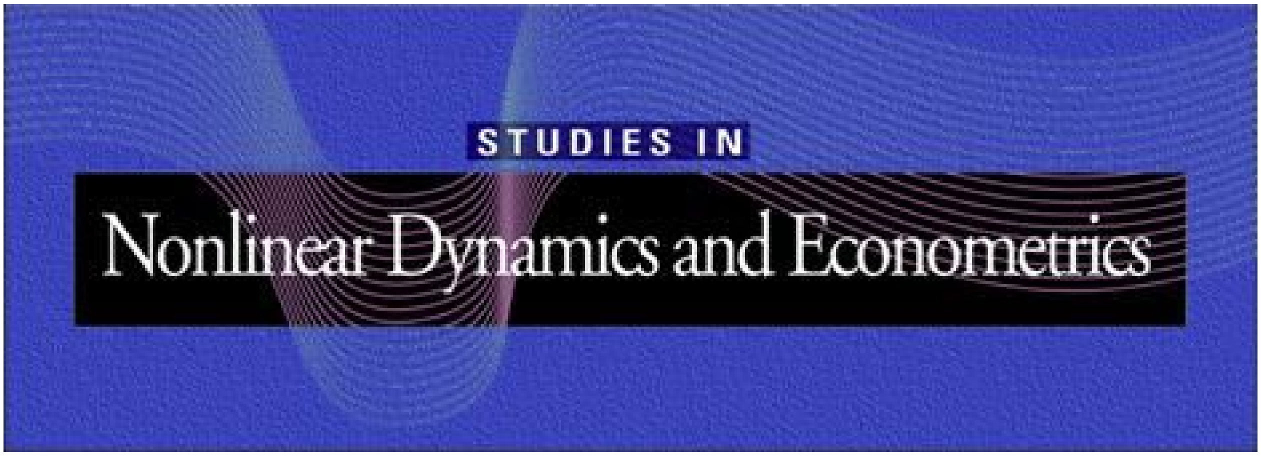

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Optimal Cycles and Chaos: A Survey

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Abstract. *This paper surveys the literature on cyclical and chaotic equilibrium paths in deterministic optimal growth models with infinitely lived agents. We focus on discrete time models but also briefly mention results for continuous time models. We start by reviewing those results that have been proved for optimal growth models in reduced form. Then we discuss results for two-sector optimal growth models in primitive form. Finally, we summarize a few results that have been obtained for other variants of the model, including models with recursive preferences and models with heterogeneous agents.*

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1 Introduction

Both the interest and the research activity in economic growth theory has increased dramatically during the last decade. One can explain this development in at least two different ways, namely, (i) that the models studied in the recent endogenous growth literature are capable of explaining long-run growth and technological advancement in a more convincing way than the models from 20 or 30 years ago, and (ii) that new mathematical concepts and methods have been developed that allow us to study growth models with dynamically complicated optimal paths or equilibrium paths. In this article we try to survey a number of new results for optimal growth models with infinitely lived households, which can be attributed to fact (ii) from above.[1] Particular emphasis will be put on results that are concerned with cyclical and chaotic optimal growth paths.

The importance of the theory of nonlinear dynamical systems for the study of economic processes has already been noted by many authors. Nonlinear dynamics are ubiquitous in economics, and the reader is referred to the following survey articles or books to get an impression of how much work has been done in

[1]A recent and comprehensive survey of developments triggered by fact (i) can be found in Barro and Sala-i-Martin (1995).

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this area over the last couple of years: Benhabib and Baumol (1989), Boldrin and Woodford (1990), Brock (1993), Brock and Dechert (1991), Brock and Malliaris (1989), Chiarella (1990), Day (1994), Frank and Stengos (1988), Lorenz (1989), Medio (1993), and Scheinkman (1990). Although models from almost any field in economics have been analyzed by methods from nonlinear dynamical systems theory, optimal growth theory is probably the one field that has received the most attention. This is not surprising, since one of the main objectives of growth theory is the explanation of the fluctuations of economic variables around their general upward trends. For the purpose of this explanation, endogenous cycles and chaos seem to be at least as appropriate as the exogenous shocks commonly used in real business-cycle theory.[2]

The two most commonly considered paradigms in the theory of economic growth are the overlapping generations model going back to Samuelson (1958) and Diamond (1965), and the model with infinitely lived households originating from Ramsey (1928). In this survey we exclusively consider models of the second variety.[3] Among the models using infinitely lived agents we can again distinguish between those that use a continuous-time framework and those that use discrete time periods. This distinction is not innocuous for the subject of the survey, because it is well known that complicated dynamics can arise more easily in low-dimensional systems of difference equations than in corresponding systems of differential equations. We shall focus our attention on discrete time models and mention possible generalizations or extensions to the continuous-time framework only briefly in Section 4.3.

Most of the results on cycles and chaos in economic growth theory deal with models in which the dynamics can be reduced to a one-dimensional difference equation. This emphasis on one-dimensional dynamics is also reflected in the present survey. It seems, therefore, worthwhile to point out those results that deal with higher dimensional dynamical systems. The general possibility theorem (Theorem 2.2) from Boldrin and Montrucchio (1986) and the result on topological entropy (Theorem 2.3) from Montrucchio and Sorger (1996) hold for state spaces of arbitrary (finite) dimension, and some of the heterogeneous agent models discussed in Section 4.1 lead to two-dimensional difference equations. As will be discussed in Section 4.3, one also needs higher dimensional state spaces to generate cycles and chaos in continuous-time models.

The general outline of the paper is as follows. In Section 2 we discuss results for models in reduced form. These models do not contain any control variables such as consumption rates or investment rates, but are stated in state variables (capital stocks) only. The advantage of the reduced form is that it has a simpler mathematical structure than nonreduced models, and is therefore more amenable to a clear-cut characterization of those properties that generate cycles and chaos. We start with definitions and basic properties of reduced utility function models in Section 2.1, discuss optimal cycles in Section 2.2, and optimal chaos in Section 2.3. Section 2.4 studies the influence of the rate of depreciation on the structure of optimal growth paths, whereas Section 2.5 analyzes the relationship between the size of the discount factor and the occurrence of complicated dynamics. Section 3 deals with nonreduced models. Since cycles and chaos cannot occur in the standard one-sector model (see Dechert and Nishimura [1983] and Dechert [1984]), a nonreduced optimal growth model that can generate complicated dynamics must have at least two production sectors. The existing literature on cyclical or chaotic optimal growth has focused almost entirely on the simplest such case, which is the case of a two-sector model, and we follow this tradition. We introduce the basic framework in Section 3.1, discuss optimal cycles in Section 3.2, and chaos in Section 3.3. Section 4 summarizes results for models that did not fit easily into Sections 2 or 3. More specifically, these are models with heterogeneous agents (Section 4.1), models with nonadditive utility functions (Section 4.2), and continuous-time models (Section 4.3). Finally, Section 5 presents some concluding remarks.

2 Reduced-Form Utility Function Models

2.1 Definitions and assumptions

Reduced-form utility function models have been used by many authors because of their simple mathematical structure and their wide applicability in economics. For a comprehensive survey of methods for and applications of such models, we refer to McKenzie (1986) and Stokey and Lucas (1989). In this subsection we briefly summarize the definitions and results that are needed in the remainder of the paper.

