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Characterizing Asymmetries in Business Cycles Using Smooth-Transition Structural Time-Series Models

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Abstract. *This paper aims at testing and modeling business-cycle asymmetries within a structural time-series framework, allowing for smooth transition in the parameters characterizing the cyclical component, namely, the damping factor and the frequency. An LM test of linearity is derived, and illustrations are provided with reference to a set of quarterly U.S. industrial production series for two-digit manufacturing industries.*

Keywords. nonlinearity, business cycles, asymmetry, LM test

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1 Introduction

Detecting and modeling asymmetry constitutes an important issue in the study of business-cycle fluctuations. A first type of asymmetry, recognized in the very early stage of business-cycle research, occurs when contractions are steeper, but shorter, than expansions, so that the average duration and the dynamics of the two phases of the business cycle differ. That contraction is a more violent change than expansion is already regarded by Burns and Mitchell (1946, p. 134) as a common empirical finding.

A second type, introduced by Sichel (1993) and referred to as *steepness*, occurs when troughs are deeper than peaks: recessions and expansions are characterized by the same duration (symmetry along the time axis), but the cycle undergoes a steep fall and a steep recovery, then it peaks at a slower rate and starts falling at a slow, but accelerating, rate; as a result, the distribution of the cyclical component is negatively skewed, with a positive mode.

A third way in which asymmetric behavior can arise is due to an amplitude-frequency relationship (Tong 1990), implying, for instance, that the period of the cycle is lower when the amplitude of the oscillation is high.

Business-cycle asymmetry is a specialized departure from linearity that is accommodated by a transition or switching model, capturing the notion that the phenomenon under investigation behaves differently according to the state of the system defined in terms of a function of a transition variable. Along with the latter, which can be observable or unobserved, the transition mechanism, i.e., the way the system moves from one state to another, needs to be specified. The choices of the transition variable and the transition mechanism have given rise to different approaches to modeling the feature under consideration.

In Hamilton's Markov-switching models (Hamilton 1989, 1993) the parameters of the autoregressive data-generating process vary according to the states of a latent first-order Markov chain; in empirical applications to output series, the feature modeled is that the mean rate of drift is higher in expansions than in recessions. The transition across regimes can be made smooth, allowing for time variation in the transition probabilities. On the contrary, the class of threshold autoregressive models (TAR) is such that the transition variable is observed—usually as linear combinations of lagged values of the series. The transition mechanism can be discrete or smooth, giving rise, in the latter case, to smooth-transition AR models (STAR), which postulate a continuum of states between two extremes.

Smooth transition is usually deemed preferable to discrete transition. Apart from being more realistic since "economic agents may not all act promptly and uniformly at the same moment" (Teräsvirta 1998), it is also encompassing, as it nests discrete switching as a limiting case, and more flexible: Sichel (1994) has recently criticized the emphasis on the two-phase characterization of business-cycle fluctuations, documenting the presence of three phases: recessions are followed by high growth recovery, and a moderate growth period follows. The smooth-transition approach has the flexibility to account for these patterns.

Other approaches have been proposed in the literature, such as AutoRegressive Asymmetric Moving Average (ARAMA) models (Brännas and de Gooijer 1994; Brännas, de Gooijer, and Teräsvirta 1998), according to which the response to negative shocks differs from the response to positive ones. However, ARAMA models are less well suited to modeling a specific business-cycle asymmetry, and thus will not be pursued further here.

This paper aims at modeling business-cycle asymmetries from a structural (Harvey 1989) perspective, by which the series is decomposable into components of direct relevance to the analysis, such as trends, seasonals, and cycles. In this framework, it is quite natural to model directly possible nonlinearities concerning the cycle itself. The latter represents the most obvious transition variable, which is used to define events or states such as recessions and expansions. This choice is somewhat a compromise between the Markov-switching and TAR approaches, as the regimes are defined according to an unobserved variable, for which, however, the minimum least-squares linear estimator is delivered by the conditionally Gaussian Kalman filter (KF). This course of action enhances the interpretability of the model, by a thorough understanding of its properties and the type of asymmetry captured. Moreover, we do not commit ourselves to an autoregressive representation, and the reduced form of the model will be such that both the autoregressive and moving-average parameters vary.

The paper is organized as follows: Section 2 is devoted to a brief review of TAR models that serves to discuss the issues of selecting a transition mechanism and linearity testing against a specific alternative. Section 3 defines the linear cyclical model and its properties, whereas in the next section we set up its enhancements in order to model business-cycle asymmetries. Likelihood inference for the models considered in the paper is discussed in Section 5, and in Section 6 we derive LM linearity tests against smooth transition in the parameters characterizing the behavior of the cycle. Finally, for illustrative purposes, we present the empirical evidence concerning a set of U.S. quarterly industrial production series (Section 8), and Section 9 concludes.

2 Threshold Autoregressive Models

In this section, we briefly review the TAR approach to testing and modeling asymmetry, since some of its ideas will be exploited later. A TAR model is specified as follows:

$$y_t = \mu_1 + \mu_2 F(z_{t-d}) + \sum_{j=1}^{p} [\phi_{1j} + \phi_{2j} F(z_{t-d})] y_{t-j} + [\sigma_1 + \sigma_2 F(z_{t-d})]\epsilon_t,$$

where $\epsilon_t \sim \text{WN}(0, 1)$, and the parameters shift according to the regime defined in terms of a function of the observable transition variable z_{t-d}, where d is the delay parameter. The transition mechanism is a function, $F(\cdot)$, usually bounded between 0 and 1, so that the range of the parameters is $(\mu_1, \mu_1 + \mu_2)$, $(\phi_{1j}, \phi_{1j} + \phi_{2j})$ and $(\sigma_1, \sigma_1 + \sigma_2)$.

As far as the definition of the transition variable is concerned, we shall restrict attention to univariate threshold models, such that z_{t-d} is a scalar variable; usually $z_{t-d} = y_{t-d}$, although Beaudry and Koop (1993), in modeling the relative changes in y_t, Δy_t, defined $z_t = y_t - \max\{y_t, y_{t-1}, y_{t-2}, \dots, y_1\}$, which captures the depth of contractions in the levels of y_t. Teräsvirta (1998) considered the case $z_{t-d} = t$, for modeling and testing time variation in the parameters of the autoregression.

Several specifications have been proposed for the transition mechanism, $F(z_{t-d})$:

- SETAR (Self-Exciting Threshold Autoregressive) models: for a threshold parameter, c, $F(z_{t-d}) = I(y_{t-d} > c)$, where $I(a)$ is the *indicator* function taking the value of 1 if the event a occurs, and 0 otherwise (Potter 1995).

- STAR models whereby $F(z_{t-d})$ is an odd monotonically increasing function with $F(-\infty) = 0$ and $F(+\infty) = 1$. Examples are the standard normal distribution function, $\Phi((y_{t-d} - c)/\delta)$, where δ is a scale parameter, and the logistic function or LSTAR models (Granger and Teräsvirta 1993),

$$F(z_{t-d}) = \frac{1}{1 + \exp[-\tau(y_{t-d} - c)]}.$$

The parameter $\tau > 0$ is the smoothness parameter determining how rapid the transition is. If $\tau \to \infty$, $F(z_{t-d}) \to I(z_{t-d})$.

- STAR models whereby $F(z_{t-d})$ is an even function; e.g., the normal density function

$$F(z_{t-d}) = \frac{1}{\sqrt{2\pi}\delta} \exp[-(y_{t-d} - c)^2/2\delta^2]$$

or

$$F(z_{t-d}) = 1 - \exp[-\tau(y_{t-d} - c)^2],$$

as in exponential STAR (ESTAR) models. For small or large values of y_{t-d}, F is close to 1; when $\tau \to \infty$, $F \to 1 - I(y_{t-d} = c)$. This is used to model amplitude-dependent phenomena: recessions and expansions have similar dynamics, but the middle ground is subject to different dynamics. As a matter of fact, the ESTAR model is a generalization of Ozaki's exponential AR models ($d = 1$ and $c = 0$); see the work of Tong (1990).

Linearity testing against a SETAR specification is a highly nonstandard inferential problem that has been recently dealt with by Hansen (1996). More convenient approximate test procedures are available for STAR alternatives. When the irregular variance is constant, the LM test of linearity against LSTAR and ESTAR alternatives is a test of $\tau = 0$ in

$$y_t = \mu_1 + \mu_2 F(y_{t-d}) + \sum_{j=1}^{p} [\phi_{1j} + \phi_{2j} F(y_{t-d})] y_{t-j} + u_t,$$

where $u_t \sim \mathrm{WN}(0, \sigma^2)$. This testing problem is also nonstandard, since under H_0 the parameters μ_2, ϕ_{2j}, and c are not identified, and the block of the information matrix corresponding to these parameters is null, which violates the standard regularity conditions under which the LM test is derived.

A way of getting around the problem is to adopt the strategy suggested by Davies (1977, 1987), which leads to a convenient test procedure. This consists of first deriving the LM statistic as a function of the unidentified parameters $\mathrm{LM}(\mu_2, \phi_{2j}, c)$, and then basing the test on $\max \mathrm{LM}(\mu_2, \phi_{2j}, c)$ over all possible values (μ_2, ϕ_{2j}, c). When the transition variable is known, this yields the test statistic TR^2, where R^2 is the coefficient of determination in the regression of the LS residuals $\hat{u}_t = y_t - \hat{\mu}_1 - \sum \hat{\phi}_{1j} y_{t-j}$ on a constant, $y_{t-1}, \ldots, y_{t-p}$, and $y_{t-1}y_{t-d}, \ldots, y_{t-p}y_{t-d}$. The coefficients associated to the cross-product terms depend on ϕ_{2j} and c, but not on μ_2, and the null is reformulated so that they are zero. Hence, under H_0 the test has an asymptotic $\chi^2(p)$ distribution, but the fact that μ_2 does not enter the coefficients is responsible for the low power of the test when the nonlinearity is mainly due to the intercept (i.e., μ_2 is large and the ϕ_{2j} values are small).

