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Wavelet Analysis of the Cost-of-Carry Model

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Abstract. *In this paper, it is shown how one can employ the wavelet analysis to reconstruct data based only on the subset of information that differentiates the two fundamentally related time series: spot and futures indices. Such an analysis allows researchers to focus on examining the relationship between the two price series. Furthermore, it also enables examination and comparison of reconstructed prices based on different levels of information detail. It is found that the lead-lag relationship described in the empirical literature still exists between the spot and the futures index prices. Such a relationship is more persistent when more detailed information is used for price reconstruction. This implies that, if market imperfection is to be blamed for the noncontemporaneous relationship between the spot and the futures indices, one should concentrate solely on those imperfections that are likely to occur within very short time horizons.*

Keywords. wavelet analysis, cost-of-carry model

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1 Introduction

Prices in spot and futures markets are contemporaneously related according to the famous cost-of-carry (COC) model in the theory of finance. A primary assumption of the COC model is that the two markets are perfectly efficient and frictionless and act as perfect substitutes. As new information arrives simultaneously to both markets and is reflected immediately in both the spot and the futures prices, profitable arbitrage should not exist. Consequently, the theoretical COC model suggests that the "fair value" of a futures contract is equal to the "fair value" of the underlying spot index, plus the cost of carrying the spot index for the duration of the contract (see Cornell and French 1983). This model is widely used by traders to determine whether futures are correctly priced.

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In the empirical literature, however, numerous studies have established the existence of a lead-lag relationship between price changes in most international futures and spot markets. Namely, futures index price changes are usually found to lead spot index price changes by up to 45 min. On the other hand, when information is firm specific, the underlying index price may lead the futures index price. These studies include Kawaller, Koch, and Koch 1987, Stoll and Whaley 1990, Harris 1989, Wahab and Lashgari 1993, and Tse 1995, to name a few. It has been argued that persistence in the lead-lag relationship between the spot index and the futures index prices may be caused by one or more market imperfections, such as transaction costs, liquidity differences between the two markets, nonsynchronous trading effects, the automation of one or the other market, short-selling restrictions, different taxation regimes on futures and stocks, dividend uncertainties, the market-to-market cash flows, and nonstochastic interest rates. Interestingly, one common feature of most of the above empirical studies is that they use *price changes* (or equivalently *returns*) rather than *prices* in the empirical studies. Although the relationship between the spot and the futures market returns is itself an interesting topic to investigate, it is worth noting that it is *prices*, rather than *price changes*, that are the variables of interest in the COC model. Therefore, instead of trying to identify whether or not any form of market imperfection contributes to the documented lead-lag relationship among price changes, and if so, by how much, the focus of this paper is to offer an approach that would allow one to examine price series directly.

Since the futures index is a derivative security of the spot market index, it is safe to say that both are subject to the same impact from changes in market fundamentals. Also, as a consequence, it is very likely that these two time series are highly co-integrated. The main thrust of this paper is to focus on the information subset that differentiates the two markets in reconstructing the two price series. By doing so, we need not take first differences on the price series and would be able to maintain discussion on the variables of interest, prices, as suggested in the COC model. Wavelet analysis provides an ideal tool for the kind of question raised here. Specifically, by examining data at different time locations and levels of resolution, wavelet analysis enables data reconstruction based only on the subset of information that is of interest to researchers. Such an analysis not only allows a focus on examining prices, but also enables examination and comparison of reconstructed prices based on different levels of information detail.

In the past decade or so, wavelet analysis has been a rapidly evolving topic in many scientific fields, such as mathematics, quantum physics, electrical engineering, seismology, and signal processing. Wavelet analysis is a new development, however, in the area of computational economics, econometrics, and finance. To the best of our knowledge, applications in these fields include outlier detection (Greenblatt 1995), processing of nonstationary data (Goffe 1994), study of the statistical self-similarity of financial data (Ramsey, Zaslavsky, and Usikov 1995), discussion of potential difficulties involved in shifting from deterministic models to stochastic processes in the analysis of economic and financial data (Priestly 1996), examination of the time-frequency (Wigner) distributions (Ramsey and Zhang 1997), examination of the relationship among key macroeconomic variables (Ramsey and Lampart 1998), analysis of commodity price behavior (Davidson, Labys, and Lesourd 1998), investigation of the potential stochastic relationship between the S&P 500 index price and S&P dividend yields in a state space framework (Pan and Wang 1998), and estimation of long-memory processes (Jensen 1999a, 1999b, 2000). In addition, Ramsey (1999) reviews numerous applications of the wavelet technique in the economics and finance literature and suggests possible future research potential in these fields. It is worth pointing out that Davidson, Labys, and Lesourd (1998) have contributed to this line of research by placing wavelet analysis in a semi-nonparametric regression framework. Not only does doing this allow one to easily handle some inherent restrictions in wavelet analysis, but it also allows one to utilize the well-established statistical properties of semi-nonparametric regression to assess output from such an analysis. It is the algorithm of Davidson, Labys, and Lesourd that we use in conducting wavelet analysis in this paper.

The rest of the paper is organized as follows. In Section 2, an introduction is given to wavelets and wavelet analysis. In particular, we demonstrate how one could use the algorithm of Davidson, Labys, and Lesourd

(1998) to construct information-based prices. In Section 3, data is presented and used to reconstruct both the spot and the futures indices based on information existing at different time horizons according to the wavelet analysis introduced in Section 2. In Section 4, a vector autoregression (VAR) model is set up to examine the relationship between the wavelet-reconstructed prices. It is found that the lead-lag relationship still exists between spot and futures index prices. Such a relationship is more pronounced and persistent when more detailed information is used for reconstructing prices. The implication of this result is that, if market imperfection is to be blamed for the noncontemporaneous relationship between index prices from the two markets, one should concentrate solely on those imperfections that are likely to occur at very short time horizons. Section 5 incorporates a discussion of the empirical implications and the conclusions of the paper.

