of techniques for the identification of special classes of nonlinear systems. These are developed under the assumption that sufficient a-priori information about the system is available and the identification procedure is reduced to the estimation of the parameters of the governing differential equation. While most of the techniques developed using this parametric approach are based on single degree of freedom (SDOF) models there are few methods applicable to multi-degree of freedom (MDOF) models. The principal reason for this is that these methods require the exact form of the governing differential equations to be known a-priori. These may be difficult to obtain in practice. However, in recent years powerful diagnosis methods have been developed, for example the Hilbert transforms \[1\] and the carpet plot \[2\]. These may provide an insight into the parametric approach to system identification of nonlinear vibrating structures.

The parametric identification method described here is based on the Describing Function (DF) Technique. It uses frequency response data at resonance, obtained from different harmonic forcing levels to identify the stiffness and mass parameters of a MDOF system incorporating cubic stiffness elements. The method assumes that the viscous damping is small so that amplitude and phase resonance occur simultaneously. It is also assumed that the damping may be modeled as proportional damping and so the mode shapes are real. If the damping is constant for a particular structure then nonlinear stiffness will invalidate this assumption as the response level increases but the assumption of a low level of damping will reduce the significance of this effect. The essential features of the method have been reported for the case where the system was restricted to one possessing 'mode shapes' that were only marginally sensitive to changes in forcing levels \[3\] and also for the more general case where this restriction is removed \[4\]. In this paper the basic procedure is reviewed and extended to situations where the measurement data is noisy. In addition, the procedure is used to locate nonlinear elements situated within the structure.

The effect of cubic stiffness nonlinearity on the frequency response functions of a system is to bend the resonant peak to a higher or lower frequency depending on whether the nonlinearity represents a hardening or a softening spring. However, other nonlinearities may cause similar effects and care must be taken to ensure these are not confused with cubic stiffening. It has been shown by simulation that in a system with three degrees of freedom in which masses are interconnected chain-wise by dampers and hardening cubic springs, all resonant peaks are bent to a higher natural frequency. Huang \[5\] and Arnold \[6\] have shown that in systems with two degrees of freedom, both peaks are bent to a higher or lower frequency if one or more springs are nonlinear. Hence with each mode of vibration there is a frequency-amplitude relation which is found to be characteristic of the type of nonlinearity present in the system.

In order to fix ideas we will begin by considering the identification of a single degree of freedom system with a cubic stiffening spring. The system is governed by the equation

$$m\ddot{x} + c\dot{x} + kx + bx^3 = f(t)$$

Since an exact solution to Eq. \((1)\) is not known, many authors have resorted to approximate methods such as the perturbation or the Describing Function (DF) method \[7\] to solve this equation. The DF method has been found to give acceptable results within the resonant regimes of structures which include amplitude
dependent nonlinearities [8]. Using the DF method, and ignoring harmonics, gives

\[ X \left[ k \left\{ 1 + \left( \frac{3}{4} \right) \alpha X^2 \right\} - m \omega^2 \right] = F \cos \phi \quad (2) \]

\[ c \omega X = F \sin \phi \quad (3) \]

where \( X = \frac{x(t)}{\sin \omega t}, F = \frac{f(t)}{\sin(\omega t + \phi)}, \alpha = b/k \) and \( \phi \) is the phase angle between the input force and the response. At resonance \( \phi = 90^\circ \) (see section 3) and substituting in Eq. (2) we have

\[ \omega_b(X) = \sqrt{\frac{k}{m} \left\{ 1 + \frac{3\alpha}{4} X^2 \right\} } \quad (4) \]

Note that \( \omega \) has been replaced by \( \omega_b(X) \) to indicate that Eq. (4) represents the backbone curve, the line on the amplitude-frequency plane that passes through the resonant peak for different values of \( X \). If we square both sides of this equation we obtain a straight line in the amplitude squared-frequency squared plane.

\[ \omega_b^2(X) = \frac{k}{m} + \frac{3b}{4m} X^2 \]

Substituting for \( \phi = 90^\circ \) in Eq. (3) gives

\[ X_L(\omega) = \frac{F}{c \omega} \quad (6) \]

This is the limit curve and defines the region outside which the forced amplitude response cannot extend. When \( \omega = \omega_b \) this line intersects the backbone curve.

Identification of the system parameters is as follows. A series of frequency response tests are carried out in the region of the resonant frequency with different values of applied force. For each force, the resonant frequency and amplitude of the response are measured. The experimental values of resonant frequency squared and amplitude squared are plotted against each other and fitted to a straight line using least squares to minimize the effect of experimental errors. From Eq. (5) it is seen that the slope of this line is \( 3b/4m \) and the intercept on the frequency squared axis is \( k/m \). If \( m \) is known \( k \) and \( b \) may be deduced. If the damping, \( c \), is required then it can be obtained from Eq. (6). The assumptions implicit in this analysis are discussed in the context of a multi-degree of freedom system.