A general way for circumventing the lack of identifiability while enhancing the power properties of the test against the alternative has been proposed by Luukkonen, Saikkonen, and Teräsvirta (1988), and amounts to replacing $F(y_{t-d})$ by a third- and a first-order Taylor approximation around $\tau = 0$, respectively, for the LSTAR and the ESTAR models. For the former, when the transition variable is known, the LM test takes the usual TR^2 form, where R^2 is the coefficient of determination in the regression

$$\hat{u}_t = \mu + \sum_j b_{1j} y_{t-j} + \sum_j b_{2j} y_{t-j} y_{t-d} + \sum_j b_{3j} y_{t-j} y_{t-d}^2 + \sum_j b_{4j} y_{t-j} y_{t-d}^3 + e_t.$$

The null hypothesis is $H_0 : b_{2j} = b_{3j} = b_{4j} = 0,\ j = 1, \ldots, p$. The test statistic has an asymptotic $\chi^2(3p)$ distribution. Alternatively, the modified LM test,

$$F = \frac{(SSE_0 - SSE_1)/3p}{SSR_1/(T - 4p - 1)},$$

is suggested, since it yields an F-test statistic with better size properties.

As far as the ESTAR model is concerned, the first-order Taylor expansion leads to the same regression for $\hat{u}_t$ with $b_{4j} = 0, \forall j$. Hence, Teräsvirta (1994) suggests first testing linearity versus STAR by F, and using the results of the sequence of tests of nested hypotheses, H_{01}: $b_{4j} = 0$, H_{02}: $b_{3j} = 0 \mid b_{4j} = 0$, H_{03}: $b_{2j} = 0 \mid b_{3j} = b_{4j} = 0,\ j = 1, \ldots, p$, to select the relevant alternative. The strategy of approximating $F(z_{t-d})$ by a Taylor-series expansion is adopted in Section 7 for testing linearity in a structural framework.

3 The Linear Stochastic Cycle Model and the Basic Structural Model

In the structural framework, the model is specified directly in terms of the stylized facts concerning an economic time series: for a quarterly economic time series, the basic model we entertain is the additive decomposition,

$$y_t = \mu_t + \gamma_t + \psi_t + \epsilon_t, \tag{1}$$

where μ_t is the trend component, γ_t is the seasonal component, ψ_t is the cyclical component, and $\epsilon_t \sim \mathrm{WN}(0, \sigma^2_\epsilon)$ is the irregular.

Throughout the paper we are holding constant the representations for the trend and the seasonal. The former is generated by $\mu_{t+1} = \mu_t + \beta_t + \eta_t$, and $\beta_{t+1} = \beta_t + \zeta_t$, with η_t and ζ_t mutually uncorrelated white noises with variances σ^2_η and σ^2_ζ, respectively. The latter has a trigonometric representation as the the sum of two stochastic cycles defined at the seasonal frequencies $\pi/2$ and π.

We now turn our attention to the linear representation of the cyclical component, which is capable of interpreting essential features, such as the presence of strong autocorrelation, determining the alternation of phases, and the damping of the fluctuations (zero long-run persistence), which are commonly recognized as pertaining to business cycles. The stochastic cycle, ψ_t, is modeled as a stationary ARMA(2,1) process, subject to the following restrictions on the parameter space:

$$\begin{aligned} \psi_t &= [1\ 0]\boldsymbol{\alpha}_t, \quad \boldsymbol{\alpha}_t = [\psi_t, \psi^*_t]', \\ \psi_{t+1} &= \rho\cos\lambda\psi_t + \rho\sin\lambda\psi^*_t + \kappa_t, \\ \psi^*_{t+1} &= -\rho\sin\lambda\psi_t + \rho\cos\lambda\psi^*_t + \kappa^*_t, \end{aligned} \tag{2}$$

where $\kappa_t \sim \mathrm{WN}(0, \sigma^2_\kappa)$ and $\kappa^*_t \sim \mathrm{WN}(0, \sigma^2_\kappa)$ are mutually uncorrelated; $\rho \in [0, 1]$ is the damping factor, and $\lambda \in [0, \pi]$ is the frequency of the cycle (the period is $2\pi/\lambda$). The reduced form of Equation 2 is ARMA(2,1),

$$(1 - 2\rho\cos\lambda L + \rho^2 L^2)\psi_{t+1} = (1 - \rho\cos\lambda L)\kappa_t - \rho\sin\lambda L\kappa^*_t,$$

such that the roots of the AR polynomial are a pair of complex conjugates with a modulus of ρ^{-1} and a phase of λ. The spectral density is everywhere positive, and displays a peak at λ; furthermore, $\mathrm{E}(\psi_t) = 0, \sigma^2_\psi = \mathrm{Var}(\psi_t) = \sigma^2_\kappa/(1 - \rho^2)$, and the autocorrelation at lag j is $\rho^j\cos\lambda j$.

An equivalent representation is obtained as follows:

$$\begin{aligned} \psi_t &= [\cos\lambda t, \sin\lambda t]\boldsymbol{A}_t, \quad \boldsymbol{A}_t = [A_{1t}, A_{2t}]', \\ A_{1,t+1} &= \rho A_{1t} + \tilde{\kappa}_{1t}, \\ A_{2,t+1} &= \rho A_{2t} + \tilde{\kappa}_{2t}, \end{aligned} \tag{3}$$

where $\tilde{\kappa}_{1t} \sim \mathrm{WN}(0, \sigma^2_\kappa)$ and $\tilde{\kappa}_{2t} \sim \mathrm{WN}(0, \sigma^2_\kappa)$ are mutually uncorrelated. By trigonometric identities, it is possible to prove that there is a one-to-one mapping between the representations of Equations 2 and 3; in particular,

$$\begin{bmatrix} \psi_{t+1} \\ \psi^*_{t+1} \end{bmatrix} = \begin{bmatrix} \cos\lambda t & \sin\lambda t \\ -\sin\lambda t & \cos\lambda t \end{bmatrix} \begin{bmatrix} A_{1,t+1} \\ A_{2,t+1} \end{bmatrix}; \quad \begin{bmatrix} \kappa_t \\ \kappa^*_t \end{bmatrix} = \begin{bmatrix} \cos\lambda t & \sin\lambda t \\ -\sin\lambda t & \cos\lambda t \end{bmatrix} \begin{bmatrix} \tilde{\kappa}_{1t} \\ \tilde{\kappa}_{2t} \end{bmatrix}.$$

The A_{it}, $i = 1, 2$ values are related to the amplitude of the oscillation, as ψ_t can be rewritten as

$$\psi_t = \varphi_t \cos(\lambda t - \vartheta_t),$$

where $\varphi_t = (A_{1t}^2 + A_{2t}^2)^{.5}$ is the time-varying amplitude, and $\vartheta_t = \tan^{-1}(A_{2t}/A_{1t})$ is the phase shift. Note that $\vartheta_t \sim U(-\pi/2, \pi/2)$; i.e., is uniformly distributed in the interval $(-\pi/2, \pi/2)$ if the A_{it} were observed. This suggests that the histogram of ϑ_t can be used as a descriptive device for nonlinearity detection (deviations from uniform distribution). Unfortunately, when the A_{it}, $i = 1, 2$, are replaced by their smoothed estimates, $\mathrm{E}(A_{it} \mid Y_T)$, where Y_t denotes information up to time t, $Y_t = \{y_1, \ldots, y_t\}$, the result is no longer valid; moreover, it depends on the normality assumption for $\tilde{\kappa}_{1t}$ and $\tilde{\kappa}_{2t}$.

In order to complete the model specification, the seasonal component is $\gamma_t = \gamma_{1t} + \gamma_{2t}$, where γ_{1t} has the same representation as Equation 2 with $\rho = 1$, $\lambda = \pi/2$, $\sigma_\kappa^2 = \sigma_\omega^2$, and $\gamma_{2t+1} = -\gamma_{2,t-1} + \omega_t$, $\omega_t \sim \mathrm{WN}(0, \sigma_\omega^2)$. Finally, all random innovations are mutually uncorrelated.

4 Smooth-Transition Cyclical Models

The linear-cycle model introduced in the previous section cannot generate the types of asymmetries we are interested in. We will now investigate ways of allowing a (possibly) smooth variation of its hyperparameters according to the values taken by a state variable, namely ψ_t. The resulting models are thereby called ST^2 (structural smooth transition) models; of course, the transition variable is nonobservable, but we know at least some aspects of its distribution, conditional upon the data.

An ST^2 model for the cycle may be obtained by letting the damping factor and the frequency vary as $\lambda_t = \lambda_1 + F(Y_{t-1})\lambda_2$, and $\rho_t = \rho_1 + F(Y_{t-1})\rho_2$, $\sigma_{\kappa,t}^2 = \sigma_\psi^2(1 - \rho_t^2)$. However, since these parameters are subject to constraints, we shall adopt the parameterization

$$\rho_t = \frac{|r_t|}{\sqrt{1 + r_t^2}}, \quad r_t = r_1 + F(Y_{t-1})r_2, \tag{4}$$

$$\lambda_t = \frac{2\pi}{2 + \exp\bar{\lambda}_t}, \quad \bar{\lambda}_t = \bar{\lambda}_1 + F(Y_{t-1})\bar{\lambda}_2, \tag{5}$$

where $r_1, r_2, \bar{\lambda}_1, \bar{\lambda}_2$ are unconstrained parameters. These transformations ensure that ρ_t and λ_t lie in their admissible range for all t.

There are some advantages in investigating nonlinearity in the structural framework, since some economic hypotheses are more easily spelled out in terms of the structural parameters. In the STAR and Markov-switching approaches, at the estimation stage the modeler is rather blind on the type of asymmetry captured by the variation in the AR parameters: for instance, in order to check that the model is sensible, Teräsvirta and Anderson (1992, p. S125) suggest checking for the dynamic properties of the estimated model by looking at the roots of the characteristic polynomial associated with the different regimes and at the long-run properties of the model. A further cause of concern is the uncertainty surrounding the choice of the transition variable.