2 Wavelets and Wavelet Analysis

A wavelet is a "small wave" that has its energy concentrated in a short interval of time.[1] Wavelet analysis allows researchers to decompose signals into a parsimoniously countable set of basis functions at different time locations and resolution levels. Because of the compact support property of wavelets, wavelet analysis is capable of capturing short-lived, transient components of data in shorter time intervals, as well as capturing trends and patterns in longer time intervals. Consequently, wavelet analysis is ideal for analyzing nonstationary data, which certainly include most economic and financial time series. The following gives a brief account of wavelets and wavelet analysis and an introduction to the algorithm of Davidson, Labys, and Lesourd (1998). How one can apply this algorithm in reconstructing price series based on a subset of information that is of interest to researchers is then described.

2.1 Wavelet functions

The building blocks of wavelet analysis consist of father wavelets (often called scaling functions in the wavelet literature) and mother wavelets. Similar to sine and cosine functions in spectral analysis, wavelet functions are functional bases onto which data are projected to extract information that is not available in the time domain. Unlike sine and cosine functions, however, wavelet functions usually do not have a closed functional form. Instead, the basis functions are usually numerically derived based on some desired mathematical properties and characteristics imposed by researchers.

Consider the square-integrable functional space, $L^2(\Re)$ (i.e., $\int |f(t)|^2 dt < \infty, \forall f(t) \in L^2(\Re)$). Based on the basis function $\phi(t)$, the father wavelets, $\phi_k(t) \equiv \phi(t-k)$, are a set of functions that span the subspace V_0 of $L^2(\Re)$, that is,

$$V_0 = \overline{\underset{k}{\text{Span}} \ \{\phi_k(t)\}} \tag{2.1}$$

where the subscript $k \in \boldsymbol{Z}$ is called the *translation parameter* and is associated with a shift in the time location t. This means that any function in the V_0 subspace can be represented as a linear combination of the father wavelets

$$f(t) = \sum_k \alpha_k \phi_k(t), \ \forall f(t) \in V_0 \tag{2.2}$$

To make these father wavelet functions useful, it is required that if a set of signals can be represented by a weighted sum of $\phi(t-k)$, then a larger set of signals can be represented by a weighted sum of $\phi(2t-k)$. This is usually called the *multiresolution condition*. To show this, we define the scaled father wavelet function as the following:

$$\phi_{j,k}(t) \equiv \phi(2^j t - k) \tag{2.3}$$

[1]Both Chui (1992) and Daubechies (1992) provide a thorough review of wavelet analysis. For a good review of the basics, see Strang 1989 and Graps 1995.

where the subscript j, called the *dilation parameter*, is associated with the level of time resolution and where a larger number indicates a higher level of resolution. For $j > 0$, the scaled father wavelet $\phi_{j,k}(t)$ has a narrower support and is translated by a smaller step, $2^{-j} \cdot k$.[2] They are therefore capable of capturing more detail from signals than $\phi_{0,k}(t)$. The converse is true for $j < 0$. Denote the subspace V_j of $L^2(\Re)$, spanned by these scaled father wavelet functions, as

$$V_j = \overline{\underset{k}{\text{Span}} \ \{\phi_{j,k}(t)\}}$$

The *multiresolution condition* requires that

$$V_j \subset V_{j+1} \ \forall \ j \in \boldsymbol{Z} \tag{2.4a}$$

$$V_{-\infty} = \varnothing \tag{2.4b}$$

$$V_{\infty} = L^2(\Re) \tag{2.4c}$$

A direct implication of this condition is that if any basis function $\phi(t)$ is in the V_0 subspace, it is also in the V_1 subspace. This means that $\phi(t)$ can be represented as a linear combination of $\phi(2t)$ as in the following:

$$\phi(t) = \sum_k h(k)\phi(2t - k), \ \forall \ k \in \boldsymbol{Z} \tag{2.5}$$

where $h(k)$ is the coefficient attached to the basis function $\phi(2t - k)$.[3] To complete the wavelet system, define the orthogonal complement of V_j in V_{j+1} as W_j, that is

$$V_{j+1} = V_j \oplus W_j \tag{2.6}$$

This, together with Equation (2.4c), implies that the $L^2(\Re)$ space can be decomposed such that

$$L^2(\Re) = V_0 \oplus W_0 \oplus W_1 \oplus \cdots \tag{2.7}$$

Essentially, one can think of $V_0, W_0, W_1, W_2, \ldots$ as a partition of the information set, as demonstrated in Figure 1. It is this information decomposition that allows us to reconstruct variables of interest based on the subset of relevant information. Let's define the mother wavelet function, $\psi(t)$, as the functions that span the subspace W_0. Since $W_0 \subset V_1$, $\psi(t)$ can also be written as a linear combination of the father wavelet function $\phi(2t)$, that is

$$\psi(t) = \sum_k g(k)\phi(2t - k) \ \forall \ k \in \boldsymbol{Z} \tag{2.8}$$

where $g(k)$ is the coefficient attached to the basis function $\phi(2t - k)$ and is related to $h(k)$ in Equation (2.5) by the following equation:

$$g(k) = (-1)^k h(1 - k) \tag{2.9}$$

Equations (2.5) and (2.8) are called *dilation equations* of the wavelet system. As mentioned earlier, wavelet functions rarely have analytical forms. Indeed, numerous wavelet systems can be built by recursively

[2] Assume that $\phi(t)$ has a finite support $[l, m]$. Since $\phi_k(t) \equiv \phi(t - k)$, the support of $\phi_k(t)$ is the same as the support of $\phi(t)$. Similarly, since $\phi_{j,k}(t) \equiv \phi(2^j t - k)$, the support of $\phi_{j,k}$ is given by $[\frac{l+k}{2^j}, \frac{m+k}{2^j}]$. Therefore, the support of $\phi_{j,k}$ is shorter than the support of ϕ_k.

[3] In the wavelet literature, different normalization has been used. In particular

$$\phi(t) = \sum_k \sqrt{2}h(k)\phi(2t - k), \ \forall \ k \in \boldsymbol{Z}$$

where the factor $\sqrt{2}$ merely serves to normalize the functions.