We will now consider how the approach outlined above can be extended to identify the parameters of a multi-degree of freedom system with one or more cubic hardening or softening springs. There are no restrictions on the discrete model connectivity, provided it is known, nor on the number of nonlinear elements present in the structure. It is, however, assumed that both the character and location of the nonlinear elements and the individual masses of the system are known prior to the identification. In due course, these assumptions will be relaxed providing the total mass of the system is known. Here the
identification procedure is illustrated by applying it to \( n \) masses interconnected by nonlinear hardening springs, linear springs and viscous dampers as shown in Fig. 1.

If only coordinate 1 is acted upon by a harmonic excitation, then a system of this kind may be described by the following differential equations

\[
\begin{align*}
 m_1 \ddot{x}_1 + c_1 \dot{x}_1 + c_2 (\dot{x}_1 - \dot{x}_2) + k_1 x_1 + b_1 x_1^3 + k_2 (x_1 - x_2) + b_2 (x_1 - x_2)^3 &= f(t) \\
 m_k \ddot{x}_k + c_k (\dot{x}_k - \dot{x}_{k-1}) + c_{k+1} (\dot{x}_k - \dot{x}_{k+1}) + k_k (x_k - x_{k-1}) \\
 &+ b_k (x_k - x_{k-1})^3 + k_{k+1} (x_k - x_{k+1}) + b_{k+1} (x_k - x_{k+1})^3 = 0
\end{align*}
\]

(7)

where subscript \( k = 2, \ldots, n \), the fictitious displacement \( x_{n+1} = 0 \), and

\[
f(t) = F \sin(\omega t + \phi)
\]

(8)

Once again the identification procedure will be based on resonant frequency and amplitude data for various force levels but this is now extended to make use of frequency-amplitude data associated with each vibration mode that is capable of adequately predicting resonant conditions in MDOF systems incorporating cubic stiffness elements. In order to achieve this objective, the nonlinear differential equations of motion for the system must be solved and again we use the DF method. Further we shall make the following assumptions:

(i) The damping is small, so that phase resonance and amplitude resonance can be taken to occur at the same frequency. Resonance can be defined in two ways. Amplitude resonance occurs at the frequency which gives a maximum amplitude in the FRF and, in a nonlinear system, occurs just before the jump, should one occur. This is the most convenient quantity to measure. Phase resonance occurs when the response is in quadrature with the input force and this definition is used in the analysis. In lightly damped structures the difference between these two resonance definitions is negligible and so we will assume that amplitude resonance occurs when the response is in quadrature with force.

(ii) Proportional damping is assumed so that the mode shapes are real. In general non-linear systems do not exhibit natural modes of vibration but at a fixed response level, using the DF technique, the model system will appear to be linear and thus possess modes. If the amplitude of vibration
changes then so will the mode. However, accepting that the modes are real, all masses will attain maximum displacements at the same instant. Modal relations can thus be determined from the ratios of the resonant displacements at each coordinate.

(iii) Harmonics present in the solution are small and can be neglected.

In view of assumption (i) above, the backbone curve and the FRF, which is bent to the higher frequency due to the presence of hardening nonlinearities, intersect each other at a point for which $\phi = 90^\circ$. When the effects of such nonlinearities become large, the jump in the frequency curve occurs suddenly after reaching resonant conditions and it is reasonable to assume that the amplitude jump occurs at resonance. Jump points were found experimentally to be well defined and have been used for parameter identification [8]. These three conditions form the basis of the mathematical formulation of the proposed method.

In view of assumption (ii) and (iii), the solution to Eq. (7) above may be approximated by

$$r_k = X_k \sin \omega \quad (k = 1, \ldots n)$$

(9)

Substituting Eqs. (9) and (8) in Eq. (7), and using the DF technique, leads to the following set of coupled nonlinear algebraic equations

$$-m_1 \omega^2 X_1 + k_1 X_1 + k_2 (X_1 - X_2) + (3/4)b_1 X_1^3 + (3/4)b_2 (X_1 - X_2)^3 = \cos \phi$$

$$-m_k \omega^2 X_k + k_k (X_k - X_{k-1}) + k_{k+1} (X_k - X_{k+1})$$

$$+ (3/4)b_k (X_k - X_{k-1})^3 + (3/4)b_{k+1} (X_k - X_{k+1})^3 = 0 \quad \text{(subscript } k = 2, \ldots n)$$

(10)

where $X_{n+1} = 0$. Similarly, the DF method yields the following equations, which contain damping information.

$$\omega [c_1 X_1 + c_2 (X_1 - X_2)] = F \sin \phi$$

$$\omega [c_k (X_k - X_{k-1}) + c_{k+1} (X_k - X_{k+1})] = 0 \quad \text{(subscript } k = 2, \ldots n)$$

(11)

Since we are only concerned in this work with the identification of the stiffness and mass parameters, Eq. (11) will not be considered further.