On the contrary, in the structural framework, we let the cycle "speak for itself," and hereby list a few sensible selections for the variable defining the regimes:

- $\mathrm{E}(\psi_t - \psi_{t-1} \mid Y_{t-1}) = \hat{\psi}_t - \tilde{\psi}_{t-1|t-1}$, where $\hat{\psi}_t = \mathrm{E}(\psi_t \mid Y_{t-1})$ and $\tilde{\psi}_{t|t} = \mathrm{E}(\psi_t \mid Y_t)$. A recessionary pattern arises when $\psi_t - \psi_{t-1}$ is negative, whereas an expansion occurs when $\psi_t - \psi_{t-1} > 0$. Of course, ψ_t is unobservable, but we can construct its minimum least-square linear prediction based on Y_t (see Section 5).
- $\mathrm{E}(\psi_t \mid Y_{t-1}) = \hat{\psi}_t$. The regimes are defined according to the level of the cycle.
- $\mathrm{E}(\psi_t^2 \mid Y_{t-1}) = \hat{\psi}_t^2 + \mathrm{Var}(\psi_t \mid Y_{t-1})$. Recessions and expansions receive a symmetric treatment, and the regimes are defined in terms of the amplitude of the fluctuation.
- $\mathrm{E}[(\psi_t - \psi_{t-1})^2 \mid Y_{t-1}] = (\hat{\psi}_t - \tilde{\psi}_{t|t})^2 + \mathrm{Var}(\psi_t - \psi_{t-1} \mid Y_{t-1})$. The regimes are defined according to the depth of recessions and the strength of expansions.
- $\hat{\varphi}_t = \mathrm{E}[(A_{1t}^2 + A_{2t}^2)^{.5} \mid Y_{t-1}]$. The regimes are defined in terms of the amplitude of the cycle.

It should be noticed that for the threshold, c, the choice $c = 0$ is quite natural, which is another advantage of working with the realization of the cycle as the transition variable.

Harvey (1989), in order to model business-cycle asymmetry between expansions and recessions, proposed the threshold cyclical model specifying $F(Y_{t-1}) = I\{\mathrm{E}(\psi_t - \psi_{t-1} \mid Y_{t-1}) > 0\}$. The rationale is that when there is an upswing, the frequency of the cycle is $\lambda_1 + \lambda_2$, whereas in the presence of a downswing, the frequency of the cycle is λ_1. The resulting model is conditionally Gaussian, since the system matrices depend on the information available at time $t - 1$.

In the sequel, we shall concentrate on the following transition mechanisms:

- The monotonic transition mechanism. For modeling type-I asymmetries, $F(Y_{t-1})$ must monotonically increase in the range (0,1) as we move from recession to expansion. This can be achieved by

$$F(Y_{t-1}) = \mathrm{pr}(\psi_t - \psi_{t-1} > 0 \mid Y_{t-1}) = \Phi\left(-(\hat{\psi}_t - \tilde{\psi}_{t-1|t-1})/\sqrt{\mathrm{Var}(\hat{\psi}_t - \tilde{\psi}_{t-1|t-1})}\right),$$

but in the sequel we are concerned with the logistic transition mechanism (LgstTrM I),

$$F(Y_{t-1}) = \frac{1}{1 + \exp[-\tau(\hat{\psi}_t - \tilde{\psi}_{t-1|t-1})]}. \tag{6}$$

An LgstTrM can be specified also with respect to the transition variable $\hat{\psi}_t$, in which case type-II asymmetry (steepness) is modeled (LgstTrM II),

$$F(Y_{t-1}) = \frac{1}{1 + \exp[-\tau\hat{\psi}_t]}. \tag{7}$$

Logistic transition mechanisms for the transition variables $\mathrm{E}(\psi_t^2 \mid Y_{t-1})$ and $\mathrm{E}[(\psi_t - \psi_{t-1})^2 \mid Y_{t-1}]$ do not correspond to any type of asymmetry dealt with in the literature, and are not considered here.

- Symmetric transition mechanism. When the transition variable is $\mathrm{E}(\psi_t^2 \mid Y_{t-1})$, we can use an exponential transition mechanism (ExpTrM I),

$$F(Y_{t-1}) = 1 - \exp[-\tau \mathrm{E}(\psi_t^2 \mid Y_{t-1})]. \tag{8}$$

This specification is able to detect amplitude-frequency relationships, but it can also detect type-II asymmetries, since the cycle dynamics are different when a steep trough takes place (ψ_t^2 is high) and in the vicinity of a peak (ψ_t^2 is small).

When the exponential transition mechanism is applied with transition variable $\mathrm{E}[(\psi_t - \psi_{t-1})^2 \mid Y_{t-1}]$, i.e.,

$$F(Y_{t-1}) = 1 - \exp\{-\tau \mathrm{E}[(\psi_t - \psi_{t-1})^2 \mid Y_{t-1}]\}, \tag{9}$$

we deal with a situation in which contractions and expansions have similar dynamics, but the middle ground behaves differently. This proves useful when the cycle is characterized by the presence of more than two phases (Sichel 1994), and will be referred to as ExpTrM II.

5 State-Space Representation and Likelihood Inference

In this section, we review likelihood theory for state-space models (SSM), as derived by de Jong (1991), with respect to a general vector time series $\boldsymbol{y}_t$, $t = 1, \ldots, T$, with N elements; let $\boldsymbol{Y}_t = \{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_t\}$ denote the information up to and including time t.

The SSM for $\boldsymbol{y}_t$ is written as

$$\begin{aligned} \boldsymbol{y}_t &= \boldsymbol{Z}_t\boldsymbol{\alpha}_t + \boldsymbol{X}_t\boldsymbol{\beta} + \boldsymbol{G}_t\varepsilon_t, \qquad t = 1, 2, \ldots, T, \\ \boldsymbol{\alpha}_{t+1} &= \boldsymbol{W}_t\boldsymbol{\beta} + \boldsymbol{T}_t\boldsymbol{\alpha}_t + \boldsymbol{H}_t\varepsilon_t, \qquad t = 0, 1, 2, \ldots, T, \end{aligned} \tag{10}$$

where $\boldsymbol{\alpha}_t$ is the $m \times 1$ state vector, with $\boldsymbol{\alpha}_0 = \boldsymbol{0}$, and $\boldsymbol{\alpha}_1 = \boldsymbol{W}_0\boldsymbol{\beta} + \boldsymbol{H}_0\varepsilon_0$. Further, $\boldsymbol{\beta} = \boldsymbol{b} + \boldsymbol{B}\boldsymbol{\delta}$, $\boldsymbol{\delta} \sim \mathrm{N}(\boldsymbol{\mu}, \sigma^2\boldsymbol{\Lambda})$, $\boldsymbol{b}$ is a nonrandom vector, $\boldsymbol{B}$ is full column rank, and the specification is completed by the assumption that

ε_t, $t = 0, 1, \ldots, T$, is NID$(\mathbf{0}, \sigma^2 \boldsymbol{I})$. Regression effects enter the SSM via $\boldsymbol{X}_t\boldsymbol{\beta}$ and $\boldsymbol{W}_t\boldsymbol{\beta}$; the system matrices, $\boldsymbol{Z}_t, \boldsymbol{G}_t, \boldsymbol{T}_t$, and $\boldsymbol{H}_t$ are nonstochastic, although they may vary over time.

In particular, Equation 1 is an SSM with $N = 1, \sigma^2 = 1, m = 7$, and $\boldsymbol{\alpha}_t = [\mu_t, \beta_t, \gamma_{1t}, \gamma_{1t}^*, \gamma_{2t}, \psi_t, \psi_t^*]'$, and system matrices are $\boldsymbol{Z}_t = \boldsymbol{Z} = [1, 0, 1, 0, 1, 1, 0]$, $\boldsymbol{X}_t = \mathbf{0}$, $\boldsymbol{G}_t = \boldsymbol{G} = [1\ \mathbf{0}_m']$, $\boldsymbol{W}_t = \mathbf{0}_m$, $t = 1, \ldots, T$, $\boldsymbol{H}_t = [\mathbf{0}_m \tilde{\boldsymbol{H}}]$, with $\tilde{\boldsymbol{H}} = \text{diag}(0, \sigma_\eta, \sigma_\zeta, \sigma_\omega, \sigma_\omega, \sigma_\omega, \sigma_\kappa, \sigma_\kappa)$, and $\boldsymbol{T}_t = \text{diag}(\boldsymbol{T}_\mu, \boldsymbol{T}_\gamma, \boldsymbol{T}_{\psi,t})$, where

$$\boldsymbol{T}_\mu = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{T}_\gamma = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \boldsymbol{T}_{\psi,t} = \rho_t \begin{bmatrix} \cos\lambda_t & \sin\lambda_t \\ -\sin\lambda_t & \cos\lambda_t \end{bmatrix},$$

and

$$\boldsymbol{H}_0 = \begin{bmatrix} \mathbf{0} \\ \sigma_\psi \boldsymbol{I}_2 \end{bmatrix}, \quad \boldsymbol{W}_0 = \begin{bmatrix} \boldsymbol{I}_5 \\ \mathbf{0} \end{bmatrix}.$$

Note that $\boldsymbol{H}_t\boldsymbol{G}_t' = \mathbf{0}$; i.e., the measurement and transition equation noise are uncorrelated; ρ_t and λ_t are formulated as Equations 4 and 5, where the transition function is given by Equations 6 and 8. Therefore, $\boldsymbol{T}_t = \boldsymbol{T}(\boldsymbol{Y}_{t-1})$ and $\boldsymbol{H}_t = \boldsymbol{H}(\boldsymbol{Y}_{t-1})$, the latter being so since $\sigma_\kappa^2 = \sigma_\psi^2(1 + \rho_t^2)$, and the resulting SSM is conditionally Gaussian (see Harvey 1989, sec. 3.7.1.; Lipster and Shiryaev 1978, ch. 11).

In the general SSM, the random vector $\boldsymbol{\delta}$, with $d + k$ elements, allows the unified treatment of k regression and d nonstationary effects; the latter arise, for instance, when trends and seasonals are present; $\boldsymbol{\delta}$ is said to be diffuse if $\boldsymbol{\Lambda}^{-1} \to \mathbf{0}$ in the Euclidean norm.