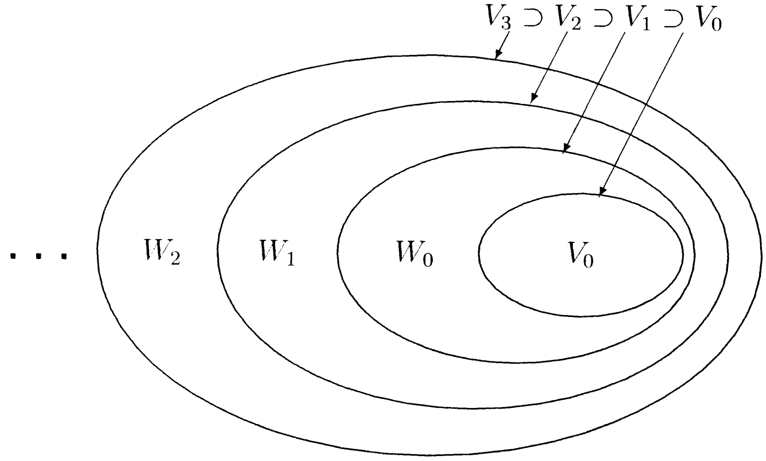

Figure 1
Decomposition of information.

evaluating those two dilation equations together with Equations (2.3) and (2.9).[4] Since the above conditions do not exhaust the degrees of freedom, there generally exists an infinite number of very different wavelet systems. As suggested in Davidson, Labys, and Lesourd 1998, we use the Daubechies wavelets in our application. What makes Daubechies wavelets really useful is that they have compact support and a certain number of vanishing moments.[5] The design of $\{h(k)\}$ in the Daubechies wavelet functions is to transform the functions $\phi(t-k)$ and $\psi(t-k)$ into an orthonormal basis of the space they span. For each set of Daubechies wavelets, there is an integer m such that, for any integer l with $0 \leq l \leq m$,

$$\int_{-\infty}^{\infty} t^l \psi(t)\,dt = 0 \tag{2.10}$$

In other words, the moment properties of the first few orders (up to m) are completely captured by the coarsest-level father wavelets. This allows one to intuitively explain that the father wavelets are capable of capturing the underlying patterns or trends of signals. There is a trade-off, however, between the length of support of the wavelets and the number of vanishing moments. An increase in m is bought at the cost of extending the length of support; that is, lessening the degree of localization achieved by the wavelets. We use $m = 2$, as suggested in Davidson, Labys, and Lesourd 1998, to allow the first two moment properties of the data to be captured by the coarsest father wavelets.

2.2 Wavelet analysis

Based on the multiresolution decomposition in Equation (2.7) and the constructed father and mother wavelets, any discrete signal $f(t) \in L^2(\Re)$ can be written in terms of wavelet functions as the following:

$$f(t) = \sum_{k=-\infty}^{\infty} \alpha_{j_0,k}\phi_{j_0,k}(t) + \sum_{j \geq j_0} \sum_{k=-\infty}^{\infty} \beta_{j,k}\psi_{j,k}(t) \tag{2.11}$$

where j_0 is a positive integer that represents the coarsest level of resolution chosen by researchers, and $\alpha_{j_0,k} = \int f(t)\phi_{j_0,k}(t)\,dt$, and $\beta_{j,k} = \int f(t)\psi_{j,k}(t)\,dt$ are the expansion coefficients.[6] In the wavelet literature, the

[4]Two popular numerical approaches of building wavelet functions are the forward and the backward pyramid algorithms. For more detail about these numerical algorithms, refer to Strang 1989, Daubechies 1992, and Chui 1992.

[5]Symmelets, another wavelet system constructed by Daubechies, have the same support length, number of vanishing moments, and number of derivatives. The main difference between Daubechies wavelets and symmelets is that symmelets are nearly symmetric, whereas Daubechies wavelets are highly asymmetric. Ramsey and Lampart (1998) use symmelets in their study of expenditure and income relationship.

[6]The scale, j_0, of the initial subspace is arbitrary. In practice, it is usually chosen to represent the coarsest details of interest in a signal.

procedure of obtaining the expansion coefficients is called the *discrete wavelet transform* (DWT). In reality, the magnitude of $\alpha_{j,k}$ and $\beta_{j,k}$ drops off rapidly with j and k for a large class of signals. Therefore, the dimensions of the summation in Equation (2.11) are practically finite. In other words, only a parsimoniously countable set of wavelet functions and the associated transformation coefficients need to be evaluated. The DWT procedure usually requires, however, that discrete data be equally spaced and that the number of observations be some integer power of two. These requirements turn out to be too restrictive to be applicable for analyzing most economic and financial data. Furthermore, what is more interesting is when the observed data are contaminated with noise, that is,

$$y_t = f(x_t) + \varepsilon_t \tag{2.12}$$

where y_t is the dependent variable, $f(x_t)$ represents the signal, x_t is the explanatory variable (which could be time t if no other explanatory variable is imposed), and ε_t represents noise. In such a situation, we would be interested in removing as much noise ε_t while retaining the main features of the data $f(x_t)$. *Wavelet shrinkage*, pioneered by Donoho and Johnstone (1994), is popular for this task in scientific applications. It involves nonparametrically estimating the signal $f(x_t)$ by significantly reducing or completely eliminating some wavelet coefficients ϕ_{j_0k} and $\psi_{j,k}$ that are attributed to noise (see Donoho and Johnstone 1994, 1995; Donoho 1995; and Donoho et al. 1995 for detail principles of this technique). Since it involves applications of the DWT, however, wavelet shrinkage inherits the same restrictions of the DWT. Hence, it would not be most suitable for analyzing our data.

Alternatively, Davidson, Labys, and Lesourd (1998) suggest a semi-nonparametric regression to overcome the problems of unequally spaced data and the number of observations being a power of two. The algorithm of Davidson, Labys, and Lesourd involves the following steps:

1. To circumvent the problems of potential edge effects or the periodic nature of data, Davidson, Labys, and Lesourd suggest padding some m observations at the beginning and the end of the sample in the following way:

$$y_m, y_{m-1}, \ldots, y_1, y_1, y_2, \ldots, y_{n-1}, y_n, y_n, y_{n-1}, \ldots, y_{n-m+1} \tag{2.13}$$

   They claim this ensures continuity at the edge and helps to fit the data better.