The resonant amplitudes $X_k (k=1, \ldots n)$, which are required in Eq. (10), are available from measurement. Thus for the $j$th forcing level and the $i$th mode of vibration, the following modal relations can be established

$$\left(\Delta_{i,k}\right)_{f=j} = \left(\frac{X_{i,k+1}}{X_{i,k}}\right)_{f=j} \quad (k = 1, \ldots n-1)$$

$$\left(\Delta_{i,0}\right)_{f=j} = \left(\frac{X_{i,1}}{X_{i,0}}\right)_{f=j} = \left(\frac{X_{i,1}}{0}\right)_{f=j} = \infty$$

$$\left(\Delta_{i,n}\right)_{f=j} = \left(\frac{X_{i,n+1}}{X_{i,n}}\right)_{f=j} = \left(\frac{0}{X_{i,n}}\right)_{f=j} = 0$$

(12)

Note that in the above equations the displacements $X_{i,0}$ and $X_{i,n+1}$ are fictitious and are set to zero. For the sake of clarity, the subscripts denoting the different forcing levels $j = 1, \ldots s$ in one particular mode $i$ will be
omitted. Equations (10) may conveniently be written to express the dependence of resonant frequency on amplitude, which is suitable for parameter estimation. This dependence describes the so-called backbone curves which are defined by Eq. (10) at resonance (φ = 90°). Thus

\[ \omega_{B(i,j)} = \left[ \frac{k_i}{m_i} + \frac{k_j}{m_i} (1 - \Delta_{i,j}) \right] + \frac{3}{4} \left( \frac{b_i}{m_i} + \frac{b_j}{m_i} (1 - \Delta_{i,j})^3 \right) X_{i,j}^2 \]

(13)

\[ \omega_{B(i,k)} = \left[ \frac{k_k}{m_k} \left( 1 - \frac{1}{\Delta_{i,k-1}} \right) + \frac{k_{k+1}}{m_k} (1 - \Delta_{i,k}) \right] \]

\[ + \frac{3}{4} \left( \frac{b_k}{m_k} \left( 1 - \frac{1}{\Delta_{i,k-1}} \right)^3 + \frac{b_{k+1}}{m_k} (1 - \Delta_{i,k})^3 \right) X_{i,k}^2 \]

(subscript \( k = 2, \ldots, n \))

Some nonlinear systems have modal relations that are essentially constant for a range of force excitation magnitudes. In this case the backbone curve is approximately linear when frequency squared is plotted against amplitude squared as in the SDOF system. This greatly simplifies the estimation procedure. The parameters of the system are readily obtainable from the slope and the intercept of the ‘best’ straight line which is derived in a least squares sense from data obtained from several forcing levels [3].

For a nonlinear system with mode shapes that vary with the strength of excitation, the modal relations must be evaluated at each forcing level. This produces frequency amplitude relations that are specific to the level of excitation. Experimentally derived backbone curves are so distorted that any attempt to linearize them by squaring both terms of Eq. (13) is not possible. This is illustrated using simulated data in Fig. 2.

The slopes of Eqs. (13) squared are a function of \( \Delta_{i,k} \) and this parameter now varies with the strength of excitation. Thus the simple curve fitting approach of Ref. [3] cannot be applied to systems of this type. It should be pointed out that a significant change in force from one test to the other is often necessary in order to yield linearly independent equations. Therefore the more forcing tests one carries out the more

![Fig.2  Backbone curve for system with varying mode shapes](image-url)
equations are generated. This process can be repeated for as many modes as required. The resulting problem is overdetermined and is solved here using the least squares principle, applied at each coordinate in turn and using the corresponding frequency-amplitude equation. Hence, for the \( k \)th coordinate the following matrix equation is obtained

\[
\{ y \} = [T]\{ p \}
\]  

(14)

The coefficient matrix \([T]\) is an \( s \times 4 \) array and the elements are given by

\[
T_{j,1} = \frac{1}{m_k} \left( 1 - \frac{1}{\Delta_{i,k-1}} \right) \quad T_{j,2} = \frac{1}{m_k} \left( 1 - \Delta_{i,k} \right)
\]

\[
T_{j,3} = \frac{3}{4m_k} \left( 1 - \frac{1}{\Delta_{i,k-1}} \right)^3 X_{i,k}^2 \quad T_{j,4} = \frac{3}{4m_k} \left( 1 - \Delta_{i,k} \right)^3 X_{i,k}^2
\]

(15)

where the excitation force is \( f_j, j = 1,..s \).

The observation vector \( \{ y \} \), containing resonant or jump frequencies squared corresponding to \( s \) forcing levels is given by Eq. (16): the vector \( \{ p \} \) contains the parameters to be identified and is given by Eq. (17).

\[
\{ y \} = \begin{bmatrix}
\omega_i^2 \big|_{f = f_1} \\
\vdots \\
\omega_i^2 \big|_{f = f_s}
\end{bmatrix}
\]

(16)

\[
\{ p \} = \begin{bmatrix}
k_k \\
k_{k+1} \\
b_k \\
b_{k+1}
\end{bmatrix}
\]

(17)

When applying the least squares method an estimation of \( \{ p \} \), denoted \( \{ p_{LS} \} \), is chosen so as to minimize the inner product \( \{ r \}^T \{ r \} \) where \( \{ r \} \) is the residual vector, i.e.,

\[
\{ r \} = \{ y \} - [T]\{ p_{LS} \}
\]

(18)