The Kalman filter is a recursive algorithm for computing the minimum mean-square linear estimator (MMSLE) of $\boldsymbol{\alpha}_t$ and its mean-square error (MSE) matrix conditional on $\boldsymbol{Y}_{t-1}$ and $\boldsymbol{\delta}$, denoted $\boldsymbol{a}_t$ and $\sigma^2\boldsymbol{P}_t$, respectively:

$$\begin{aligned} \boldsymbol{\nu}_t &= \boldsymbol{y}_t - \boldsymbol{X}_t\boldsymbol{\beta} - \boldsymbol{Z}_t\boldsymbol{a}_t, & \boldsymbol{F}_t &= \boldsymbol{Z}_t\boldsymbol{P}_t\boldsymbol{Z}_t' + \boldsymbol{G}_t\boldsymbol{G}_t', \\ q_t &= q_{t-1} + \boldsymbol{\nu}_t'\boldsymbol{F}_t^{-1}\boldsymbol{\nu}_t, & \boldsymbol{K}_t &= (\boldsymbol{T}_t\boldsymbol{P}_t\boldsymbol{Z}_t' + \boldsymbol{H}_t\boldsymbol{G}_t')\boldsymbol{F}_t^{-1}, \\ \boldsymbol{a}_{t+1} &= \boldsymbol{T}_t\boldsymbol{a}_t + \boldsymbol{W}_t\boldsymbol{\beta} + \boldsymbol{K}_t\boldsymbol{\nu}_t, & \boldsymbol{P}_{t+1} &= \boldsymbol{T}_t\boldsymbol{P}_t\boldsymbol{L}_t' + \boldsymbol{H}_t\boldsymbol{M}_t', \end{aligned} \tag{11}$$

with $\boldsymbol{L}_t = \boldsymbol{T}_t - \boldsymbol{K}_t\boldsymbol{Z}_t$, $\boldsymbol{M}_t = \boldsymbol{H}_t - \boldsymbol{K}_t\boldsymbol{G}_t$, and starting conditions $\boldsymbol{a}_1 = \boldsymbol{W}_0\boldsymbol{\beta}$, $\boldsymbol{P}_1 = \boldsymbol{H}_0\boldsymbol{H}_0'$, and $q_0 = 0$. Note that, unlike the unconditionally Gaussian case, $\boldsymbol{a}_t$ is nonlinear in $\boldsymbol{Y}_{t-1}$, and its conditional MSE matrix depends on the particular realization $\boldsymbol{Y}_{t-1}$.

When $\boldsymbol{\delta}$ is diffuse, the relevant algorithm for likelihood evaluation is the diffuse KF (DKF) proposed by DeJong (1991), which replaces the recursions for $\boldsymbol{\nu}_t$, $\boldsymbol{a}_{t+1}$, and q_t, respectively, with

$$\begin{aligned} \boldsymbol{V}_t &= [\boldsymbol{y}_t, \mathbf{0}] - \boldsymbol{Z}_t\boldsymbol{A}_t - \boldsymbol{X}_t[\boldsymbol{b}, \boldsymbol{B}], \\ \boldsymbol{A}_{t+1} &= \boldsymbol{T}_t\boldsymbol{A}_t + \boldsymbol{W}_t[\boldsymbol{b}, \boldsymbol{B}] + \boldsymbol{K}_t\boldsymbol{V}_t, \\ \boldsymbol{Q}_t &= \boldsymbol{Q}_{t-1} + \boldsymbol{V}_t'\boldsymbol{F}_t^{-1}\boldsymbol{V}_t, \end{aligned} \tag{12}$$

with starting conditions $\boldsymbol{A}_1 = \boldsymbol{W}_0[\boldsymbol{b}, \boldsymbol{B}]$ and $\boldsymbol{Q}_0 = \mathbf{0}$.

Then, partitioning

$$\boldsymbol{Q}_t = \begin{bmatrix} q_t^\dagger & \boldsymbol{s}_t' \\ \boldsymbol{s}_t & \boldsymbol{S}_t \end{bmatrix},$$

de Jong (1991) shows that the diffuse log-likelihood is

$$\mathcal{L}(\boldsymbol{y} \mid \boldsymbol{\theta}) = -\frac{1}{2}\left[N(T - d - k)\ln\sigma^2 + \sum_{t=1}^{T}\ln|\boldsymbol{F}_t| + \ln|\boldsymbol{S}_T| + \sigma^{-2}(q_T^\dagger - \boldsymbol{s}_T'\boldsymbol{S}_T^{-1}\boldsymbol{s}_T)\right].$$

The DKF is computationally more demanding than the KF, since it is based on matrix recursion for $\boldsymbol{V}_t$ $(N \times d + k + 1)$, $\boldsymbol{A}_t$ $(m \times d + k + 1)$, and $\boldsymbol{Q}_t$ $(d + k + 1 \times d + k + 1)$. The computational burden can be significantly reduced by collapsing the filter; i.e., by reducing to one the column dimension of $\boldsymbol{A}_t$, $\boldsymbol{V}_t$, and $\boldsymbol{Q}_t$.

A full collapse to the KF can take place when $k = 0$; i.e., $\boldsymbol{X}_t = \boldsymbol{0}$ and $\boldsymbol{W}_t = \boldsymbol{0}$, $t = 1, \ldots, T$. Suppose that after d runs of the filter the matrix $\boldsymbol{S}_d$ is invertible; it is then possible to compute the diffuse predictions as follows:

$$\begin{aligned}
\hat{\boldsymbol{\delta}}_d &= -\boldsymbol{S}_d^{-1}\boldsymbol{s}_d, \\
\boldsymbol{a}_{d+1} &= \boldsymbol{A}_{d+1}[1, \hat{\boldsymbol{\delta}}'_d]', \\
\boldsymbol{P}_{d+1} &= \boldsymbol{P}_{d+1} + \boldsymbol{A}^{\dagger}_{d+1}\boldsymbol{S}_d^{-1}\boldsymbol{A}^{\dagger\prime}_{d+1},
\end{aligned}$$

and $q_d = q_{1d} - \boldsymbol{s}'_d\boldsymbol{S}_d^{-1}\boldsymbol{s}_d$, where $\boldsymbol{A}^{\dagger}_{d+1}$ denotes the last d columns of $\boldsymbol{A}_{d+1}$. Then a switch is made to the ordinary KF, which is run for $t = d+1, \ldots, T$, and the log-likelihood function is

$$\mathcal{L}(\boldsymbol{y} \mid \boldsymbol{\theta}) = -\frac{1}{2}\left[N(T-d)\ln\sigma^2 + \sum_{t=1}^{T}\ln|\boldsymbol{F}_t| + \ln|\boldsymbol{S}_d| + \sigma^{-2}q_T\right].$$

The variance σ^2 can be estimated as $\hat{\sigma}^2 = \hat{q}_T/(T-d)$, and the concentrated LF is

$$\mathcal{L}_{\sigma^2}(\boldsymbol{y} \mid \boldsymbol{\theta}) = -\frac{1}{2}\left[N(T-d)\ln\hat{\sigma}^2 + \sum_{t=1}^{T}\ln|\boldsymbol{F}_t| + \ln|\boldsymbol{S}_d|\right].$$

A further simplification arises for the model considered in this paper, as $\sum_{t=1}^{d}\ln|\boldsymbol{F}_t| = -\ln|\boldsymbol{S}_d|$ and $q_d = 0$ (DeJong and Chu-Chun-Lin 1994, p. 139), so that, assuming without loss of generality $\sigma^2 = 1$, we can rewrite

$$\mathcal{L}(\boldsymbol{\theta}) = -\frac{1}{2}\sum_{t=d+1}^{T} l_t(\boldsymbol{\theta}), \tag{13}$$

where $l_t(\boldsymbol{\theta}) = \ln|\boldsymbol{F}_t| + \boldsymbol{\nu}'_t\boldsymbol{F}_t^{-1}\boldsymbol{\nu}_t$.

6 The LM Test for a Structural Model

Let $\boldsymbol{C}$ be any $m \times n$ matrix functionally related to the $q \times 1$ parameter set $\boldsymbol{\theta}$; following Magnus and Neudecker (1988), we denote by $\mathsf{D}\boldsymbol{C}$ the $mn \times q$ matrix $\partial\text{vec}(\boldsymbol{C})/\partial\boldsymbol{\theta}'$, whereas by $\mathsf{H}\boldsymbol{C}$ we denote the $mnq \times q$ Hessian matrix $\mathsf{D}(\mathsf{D}\boldsymbol{C})' = \partial\text{vec}(\mathsf{D}\boldsymbol{C})/\partial\boldsymbol{\theta}'$. Further, let K_{mn} denote the *commutation matrix*, such that $\mathsf{K}_{mn}\text{vec}\boldsymbol{C} = \text{vec}\boldsymbol{C}'$, $\mathsf{K}_m = \mathsf{K}_{mm}$, and $\mathsf{N}_m = \frac{1}{2}(\boldsymbol{I}_{m^2} + \mathsf{K}_m)$.

The LM test of the restriction H_0: $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ takes the form

$$\text{LM} = \mathsf{D}\mathcal{L}(\boldsymbol{\theta}_0)\mathcal{I}(\boldsymbol{\theta}_0)^{-1}\mathsf{D}\mathcal{L}(\boldsymbol{\theta}_0)',$$

where $\mathsf{D}\mathcal{L}(\boldsymbol{\theta}_0)$ is a $1 \times q$ vector containing the partial derivatives with respect to the parameters evaluated at the null, and $\mathcal{I}(\boldsymbol{\theta}_0)$ is the information matrix evaluated at $\boldsymbol{\theta}_0$, $\mathcal{I}(\boldsymbol{\theta}) = -\text{E}[\mathsf{H}\mathcal{L}(\boldsymbol{\theta}_0)]$.