2. Perform the same padding to the independent variable x_t as done to the dependent variable y_t. Compress the padded series x_t into the [0, 1] interval.[7]

3. For each level of resolution, starting from j_0 and ending with J, select only those wavelet functions that intersect the [0, 1] interval.[8] Evaluate these functions at 2^i grid points, where i is some positive integer that gives a much finer partition of the [0, 1] interval than x_t does.[9]

4. Evaluate $f(x_t)$ by a set of father wavelet functions and several sets of mother wavelet functions, all of which are evaluated at the grid points $\tilde{x}_t$ that are closest to x_t. Based on this, construct a regression vector

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[7]When the data are equally spaced, one can use the sequence $(\frac{1}{n+2m}, \frac{2}{n+2m}, \ldots, 1)$ as the associated grid points at which the function $f(x_t)$ is evaluated. This is certainly the case in Davidson, Labys, and Lesourd 1998, where y_t is equally spaced and $x_t = t$ (time).

[8]J is the highest level of resolution. Despite the formula in Equation (2.11), J is practically a finite number in applications, because of the finite nature of the available data (hence concerns about the degrees of freedom) and the fact that the significance of wavelet coefficients actually drops quite rapidly as the level of resolution increases. Therefore, researchers can judiciously choose a finite number for the desired highest resolution level J.

[9]The numerical evaluation of the Daubechies wavelet function at numerous grid points is completed by employing the S-Plus command "wavelet."

of the following form:

$$\begin{aligned}\mathbf{\Psi}_t^T = (&\phi_{j_0,a_0}(\tilde{x}_t), \ldots, \phi_{j_0,b_0}(\tilde{x}_t)\\ &\psi_{j_0,a_0}(\tilde{x}_t), \ldots, \psi_{j_0,b_0}(\tilde{x}_t)\\ &\psi_{j_0+1,a_1}(\tilde{x}_t), \ldots, \psi_{j_0+1,b_1}(\tilde{x}_t)\\ &\vdots\\ &\psi_{J,a_{J-j_0}}(\tilde{x}_t), \ldots, \psi_{J,b_{J-j_0}}(\tilde{x}_t))\end{aligned} \tag{2.14}$$

where the newly introduced subscripts a_i and b_i are respectively the upper bound and the lower bound of translation at resolution level $j_0 + i$ such that the support of the associated wavelet function intersects the [0, 1] interval.

5. Regress y_t on $\mathbf{\Psi}_t$.

By employing Davidson, Labys, and Lesourd's algorithm, as illustrated by the above five steps, not only are the restrictions associated with the DWT and the wavelet shrinkage accounted for, but statistical properties of semi-nonparametric regression can also be directly applied to assess output from such an analysis.

In essence, the main contribution of Davidson, Labys, and Lesourd lies in placing the wavelet shrinkage procedure into an econometric framework to enable application of wavelet analysis to nonstationary economic and financial data. By following their algorithm, one can reconstruct variables of interest according to information contained in each of the V_0, W_0, W_1, $W_2, \ldots,$ subspaces as described in Equation (2.7) and Figure 1. In other words, instead of regressing y_t on $\mathbf{\Psi}_t$, we regress y_t on subsets of $\mathbf{\Psi}_t$ corresponding to V_0, W_0, W_1, $W_2, \ldots,$ etc.[10] Alternatively, we can regress y_t on V_0, V_1, V_2, $V_3, \ldots,$ if we are interested in examining the impact of information contained at a particular frequency and above rather than the marginal information contained in the information subsets W_0, W_1, $W_2, \ldots$. This allows examination of the impact of the level of information detail on the economic relationship under investigation. A description of our data and the reconstructed data series is presented in the next section.

3 Original and Wavelet-Reconstructed Data

Data used in this study are 5-min price observations for four Share Price Index (SPI) contracts maturing in March, June, September, and December 1995 and the correspondingly recorded prices of the All Ordinaries Index (AOI).[11] SPI futures contracts began trading on the Sydney Futures Exchange (SFE) in 1983. Since these contracts track the movement of the share market, they serve as a substitute for owning a diversified portfolio of the shares that form the AOI.[12] The trading floor of the SFE operates from 9:50 A.M. to 12:30 P.M. and from 2:00 P.M. to 4:10 P.M. Australian Eastern Standard Time (EST). As most trading occurs in the nearest expiry month, we use the index futures time series based on the near contract, shifting to the next near contract on expiration day. Trading on the Australian Stock Exchange (ASX) is fully automated and operates from 10:00 A.M. (EST) to the close of trade at 4:00 P.M. Opening times for trading are staggered, with all stocks trading by 10:10 A.M.

[10] "Subset of $\mathbf{\Psi}_t$ corresponding to $V_i, \forall i$" refers to a collection of father wavelets at resolution level $j_0 + i$, that is, $(\phi_{j_0+i,a_0}(\tilde{x}_t), \ldots, \phi_{j_0+i,b_0}(\tilde{x}_t))$; whereas "subset of $\mathbf{\Psi}_t$ corresponding to $W_i, \forall i$" refers to a collection of mother wavelets at resolution level $j_0 + i$, that is, $(\psi_{j_0+i,a_0}(\tilde{x}_t), \ldots, \psi_{j_0+i,b_0}(\tilde{x}_t))$.

[11] Intraday price observations for both series used in this study have been obtained from the Securities Industry Research Centre of Asia-Pacific (SIRCA).

[12] At maturity, the value of a contract in 1995 was the actual AOI on the last day of trading, multiplied by 25 Australian dollars.