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[2]See Stadler (1994) for a recent survey of real business-cycle theory.

[3]The occurrence of chaos and cycles in models with overlapping generations of finitely lived agents has been discussed, for example, in Grandmont (1985).

The basic structure of a reduced-form utility function model is:

$$\text{Maximize} \quad \sum_{t=1}^{\infty} \rho^{t-1} v(k_{t-1}, k_t) \tag{1}$$

$$\text{subject to} \quad (k_{t-1}, k_t) \in D \tag{2}$$

$$k_t \in K \tag{3}$$

where K is the state space, $D \subseteq K \times K$ is the constraint set, v is a real-valued utility function defined on D, and $\rho \in (0, 1)$ is the discount factor. The state-space K is the set of all possible states of the economy.

In the context of optimal growth theory, the state of the economy is usually described by a vector of capital stocks. In this case, D is called the *production possibility set*, and contains all pairs of states (x, y) such that x is a possible vector of capital stocks and y is producible from x within one period. The utility function $v(x, y)$ measures the maximal utility that can be derived in one period when the capital stock at the beginning of that period is equal to x and the capital stock at the end of the period is required to be y.

A sequence of states $(k_0, k_1, \ldots)$ is called a *feasible path* if conditions (2) and (3) are satisfied for all $t \geq 1$. The planner tries to maximize the present value of utility over all feasible paths which satisfy the initial condition $k_0 = k$, where $k \in K$ is a given initial state. A feasible path that attains the maximum in (1) is called an *optimal path*. The optimal value of the optimization problem (1)–(2) with initial state k will be denoted by $V(k)$, and we shall refer to V as the optimal value function of the model.

Throughout this section we shall assume without further mentioning it that K and D are nonempty, closed, and convex subsets of Euclidean spaces, that the utility function v is continuous and concave, that the set $D_x = \{y | (x, y) \in D\}$ is nonempty for all $x \in K$, and that the infinite sum in (1) converges to a finite real number for every feasible path $(k_0, k_1, \ldots)$. Under these assumptions it can be shown that the optimal value function satisfies the Bellman equation

$$V(x) = \sup \left\{ v(x, y) + \rho V(y) \mid (x, y) \in D \right\}$$

for all $x \in K$, and that any optimal program $(k_0, k_1, \ldots)$ satisfies $V(k_{t-1}) = v(k_{t-1}, k_t) + \rho V(k_t)$ for all $t \geq 1$. In other words, the optimal capital stock k_t maximizes the right-hand side of the Bellman equation when $x = k_{t-1}$. The maximizer may not be unique or the maximum may not be attained at all, but in many important examples it is attained at a unique point $y = h(x)$.[4] The function $h : K \to K$ defined in that way is called the optimal policy function. For each state $x \in K$, the value $h(x)$ determines the unique optimal successor state of x.

Another important optimality condition is the Euler equation. It states that if the utility function v is continuously differentiable, then every interior optimal path (that is, any optimal path with $k_t \in \text{int } D_{k_{t-1}}$) must satisfy

$$v_2(k_{t-1}, k_t) + \rho v_1(k_t, k_{t+1}) = 0.$$

If a constant feasible path $(\bar{k}, \bar{k}, \ldots)$ is an optimal path, then we call $\bar{k}$ an *optimal steady state*. Optimal steady states play an important role in the analysis of reduced-form utility models (see, e.g., McKenzie [1986]).

In addition to the basic assumptions mentioned above, we shall make use of one or more of the following assumptions whenever necessary.

- **A1:** The state space K is compact.
- **A2:** The function $v(x, \cdot)$ is strictly concave for all $x \in K$.
- **A3:** The function $v(x, \cdot)$ is decreasing for all $x \in K$, and the function $v(\cdot, y)$ is increasing for all $y \in K$.
- **A4:** The function v is twice continuously differentiable.

[4] In particular, this is the case if the set D_x is compact for all $x \in K$ and if the utility function v is strictly concave with respect to its second argument; see assumption A2 below.

2.2 Optimal cycles

One of the earliest examples of cycles as optimal solutions of reduced-form utility function models is due to Sutherland (1970).[5] In his paper, the state space is $K = [0, 1]$ and the reduced utility function is defined on $D = K \times K$ by

$$v(x, y) = -9x^2 - 11xy - 4y^2 + 43x. \tag{4}$$

There exists a unique optimal steady state for each discount factor $\rho \in (0, 1)$. For $\rho = \frac{1}{3}$, the steady state is $\bar{k} = \frac{1}{2}$ and is unstable. It has been shown by Sutherland (1970) that in this case ($\rho = \frac{1}{3}$) there exists a periodic optimal solution $(0, 1, 0, 1, 0, \ldots)$.

Subsequently, Samuelson (1973) reported an example due to Weitzman. It was later generalized by McKenzie (1983) and Benhabib and Nishimura (1985). These examples use the same state space $K = [0, 1]$, but Cobb-Douglas-type utility functions defined on $D = K \times K$ by

$$v(x, y) = x^{\alpha}(1 - y)^{\beta}, \tag{5}$$

with $\alpha > 0$, $\beta > 0$, and $0 < \alpha + \beta \leq 1$. Samuelson (1973) assumes $\alpha = \beta = \frac{1}{2}$. McKenzie (1983) studies the linearly homogeneous case with $\alpha + \beta = 1$. Benhabib and Nishimura (1985) allow the strictly concave case with $0 < \alpha + \beta \leq 1$. The optimal steady-state value is $\bar{k} = \rho\alpha / (\rho\alpha + \beta)$. If $\alpha \in (\frac{1}{2}, 1)$ and $\beta \in (0, \frac{1}{2})$, then it can be shown that the steady state is stable for $\rho_0 < \rho < 1$ and unstable for $0 < \rho < \rho_0$ where $\rho_0 = \beta(2\alpha - 1) / [\alpha(1 - 2\beta)]$. Under the same restrictions on α and β, Benhabib and Nishimura (1985) also prove the existence of optimal period-two cycles in this model.

In the above examples the optimal policy functions have negative slopes at the steady states. This may be checked by solving the characteristic equations of the Euler equations. A result characterizing the global monotonicity properties of the optimal policy function h in models with a one-dimensional state space is the following theorem, due to Benhabib and Nishimura (1985).

Theorem 2.1. *Let h be the optimal policy function of a reduced utility function model with a one-dimensional state space such that assumptions A1, A2, and A4 are satisfied.*

(i) If $(k, h(k)) \in \operatorname{int} D$ and if $v_{12}(x, y) > 0$ for all $(x, y) \in \operatorname{int} D$, then h is strictly increasing at $x = k$.

(ii) If $(k, h(k)) \in \operatorname{int} D$ and if $v_{12}(x, y) < 0$ for all $(x, y) \in \operatorname{int} D$, then h is strictly decreasing at $x = k$.