Now, by straightforward differentiation of Equation 13,

$$\mathsf{D}\mathcal{L}(\boldsymbol{\theta}) = -.5\sum_{t=d+1}^{T}\mathsf{D}l_t(\boldsymbol{\theta}) = -.5\sum_{t=d+1}^{T}\left\{\text{vec}'[\boldsymbol{F}_t^{-1}(\boldsymbol{I}_N - \boldsymbol{\nu}_t\boldsymbol{\nu}'_t\boldsymbol{F}_t^{-1})]\mathsf{D}\boldsymbol{F}_t + 2\boldsymbol{\nu}'_t\boldsymbol{F}_t^{-1}\mathsf{D}\boldsymbol{\nu}_t\right\},$$

and

$$\mathsf{H}\mathcal{L}(\boldsymbol{\theta}) = -.5\sum_{t=d+1}^{T}\mathsf{H}l_t(\boldsymbol{\theta}),$$

where

$$\begin{aligned}
\mathsf{H}l_t(\boldsymbol{\theta}) = {} & -\mathsf{D}\boldsymbol{F}'_t[(\boldsymbol{I}_N - \boldsymbol{F}_t^{-1}\boldsymbol{\nu}_t\boldsymbol{\nu}'_t)\boldsymbol{F}_t^{-1} \otimes \boldsymbol{F}_t^{-1}]\mathsf{D}\boldsymbol{F}_t - \mathsf{D}\boldsymbol{F}'_t(\boldsymbol{F}_t^{-1} \otimes \boldsymbol{F}_t^{-1})2\mathsf{N}_N(\boldsymbol{\nu}_t \otimes \boldsymbol{I}_N)\mathsf{D}\boldsymbol{\nu}_t \\
& + \mathsf{D}\boldsymbol{F}'_t(\boldsymbol{F}_t^{-1} \otimes \boldsymbol{F}_t^{-1}\boldsymbol{\nu}_t\boldsymbol{\nu}'_t\boldsymbol{F}_t^{-1})\mathsf{D}\boldsymbol{F}_t + 2(\boldsymbol{\nu}_t\boldsymbol{F}_t^{-1} \otimes \boldsymbol{I}_q)\mathsf{H}\boldsymbol{\nu}_t \\
& - 2(\boldsymbol{\nu}_t\boldsymbol{F}_t^{-1} \otimes \mathsf{D}\boldsymbol{\nu}'_t\boldsymbol{F}_t^{-1})\mathsf{D}\boldsymbol{F}_t + 2\mathsf{D}\boldsymbol{\nu}'_t\boldsymbol{F}_t^{-1}\mathsf{D}\boldsymbol{\nu}_t,
\end{aligned}$$

$$\begin{aligned}\mathrm{E}(\mathsf{H}l_t(\boldsymbol{\theta})) &= \mathrm{E}[\mathrm{E}(\mathsf{H}l_t(\boldsymbol{\theta}) \mid \boldsymbol{Y}_{t-1})] \\ &= -\mathrm{E}[2\mathsf{D}\boldsymbol{\nu}_t' \boldsymbol{F}_t^{-1}\mathsf{D}\boldsymbol{\nu}_t - \mathsf{D}\boldsymbol{F}_t'(\boldsymbol{F}_t^{-1} \otimes \boldsymbol{F}_t^{-1})\mathsf{D}]\boldsymbol{F}_t.\end{aligned}$$

Dropping the expectation operator yields the following approximation of the information matrix:

$$\hat{\mathcal{I}}(\boldsymbol{\theta}) = .5 \sum_{t=d+1}^{T} \left[2\mathsf{D}\boldsymbol{\nu}_t'\boldsymbol{F}_t^{-1}\mathsf{D}\boldsymbol{\nu}_t - \mathsf{D}\boldsymbol{F}_t'(\boldsymbol{F}_t^{-1} \otimes \boldsymbol{F}_t^{-1})\mathsf{D}\boldsymbol{F}_t\right],$$

which is appealing from the computational standpoint, since it involves only first derivatives.

Analytic derivatives can be obtained by running a set in parallel to the DKF with full collapsing at d: assuming $\boldsymbol{H}_t\boldsymbol{G}_t' = \mathbf{0}$, $\boldsymbol{W}_t = \mathbf{0}$, and $\boldsymbol{X}_t = \mathbf{0}$ for $t = 1, \ldots, T$, and given the matrices $\mathsf{D}\boldsymbol{Z}_t$, $\mathsf{D}\boldsymbol{G}_t$, $\mathsf{D}\boldsymbol{T}_t$, $\mathsf{D}\boldsymbol{H}_t$, $\mathsf{D}\boldsymbol{W}_0$, and $\mathsf{D}\boldsymbol{H}_0$, we have the following algorithm:

- Initialization

$$\begin{aligned}\boldsymbol{A}_1 &= \boldsymbol{W}_0[\boldsymbol{b}, \boldsymbol{B}], & \mathsf{D}\boldsymbol{A}_1 &= [(\boldsymbol{b}, \boldsymbol{B}) \otimes \boldsymbol{I}_m]\mathsf{D}\boldsymbol{W}_0, \\ \boldsymbol{P}_1 &= \boldsymbol{H}_0\boldsymbol{H}_0', & \mathsf{D}\boldsymbol{P}_1 &= 2\mathsf{N}_m(\boldsymbol{H}_0 \otimes \boldsymbol{I}_m)\mathsf{D}\boldsymbol{H}_0, \\ \boldsymbol{Q}_0 &= \mathbf{0}, & \mathsf{D}\boldsymbol{Q}_0 &= \mathbf{0}.\end{aligned}$$

- For $t = 1, \ldots, d$ run the filter

$$\begin{aligned}\boldsymbol{V}_t &= [\boldsymbol{y}_t, \mathbf{0}] - \boldsymbol{Z}_t\boldsymbol{A}_t, \\ \mathsf{D}\boldsymbol{V}_t &= -(\boldsymbol{A}_t \otimes \boldsymbol{I}_N)\mathsf{D}\boldsymbol{Z}_t - (\boldsymbol{I}_{d+1} \otimes \boldsymbol{Z}_t)\mathsf{D}\boldsymbol{A}_t, \\ \boldsymbol{F}_t &= \boldsymbol{Z}_t\boldsymbol{P}_t\boldsymbol{Z}_t' + \boldsymbol{G}_t\boldsymbol{G}_t', \\ \mathsf{D}\boldsymbol{F}_t &= (\boldsymbol{Z}_t \otimes \boldsymbol{Z}_t)\mathsf{D}\boldsymbol{P}_t + 2\mathsf{N}_N(\boldsymbol{Z}_t\boldsymbol{P}_t \otimes \boldsymbol{I}_N)\mathsf{D}\boldsymbol{Z}_t + 2\mathsf{N}_N(\boldsymbol{G}_t \otimes \boldsymbol{I}_N)\mathsf{D}\boldsymbol{G}_t, \\ \boldsymbol{Q}_t &= \boldsymbol{Q}_{t-1} + \boldsymbol{V}_t'\boldsymbol{F}_t^{-1}\boldsymbol{V}_t, \\ \mathsf{D}\boldsymbol{Q}_t &= \mathsf{D}\boldsymbol{Q}_{t-1} - (\boldsymbol{V}_t'\boldsymbol{F}_t^{-1} \otimes \boldsymbol{V}_t'\boldsymbol{F}_t^{-1})\mathsf{D}\boldsymbol{F}_t + 2\mathsf{N}_{d+1}(\boldsymbol{I}_{d+1} \otimes \boldsymbol{V}_t'\boldsymbol{F}_t^{-1})\mathsf{D}\boldsymbol{V}_t, \\ \boldsymbol{K}_t &= \boldsymbol{T}_t\boldsymbol{P}_t\boldsymbol{Z}_t'\boldsymbol{F}_t^{-1}, \\ \mathsf{D}\boldsymbol{K}_t &= (\boldsymbol{F}_t^{-1}\boldsymbol{Z}_t\boldsymbol{P}_t \otimes \boldsymbol{I}_m)\mathsf{D}\boldsymbol{T}_t + (\boldsymbol{F}_t^{-1}\boldsymbol{Z}_t \otimes \boldsymbol{T}_t)\mathsf{D}\boldsymbol{P}_t \\ &\quad + (\boldsymbol{F}_t^{-1} \otimes \boldsymbol{T}_t\boldsymbol{P}_t)\mathsf{K}_{Nm}\mathsf{D}\boldsymbol{Z}_t - (\boldsymbol{F}_t^{-1} \otimes \boldsymbol{T}_t\boldsymbol{P}_t\boldsymbol{Z}_t'\boldsymbol{F}_t^{-1})\mathsf{D}\boldsymbol{F}_t, \\ \mathsf{D}\boldsymbol{A}_{t+1} &= (\boldsymbol{I}_{d+1} \otimes \boldsymbol{T}_t)\mathsf{D}\boldsymbol{A}_t + (\boldsymbol{A}_t \otimes \boldsymbol{I}_m)\mathsf{D}\boldsymbol{T}_t + (\boldsymbol{I}_{d+1} \otimes \boldsymbol{K}_t)\mathsf{D}\boldsymbol{V}_t + (\boldsymbol{V}_t' \otimes \boldsymbol{I}_m)\mathsf{D}\boldsymbol{K}_t, \\ \boldsymbol{A}_{t+1} &= \boldsymbol{T}_t\boldsymbol{A}_t + \boldsymbol{K}_t\boldsymbol{V}_t, \\ \boldsymbol{L}_t &= \boldsymbol{T}_t - \boldsymbol{K}_t\boldsymbol{Z}_t, \\ \mathsf{D}\boldsymbol{L}_t &= \mathsf{D}\boldsymbol{T}_t - (\boldsymbol{Z}_t' \otimes \boldsymbol{I}_m)\mathsf{D}\boldsymbol{K}_t - (\boldsymbol{I}_m \otimes \boldsymbol{K}_t)\mathsf{D}\boldsymbol{Z}_t, \\ \mathsf{D}\boldsymbol{P}_{t+1} &= (\boldsymbol{L}_t \otimes \boldsymbol{T}_t)\mathsf{D}\boldsymbol{P}_t + (\boldsymbol{L}_t\boldsymbol{P}_t \otimes \boldsymbol{I}_m)\mathsf{D}\boldsymbol{T}_t + (\boldsymbol{I}_m \otimes \boldsymbol{T}_t\boldsymbol{P}_t)\mathsf{K}_m\mathsf{D}\boldsymbol{L}_t + 2\mathsf{N}_m(\boldsymbol{H}_t \otimes \boldsymbol{I}_m)\mathsf{D}\boldsymbol{H}_t, \\ \boldsymbol{P}_{t+1} &= \boldsymbol{T}_t\boldsymbol{P}_t\boldsymbol{L}_t' + \boldsymbol{H}_t\boldsymbol{H}_t'.\end{aligned} \tag{14}$$