Table 1
Descriptive statistics of time series examined

	N	$\hat{\mu}$	$\hat{\sigma}^2$	$\hat{k}_3$	$\hat{k}_4$	BJ	ACF
Mar95aomg	1764	1877.3	608.6	−0.11	−0.98	74.1	0.995
Mar95spmg	1764	1890.5	746.4	−0.09	−0.79	48.3	0.993
Mar95aoaf	1512	1876.5	615.1	−0.07	−1.02	66.8	0.994
Mar95spaf	1512	1889.0	743.5	−0.06	−0.91	53.1	0.992
Jun95aomg	1708	2014.0	1209.3	−0.76	0.82	212.3	0.993
Jun95spmg	1708	2038.3	1489.0	−0.48	−0.07	65.9	0.992
Jun95aoaf	1464	2013.4	1197.2	−0.76	0.82	182.0	0.993
Jun95spaf	1464	2037.3	1485.9	−0.51	0.00	63.5	0.993
Sep95aomg	1820	2118.4	1268.6	−1.51	2.31	1096.3	0.995
Sep95spmg	1820	2143.0	1233.3	−1.53	2.53	1195.5	0.993
Sep95aoaf	1560	2118.2	1268.4	−1.64	2.91	1249.7	0.994
Sep95spaf	1560	2142.0	1215.2	−1.75	3.37	1534.4	0.991
Dec95aomg	1736	2137.4	2510.1	0.25	−1.19	120.5	0.998
Dec95spmg	1736	2153.7	2521.7	0.07	−1.10	88.9	0.997
Dec95aoaf	1488	2137.6	2510.5	0.21	−1.22	103.2	0.998
Dec95spaf	1488	2153.2	2505.4	0.00	−1.12	77.8	0.997

Note: aomg: All Ordinaries A.M.; spmg: Sydney Futures Index A.M.; aoaf: All Ordinaries P.M.; spaf: Sydney Futures Index P.M.; N: number of observations; $\hat{\mu}$: sample mean; $\hat{\sigma}^2$: sample variance; $\hat{k}_3$: sample skewness; $\hat{k}_4$: sample kurtosis; BJ: Jarque-Bera statistics. ACF: first-lag autocorrelation.

Because of the break between 12:30 P.M. and 2:00 P.M. on the SFE, there is a problem of matching up the index prices from the two markets. Rather than assuming that the data-generating process is the same for both morning and afternoon trading sessions, we divide our analysis between mornings and afternoons over the four 1995 futures contracts. This enables a determination to be made as to whether the relationship between index prices differs systematically over the two sessions, while at best aiding compliance with the time-series assumption of continuous, equally spaced observations. Consequently, data are used from both exchanges from 10:15 A.M. to 12:30 P.M. to cover each morning trading session, discarding the first 5 min of trading from 10:10 A.M. to account for possible anomalous trading after the start-up procedure for the AOI. For a similar reason, we discard the first 5 min of trading in the SPI in the afternoon session, which runs from 2:05 P.M. to 4:00 P.M. The nearby futures prices and those of the underlying index are matched for each 5-min interval on a morning and afternoon basis for each of the four futures contracts during 1995. This pairing provides 28 paired observations for each morning and 24 for each afternoon.

As reported in Table 1, the number of 5-min price observations for the morning series totaled 1,764, 1,708, 1,820, and 1,736 for the March, June, September, and December contracts, respectively. For the afternoon series, there were 1,512, 1,464, 1,560, and 1,488 5-min price observations, respectively. These index prices from both markets are clearly non-normally distributed and have large first order autocorrelation. Figure 2 presents time-series plots for both the AOI and SPI price series for the March 1995 contract. From Figure 2, it is clear that these two price series are co-integrated. This comes as no surprise, given that the futures index is a derivative security of the spot market index, with both subject to the same impact from changes in market fundamentals.

As described in Equation (2.10), we now reconstruct our data using the Daubechies wavelet with two vanishing moments, that is, $m = 2$. The support of this particular wavelet function lies in the interval [0, 5]. Since the time domain of data is compressed into the [0, 1] interval, we choose the coarsest level of resolution (j_0) to be three, with the associated support length $\frac{5}{8} < 1$, and translation step $\frac{1}{8}$.[13] This would correspond to

[13]As discussed in Section 2, the choice of a zero subscript for the coarsest level of time resolution was arbitrary. In the discussion of the reconstruction procedure here, we chose this level to be three based solely upon our choice of wavelet function and its associated support.

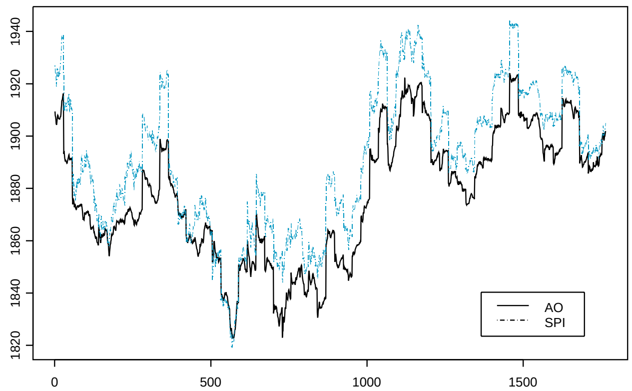

Figure 2
1995 March AO and SPI (A.M.).

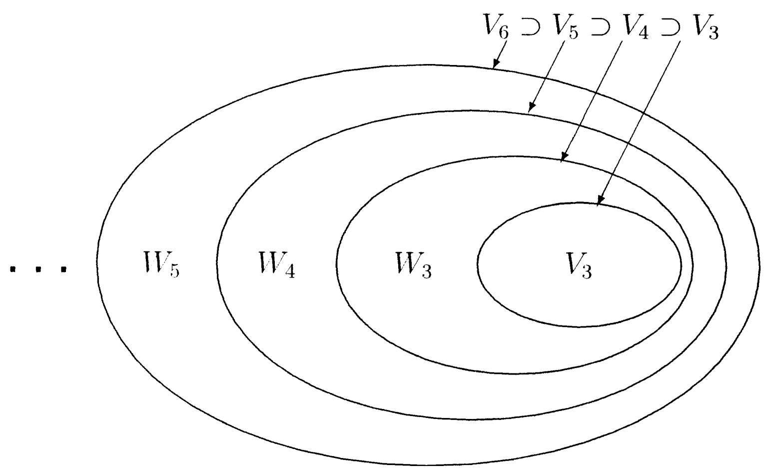

Figure 3
Decomposition of information starting from level 3.

an information subset with a time horizon of roughly eight morning or afternoon trading sessions ((13 weeks × 5 sessions per week)/2^3). Just enough translations of the father and mother wavelet functions are constructed at this level of resolution to cover the [0, 1] interval. Then, we increase the level of resolution by one and construct the mother wavelet function and all of its associated translations at this higher level. This process is repeated until the level of resolution reaches seven. This would correspond to an information subset with a time horizon of roughly one-half of a trading section ((13 × 5 sessions) /2^7), or 70 min for morning sessions and 60 min for afternoon sessions. At each level of time resolution, these constructed wavelet functions span the information subset as shown in Figure 3. We then conduct semi-nonparametric regressions of y_t on subsets of $\mathbf{\Psi}_t$ corresponding to V_3, V_4, V_5, $V_6, \ldots,$ as described at the end of Section 2.