The cross-partial derivatives of the instantaneous utility functions are -11 for the example in (4) and $-\alpha\beta x^{\alpha-1}(1 - y)^{\beta-1}$ for the example in (5). Optimal policy functions are therefore strictly decreasing by Theorem 2.1(ii), and either no cycles at all or period-two cycles arise in these examples. A generalization of Theorem 2.1 to higher dimensional state spaces using the lattice theoretic concept of supermodularity was developed in Amir (1996).

If one slightly modifies (5) and considers the utility function

$$v(x, y) = (x - ay)^{\alpha}(1 - y)^{\beta},$$

then more complicated dynamics may arise. This was suggested by Scheinkman (1984) and proved by Boldrin and Deneckere (1990).

On the other hand, Nishimura and Yano (1994a,b) show that the utility function (5) gives rise to a chaotic solution if one modifies the domain D. Their method will be discussed in Section 2.4 below.

2.3 Optimal chaos

An optimal policy function may be a chaotic function. For example, assume that the state space is $K = [0, 1]$, and consider the reduced utility function defined on $D = K \times K$ by

$$v(x, y) = xy - x^2y - \frac{1}{3}y - \frac{3}{40}y^2 + \frac{100}{3}x - 7x^2 + 4x^3 - 2x^4. \tag{6}$$

[5] This example has a unique interior steady state that is unstable. The instability arising from multiplicity of steady states is reported in Kurz (1968a). The instability arising from noninterior steady states is reported in Kurz (1968b).

This is a strictly concave function with $v_1(x, y) > 0$ and $v_2(x, y) < 0$, so that assumptions A1–A4 are satisfied. It has been shown by Deneckere and Pelikan (1986) that this optimal growth problem has the logistic function $h(x) = 4x(1 - x)$ as its optimal policy function when the discount factor is given by $\rho = \frac{1}{100}$. The logistic function is probably the most famous example of a chaotic map (see, e.g., Benhabib and Baumol [1989]).

A constructive method to find an optimal growth model with a given optimal policy function was provided by Boldrin and Montrucchio (1986). A somewhat simplified construction is presented in Sorger (1992a), where it is shown that for any given function $h : K \to K$, and any discount factor $\rho \in (0, 1)$, the model defined by $D = K \times K$ and

$$v(x, y) = -\frac{1}{2}|y - h(x)|^2 - \frac{a}{2}|x|^2 + \frac{a\rho}{2}|y|^2 + bx - b\rho y \tag{7}$$

has the function h as its optimal policy function. Here a and b are real parameters with $a > 0$, and K is any compact state space of arbitrary finite dimension. The optimal value function for this problem is given by $V(x) = -(\frac{a}{2})|x|^2$. The reduced-utility function defined above is as many times differentiable as the function h. However, it does not necessarily satisfy assumptions A2 and A3. To ensure that these assumptions hold, the function h must be twice continuously differentiable, and one has to restrict the parameter values a, b, and ρ. In particular, ρ cannot be too large. Given the explicit formula for v from (7), it is actually not hard to compute an upper bound on the set of discount factors for which this construction yields a utility function v satisfying A2–A4. This upper bound ρ^* depends on the first two derivatives of the function h, more precisely, on the numbers $H_1 = \max\{|h'(x)| \mid x \in K\}$ and $H_2 = \max\{|h''(x)| \mid x \in K\}$, as well as on the diameter of the state space $\delta(K) = \max\{|x - y| \mid x, y \in K\}$.

The main result of Boldrin and Montrucchio (1986) can thus be stated as follows.[6]

Theorem 2.2. *Let assumption A1 be satisfied, and let $h : K \to K$ be any twice continuously differentiable function. There exists a number $\rho^* > 0$ such that for all discount factors $\rho \in (0, \rho^*)$ one can find a reduced-utility function model satisfying A2–A4 which has the given function h as its optimal policy function.*

The smoothness assumption on h in this theorem can be relaxed a little bit if one replaces A4 by a weaker assumption (see Neumann et al. [1988]). The upper bound ρ^* is usually quite low if the mapping h is nonlinear. For example, if h is the logistic function $h(x) = 4x(1 - x)$ mentioned above, then the upper bound ρ^* for the construction described by Boldrin and Montrucchio (1986) is approximately 0.01. Using a more sophisticated approach, Sorger (1992a) has found models with discount factors as large as $\frac{1}{24}$ which have the logistic map as their optimal policy function.

2.4 Slow depreciation

In the examples presented so far we have always used the production possibility set $D = K \times K$ and the state space $K = [0, 1]$. In particular, this implies that capital fully depreciates within one period. Now, let us suppose that the capital stock k_t depreciates to δk_t after one period when it is used in production. If δ is strictly positive, then the depreciation rate $1 - \delta$ is less than 1, and capital depreciates only slowly. In this case, the domain of a reduced-form utility function becomes $D = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 1, \delta x \le y \le 1\right\}$.[7] Boldrin and Deneckere (1990) discuss the possibility that an optimal path may be chaotic in the case of slow depreciation. For the same situation, Nishimura and Yano (1995a) provide a sufficient condition for an optimal path to be periodic with period three which, according to the famous paper by Li and Yorke (1975), is sufficient for the existence of chaotic optimal paths. They consider the functions

$$\Gamma\left(k_{t-1}, k_t, k_{t+1}\right) = v_2\left(k_{t-1}, k_t\right) + \rho v_1\left(k_t, k_{t+1}\right)$$
$$\Gamma_1(k) = \Gamma\left(\delta k, \delta^2 k, k\right)$$
$$\Gamma_2(k) = \Gamma\left(k, \delta k, \delta^2 k\right) + \rho\Gamma_1(k)\delta$$
$$\Gamma_3(k) = \Gamma\left(\delta^2 k, k, \delta k\right) + \rho\Gamma_2(k)\delta,$$

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[6]Because of the title of Boldrin and Montrucchio's paper, this result has often been termed "indeterminacy theorem." Since "indeterminacy" has also a quite distinct meaning in equilibrium theory, we prefer the term "possibility theorem," which describes the contents of the result less ambiguously.

[7]We continue to assume that the state space is given by $K = [0, 1]$.

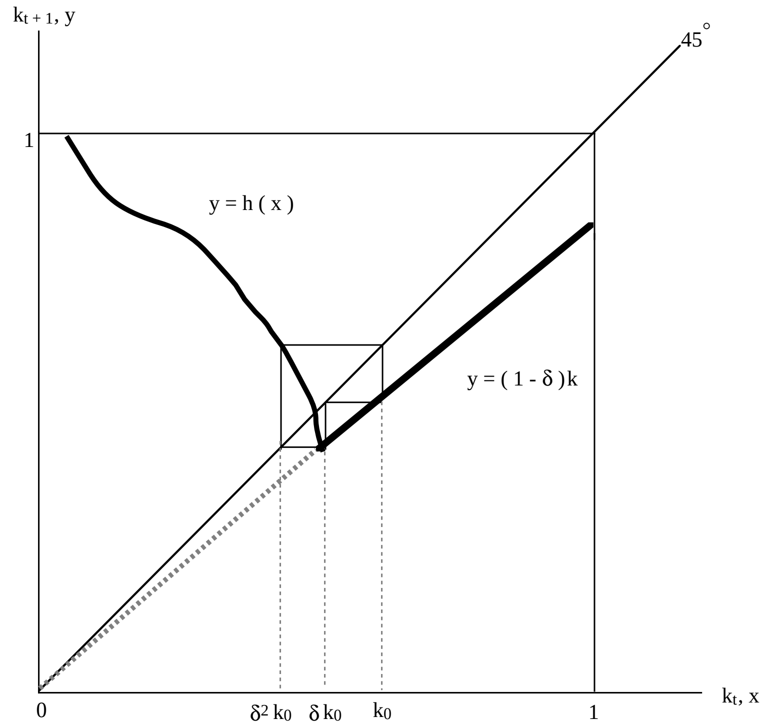

Figure 1
A period three cycle.

and show that the period-three cycle $(k_0, \delta k_0, \delta^2 k_0, k_0, \delta k_0, \ldots)$ is optimal if the conditions $\Gamma_1(k_0) < 0$, $\Gamma_2(k_0) < 0$ and $\Gamma_3(k_0) = 0$ are satisfied.