- Fully collapse the DKF filter at $t = d$

$$\begin{aligned}\hat{\boldsymbol{\delta}}_d &= -\boldsymbol{S}_d^{-1}\boldsymbol{s}_d, \\ \mathsf{D}\hat{\boldsymbol{\delta}}_d &= -\boldsymbol{S}_d^{-1}\mathsf{D}\boldsymbol{s}_d + (\boldsymbol{s}_d'\boldsymbol{S}_d^{-1} \otimes \boldsymbol{S}_d^{-1})\mathsf{D}\boldsymbol{S}_d, \\ \boldsymbol{P}_{d+1} &= \boldsymbol{P}_{d+1} + \boldsymbol{A}_{d+1}^{\dagger}\boldsymbol{S}_d^{-1}\boldsymbol{A}_{d+1}^{\dagger\prime}, \\ \mathsf{D}\boldsymbol{P}_{d+1} &= \mathsf{D}\boldsymbol{P}_{d+1} - (\boldsymbol{A}_{d+1}^{\dagger}\boldsymbol{S}_d^{-1} \otimes \boldsymbol{A}_{d+1}^{\dagger}\boldsymbol{S}_d^{-1})\mathsf{D}\boldsymbol{S}_d + 2\mathsf{N}_m(\boldsymbol{A}_{d+1}^{\dagger}\boldsymbol{S}_d^{-1} \otimes \boldsymbol{I}_m)\mathsf{D}\boldsymbol{A}_{d+1}^{\dagger}, \\ \mathsf{D}\boldsymbol{a}_{d+1} &= \boldsymbol{A}_{d+1}\begin{bmatrix}\mathbf{0} \\ \mathsf{D}\hat{\boldsymbol{\delta}}_d\end{bmatrix} + \left([1, \hat{\boldsymbol{\delta}}_d'] \otimes \boldsymbol{I}_m\right)\mathsf{D}\boldsymbol{A}_{d+1},\end{aligned}$$

$$\begin{aligned} \boldsymbol{a}_{d+1} &= \boldsymbol{A}_{d+1}[1, \hat{\boldsymbol{\delta}}_d']', \\ q_d &= q_{1d} - \boldsymbol{s}_d' \boldsymbol{S}_d^{-1} \boldsymbol{s}_d, \\ \mathsf{D}q_d &= \mathsf{D}q_{1d} + \boldsymbol{s}_d' \mathsf{D}\hat{\boldsymbol{\delta}}_d + \hat{\boldsymbol{\delta}}_d \mathsf{D}\boldsymbol{s}_d, \end{aligned}$$

where $\mathsf{D}\boldsymbol{s}_d = (\boldsymbol{R} \otimes \boldsymbol{C})\mathsf{D}\boldsymbol{Q}_d$, $\mathsf{D}\boldsymbol{S}_d = (\boldsymbol{C} \otimes \boldsymbol{C})\mathsf{D}\boldsymbol{Q}_d$, $\mathsf{D}q_{1d} = (\boldsymbol{R} \otimes \boldsymbol{R})\mathsf{D}\boldsymbol{Q}_d$, and $\mathsf{D}\boldsymbol{A}_{d+1}^{\dagger} = (\boldsymbol{I}_d \otimes \boldsymbol{C})\mathsf{D}\boldsymbol{A}_{d+1}$, with $\boldsymbol{R} = [1, \boldsymbol{0}]$, and $\boldsymbol{C} = [\boldsymbol{0}, \boldsymbol{I}_d]$.

- Run the KF filter for $t = d+1, \ldots, T$, dropping the recursion for $\mathsf{D}\boldsymbol{Q}_t$ and replacing the recursions for $\boldsymbol{V}_t$, $\mathsf{D}\boldsymbol{V}_t$, $\boldsymbol{Q}_t$, $\mathsf{D}\boldsymbol{A}_{t+1}$, and $\boldsymbol{A}_{t+1}$, respectively, by

$$\begin{aligned} \boldsymbol{\nu}_t &= \boldsymbol{y}_t - \boldsymbol{Z}_t \boldsymbol{a}_t, \\ \mathsf{D}\boldsymbol{\nu}_t &= -(\boldsymbol{a}_t \otimes \boldsymbol{I}_N)\mathsf{D}\boldsymbol{Z}_t - \boldsymbol{Z}_t \mathsf{D}\boldsymbol{A}_t, \\ q_t &= q_{t-1} + \boldsymbol{\nu}_t' \boldsymbol{F}_t^{-1} \boldsymbol{\nu}_t, \\ \mathsf{D}\boldsymbol{a}_{t+1} &= \boldsymbol{T}_t \mathsf{D}\boldsymbol{a}_t + (\boldsymbol{a}_t \otimes \boldsymbol{I}_m)\mathsf{D}\boldsymbol{T}_t + \boldsymbol{K}_t \mathsf{D}\boldsymbol{\nu}_t + (\boldsymbol{\nu}_t' \otimes \boldsymbol{I}_m)\mathsf{D}\boldsymbol{K}_t, \\ \boldsymbol{a}_{t+1} &= \boldsymbol{T}_t \boldsymbol{a}_t + \boldsymbol{K}_t \boldsymbol{\nu}_t. \end{aligned} \quad (15)$$

Moreover, compute and accumulate

$$\mathsf{D}l_t(\boldsymbol{\theta}) = \mathrm{vec}'[\boldsymbol{F}_t^{-1}(\boldsymbol{I}_N - \boldsymbol{\nu}_t \boldsymbol{\nu}_t' \boldsymbol{F}_t^{-1})]\mathsf{D}\boldsymbol{F}_t + 2\boldsymbol{\nu}_t' \boldsymbol{F}_t^{-1} \mathsf{D}\boldsymbol{\nu}_t$$

and

$$2\mathsf{D}\boldsymbol{\nu}_t' \boldsymbol{F}_t^{-1} \mathsf{D}\boldsymbol{\nu}_t - \mathsf{D}\boldsymbol{F}_t'(\boldsymbol{F}_t^{-1} \otimes \boldsymbol{F}_t^{-1})\mathsf{D}\boldsymbol{F}_t.$$

- Compute the LM test.

7 The LM Test for Linearity in the Business Cycle

Let us turn back now to the issue of testing linearity of the cyclical component within a structural model for quarterly economic time series displaying trends and seasonals. The null is formulated as H_0: $\tau = 0$ against H_1: $\tau > 0$ in Equations 6–9. As with STAR models, the problem is nonstandard, since r_2 and $\bar{\lambda}_2$ are not identified under the null; however, we follow the strategy of replacing $F(Y_{t-1})$ by its first Taylor-series expansion around $\tau = 0$. This amounts to reparameterizing the alternative model in such a way that the identification problem disappears. For the LgstTrM I model, we get $r_t \approx r_1^* + (\hat{\psi}_t - \tilde{\psi}_{t-1|t-1}) r_2^*$ and $\bar{\lambda}_t \approx \bar{\lambda}_1^* + (\hat{\psi}_t - \tilde{\psi}_{t-1|t-1})\bar{\lambda}_2^*$. Hence, we reformulate the testing problem as follows: H_0: $r_2^* = 0, \bar{\lambda}_2^* = 0$ versus H_1: $r_2^* \neq 0, \bar{\lambda}_2^* \neq 0$. Note that we do not need higher-order terms in the Taylor expansion.

The LM test of this hypothesis can be implemented according to the general theory set forth in the previous section: if we use the parameterization $\sigma_a^2 = \exp(2\theta_a)$ for a generic variance parameter, $\boldsymbol{\theta} = [\theta_\epsilon, \theta_\eta, \theta_\zeta, \theta_\omega, \theta_\psi, r_1^*, r_2^*, \bar{\lambda}_1^*, \bar{\lambda}_2]$, so that under the null, $\boldsymbol{\theta}_0 = [\theta_\epsilon, \theta_\eta, \theta_\zeta, \theta_\omega, \theta_\psi, r_1^*, 0, \bar{\lambda}_1^*, 0]$.

The first step is to compute the derivatives of the system matrices with respect to $\boldsymbol{\theta}$ and evaluate them at $\boldsymbol{\theta}_0$: the only system matrices depending on r_2^* and $\bar{\lambda}_2^*$ are $\boldsymbol{T}_t$ and $\boldsymbol{H}_t$. For instance, by the chain rule,

$$\frac{\partial \boldsymbol{T}_\psi}{\partial \bar{\lambda}_2^*} = \rho_t \begin{bmatrix} -\sin\lambda_t & \cos\lambda_t \\ -\cos\lambda_t & -\sin\lambda_t \end{bmatrix} \frac{\partial \lambda_t}{\partial \bar{\lambda}_2^*} (\hat{\psi}_t - \tilde{\psi}_{t-1|t-1}),$$

where $\partial\lambda_t/\partial\bar{\lambda}_2^* = -2\pi \exp\bar{\lambda}_2^*(2 + \exp\bar{\lambda}_2^*)^{-2}$. Under the null, $\rho_t = |r_1^*|/\sqrt{1 + r_1^{*2}}$, $\lambda_t = 2\pi/(2 + \exp\bar{\lambda}_1^*)$, and $\partial\lambda_t/\partial\bar{\lambda}_2^* = -2\pi/9$, and so forth. Then, in running the algorithm described in the previous section, for ease of computation we cast $\hat{\psi}_t - \tilde{\psi}_{t-1|t-1} = 0$ for $t < d$, and start computing the transition variable after processing d observations. This avoids the calculation of diffuse predictions by the DKF, and amounts to keeping the values of ρ_t and λ_t fixed for the first d observations. To get the updated estimate of the state vector, $\boldsymbol{a}_{t|t}$, we use the KF-updating equation: $\boldsymbol{a}_{t|t} = \boldsymbol{a}_t + \boldsymbol{P}_t \boldsymbol{Z}_t' \boldsymbol{F}_t^{-1} \boldsymbol{\nu}_t$.

In the ExpTrM I case, the first Taylor-series expansion gives, with reference to the damping factor, $r_t \approx r_1^* + (\hat{\psi}_t^2 + \mathrm{Var}(\hat{\psi}_t \mid Y_{t-1})) r_2^*$, but since the null model is time-invariant, detectable, and stabilizable, $\boldsymbol{P}_t$ converges to a steady-state matrix and $\mathrm{Var}(\hat{\psi}_t \mid Y_{t-1})$ is constant, so we can equivalently perform the LM test using the approximation $r_t \approx r_1^* + \hat{\psi}_t^2 r_2^*$.