This yields the equation

\[
\{ p_{LS} \} = ([T]^T[T])^{-1}[T]^T \{ y \}
\]

(19)

The coefficient matrix \([T]\) given by Eq. (15) is constructed using \( s \) equations obtained from one mode. In general, however, more modes are used to construct \([T]\). Although data from \( s \) forcing levels (\( s \) greater than the number of parameters to be identified) obtained from one mode is theoretically sufficient to estimate the unknown parameters in a least square sense, experience has shown that at least two or three modes are usually required for an accurate identification. This also depends on other factors such as the number of forcing levels used in each mode and also the sensitivity of the modes used in the identification to the force level.
The least squares solution, Eq. (19), is carried out at each coordinate in turn. This formulation will be referred to here as identification on a coordinate basis. It yields the estimation of the stiffness parameters which lie on either side of the coordinate under consideration. For example, if the identification is carried out at the $k$th coordinate then the estimation will yield the $k$th and $(k+1)$th linear and nonlinear stiffness parameters. Therefore this formulation yields a double estimation of all parameters that are coupled between any two coordinates. Both estimations are usually close and are often averaged in the identification process. Stiffness parameters that are dependent on the amplitude of a single coordinate can only be estimated once.

It is possible to force an identification which yields only one estimation per parameter. Such an approach can be carried out by formulating a coefficient matrix which is constructed from all individual coefficient matrices that are obtained on a coordinate basis. These are then arranged in such a way that parameter estimation according to Eq. (19) yields only one estimation per parameter. However, this formulation has some disadvantages. The computational time is greatly increased due to the resulting bigger coefficient matrix and the procedure is restricted to identification cases where the individual masses of the system are known. This is not the case with identification on a coordinate basis, which can easily be modified in order to provide an estimation of individual masses estimation when only the total mass of the system is known. This latter approach is described in section 4 below.

Numerical Example 1: To illustrate the identification technique described, consider the three degree of freedom nonlinear system shown in Fig.3. This system contains two cubic stiffness elements, the first one coupling coordinates 1 & 2 and the second coupling coordinate 3 to ground. The mass and linear stiffness parameters of the system were chosen so that the modes are well separated and the damping was almost proportional to linear stiffness. Simulated frequency response data was generated by numerically solving the equations of motion of the system when forced by stepped sine excitation. The identification technique was carried out under the assumption that the location of nonlinearities is known a-priori. The system masses and damping coefficients are as follows: $m_1 = 3$ kg, $m_2 = 2$ kg, $m_3 = 1$ kg, $c_1 = 3$ Ns/m, $c_2 = 5$ Ns/m, $c_3 = 7.5$ Ns/m, $c_4 = 10$ Ns/m. The actual and identified linear and nonlinear spring stiffness coefficients are shown in Table 1. The identified coefficients were obtained using 18 data pairs and two modes. Each pair of coordinates of the resonant or jump amplitude and frequency corresponds to a different forcing level.

Table 1 shows some discrepancies between the identified and the actual parameters, even though no measurement errors are present. These may be attributed to the following:

(i) the model damping was proportional to the linear stiffness. Since the stiffness is amplitude dependent, at higher forcing levels the simulated model will yield complex mode shapes that are not accounted for by the underlying theory. This results in modeling errors.

Fig.3 An example 3 degree of freedom system
(ii) the data obtained from the first mode doesn’t show significant changes in resonant frequencies from one level of forcing to the other. One possible explanation is that the nonlinearities are not fully excited, therefore creating mode shapes that are not sufficiently sensitive to forcing. This has the effect of producing equations in the coefficient matrix that are close to linear dependency. In severe situations this may result in an ill-conditioned coefficient matrix. This problem is caused when the two coordinates coupling a nonlinearity have little or no relative movement between them.

(iii) the simulation data is affected by numerical errors and these also contribute to the above discrepancies.

(iv) the DF method provides an approximate solution for the governing differential equations of the system. In particular, the chosen model ignores the effects of harmonic terms that are present in the solution.

In the above numerical example the identification has been carried out on a coordinate basis with the individual masses of the system known in advance. As already stated, this approach can be easily extended to estimate the individual masses of the system provided the total mass is known. This is achieved by reformulating the problem given by Eq. (14) so that the identification yields estimates of ratios between stiffness parameters (linear and nonlinear) and the mass at the coordinate under consideration. The coefficient matrix in this case is similar to the one given by Eq. (15) but without the mass parameters.

Let the total mass of the system given by

$$M = \sum_{k=1}^{n} m_k$$  \hspace{1cm} (20)

A mass ratio between two adjacent masses, denoted $\mu_k$, can be obtained by dividing the identified ratio $k_{k+1}/m_k$ (or $k_{k+1}/m_k$) at coordinate $k$ by the corresponding identified ratio $k_{k+1}/m_{k+1}$ (or $b_{k+1}/m_{k+1}$) at coordinate $k+1$. In the case of ratios corresponding to linear stiffness parameters we obtain the following

$$\mu_k = \frac{k_{k+1}/m_k}{k_{k+1}/m_{k+1}} = \frac{m_{k+1}}{m_k}$$  \hspace{1cm} (21)

where subscript $k=1,..,n-1$.