In the above example, a period-three cycle appears since a path hits the lower boundary of the domain in two consecutive periods (see Figure 1). If the upper boundary of the domain is described by a strictly increasing and concave function, then a path that hits the upper boundary in two consecutive periods may also form a period-three cycle. A sufficient condition for this scenario to occur, as well as a numerical example, are given in Nishimura and Yano (1994a).

2.5 Complex dynamics and impatience

Boldrin and Montrucchio's (1986) theorem discussed in Section 2.3 shows that optimal paths in reduced-utility function models can exhibit arbitrarily complicated dynamics. However, we have also seen that the constructions used to prove this theorem require a sufficiently small discount factor. It is therefore quite natural to ask if there is a general provable relation between the size of the discount factor and the dynamic complexity of the optimal paths. First results in this direction were derived by Sorger (1992a,b and 1994a). These theorems, however, involved rather unusual measures of dynamic complexity. Only later it became possible to relate the discount factor to generally accepted measures of complexity such as the topological entropy. We state here the main result from Montrucchio and Sorger (1996), which is a generalization of an earlier result by Montrucchio (1994). As is Theorem 2.2, this result is valid for models with state spaces of arbitrary finite dimension n.

Theorem 2.3. *Let $h : K \mapsto K$ be the optimal policy function of a reduced-utility function model satisfying A2, and assume that the set $A \subseteq int\ K$ is compact and invariant under h, i.e., $h(A) \subseteq A$. Then it holds that $\kappa(h, A) \leq -(ln\ \rho)c_+(A)$, where $\kappa(h, A)$ denotes the topological entropy of h on the invariant set A and $c_+(A)$ is the capacity dimension of A.*

The capacity dimension of A, $c_+(A)$, is one of the possible dimensions used for fractal sets. It is related to the more popular Hausdorff dimension $H(A)$ by $H(A) \leq c_+(A) \leq n$, where n is the dimension of the state space

K.[8] It therefore follows from the above theorem that $\kappa(h, A) \leq -n \ln \rho$. The topological entropy of a dynamical system is one of the most fundamental measures of dynamic complexity. It measures the rate at which new information is revealed by the iterations of h. If $\kappa(h, A) > 0$, then one says that the dynamical system h exhibits complicated dynamics on A or that it exhibits topological chaos. If the discount factor ρ is close to 1, then it follows that the optimal policy function cannot be too complicated in the sense that its topological entropy cannot be too large.

Theorem 2.3 does not rule out the occurrence of chaos or other types of complicated dynamics for small discounting. It only says that weaker discounting implies less-complicated solutions. As a matter of fact, it has been shown in Nishimura, Sorger, and Yano (1994) that for any discount factor $\rho \in (0, 1)$, one can find a reduced-utility function model with an optimal policy function exhibiting ergodic chaos and positive topological entropy.

Positive topological entropy is not the only indication of complicated dynamics. To explain this in more detail, consider the case of a one-dimensional state space in which there exists a simple characterization of the occurrence of positive topological entropy (topological chaos). If $h : K \mapsto K$ is a continuous function and K is an interval on the real line, then it follows that h exhibits topological chaos if and only if there exists a periodic point of h with a minimal period that is different from a power of 2 (see, e.g., Alseda et al. [1993, p. 231]). The special case in which h has a periodic point with minimal period 3 has been investigated by Li and Yorke (1975). It is therefore quite often referred to in the literature as Li-Yorke chaos, although the terminology is not uniform in this respect. Note that Li-Yorke chaos is by definition stronger than topological chaos. Coming back to reduced-utility function models, it turns out that there exists a sharp upper bound on the set of discount factors that are compatible with Li-Yorke chaos. This result can be stated as follows.

Theorem 2.4. *Let $h : K \mapsto K$ be the optimal policy function of a reduced-utility function model satisfying A2, and assume that the state space K is an interval on the real line. If h exhibits Li-Yorke chaos then $\rho < (\sqrt{5} - 1)^2/4 \approx 0.382$. Conversely, if $\rho < (\sqrt{5} - 1)^2/4$, then one can find a reduced-utility function model satisfying A1–A3 with discount factor ρ such that the optimal policy function of this model exhibits Li-Yorke chaos.*

Theorem 2.4 was independently discovered by Mitra (1996) and Nishimura and Yano (1996), and generalizes a weaker result from Sorger (1994b).

To interpret the results of this subsection, let us assume that the model's time period is equal to m years. In other words, period $t + 1$ starts m years after the beginning of period t, and decisions can be made once every m years. Furthermore, let us denote by r the long-run annual real interest rate. The two numbers m and r are related in the following way: $(1 + r)^m = \rho^{-1}$. If we, therefore, choose a long-run annual real interest rate inside the range of typical interest rates existing in the real world, then we can get a relation between the decision periods that are compatible with various forms of chaotic behavior. For example, suppose that the long-run annual real interest rate is 5%. Then, the length of the model's time period is 1 year if $\rho = 0.952$, 10 years if $\rho = 0.614$, and 20 years if $\rho = 0.377$. Therefore, discount factors smaller than 0.5 imply decision periods of more than 14 years, which is much too long to be realistic.

3 Two-Sector Models

3.1 The basic framework

The model we consider is a discrete time version of the two-sector optimal growth model from Uzawa (1964). There are two goods: the pure consumption good, C, and the pure capital good, K. Each sector uses both capital K and labor L as input. Capital input must be made one period prior to the period in which output is produced. Labor input is made in the same period as output is produced. Denote by $c = F_C(K_C, L_C)$ and $y = F_K(K_K, L_K)$ the production functions of sectors C and K, respectively. The production functions are assumed to have all the standard neoclassical properties.

Denote by c_t and y_t the outputs of sectors C and K, respectively, in period t. Moreover, denote by $K_{C,t-1}$ and $L_{C,t}$ the factor inputs used in sector C for the production of c_t and by $K_{K,t-1}$ and $L_{K,t}$ those used in sector

[8]For a detailed discussion of various dimensions we refer to Falconer (1991). For the capacity dimension, see also Brock and Dechert (1991).