Figures 4 and 5 demonstrate examples of such reconstruction.[14] From these figures, it is clear that the coarsest-level father wavelet functions trace out the underlying low-frequency trend of the data very well.

[14]Because of space limitations, figures of other contracts are not reproduced here but are available from the authors upon request.

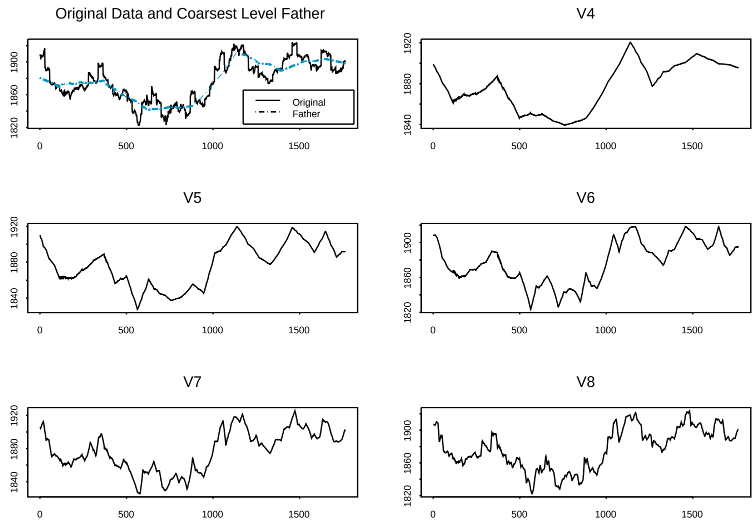

Figure 4
Original and reconstructed 1995 March AOI (A.M.).

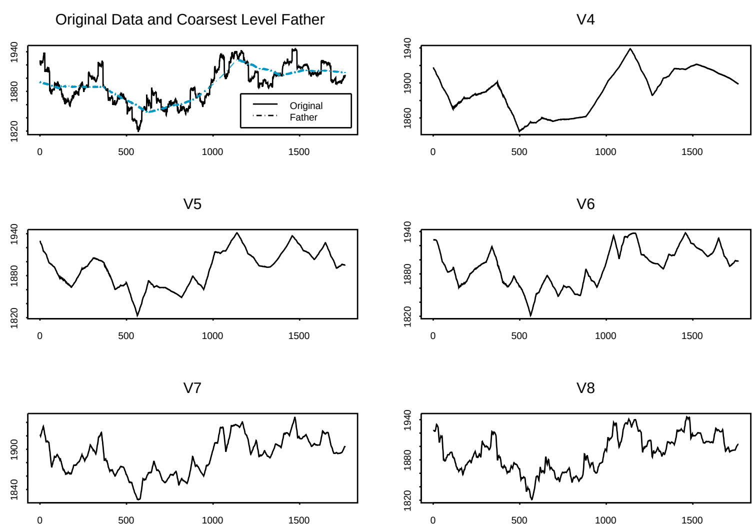

Figure 5
Original and reconstructed 1995 March SPI (A.M.).

Furthermore, as the level of resolution being used is increased, the reconstructed data more closely track the original data. In particular, when the top-level resolution is approached, there is virtually no difference between the original and the reconstructed data by visual inspection. Intensities of variation across different time scales are compared by determining the overall contribution of any time scale by its marginal contribution to the explained sum of squares of the regression. We also measure the importance of the residual variation at time scales shorter than that of the highest resolution actually considered. The R^2 is calculated, as in Davidson, Labys, and Lesourd 1998, by ignoring the variation in the trend of the series. Thus, the total "sum of squares" is the increase in the explained sum of squares when going from level 3 to level 7. The results reported in Table 2 confirm what we have just found by visually inspecting Figures 4 and 5.

Table 2
Contribution to the explained sum of squares and the goodness of fit (R^2)

	M_3	M_4	M_5	M_6	M_7	R^2
Mar95aomg	0.389	0.344	0.159	0.078	0.030	0.968
Mar95spmg	0.402	0.357	0.138	0.074	0.029	0.965
Mar95aoaf	0.410	0.298	0.176	0.089	0.027	0.969
Mar95spaf	0.435	0.311	0.152	0.077	0.026	0.969
Jun95aomg	0.435	0.263	0.210	0.065	0.027	0.972
Jun95spmg	0.405	0.258	0.241	0.067	0.030	0.968
Jun95aoaf	0.513	0.155	0.275	0.035	0.022	0.981
Jun95spaf	0.461	0.180	0.304	0.033	0.022	0.977
Sep95aomg	0.816	0.010	0.079	0.064	0.031	0.977
Sep95spmg	0.738	0.143	0.055	0.042	0.022	0.980
Sep95aoaf	0.801	−0.007	0.120	0.054	0.031	0.977
Sep95spaf	0.687	0.133	0.101	0.054	0.025	0.976
Dec95aomg	−0.044	0.516	0.327	0.132	0.069	0.942
Dec95spmg	0.207	0.441	0.214	0.083	0.055	0.954
Dec95aoaf	0.103	0.405	0.323	0.097	0.071	0.953
Dec95spaf	0.293	0.364	0.210	0.079	0.053	0.961

Note: M_i corresponds to contribution to the explained sum of squares from the information subset V_i. Other abbreviations are as in Table 1.