### Table 1: Actual and Identified Stiffness Parameters for the 3 DOF Example.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$k_1$ (N/m)</th>
<th>$k_2$ (N/m)</th>
<th>$k_3$ (N/m)</th>
<th>$k_4$ (N/m)</th>
<th>$b_2$ (N/m$^3$)</th>
<th>$b_4$ (N/m$^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Actual</td>
<td>5000</td>
<td>10000</td>
<td>15000</td>
<td>20000</td>
<td>200000</td>
<td>200000</td>
</tr>
<tr>
<td>Identified</td>
<td>5130</td>
<td>9930</td>
<td>14600</td>
<td>20970</td>
<td>201130</td>
<td>177390</td>
</tr>
<tr>
<td>% Error</td>
<td>2.52</td>
<td>-0.7</td>
<td>-2.63</td>
<td>4.84</td>
<td>0.57</td>
<td>-11.31</td>
</tr>
</tbody>
</table>
Equation (21) can be used to express each mass in terms of $m_1$, thus

\[m_2 = m_1 \mu_1 \]
\[m_3 = m_2 \mu_2 = m_1 \mu_1 \mu_2 \]

Generalizing

\[m_k = m_1 \prod_{e=1}^{k-1} \mu_e \quad \text{(22)}\]

Substituting Eq. (22) in Eq. (20) yields an estimate of $m_1$ since the total mass of the system is assumed to be known. The other masses may then be estimated using Eq. (22) and hence the stiffness parameters can be computed from the identified ratios. Similar results can be obtained by using the identified ratios containing nonlinear parameters. In both cases, however, the method yields two estimates for each stiffness parameter. The results of estimating mass is shown in Table 2. This was obtained using the data of example 1 and assuming that only the total mass of the system is known a-priori.

In many mechanical systems the nonlinearities are localized. The object of this section is to investigate the possibility of using the identification technique described above as a means for locating nonlinearities within a structure, assuming that the character of the nonlinearity is known. The location of nonlinearities has important engineering applications, for example it may offer the possibility of failure detection.

So far identification has been carried out with the location of the cubic stiffness elements known a-priori. Thus the mathematical model used as the basis for the identification is derived accordingly and nonlinear coefficients in Eq. (13) are forced to be zero when it is known that the corresponding nonlinearities do not exist. There are fewer parameters to identify and, in general, this gives a more accurate estimation of the system parameters.

Consider now a MDOF system displaying a characteristic typical of one containing one or more hardening cubic stiffnesses. To use the identification technique to locate a nonlinearity it is necessary to model the structure to be identified as one which contains nonlinearities in all its springs. Such a model is then used as the basis of the identification process previously described. Assumed nonlinear parameters that are identified as having small values probably do not exist. Such assumptions may be validated to some extent by reconstructing the original nonlinear characteristic using the identified linear and nonlinear

<table>
<thead>
<tr>
<th>Masses</th>
<th>m₁ (kg)</th>
<th>m₂ (kg)</th>
<th>m₃ (kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Actual</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Identified</td>
<td>2.97</td>
<td>1.99</td>
<td>1.05</td>
</tr>
<tr>
<td>% Error</td>
<td>-1.16</td>
<td>-0.51</td>
<td>4.52</td>
</tr>
</tbody>
</table>
parameters, but assigning a zero value to any nonlinear parameter which is suspected to be so. If the reconstructed results closely match the measured ones, then the assumptions are confirmed. A simulated experiment is given below to illustrate these effects.

Further checks on the accuracy of the identification may be possible when the individual masses of the system are known. As shown previously, parameters identification based on a coordinate approach yields two estimates per parameter (except for grounded elements). This characteristic may be used to advantage when applying the identification method as a location procedure, since this provides an opportunity to compare results identified from two adjacent coordinates. A further advantage of using the location procedure performed on a coordinate basis is that in order to undertake a location test it is only necessary to test the coordinates which are believed to couple a suspected nonlinearity. This is useful when testing a nonlinear system having several degrees of freedom but where the nonlinearity appears to be concentrated in a specific part of the system. In such cases measurements are only required in the region of the suspected nonlinearity.

Numerical Example 2: In order to illustrate the identification method when used to locate a nonlinearity, the response to stepped sine excitation of a seven degree of freedom system has been simulated. The system comprises a series of masses, all of 0.5kg, connected chain-wise by both springs and damping elements. All springs have linear component of 10kN/m and some have, in addition, cubic nonlinearities. No information concerning the location or number of nonlinear elements present in the structure is assumed to be available a-priori, only the character of nonlinearity which is a hardening cubic law spring. The damping elements all have a coefficient of 1N s/m. The results of this simulation are shown in Table 3. The least squares solution yields a small numerical value for each of the nonlinear stiffness coefficients in springs 2, 4 and 8, indicating that these springs have essentially linear characteristics. On the other hand, the numerical values for the other nonlinear coefficients are quite large suggesting that these springs are indeed nonlinear.