K to produce y_t, i.e.,

$$c_t = F_C\left(K_{C,t-1}, L_{C,t}\right), \tag{8}$$

$$y_t = F_K\left(K_{K,t-1}, L_{K,t}\right). \tag{9}$$

Denote by k_{t-1} the aggregate capital input, i.e.,

$$K_{C,t-1} + K_{K,t-1} = k_{t-1}. \tag{10}$$

The per-capita output of the capital good, y_t, represents the gross accumulation of capital, given as

$$y_t = k_t - \delta k_{t-1}, \tag{11}$$

where $1 - \delta$ is the rate of depreciation with $0 < \delta < 1$. The labor endowment of the economy is constant and time-independent. Without loss of generality we normalize it to 1, i.e.,

$$L_{C,t} + L_{K,t} = 1. \tag{12}$$

Denote by $u(c)$ the representative consumer's instantaneous utility when he consumes c units of the consumption good. With these notations, the two-sector optimal growth model is described by the following maximization problem.

$$\text{Maximize} \qquad \sum_{t=1}^{\infty} \rho^{t-1} u\left(c_t\right) \tag{13}$$

$$\text{subject to} \quad k_0 \le k \text{ and constraints (8)–(12)}$$

where $\rho \in (0, 1)$ is the discount factor.

In order to analyze the dynamics of the above model, it is convenient to express, for each given amount of capital input, k, the trade-off between the two outputs by $c = T\left(y, k\right)$, i.e.,

$$\begin{aligned} T\left(y, k\right) = \ \max \ & F_C\left(K_C, L_C\right) \\ \text{s.t.} \ & F_K\left(K_K, L_K\right) = y, \\ & L_C + L_K = 1, \\ & K_C + K_K = k. \end{aligned} \tag{14}$$

Note that for a given amount of capital input, k_{t-1}, the relation $c_t = T\left(y_t, k_{t-1}\right)$ captures the production-possibility frontier in the output plane (see Figure 2). With this definition, the optimal growth model described by (13) can be transformed into the following form.

$$V(k) := \qquad \max \sum_{t=1}^{\infty} \rho^{t-1} v\left(k_{t-1}, k_t\right) \tag{15}$$

$$\text{s.t.} \ \ k_0 = k \text{ and } k_t \le F_K(k_{t-1}, 1) + \delta k_{t-1}$$

where $v\left(x, y\right) = u\left(T\left(y - \delta x, x\right)\right)$. Note that (15) is a reduced model of the form (1)–(3).

Given an arbitrary function $v\left(x, y\right)$, we can construct two production functions (one for each sector) and an instantaneous utility function such that the corresponding reduced-utility function is given by $v(x, y)$. We shall explain this fact using an idea from Deneckere and Pelikan (1984) (see also Deneckere and Pelikan [1986] or Boldrin and Montrucchio [1986]). Suppose that a direct utility function is linear, i.e., $u(c) = c$, and that the technology of the capital good industry is described by the Leontief production function $F_K\left(K_K, L_K\right) = \min\left\{K_K/a, L_K\right\}$. Then it must hold that $K_K = ay$, and $L_K = y$. Using the resource constraints (10) and (12) and the relation (11), we can rewrite the utility function as follows:

$$v\left(k_{t-1}, k_t\right) = v\left(K_{C,t-1} + a\left(1 - L_{C,t}\right), 1 - L_{C,t} + \delta\left(K_{C,t-1} + a\left(1 - L_{C,t}\right)\right)\right). \tag{16}$$

Then we define a production function of the consumption sector by (16). However, the production function (16) is not necessarily linearly homogeneous.

Note that the above formulation does not allow for external effects. The introduction of production externalities may quite easily lead to models that generate complicated dynamics. These approaches will not be covered here in detail. We just mention that Boldrin and Rustichini (1994) point out the possibility of chaotic equilibria in two-sector models with external effects.

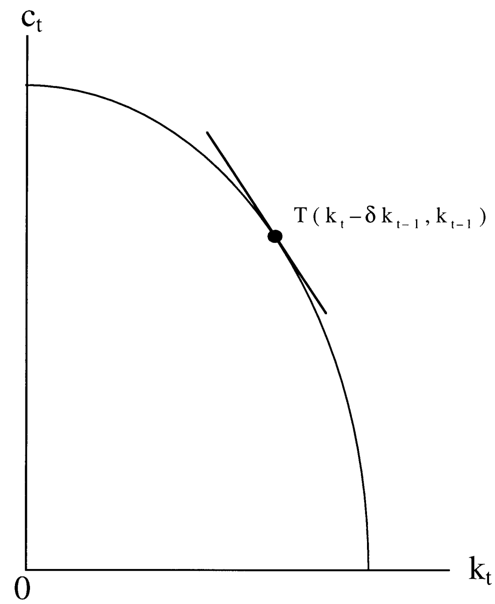

Figure 2
The production possibility frontier.

3.2 Optimal cycles

In the following, we assume that the instantaneous utility function is linear, i.e., $u(c) = c$, and that capital fully depreciates within one period, i.e., $\delta = 0$. Then the reduced-form utility function is identical to the social production function, i.e., $v(k, y) = T(y, k)$. Moreover, the sign of the cross-partial derivative $v_{12}(k, y)$ is determined by the factor-intensity difference of the consumption good sector and the capital good sector. That is,

$$v_{12}(k, y) \begin{cases} > 0 & \text{if } K_K/L_K > K_C/L_C \\ < 0 & \text{if } K_K/L_K < K_C/L_C \end{cases} \tag{17}$$

(see, for example, Benhabib and Nishimura [1985]).

We first consider the case that both sectors have Cobb-Douglas production functions:

$$F_C\,(K_C, L_C) = (K_C)^{\alpha_1}\,(L_C)^{\alpha_2}, \alpha_1 > 0, \alpha_2 > 0, \alpha_1 + \alpha_2 = 1, \tag{18}$$

$$F_K\,(K_K, L_K) = (K_K)^{\beta_1}\,(L_K)^{\beta_2}, \beta_1 > 0, \beta_2 > 0, \beta_1 + \beta_2 = 1. \tag{19}$$

Solving the problem (14) we can derive the following relation from the first-order conditions: $(K_K/L_K)\,/\,(K_C/L_C) = (\beta_1/\beta_2)\,/\,(\alpha_1/\alpha_2)$. Hence

$$(K_K/L_K) - (K_C/L_C) \begin{cases} > 0 & \text{if } \beta_1 > \alpha_1 \\ < 0 & \text{if } \beta_1 < \alpha_1 \end{cases}. \tag{20}$$

From (17), (20), and Theorem 2.1, an optimal policy function h is strictly increasing on D if $\beta_1 > \alpha_1$, and strictly decreasing on the interior of D if $\beta_1 < \alpha_1$. The period-two cycles appear in the case of $\beta_1 < \alpha_1$. For example, given $\beta_1 = \frac{1}{2}$ and $1 > \alpha_1 > \frac{3}{4}$, the steady state $\bar{k} = \alpha_1 \rho^2 / \left[2\alpha_2 \rho\,(\alpha_1 - \alpha_2)\right]$ is totally unstable, and period-two cycles appear for ρ smaller than the value $2\alpha_2/\,(\alpha_1 - \alpha_2)$. This example is provided in Nishimura and Yano (1995b). Subsequently, Baierl, Nishimura, and Yano (1995) characterized the dynamics of optimal paths in two-sector Cobb-Douglas models with slowly depreciating capital stocks.