This study seeks to examine the relationship between index prices from the spot and futures markets. In particular, it is interested in the contribution to price formation from information contained within a certain frequency band. To examine this, we denote the information subset that is complementary to V_i as CI_i. It follows that

$$CI_i = I_\Omega - V_i \tag{3.1}$$

where I_Ω is the information universe. Further, denote the price associated with information CI_i by PCI_i, which can be calculated as the fitted residual from regressing y_t on wavelet functions that span V_i. By construction, the PCIs at lower levels of time resolution are dominated by economic fundamentals driving both markets. We expect the contemporaneous price assumptions of the COC to hold at these lower levels of resolution. As one moves to higher levels of resolution, however, where the effect of market imperfections is likely to dominate, a more noncontemporaneous relationship between prices would be expected. These hypotheses are examined in the next section.

4 Econometric Model and Empirical Results

This section analyzes the relationship between the *PCI*s of indices from the cash and the futures markets at increasingly higher levels of time resolution. Since there is both theoretical and empirical evidence that points to bidirectional causality between spot and futures index prices, the price relationships are examined in a bivariate VAR model so that the level of persistence of the lead-lag phenomenon between the futures and underlying spot index prices can be determined. In this framework, each variable is a linear function of past realizations of itself and the other variable. Specifically, the econometric model contains two equations. In one equation, the *PCI*s for the futures index (*PCISP*) at a particular level of resolution are regressed on past realizations of themselves and the *PCI*s for the underlying index (*PCIAO*) at the same resolution level. In the other equation, the *PCIAO*s are regressed on past realizations of themselves and the *PCISP*s. This bivariate VAR model can be represented by the following equation system:

$$PCIAO_t = \alpha_1 + \sum_{j=1}^{k} \gamma_j PCIAO_{t-j} + \sum_{j=1}^{k} \beta_j PCISP_{t-j} + e_{1t} \tag{4.1a}$$

$$PCISP_t = \alpha_2 + \sum_{j=1}^{k} \pi_j PCISP_{t-j} + \sum_{j=1}^{k} \theta_j PCIAO_{t-j} + e_{2t} \quad (4.1b)$$

where α_1 and α_2 are intercept terms, γ_j and π_j are coefficients for the lagged dependent variables, β_j and θ_j are the coefficients of the lagged independent variables, and e_{1t} and e_{2t} are the random-error terms. The lag length, k, for each variable in both equations is chosen to be 15, based on the number of 5-min time periods required to span the time horizon of the information subset at the highest level of time resolution.[15]

To make sure the VAR model is correctly specified, two associated statistical issues are addressed. First, the use of a VAR model is strictly appropriate only when the variables are stationary. When variables are nonstationary, they may be valid only approximately or not at all (see Charemza and Deadman 1997). Using the Augmented Dickey-Fuller test, we tested each *PCI* series from both markets for the existence of a unit root process. From the results reported in Table 3, it is concluded that the *PCI* series for both markets, at all levels of time resolution and all contracts, are $I(0)$ processes. It follows that the *PCI* series corresponding to the *SPI* and the *AOI* at different levels of resolution are not co-integrated. As a consequence, a VAR model will be correctly specified.

The second statistical issue is related to the significantly large autocorrelation that is typical of high frequency data. As pointed out by Wahab and Lashgari (1993), if this autocorrelation is left unpurged, its presence will tend to reduce the degree of association between the spot and futures index prices. Following the procedure advocated by Stoll and Whaley (1990), we filtered our *PCI* series using autoregressive moving average (ARMA) processes. The innovations from the ARMA models resulted in time series with no significant autocorrelation. These innovations were then used as instruments for the cash and the futures price series. Further, there was no evidence of remaining autocorrelation in the residuals from the estimated bivariate VAR models.

Tables 4 and 5 summarize the estimation results of the bivariate VAR models. The persistence of a lead-lag effect in both markets is determined by the number of significant (at the 2.5% level of significance) coefficients of the lagged independent variables in each equation. Significant β_j coefficients on lags of the *PCISP* variable in Equation (4.1a) are evidence of the *PCISP* leading the *PCIAO*. Similarly, significant θ_j coefficients of lags of the *PCIAO* variable in Equation (4.1b) indicate leads in the *PCIAO* relative to the *PCISP*.

From Tables 4 and 5, it is evident that, for the three lowest resolution levels (with frequency band no smaller than 1.2 trading hours), the number of significant β_j is small for most contracts.[16] This indicates short-lived persistence in the lead of the *PCISP* over the corresponding *PCIAO* at low resolution levels. On the other hand, when information of higher resolution (with frequency band less than 1.2 trading hours) is considered, the lead-lag relationship between the two markets is markedly different. For all morning trading contracts studied, as well as the December afternoon contract, the lead of the *PCISP* over the *PCIAO* is much longer. This implies greater persistence of *PCISP* leading over corresponding *PCIAO* at higher resolution levels, although for the afternoon contracts in March, June, and September, the lead-lag persistence at the second-highest resolution level (where the frequency band of information is within 1 trading hour) is rather short-lived. Namely, the persistence is at most for two 5-min trading periods for the March afternoon contract.

Next, we examine the estimated coefficients of θ_j as defined in Equation (4.1b). This is shown in the last columns of Tables 4 and 5. Overall, we find that there is little evidence supporting a reciprocal lead of the *PCIAO* over the *PCISP* for the first three lower resolution levels (with frequency band no less than 1.2 trading hours).[17] In contrast, at the two highest resolution levels (that is, *CI* with frequency band less than 1.1 trading

[15]In practice, k is usually determined by some information criterion, such as Akaike information criterion (*AIC*) or Bayesian information criterion (*BIC*). As a result, lag lengths in the two equations are often different from one another. Sims (1980), however, argues against purging those lags with insignificant parameter estimates. Accordingly, we include the same number of lags in each equation.

[16]The only two exceptions are the March and the December morning trading sessions at the third level, where the frequency band is less than 2.3 trading hours. In those two situations, we observe much longer leads of the *PCISP* over the *PCIAO*.

[17]The December afternoon contract is the only exception. In that case, however, the persistence is at most for one 5-min trading period.