So far, the identification procedure has been carried out assuming that the coefficient matrix contains error free amplitude data. The basis of the identification was the standard least squares method. Assuming the elements forming the observation vector are free from errors or are Gaussian distributed, then the standard least squares method will yield an unbiased estimation of the parameters provided that the elements making up the coefficient matrix are free from errors. An estimate \( p_{LS} \) is said to be unbiased if its expected value is equal to its true value, i.e.,

\[
E\{p_{LS}\} = \{p\}
\]

(23)

<table>
<thead>
<tr>
<th>Nonlinearities N/m³</th>
<th>( b_1 )</th>
<th>( b_2 )</th>
<th>( b_3 )</th>
<th>( b_4 )</th>
<th>( b_5 )</th>
<th>( b_6 )</th>
<th>( b_7 )</th>
<th>( b_8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Actual</td>
<td>10000</td>
<td>0</td>
<td>10000</td>
<td>0</td>
<td>10000</td>
<td>10000</td>
<td>10000</td>
<td>0</td>
</tr>
<tr>
<td>Identified</td>
<td>10611</td>
<td>-153</td>
<td>10141</td>
<td>711</td>
<td>9947</td>
<td>9948</td>
<td>9269</td>
<td>-381</td>
</tr>
</tbody>
</table>

Table 3: The Location of Nonlinear Springs in a 7 DOF System
The bias error vector, \( \{ p_{BE} \} \), is given by

\[
\{ p_{BE} \} = E \{ \{ p_{LS} \} - \{ p \} \} = E \{ p_{LS} \} - \{ p \} \tag{24}
\]

An unbiased estimation of the parameters may also be obtained using a weighted least squares method when the coefficient matrix is error free and the elements of the observation vector are polluted with Gaussian distributed errors having zero mean. The weighting matrix in this case is usually based on the accuracy of the measurements of the elements constituting the observation vector.

The application of the standard least squares to our identification procedure has yielded acceptable results in the case of error free data. However, in a practical situation, the elements of the coefficient matrix \([T]\) are often polluted with random errors since they are functions of the measured amplitudes. This has the consequence of drastically worsening the identification results. Under these circumstances an unbiased estimation of the parameters using the standard or weighted least squares methods cannot be expected. Assuming in a first instance that only the frequency vector \( \{ y \} \) is polluted with random errors having zero mean then

\[
\{ y \} = \{ y_f \} + \{ y_e \} \tag{25}
\]

where \( \{ y_f \} \) is the error free vector and \( \{ y_e \} \) is a vector of errors. From Eqs. (19), (24) and (25)

\[
\{ p_{BE} \} = E \{ \left( [T]^T [T] \right)^{-1} [T]^T \{ y_e \} \} \tag{26}
\]

The array \([T]\) contains no errors and \( E \{ y_e \} = 0 \) so that Eq. (26) vanishes. This demonstrates that the bias error of the least squares method is zero when applied under these circumstances. Unfortunately this is not the case when \([T]\) contains errors and may be expressed as

\[
[T] = [T_f] + [T_e] \tag{27}
\]

\([T_e]\) denotes an error matrix associated with the coefficient matrix whose elements are assumed to be normally distributed with zero mean. Substituting Eq. (19) into Eq. (24) yields the bias error

\[
\{ p_{BE} \} = E \left( \left( [T]^T [T] \right)^{-1} [T]^T \{ y \} \right) - \{ p \} \tag{28}
\]

Assuming observation errors in \([T]\) and \( \{ y \} \) are statistically independent then, using Eq. (25), Eq. (28) becomes

\[
\{ p_{BE} \} = E \left( \left( [T]^T [T] \right)^{-1} [T]^T \{ y \} \right) - \{ p \}
\]

or

\[
\{ p_{BE} \} = E \left( \left( [T]^T [T] \right)^{-1} [T]^T \{ y_f \} \right) - \{ p \}
\]

The bias error is zero only if

\[
E \left( \left( [T]^T [T] \right)^{-1} [T]^T \right) = \left( [T_f]^T [T_f] \right)^{-1} [T_f]^T \tag{29}
\]
If \([T]\) contains a non-zero error, even if its mean is zero, then Eq. (29) will not hold and the parameter estimate is biased. Several authors have addressed the bias problem in an attempt to reduce or eliminate it [9,10,11]. One approach is to use the instrumental variable (IV) method. The IV method reduces the bias but its usual implementation requires simulated data from the model for all measurement force levels and for all measurement locations. When applied to this identification problem it becomes a time consuming iterative process that is too lengthy to be attractive.

To overcome the problems of the IV method, our approach here in dealing with the bias problem is to attempt to reduce it by using the so-called effective variance method [12], which has been applied extensively in the field of experimental physics. The purpose of this is to make the identification procedure less sensitive to experimental errors. Unlike the IV method which theoretically eliminates the bias error, the effective variance method can only reduce it through the use of more effective weighting factors. The development of the effective variance method is in line with the weighted least squares method. The weighting factors, which have been previously computed on the basis of the accuracy of the elements making up the observation vector, are now constructed using also the accuracy of the amplitude data contained in the coefficient matrix. The computation of the weighting factors making up the new covariance matrix assumes that the errors in the response data forming the elements of the coefficient matrix are Gaussian distributed with zero mean. As in the case of the weighted least squares method, the elements of the observation vector are also assumed to be Gaussian distributed. The derivation for obtaining these new weighting factors is now briefly described.