If we assume $\beta_1 = 0$ in (19), then the social production function becomes $T\left(y, k\right) = k^{\alpha_1}\left(1 - y\right)^{\alpha_2}$, which is a special case of (5). If we replace the Cobb-Douglas function (19) by a Leontief production function $y = \min\{K_K/a, L_K\}$, then the social production function becomes $T\left(y, k\right) = \left(k - ay\right)^{\alpha_1}\left(1 - y\right)^{\alpha_2}$ which is a special case of (6).

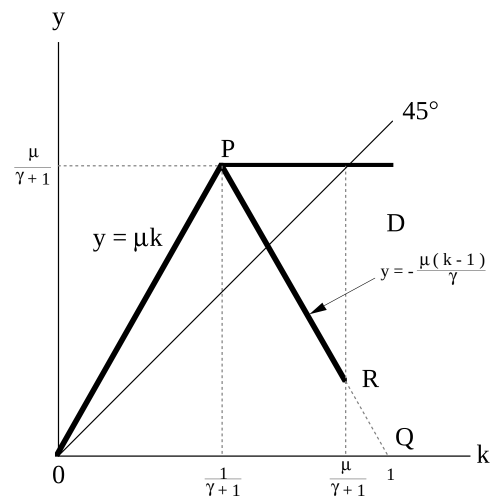

Figure 3
The policy function from (24).

3.3 Ergodic chaos with arbitrarily weak discounting

Suppose that both sectors have Leontief production functions

$$c_t = \min\left\{K_{C,t-1}, L_{C,t}\right\} \tag{21}$$

$$y_t = \mu \min\left\{K_{K,t-1}, L_{K,t}/b\right\} \tag{22}$$

where $\mu > \rho^{-1}$and $b > 1$. Note that $b > 1$ implies that the capital good sector is more labor intensive than the consumption good sector. We still assume that the utility function is linear and that capital is fully depreciated after one period. In this case the maximization problem (13) has, in general, multiple solutions. These solutions can be described by a set-valued function H. That is to say, $(k_0, k_1, \ldots)$ solves (13) if and only if $k_0 \leq k$ and

$$k_t \in H\left(k_{t-1}\right). \tag{23}$$

The set-valued mapping H may be called the generalized optimal policy function.

To answer whether or not an optimal program can exhibit chaos, we need to address the following two specific questions.

1. Under which conditions is the generalized optimal policy function in (23) in fact a single-valued function?
2. Under which conditions is the resulting dynamical system chaotic?

Nishimura and Yano (1993c, 1994b) prove that if the parameter values are suitably chosen, then the optimal paths of problem (13) with linear utility function $u(c) = c$ and Leontief production functions (21) and (22) can be described by a (single-valued) optimal policy function that is expansive and unimodal. Let $\gamma = b - 1$, and define

$$h(x) = \begin{cases} \mu x & \text{if } 0 \leq x \leq 1/(\gamma+1) \\ -(\mu/\gamma)(x-1) & \text{if } 1/(\gamma+1) \leq x \leq 1. \end{cases} \tag{24}$$

The graph of this function is shown in Figure 3. Since $\gamma > 0$ in our setting, the line $y = -(\mu/\gamma)(x-1)$ is negatively sloped. In the figure, this is the line PQ. The candidate of our optimal policy function h coincides with the kinked line OPQ. The slope of OP is strictly larger than 1 because of $\mu > 1$. Under the assumption $\mu/(1+\gamma) \leq 1$, one can show that the function h maps the unit interval [0, 1] onto itself. For all practical purposes, we may restrict h to the closed interval $I = \left[0, \mu/(1+\gamma)\right]$ and treat it as a function from I onto itself. Nishimura and Yano (1994b) prove the following result.

Theorem 3.1. *Let h_I be the function defined in (24) restricted to the interval $I = \left[0, \mu/(1+\gamma)\right]$. Suppose that the parameters μ, ρ, and γ satisfy*

$$1 > \rho > 1, \quad \gamma > 0, \quad \rho\mu > 1, \quad \textit{and} \quad \gamma + 1 > \mu. \tag{25}$$

Then it holds that the generalized optimal policy function H coincides with the single-valued function h_I if one of the following two conditions is satisfied:

$$\textit{Condition A: } \mu \le \gamma .$$

$$\textit{Condition B: } \gamma < \mu \le \min\left\{\left(\gamma + \sqrt{\gamma^2 + 4\gamma}\right)/2, \left(-1 + \sqrt{1 + 4\gamma}\right)/(2\rho)\right\}.$$

Under condition A, the slope of the line PQ is larger than or equal to -1. More specifically, if condition A is satisfied with strict inequality, then the optimal policy function, OPR, is globally stable. If, instead, condition A is satisfied with equality, then any optimal solution from $k > 0$ converges to a period-two cycle that appears on the segment PR, except for the unique path that corresponds to the fixed point.

Under condition B, the slope of line PQ is smaller than -1. In this case, h_I is expansive and unimodal. Therefore, the optimal dynamical system is chaotic. Nishimura and Yano (1994b) show that the set of parameter values (ρ, μ, γ) satisfying condition B and (25) is nonempty if $0 < \rho < 0.5$.

Condition B and (25) are sufficient conditions for h_I to be an optimal policy function that is expansive and unimodal. There may be other sufficient conditions. In fact, Nishimura and Yano (1993c, 1995c) provide an alternative and constructive method to find parameter values (μ, γ) for which h_I is optimal and ergodically chaotic. This method works for any given discount factor ρ, even if it is arbitrarily close to 1. In that paper it is shown that part of the graph of the optimal policy function lies on a von Neumann facet containing the stationary state, and that any optimal path is confined in a small neighborhood of the facet. In this respect, the result is closely related to the neighborhood turnpike theorem of McKenzie (1983), which implies that any optimal path converges to a neighborhood of the von Neumann facet. The result of Nishimura, Sorger, and Yano (1994) in Section 2.5 extends those of Nishimura and Yano (1993c) and Nishimura and Yano (1995c) to the case in which the von Neumann facet is trivial.

4 Other Models

4.1 Models with heterogeneous agents

In the previous section we considered models with a representative consumer. We shall now discuss models with heterogeneous agents. The models presented in this subsection will furthermore have the property that financial markets are incomplete or missing. The two properties together (heterogeneity and incomplete markets) allow for a rich set of dynamic behaviors including cycles and chaos.

A first example illustrating the possibility of equilibrium cycles with period two in a model with heterogeneous agents and borrowing constraints was provided by Bewley (1986). In his model there exist two types of infinitely lived households who may save capital but who are not allowed to borrow from each other. Type 1 households get a high endowment in even periods, and type 2 households get a high endowment in odd periods. The borrowing constraint is binding for the agent who has a low endowment in each period. Bewley (1986) shows that equilibrium paths may exhibit period-two cycles although the aggregate endowment is constant over time.