Table 1
U.S. index of industrial production: Lagrange multiplier linearity tests

	LgstTrM I		LgstTrM II		ExpTrM I		ExpTrM II	
	LM Test	*p*-Value	LM Test	*p*-Value	LM Test	*p*-Value	LM Test	*p*-Value
Apparel	7.2336	0.0269	12.6094	0.0018	4.6068	0.0999	25.8565	0.0000
Chemicals	1.4702	0.4795	13.4625	0.0012	5.0586	0.0797	1.4551	0.4831
Electric Machinery	17.1487	0.0002	4.8054	0.0905	16.1368	0.0003	38.5147	0.0000
Fabricated Metal	1.6252	0.4437	2.5370	0.2813	0.4212	0.8101	1.5856	0.4526
Furniture	7.2874	0.0262	4.8439	0.0887	1.2190	0.5436	2.5440	0.2803
Instruments	0.8096	0.6671	1.1986	0.5492	7.7708	0.0205	13.0270	0.0015
Leather	2.6553	0.2651	6.1942	0.0452	2.6780	0.2621	2.1872	0.3350
Lumber	0.4569	0.7958	2.9558	0.2281	0.0768	0.9623	3.3447	0.1878
Machinery	6.4962	0.0388	15.1307	0.0005	2.7320	0.2551	1.1042	0.5758
Other	3.9933	0.1358	2.0185	0.3645	3.5332	0.1709	6.0902	0.0476
Paper	5.4632	0.0651	5.3094	0.0703	18.4641	0.0001	6.3231	0.0424
Primary Metal	9.0915	0.0106	7.2190	0.0271	54.8560	0.0000	0.0728	0.9643
Printing	5.7123	0.0575	7.5271	0.0232	1.5789	0.4541	1.1113	0.5737
Rubber	1.6472	0.4388	3.9062	0.1418	1.9689	0.3736	4.0031	0.1351
Stone, etc.	3.4459	0.1785	8.5128	0.0142	10.1723	0.0062	1.3601	0.5066
Textiles	2.0573	0.3575	20.1721	0.0000	17.0789	0.0002	5.8021	0.0550
Durables	0.2799	0.8694	4.6396	0.0983	2.6243	0.2692	11.7624	0.0028
Nondurables	1.7064	0.4261	8.6521	0.0132	7.8959	0.0193	5.6925	0.0581
Total	1.5083	0.4704	4.6568	0.0975	0.8164	0.6648	11.7089	0.0029
Total (61.1–96.4)	19.9807	0.0000	2.5659	0.2772	0.8281	0.6610	10.5917	0.0050

8 Nonlinearity in U.S. Industrial Production Series

To illustrate the issue of characterizing asymmetries in the business cycle by ST^2 models, we consider a set of quarterly U.S. industrial production series for two-digit manufacturing industries, along with the total, durables, and nondurables series, for a number of 23 series altogether. This set, which was recently studied by Miron (1996) from a different perspective, is made available electronically by the Federal Reserve Board on the World Wide Web at the URL `www.bog.frb.fed.us`, and covers the period 1947.1–1996.4.

Table 1 reports the results of LM testing for linearity versus four alternative models,[1] specifying ρ_t and λ_t as in Equations 4 and 5 with LgstTrM and transition variable $\hat{\psi}_t - \tilde{\psi}_{t-1|t-1}$ (LgstTrM I), with logistic transition mechanism and transition variable $\hat{\psi}_t$ (LgstTrM II), with exponential transition mechanism and transition variable $\hat{\psi}_t^2$ (ExpTrM I), with exponential transition mechanism and transition variable $(\hat{\psi}_t - \tilde{\psi}_{t-1|t-1})^2$ (ExpTrM II). The tests were not performed on the Food, Petroleum, Tobacco, and Transportation Equipment series, since under the null model we could not extract a cyclical component.

The evidence for nonlinearity, although not overwhelming, is fairly strong: type-I asymmetry seems to characterize the Electrical Machinery and the Total series. Note that for the latter, the test conducted on the full sample is not significant, whereas it is highly so when the range of observations is restricted to 1961.1–1996.4, which is an extension of the series considered by Teräsvirta and Anderson (1992). The test against LgstTrM I is also significant at the 5% level for the Apparel, Furniture, Machinery, and Primary Metal series. The presence of type-II asymmetries is detected by the test corresponding to LgstTrM II, which turns out to be highly significant for the Apparel, Chemicals, Machinery, and Textiles series. Amplitude dependence (ExpTrM I) is detected for the Apparel, Electric Machinery, Instruments, Durables, and Total series. Finally, evidence for asymmetric behavior relating to the strength of contractions and expansions and hence for the presence of more than two phases, is fairly strong for the Electric Machinery, Paper, Primary Metal, Stone, and Textiles series.

It should be noticed that in some cases (e.g., the Apparel series), the LM test is significant for a variety of alternative specifications. This poses the question of which transition mechanism provides the best explanation of the data. One of the referees suggested that a sequence of tests could be implemented for model selection, as set out by Teräsvirta (1994, sect. 4.3).

As a matter of fact, the first-order Taylor approximations of the ST^2 models considered in the paper are

[1] All computations were carried out in GAUSS, and the programs can be made available upon request.

nested within the following specification:

$$r_t = r_1^* + \hat{\psi}_t r_2^* + \tilde{\psi}_{t-1|t-1} r_3^* + \hat{\psi}_t^2 r_4^* + \tilde{\psi}_{t-1|t-1}^2 r_5^* + \hat{\psi}_t \tilde{\psi}_{t-1|t-1} r_6^*,$$
$$\bar{\lambda}_t = \bar{\lambda}_1^* + \hat{\psi}_t \bar{\lambda}_2^* + \tilde{\psi}_{t-1|t-1} \bar{\lambda}_3^* + \hat{\psi}_t^2 \bar{\lambda}_4^* + \tilde{\psi}_{t-1|t-1}^2 \bar{\lambda}_5^* + \hat{\psi}_t \tilde{\psi}_{t-1|t-1} \bar{\lambda}_6^*.$$

This suggests that model selection in an ST^2 framework, with r_t and $\bar{\lambda}_t$ parameterized as above, can be performed by running a series of Wald tests of the following linear restrictions:

$$H_{0,L.I}:\ r_2^* = r_3^*, r_4^* = r_5^* = r_6^* = 0; \bar{\lambda}_2^* = -\bar{\lambda}_3^*, \bar{\lambda}_4^* = \bar{\lambda}_5^* = \bar{\lambda}_6^* = 0,$$
$$H_{0,L.II}:\ r_3^* = r_4^* = r_5^* = r_6^* = 0; \bar{\lambda}_3^* = \bar{\lambda}_4^* = \bar{\lambda}_5^* = \bar{\lambda}_6^* = 0,$$
$$H_{0,E.I}:\ r_2^* = r_3^* = r_5^* = r_6^* = 0; \bar{\lambda}_2^* = \bar{\lambda}_3^* = \bar{\lambda}_5^* = \bar{\lambda}_6^* = 0,$$
$$H_{0,E.II}:\ r_2^* = r_3^* = 0, r_4^* = r_5^* = -.5r_6^*; \bar{\lambda}_2^* = \bar{\lambda}_3^* = 0, \bar{\lambda}_4^* = \bar{\lambda}_5^* = -.5\bar{\lambda}_6^*,$$

so that the null model is a first-order Taylor approximation of the LgstTrM I, LgstTrM II, ExpTrM I, and ExpTrM II models, respectively, at $\tau = 0$.

Now, in deciding between LSTAR and ESTAR in a smooth-transition autoregressive framework, a sequence of F-tests is performed on the coefficients of an auxiliary regression model, which renders the procedure easy to implement. As far as ST^2 models are concerned, the above sequence is more effective in discriminating the models entertained by the null, but is also more computationally demanding, as it asks for estimation of the unrestricted model, which has 10 extra parameters ($r_2^*, \ldots, r_6^*, \bar{\lambda}_2^*, \ldots, \bar{\lambda}_6^*$) with respect to the linear model.

Comparing the relative strength of the rejections and selecting the specification that provides the greatest LM test (or minimum p-value) yields a much simpler rule, and this is what I tentatively suggest. Some experimentation is needed to judge the effectiveness of this strategy. Future research will pursue this point further.

We next illustrate the results of fitting ST^2 models (Equation 1) respectively with LgstTrM I for the Total series, 1961.1–1996.4, and with LgstTrM II for the Textiles series. Table 2, which reports the parameter estimates along with diagnostic and goodness-of-fit statistics for both linear and ST^2 models, shows that the performance of the latter is indeed satisfactory. Apart from residual correlation, the ST^2 model for the Total series substantially improves the fit, noticeably for residual skewness and kurtosis; for the Textiles series, the evidence is less clear cut. Also notice that the value of the estimated τ is in both cases seemingly high; actually, this comes as no surprise, since its size depends on the scale of the transition variable.

Figure 1 is helpful in interpreting the kind of asymmetry that is captured by the smooth-transition model. Along with the logarithms of the Total series and its seasonal differences, the time pattern of the estimated ρ_t and λ_t are displayed, where the dotted line in the background is the suitably rescaled cyclical component. It should be noticed that lower values of ρ_t and higher values of λ_t correspond to a recessionary pattern. Hence recessions are stronger but less persistent than expansions, since the period of the oscillation becomes relatively short (compared to expansions) and the damping is relatively low (i.e., the successive realizations are comparatively less correlated). On the contrary, expansions are characterized by a higher period and stronger autocorrelation.

Figure 2 compares the smoothed cyclical component extracted by the ST^2 model with that extracted by the linear model. It is interesting to notice that there is a close correspondence in the location of the turning points highlighted by the two series, although there is some difference in the dynamics from troughs to peaks and vice versa. Also, the former emphasizes the depth of the recessions at the end of the sample period.

As far as the Textiles series is concerned (Figure 3), both ρ_t and λ_t move sharply across two regimes: in the neighborhood of a trough, the damping factor moves down and the frequency rises upward, as a result of type-II asymmetry. Hence, troughs are deeper than peaks, and this behavior corresponds to fluctuations that have a lower period and damping in the vicinity of a trough, and a higher period and damping when the cycle peaks. Finally, Figure 4 displays the nonlinear and the linear smoothed cyclical component.