Equation (14) is modified below to include the error matrix associated with the coefficient matrix and error vector associated with observation vector. Using Eq. (25) and Eq. (27) in Eq. (14), we have

\[
[T_f][p] = \{y_f\} + \{e\}
\]

where

\[
\{e\} = \begin{bmatrix}
\varepsilon_1 | f = f_1 \\
\varepsilon_2 | f = f_2 \\
\vdots \\
\varepsilon_j | f = f_s
\end{bmatrix}
\]

Note that \(\{e\}\) is a total error combining the individual errors in frequency and amplitude measurements. Thus

\[
\{e\} = \{y_f\} - [T_e][p]
\]

The new covariance matrix will be derived here on the assumption that parameter estimation is carried out on a coordinate basis, in which case the associated coefficient matrix is constructed using measured amplitudes at the coordinate of interest and also at adjacent coordinates. Thus

\[
[T] = g(X_{k-1}, X_k, X_{k+1})
\]

\[X_0 = 0 \quad \text{and} \quad X_{n+1} = 0\]

where \(g\) is some function of the variables. Note that Eq. (33) is only valid in the simple case of a chain system such as that shown in Fig 1. Now since \([T] = [T_f] + [T_e]\), then \([T_e]\) can be expanded in a Taylor series to obtain (neglecting higher terms)
\[
[T_x] = \frac{\partial[T]}{\partial X_{k-1}} \delta X_{k-1} + \frac{\partial[T]}{\partial X_k} \delta X_k + \frac{\partial[T]}{\partial X_{k+1}} \delta X_{k+1}
\] (34)

The \( \delta \) in Eq. (34) is used to indicate an error in \( X \). Substituting Eq. (34) into Eq. (32) and making use of Eq. (14) results in the following expression for the error vector \( \{ \epsilon \} \):

\[
[\epsilon] = \{ y_e \} - \left[ \frac{\partial \{ y \}}{\partial X_{k-1}} \delta X_{k-1} + \frac{\partial \{ y \}}{\partial X_k} \delta X_k + \frac{\partial \{ y \}}{\partial X_{k+1}} \delta X_{k+1} \right]
\] (35)

\( \{ y_e \} \) is a column vector of length \( s \) since Eq. (35) is constructed using data obtained from \( s \) different forcing levels when the structure is vibrating in its \( i \)th mode. In general however, as already stated, in order to yield acceptable identification results, data from more than one mode is usually required. Therefore Eq. (35) is often extended to accommodate more data from other modes. For the sake of clarity however, in what follows, the equations will be constructed using data from one mode only.

The covariance matrix for vector \( \{ \epsilon \} \) is now

\[
[S_\epsilon] = E[\{ \epsilon \} \{ \epsilon \}^T]
\] (36)

and \( \{ \epsilon \} \) can be expanded to give

\[
[S_\epsilon] = E \left[ \left\{ y_e \right\} - \sum_{m=k-1}^{m=k+1} \left( \frac{\partial \{ y \}}{\partial X_m} \delta X_m \right) \right] \left[ \left\{ y_e \right\} - \sum_{m=k-1}^{m=k+1} \left( \frac{\partial \{ y \}}{\partial X_m} \delta X_m \right) \right]^T
\] (37)

which represents a diagonal matrix since the expansion of Eq. (37) yields off-diagonal terms that are products of statistically independent quantities with zero mean value and therefore the expected values of the off-diagonal terms is zero. The covariance matrix is given by Eq. (38) thus

\[
[S_\epsilon] = \begin{bmatrix}
\sigma_{\epsilon_1}^2 \\
\sigma_{\epsilon_2}^2 \\
\vdots \\
\sigma_{\epsilon_s}^2
\end{bmatrix}
\] (38)

where the \( j \)th element is given by

\[
\sigma_{\epsilon_j}^2 = (\sigma_{y_e})_j^2 + \left( \frac{\partial y_j}{\partial X_{k-1}} \right)_j^2 (\sigma_{X_{k-1}})_j^2 + \left( \frac{\partial y_j}{\partial X_k} \right)_j^2 (\sigma_{X_k})_j^2 + \left( \frac{\partial y_j}{\partial X_{k+1}} \right)_j^2 (\sigma_{X_{k+1}})_j^2
\] (39)

where \( y_j \) is the \( j \)th element of \( \{ y \} \). The inverse of the covariance matrix is inserted into Eq. (18) to weight according to the accuracy of measurements in both amplitude and frequency. However, in order to apply the effective variance method it is necessary to have initial estimates of the parameters to be identified in order to compute the values for \( \partial y_j / \partial X_x \) etc. These are required to evaluate the weighting factors according to Eq. (39). In the case where no initial estimates of the parameters are available the following iterative
procedure has proved to be effective [12].