In some sense, the result from Bewley (1986) is not too surprising, because there is an exogenous fluctuation of (individual) endowments. The following models exhibit equilibrium cycles even without such an exogenous driving force.

Consider a one-sector economy with a large number of identical firms maximizing in each period t the profit function

$$F(K_t, L_t) - w_t L_t - q_t K_t,$$

subject to non-negativity constraints for the factor inputs labor, L_t, and capital, K_t. The firms behave as perfect competitors, which means that factor prices w_t and q_t are considered as fixed data. There are $H > 1$ infinitely lived households. Each household $h \in \{1, 2, \ldots, H\}$ is characterized by a triple (u^h, ρ^h, k^h) where u^h is a strictly concave utility function, $\rho^h \in (0, 1)$ is the household's discount factor, and $k^h > 0$ is its initial endowment with capital. Households maximize the discounted utility stream

$$\sum_{t=0}^{\infty} (\rho^h)^{t-1} u^h(c_t^h)$$

subject to the budget equation $c_t^h + x_{t+1}^h = w_t + (q_t - \delta)x_t^h$ and the non-negativity and initial conditions $c_t^h \geq 0$, $x_t^h \geq 0$, and $x_0^h = k^h$. Here, c_t^h is household h's consumption in period t, x_t^h is its asset holdings (wealth) at the beginning of period t, and $1 - \delta \in [0, 1]$ is the depreciation factor of capital.

The above model was first considered in Becker (1980). Note that households can differ in all their characteristics. They can have different utility functions, different discount factors, and different endowments with capital. Moreover, note that the constraint $x_t^h \geq 0$ requires that each household must have non-negative wealth at each point in time. In other words, there is no credit market on which households can borrow money to increase their consumption beyond their current asset position plus factor income. Becker and Foias (1987) prove that under the condition that the function $F_K(K, H)K - \delta K$ is strictly increasing with respect to K, every equilibrium of the model is eventually monotonic and converges to a unique stationary state.[9] They also demonstrate by means of an example that if the condition is not satisfied, period-two cycles can emerge as equilibrium paths. Later, Becker and Foias (1994) have analyzed the existence of equilibrium cycles of period two more systematically, and have shown how they emerge from a flip bifurcation when the elasticity of substitution between capital and labor is increased beyond a critical value. The existence of equilibrium cycles of arbitrarily long period is demonstrated in an example by Sorger (1994c). It is also shown there that nontrivial stochastic equilibria (sunspot equilibria) can occur in this model. The occurrence of deterministic chaos as the equilibrium outcome is discussed in Sorger (1995). It remains to mention that the models discussed so far in this section lead to two-dimensional difference equations.

A different variety of models with heterogeneous agents and borrowing constraints was investigated by Woodford (1986 and 1988a,b). In contrast to the above models, the equilibrium dynamics here can be reduced to a one-dimensional difference equation. The characteristic feature of these models is that there are two distinct classes of households: capitalists and workers. Capitalists hold all the capital and do not earn any labor income. For workers, on the other hand, labor income is the only possible income. Since workers are assumed to be unable to accumulate capital and unwilling to accumulate debt issued by the capitalists, they are unable to save. Woodford (1986) shows that stochastic fluctuations (sunspot equilibria) are possible for reasonable parameter constellations. Woodford (1988a) discusses the possibility of equilibrium paths that exhibit cycles and chaos.

In the context of the international trade theory, Nishimura and Yano (1993a,b) study the nonlinear dynamics of other models with heterogeneous agents. The agents in their models are large countries. They derive conditions on the utility and production functions under which the capital-accumulation path of each country is oscillatory or monotone along post-trade equilibrium paths.

4.2 Recursive utility models

In studying intertemporal optimization problems, Koopmans (1960) was led to a useful class of preference orderings over a set of consumption sequences over an infinite time horizon, which generalizes the familiar class of additively separable preferences. Denote the consumption stream from time t onward by ${}_tc = (c_t, c_{t+1}, c_{t+2}, \ldots)$ where $0 \leq c_{t+i} < \infty$ for $i = 0, 1, \ldots$. Koopmans's preference orderings may be represented by a utility function $U({}_tc)$ which satisfies

$$U({}_tc) = A(c_t, U({}_{t+1}c)) \tag{26}$$

for $t = 1, 2, \ldots$. The function $A(\cdot, \cdot)$ is called an aggregator. Beals and Koopmans (1969) and Iwai (1972) studied the maximization problem of the intertemporal utility

$$A(c_1, U({}_2c)) \tag{27}$$

[9] See also Hernandez (1991) for a related turnpike theorem.

subject to the constraints

$$c_t = f(k_{t-1}) - k_t, \qquad t \geq 1 \tag{28}$$

$$k_0 = k. \tag{29}$$

Let $V(k)$ be a maximum of the intertemporal utility in the problem defined by (27)–(29). Then by the principle of optimality we obtain the following relation:

$$V(k_0) = \max A(c_1, V(k_1))$$

$$c_1 + k_1 \leq f(k_0).$$

Under certain conditions, it may be proved that the optimal sequence of capital stocks $k_0, k_1, \ldots$ is a monotone sequence and converges to a steady state. Iwai (1972) also points out that if the future utility u is an inferior good in $A(c, u)$, then the optimal capital sequence is oscillatory. Benhabib, Majumdar, and Nishimura (1987) obtain nonmonotonic optimal paths without assuming the inferiority of the future utility in the recursive utility model with a joint production function $c_t = T(k_{t-1}, k_t)$. They also study the reduced-form aggregator of intertemporal utility given by $A(k_0, k_1, V(k_1))$, where $V(k_1)$ is the maximum future utility as a function of k_1 and where (k_0, k_1) belongs to a compact and convex subset of R_+^2. This model contains a one-sector model with wealth effects in the utility function proposed by Majumdar and Mitra (1994). Majumdar and Mitra's model is described as follows:

$$\begin{array}{ll} \text{Maximize} & \displaystyle\sum_{t=1}^{\infty} \rho^{t-1} u(c_t, k_t) \\ \text{subject to} & f(k_{t-1}) - k_t = c_t, \quad t \geq 1 \\ & k_0 \text{ given.} \end{array} \tag{30}$$

By using the value function $V(\cdot)$, problem (30) may be transformed into the following:

$$\begin{array}{ll} \text{Maximize} & \left\{u\left(f(k_0) - k_1, k_1\right) + V(k_1)\right\} \\ \text{subject to} & 0 \leq k_1 \leq f(k_0). \end{array}$$

A model with heterogeneous agents and recursive preferences has been rigorously formulated and studied by Lucas and Stokey (1984). Epstein (1987) proved global stability in a continuous-time version of the Lucas-Stokey model. Benhabib, Jafray, and Nishimura (1988) characterize the local dynamics around a steady state in a model with heterogeneous agents, recursive preferences, and two production sectors in a discrete-time framework. They also show that an optimal path never oscillates in a model with a single sector, as long as future utility is a normal good in an aggregator function for every agent.

The relation between the complexity of optimal paths and impatience in both discrete time and continuous-time versions of reduced form models with recursive preferences is studied in Sorger (1992a).