Table 3
Unit root tests for prices of complementary information at different levels of time resolution for both morning and afternoon trading sessions

	Contract (1995)			
	March	June	September	December
Series	ADF	ADF	ADF	ADF
AOV3MG	−4.15	−3.36	−3.91	−3.78
SPV3MG	−4.36	−3.29	−4.02	−3.99
AOV3AF	−4.41	−2.98	−3.56	−3.74
SPV3AF	−4.66	−2.88	−3.93	−4.00
AOV4MG	−5.09	−5.47	−5.48	−4.76
SPV4MG	−5.11	−5.46	−5.42	−4.97
AOV4AF	−4.61	−4.17	−6.14	−4.63
SPV4AF	−4.76	−4.12	−6.33	−4.76
AOV5MG	−7.13	−6.78	−6.24	−5.96
SPV5MG	−7.09	−6.79	−6.56	−5.96
AOV5AF	−4.94	−6.39	−5.08	−6.65
SPV5AF	−4.99	−6.55	−5.39	−6.65
AOV6MG	−7.62	−7.34	−7.10	−7.16
SPV6MG	−7.47	−7.40	−7.21	−6.85
AOV6AF	−7.23	−6.14	−7.54	−5.96
SPV6AF	−7.75	−6.31	−7.51	−6.67
AOV7MG	−7.12	−7.27	−8.30	−8.65
SPV7MG	−6.70	−7.38	−8.30	−8.78
AOV7AF	−5.67	−3.69	−6.25	−3.75
SPV7AF	−5.30	−4.30	−6.40	−4.03

Note: Each series is indexed by three sets of double-digit symbols. For the first two digits, AO indicates the All Ordinaries Index, and SP indicates the Share Price Index. For the next two digits, Vi (i = 3, 4, 5, 6, and 7) denotes that the complementary information set CI_i is used to construct the series. For the last two digits, MG represents the morning data, and AF represents the afternoon data. For example, AOV3MG represents reconstructed morning All Ordinaries Index using the CI_3 information set.

ADF is the Augmented Dickey-Fuller test. Critical values are −3.44(1%), −2.86 (5%), and −2.57 (10%). Number of lags to remove serial correlation is 150.

hours), there is some evidence of persistence in the lead of the *PCIAO* over the *PCISP*, such as the June morning, the September afternoon, and the December afternoon contracts.

In summary, the results of this study suggest that

1. When information of lower time resolution is considered, index price levels of the futures market *PCISP* (reconstructed from complementary information subsets) lead the corresponding reconstructed index price levels of the cash market *PCIAO*. On the other hand, at the same lower levels of time resolution, there is little evidence that there is any lead from the *PCIAO* to the *PCISP*.

2. In contrast, when information of higher time resolution is considered, there is evidence of greater lead-lag persistence in both directions.

These results suggest that both market-wide and stock-specific influences as well play a role in the relationship between the two indices. The difference in the degree of persistence in the lead-lag phenomenon

Table 4
Number of leads and lags between cash and futures index prices during the morning trading session

Contract (morning)	*CI* with frequency band < trading hours	Number of 5-min periods in which *PCISP* leads *PCIAO*	*PCIAO* leads *PCISP*
	9.20	1	0
March	4.60	1	0
	2.30	7	0
($n = 1{,}764$)	1.10	7	0
	0.57	13	0
	8.90	2	0
June	4.40	2	0
	2.20	2	0
($n = 1{,}708$)	1.10	11	7
	0.56	11	7
	9.50	2	0
September	4.70	2	0
	2.40	2	0
($n = 1{,}820$)	1.20	3	0
	0.59	9	0
	9.00	2	0
December	4.50	2	0
	2.30	9	0
($n = 1{,}736$)	1.10	11	0
	0.57	11	0

Note: Coefficients on lagged variables in the VAR model were deemed significant at the 2.5% level of significance.

Table 5
Number of leads and lags between cash and futures index prices during the afternoon trading session

Contract (afternoon)	*CI* with frequency band < trading hours	Number of 5-min periods in which *PCISP* leads *PCIAO*	*PCIAO* leads *PCISP*
	7.90	0	0
March	3.90	0	0
	2.00	2	0
($n = 1{,}512$)	0.98	2	0
	0.49	8	4
	7.60	1	0
June	3.80	1	0
	1.90	1	0
($n = 1{,}464$)	0.95	1	1
	0.48	11	0
	8.10	1	0
September	4.10	1	0
	2.00	1	0
($n = 1{,}560$)	1.00	1	6
	0.51	10	4
	7.80	1	0
December	3.90	1	1
	1.90	1	1
($n = 1{,}488$)	0.97	8	1
	0.48	13	1

Note: Coefficients on lagged variables in the VAR model were deemed significant at the 2.5% level of significance.

from lower to higher levels of time resolution is unambiguous. This is evidence in favor of price discovery by both markets when dealing with information subsets defined over shorter time intervals.

To conclude, the evidence is consistent with our conjecture that the effect of market imperfection could be an explanation for the increased persistence of the lead-lag effect the higher the level of time resolution considered.

5 Implications and Conclusions

In this paper, wavelet analysis is employed to reconstruct index price series from the cash and the futures markets. These reconstructed price series are based only on subsets of information that differentiate between the two fundamentally related prices. Using a VAR approach, we modeled the interrelationship between the *PCI*s from both markets at different levels of time resolution. In the lower levels of time resolution, where market fundamentals dominate both markets, evidence was found of short-lived lead-lag persistence, where the *PCISP* led *PCIAO*. We argue that this is evidence of price discovery in the futures market. In contrast, increased lead-lag persistence of *PCI*s from both markets was found for the higher levels of time resolution. Although this indicates price discovery in both markets, on the balance of the evidence, we conclude that price discovery is greater in the futures market.

A further conclusion that can be drawn from the results of this study concerns the difficulty of examining the assumption of contemporaneous prices in the COC model by determining the persistence of the lead-lag relationship. If our conjecture that market imperfections are a possible explanation for lead-lag effects is true, to determine the appropriateness of the assumption, market imperfections will need to be controlled for when determining lead-lags. With a number of causes of market imperfections attributable to market operational differences, it is difficult to imagine how one could adequately control for effects generic to the structure of both markets. In the event that this is possible, then our results suggest that concentration should be on those imperfections that are likely to occur within very short time horizons.

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