Step 1 Form a weighting matrix based on the accuracy of the elements constituting the observation vector.
Step 2 Using the covariance matrix calculated in step 1, estimate parameters using the classical weighted least squares method.
Step 3 Calculate new covariance matrix according to Eq. (39) using parameters estimated in step 2.
Step 4 Estimate parameters using new covariance matrix, i.e., effective variance method.
Step 5 Compare actual parameter estimates with those of the last step. If agreement is good stop the iteration otherwise go to step 3.

In order to test the effectiveness of the above algorithm, the three DOF system considered previously in Example 1 is again used here. This time however, the response data obtained from numerical simulations, which was previously assumed to be noise free in the identification process, is now polluted with random numbers generated by the computer with zero mean values. The algorithm is tested using RMS values of noise to signal ratio (N/S) equal to 2% in the case of both frequency and amplitude data. Test results are shown in Table 4 which also include, for comparison purposes, the results yielded by the classical weighted least squares method applied under the same noisy conditions.

Table 4 shows that the identification results obtained by the classical least squares method have worsened considerably when compared to those of Table 1 which were obtained using error free data. This is due to the fact that the covariance matrix used in the classical approach does not take into account errors contained in the coefficient matrix, resulting in a least squares estimation which is not properly weighted.

TABLE 4 COMPARISON OF THE WLS AND EFFECTIVE VARIANCE METHODS. EXAMPLE 1 WITH 2% NOISE

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Actual</th>
<th>Weighted Least Sq.</th>
<th>% Error</th>
<th>Effective Variance</th>
<th>% Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>k1 (N/m)</td>
<td>5000</td>
<td>5800</td>
<td>16.02</td>
<td>5450</td>
<td>9.02</td>
</tr>
<tr>
<td>k2 (N/m)</td>
<td>10000</td>
<td>10810</td>
<td>8.12</td>
<td>10420</td>
<td>4.15</td>
</tr>
<tr>
<td>k3 (N/m)</td>
<td>15000</td>
<td>13150</td>
<td>-12.36</td>
<td>14590</td>
<td>-2.75</td>
</tr>
<tr>
<td>k4 (N/m)</td>
<td>20000</td>
<td>17340</td>
<td>-13.29</td>
<td>19900</td>
<td>-0.49</td>
</tr>
<tr>
<td>b2 (N/m3)</td>
<td>200000</td>
<td>166210</td>
<td>-16.89</td>
<td>188620</td>
<td>-5.68</td>
</tr>
<tr>
<td>b4 (N/m3)</td>
<td>200000</td>
<td>163750</td>
<td>-18.12</td>
<td>219910</td>
<td>9.95</td>
</tr>
</tbody>
</table>

TABLE 5 COMPARISON OF THE WLS METHOD AND ITERATIVE STEPS IN THE EFFECTIVE VARIANCE METHOD AT COORDINATE 2. EXAMPLE 1 WITH 2% NOISE

<table>
<thead>
<tr>
<th>Actual Parameters</th>
<th>Weighted Least Sq.</th>
<th>Effective Variance 1st Iteration</th>
<th>Effective Variance 2nd Iteration</th>
<th>Effective Variance 3rd Iteration</th>
</tr>
</thead>
<tbody>
<tr>
<td>k2 = 10000 (N/m)</td>
<td>11430</td>
<td>10700</td>
<td>10690</td>
<td>10690</td>
</tr>
<tr>
<td>k3 = 15000 (N/m)</td>
<td>14130</td>
<td>14110</td>
<td>14150</td>
<td>14150</td>
</tr>
<tr>
<td>b2 = 200000 (N/m3)</td>
<td>150370</td>
<td>185420</td>
<td>185320</td>
<td>185320</td>
</tr>
</tbody>
</table>

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On the other hand, the effective variance method shows a remarkable improvement in the results compared with the least squares approach and this is usually achieved after only 3 or 4 iterations. Table 5 illustrates this in the case of an identification performed at coordinate 2. There is an apparent discrepancy between the parameter values of Tables 4 and Table 5 which is explained as follows. Estimates are made on a coordinate basis so that, except for elements that are connected to ground, there are two estimates for each parameter. The values in Table 4 are the average of two estimates, those in Table 5 are a single estimate made at coordinate 2. Further tests have been carried under other noisy conditions in order to ascertain further the effectiveness of the method. These have shown that the effective variance method generally gives more accurate results than the classical least squares method.

### 7. Conclusions

A method of identification for nonlinear systems incorporating cubic stiffness elements has been developed. It is based on the use of frequency-amplitude relations, obtained by the DF method for the identification of both the linear and nonlinear stiffness parameters. The method does not require initial estimates of the parameters, only that the character of the nonlinearity is known in advance. It has been shown from simulated experiments that the method, which uses the least squares approach, generally yields acceptable results under noise free conditions. However, this method of solution has been shown to yield biased estimates when the identification is carried out under noisy conditions. In order to reduce the bias, an effective variance method has been used which showed less sensitivity to noise compared the weighted least squares method. The method may be applied to nonlinear structures not represented by Duffing’s equation, provided an analytical expression for the backbone and limit curves can be established.

### References


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