4.3 Continuous-time models

So far we have concentrated on models that are formulated in discrete time. Some of the results that we have presented only hold in this framework, whereas others have continuous-time counterparts. In this section, we briefly discuss some of the work that has been done on infinite-horizon models in continuous time.

The main obstacle in constructing continuous-time economic models that generate complicated dynamics is the necessity of a high-dimensional state space. Whereas cycles of arbitrary length and chaos are possible for one-dimensional difference equations, they are not possible in one-dimensional differential equations. As a matter of fact, a differential equation defined on a one-dimensional state space can only have monotonic solutions (at least if solutions are unique). For cycles to be possible, a dimension of at least two is required and for chaotic dynamics, one needs at least a three-dimensional system. A simple proof that optimal trajectories in continuous-time growth models with a one-dimensional state space must be monotonic can be found in Hartl (1987).[10]

[10] The assumption of uniqueness of solutions is crucial for this result.

Another difficulty is the lack of simple sufficient conditions for the existence of chaotic dynamics. Existence theorems for periodic solutions are easier to verify, in particular the Hopf bifurcation theorem which basically only requires that a pair of complex conjugate eigenvalues crosses the imaginary axis as one of the model parameters is varied. The first application of this theorem in the context of an infinite-horizon optimal growth model was presented in Benhabib and Nishimura (1979). In the framework of multisector capital accumulation models they studied general conditions for a Hopf bifurcation to occur. They also gave a numerical example of a two-sector model with Cobb-Douglas technologies in which these conditions can be verified. The Hopf bifurcation can occur in multisector optimal growth models with Cobb-Douglas production functions for arbitrary small discount rates. This point was made explicit by Benhabib and Rustichini (1990). Further investigations of the conditions that lead to Hopf bifurcations in continuous-time optimal growth models are presented in Medio (1987) and in Cartigny and Venditti (1994).

Another application of the Hopf bifurcation theorem is described in Dockner and Feichtinger (1991), where a general optimal control model with two state variables and one control variable is considered. Conditions are derived under which a Hopf bifurcation occurs, and they are interpreted in three different ways. This gives rise to three possible mechanisms that can generate cyclical solutions: complementarity over time, dominating cross effects with respect to the state variables, and positive growth at the equilibrium. Similar results are also stated in Wirl (1992).

Modifications of Theorem 2.2 (the possibility theorem) for continuous-time models of the form

$$\begin{aligned} \text{Maximize} \quad & \int_{t=0}^{\infty} e^{-rt} v(k(t), \dot{k}(t)) \\ \text{subject to} \quad & (k(t), \dot{k}(t)) \in D \\ & k(t) \in K \end{aligned}$$

have been derived in Montrucchio (1992) and Sorger (1990). Here K is again the state space of the model, which is assumed to be a nonempty, convex, and closed subset of the Euclidean space $\mathbb{R}^n$. The production possibility set D is a nonempty, closed, and convex subset of $K \times \mathbb{R}^n$. The constructive proofs of the theorems in Montrucchio (1992) and Sorger (1990) can, in principle, be used to generate examples of reduced utility function models in continuous time which have periodic or chaotic solutions. Any differential equation with a sufficiently smooth right-hand side can be the optimal policy function of such a model, provided the state space is compact and the discount rate r is sufficiently high. Theorem 2.3, which relates the discount factor to the topological entropy of the policy function, also carries over to the continuous-time framework. For the model mentioned above, it states that under assumption A2 the relation $\kappa(h, A) \leq rc_+(A)$ must hold, where h is the optimal policy function and A, $\kappa(A, h)$, and $c_+(A)$ are defined as in Theorem 2.3 (see Montrucchio and Sorger [1996]). Discount rate restrictions for continuous-time models with additively separable or recursive preferences are derived in Sorger (1992a). No result that is analogous to Theorem 2.4 (relation between the discount factor and the occurrence of Li-Yorke chaos) holds in continuous-time models.

5 Concluding Remarks

In this paper we have surveyed a number of deterministic economic models with infinitely lived agents and no exogenous perturbations, in which cycles and chaos turn out to be optimal under certain assumptions. We have not tried to discuss all papers in this field, but rather to cover those classes of models for which the most results have been derived. In this section we shall summarize our findings and discuss possible directions of future research.

First of all, the literature surveyed in the present article clearly shows that complicated dynamics can occur in infinite horizon-optimization models satisfying standard convexity and smoothness assumptions. The general possibility theorem, stated as Theorem 2.2, makes this point very clear. Critics of this theorem and of other early papers demonstrating the possibility of chaos in optimal growth models often claimed that cycles and chaos only occur for "unrealistic" parameter values (especially the discount factor, which is the only parameter explicitly used in the reduced-form model). This claim has been refuted by the results mentioned in Section 3.3. We now have a rather clear understanding of the relationship between the size of the discount factor and the possibility of complicated behavior in optimal growth models: there is a sharp discount-factor restriction for the possibility of Li-Yorke chaos in models with a single-state variable (Theorem 2.4), and there is an inequality relation between the discount factor and the topological entropy of the optimal policy function that is valid for models with arbitrary dimension (Theorem 2.3).

An open question in the study of reduced-form models is whether there is a reasonable generalization of Theorem 2.4 to higher dimensional models. Another interesting goal would be to derive a sharp relation between the topological (or some other) entropy of the optimal policy function and the discount factor. Finally, one could try to systematically study whether the dimension of the state space matters for such a relation or not. Theorem 2.3 seems to indicate that this is not the case, but that the (fractal) dimension of the attractor of the optimal policy function is the crucial parameter.

As far as two-sector models are concerned (or, in general, n-sector models), there are some other interesting research topics that remain to be studied. Since the nonreduced form contains more structure and more parameters than the reduced form, one can analyze the influence of these additional parameters on the structure of the optimal growth paths. First results in this direction concerning the rate of depreciation were discussed in Section 2.4. As far as we know, there are no published studies concerning other parameters (such as substitution rates and so on). It also remains to be seen if one can generalize the results from Section 3 to models with more than one capital good, because this would require the analysis of systems of coupled difference equations. The main obstacle to deriving further results in this field is the analytical intractability. It would be a worthwhile project to combine numerical and analytical methods in order to study parametric examples of such models. This could help in gaining more insight into the underlying economic mechanisms.

In almost all the results mentioned in this survey, convexity properties play a fundamental role. In some cases it might be interesting to further investigate this role. One could, for example, try to weaken assumption A2 in the study of reduced-form models in some way. This could lead to models with nonunique optimal paths that can be described by a set-valued optimal policy function, as briefly introduced in Section 3.3.

Finally, we would like to mention that there is only very little known about general properties of optimal policy functions. The monotonicity results stated in Theorem 2.1 are a notable exception. Whereas we know from Theorem 2.2 how to construct, for any sufficiently smooth function h, an optimal growth model that has h as its optimal policy function, there is no general procedure that allows us to derive the optimal policy function from a given model. Although it is possible to approximate the optimal policy function numerically (application of the method of successive approximation to the Bellman equation), it would be invaluable to have analytical characterization results for the optimal policy function that go beyond the monotonicity results from Theorem 2.2.

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