9 Conclusions

This paper has aimed at detecting and modeling the asymmetric features of business-cycle fluctuations within a structural framework. We have advocated that this approach is especially tailored for the problem at hand,

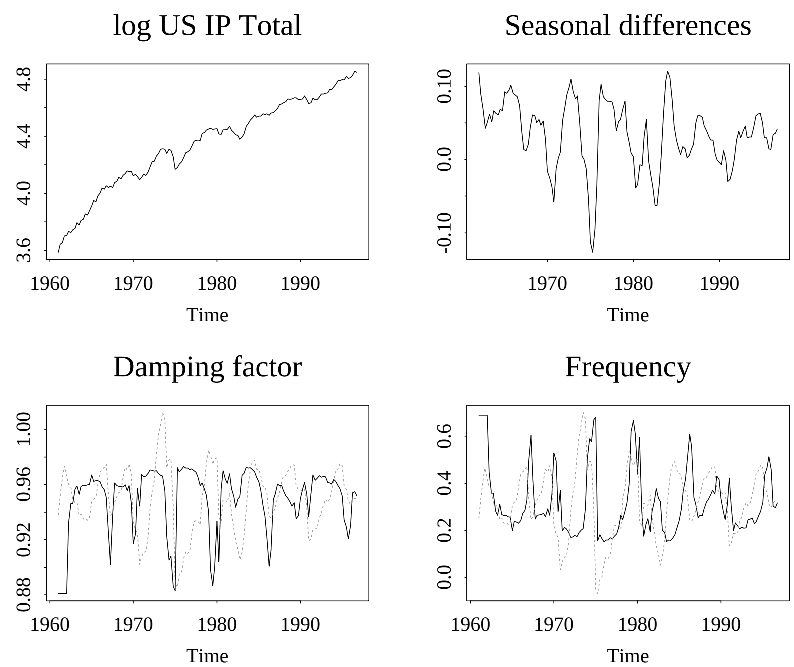

Figure 1
U.S. industrial production: Total, 1961.1–1996.4.

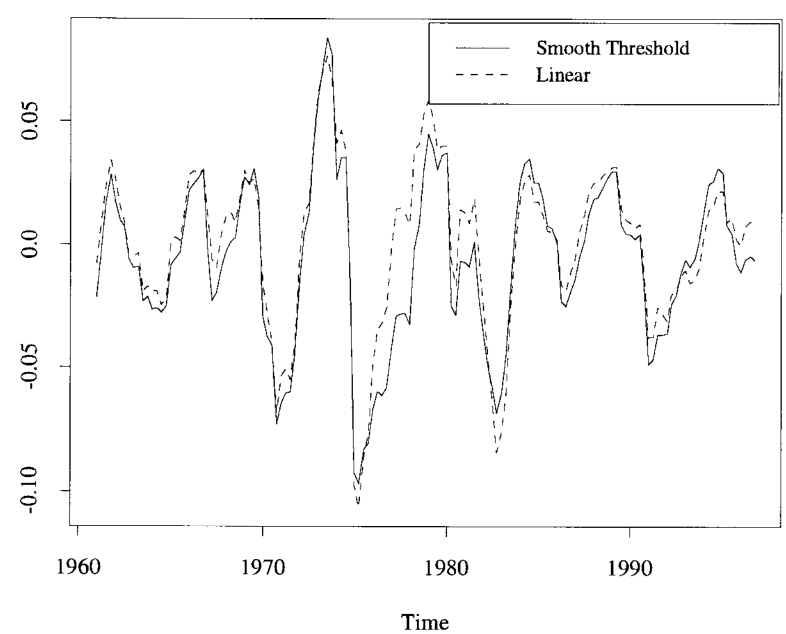

Figure 2
Cyclical component in U.S. industrial production: Total, 1961.1–1996.4.

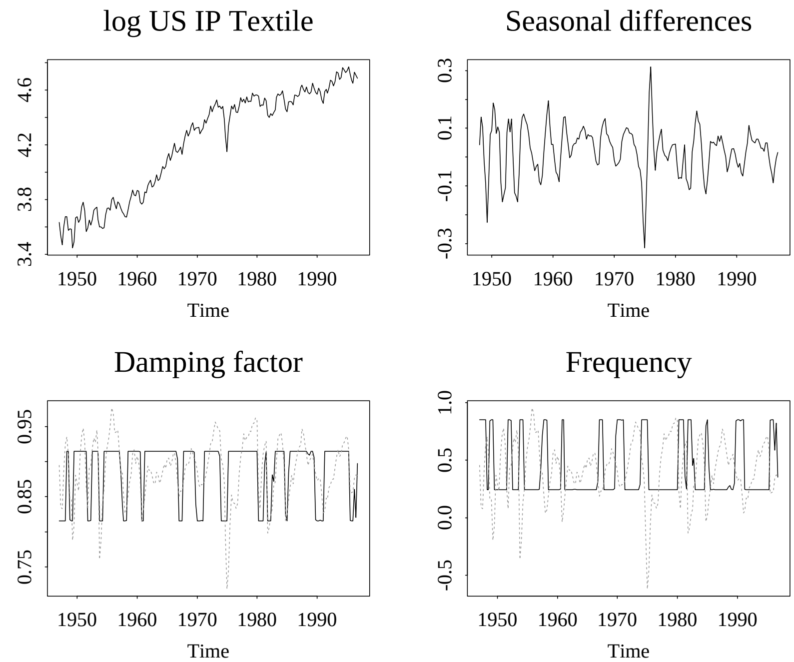

Figure 3
U.S. industrial production: Textiles, 1947.1–1996.4.

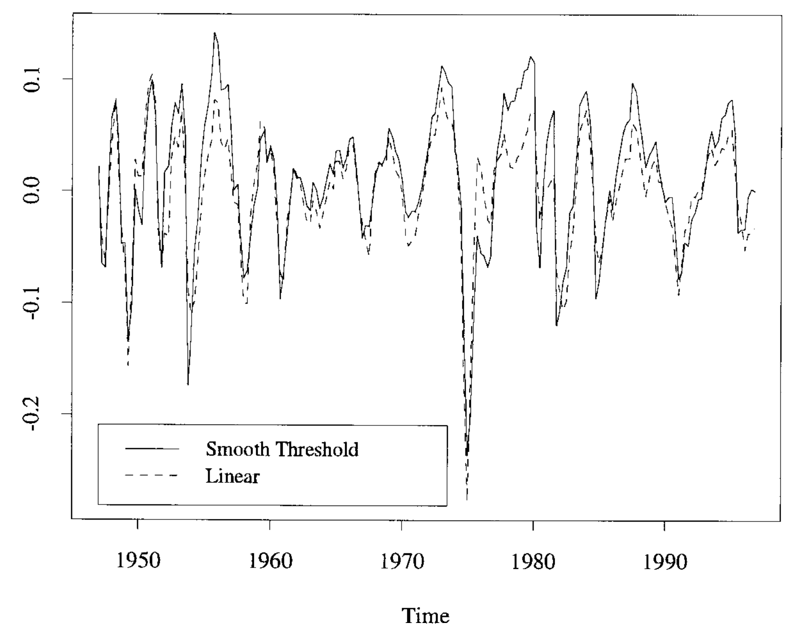

Figure 4
U.S. industrial production: Textiles, 1947.1–1996.4.

Table 2
U.S. index of industrial production: Parameter estimates, diagnostics, and goodness of fit for Total and Textiles series[a]

	Total		*Textiles*	
	Linear	LgstTrM I	Linear	LgstTrM II
σ_η^2	0	0	0	0
σ_ζ^2	18	45	13	12
σ_ω^2	4	5	33	16
σ_ψ^2	11988	12865	29359	34388
ρ	0.932		0.860	
λ	0.297		0.469	
τ		145		588
r_1		1.860		1.410
r_2		2.746		0.855
$\bar{\lambda}_1$		1.960		1.686
$\bar{\lambda}_2$		2.019		1.480
$\mathcal{L}$	559.451	568.961	629.004	648.522
N_1	23.046	5.028	20.693	33.916
N_2	53.944	3.418	113.980	84.785
N	76.990	8.446	134.673	118.701
$Q(12)$	19.800	26.234	21.025	18.102
pev	2835	2640	14390	15424
R_s^2	0.283	0.373	0.306	0.275
AIC	−8.046	−8.053	−6.453	−6.102

[a] Variance parameters and pev are multiplied by 10^7. Standard errors are not reported, since they are not meaningful in this context. $\mathcal{L}$ is the value of the natural logarithm of the maximized-likelihood function; N_1 is a test for residual skewness, based on the standardized third moment of the residuals about the mean (Harvey 1989, sect. 5.4.2.); N_2 is a test for residual kurtosis; and $N = N_1 + N_2$ is the Bowman and Shenton test for non-normality. $Q(12)$ is the Ljung-Box statistic based on 12 residual autocorrelations. pev is the prediction-error variance; and $R_s^2 = 1 - SSE/SSDSM$, where $SSE = (T - d)pev$, and $SSDSM$ is the sum of squares of first differences around the seasonal means. The Akaike information criterion in the last column is computed as AIC$= \ln(pev) + [2(m + d)/T]$, where m is the number of hyperparameters, d is the number of diffuse components, and T is the number of observations.

since by letting "the cycle speak for itself" and using a suitable transition mechanism, it is possible to keep track of the nature of the asymmetry under investigation, thereby enhancing model interpretation.

Linearity in the cyclical component has been tested against well-specified alternatives for a set of U.S. industrial production series, and it was concluded that various types of asymmetries are detectable. Also, the results of fitting structural models with smooth transition in the persistence and the period of the oscillations were satisfactory. An interesting topic for future research is the evaluation of the forecast performance of ST^2 models, in comparison with that provided by the linear model.

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Editorial Policy

The *SNDE* is formed in recognition that advances in statistics and dynamical systems theory may increase our understanding of economic and financial markets. The journal will seek both theoretical and applied papers that characterize and motivate nonlinear phenomena. Researchers will be encouraged to assist replication of empirical results by providing copies of data and programs online. Algorithms and rapid communications will also be published.

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