Macrostructures in Microeconomic Dynamics

Takuya Iimura
Tokyo Metropolitan College

Abstract. This paper investigates topologically semiconjugate dynamics as a macrorepresentation of microeconomic dynamics. The condition for its existence, its summarizing property, and its inferability property are discussed. As an example, we present a model of a temporary equilibrium price dynamic that has a topologically semiconjugate one-dimensional income dynamic, from which the nature of the original price dynamic will be inferred.

Keywords. topologically semiconjugate dynamics, temporary equilibrium dynamics, microfoundation of macroeconomics, aggregation problem

Acknowledgments. I would like to thank an anonymous reviewer for many useful comments.

1 Introduction

In this paper we consider, from a rather general viewpoint, some macrostructures that may appear in some microeconomic dynamics. Mathematically, they are topologically semiconjugate dynamics (the term is taken from Devaney 1989) to which we will call attention and refer to as macrostructures. We will consider only the discrete dynamics here, but a similar argument is possible for the continuous dynamics.

The topologically conjugate dynamics are well known in the study of dynamics; they are homeomorphic transformations of given dynamics. They are precise copies of the original dynamics and, if appropriately chosen, greatly facilitate the analysis of the originals. A famous example is the symbolic dynamic, which has been used for studies of the dynamics of logistic mapping and of horseshoe mapping (see, e.g., Smale 1967 or Devaney 1989).

Loosely speaking, the topologically semiconjugate dynamics are continuous transformations of given dynamics. They are not precise copies, but rather rough "sketches," if any, of original dynamics. Note that this is a natural generalization of the conjugate dynamics, but their existence (except in a trivial sense) is not always guaranteed unless the transformation mapping is one to one; and indeed this is the case when we speak of semiconjugacy.

Actually, we will look at a continuous transformation mapping as one that gives a classification of elements in the domain of dynamics. (We will see in the sequel that a semiconjugate dynamic exists if the classification is compatible with the original dynamic.) Thus, the semiconjugate dynamics we consider in this paper are dynamics of classes of elements, which, by nature, have lower dimensionality than the original dynamics. It would be convenient to say, in terms of group theory, that the semiconjugate dynamic is a factor of a given dynamic and that we are looking at the original through a homeomorphism (the transformation mapping) summarizing the nature of the original from a particular viewpoint.

This summarizing nature of semiconjugate dynamics may be of interest in its own right; for example, if some higher-dimensional price dynamic turns out to show a simple rule, it must be in itself an addition to our knowl-
edge. But as important as it is in itself, a semiconjugate dynamic can also serve, as does the conjugate dynamic, as a tool for inferring the nature of its original dynamic. This is a point we want to stress in this paper. Generally, the existence of a periodic point in a semiconjugate dynamic does not allow us to infer the existence of a periodic point in the original dynamic. But if some conditions are met (e.g., the compactness and convexity of the classes), we can conclude from the former the latter, including its periodicity; the nature of the original dynamic then becomes, to some extent, “inferable.” We will investigate this inferability nature in the following section.

We will look at an example of a higher-dimensional, nonlinear, microeconomic “price” dynamic that can be summarized by a one-dimensional, generally nonlinear, macroeconomics-like “income” dynamic. The nature of the price dynamic is inferable, to some extent, from the observation of the income dynamic. Hence, from the viewpoint of the significance for economics, one may say that our argument concerns a correspondence problem between microeconomics and macroeconomics, or an aggregation problem, in the context of dynamics.

The paper is organized as follows. In Section 2, we present some mathematics on the nature of topologically semiconjugate dynamics and provide some lemmata that facilitate their use. In Section 3, we construct a simple example of a temporary equilibrium with loans and consumption that has the alleged topologically semiconjugate dynamics and provide some lemmata that facilitate their use. In Section 4, we consider some possible extensions of the results. Throughout, the notations \( \mathbb{R}_+ \) and \( \mathbb{R}_{++} \) will be used to denote the set of nonnegative and positive reals, respectively.

### 2 The Macrostructures: Topologically Semiconjugate Dynamics

Let \( S \) be a topological space and \( f: S \to S \) be a continuous mapping. We call \( f \) a (discrete) dynamic on \( S \), considering the totality of the movements of \( x \in S \) under \( f(x, f(x), f^2(x), f^3(x), \ldots) \) (we denote by \( f^v \) the \( v \) iterations of \( f \)). When we study the properties of one such \( f \) (e.g., the existence of fixed or periodic points), it is sometimes more convenient to study those of its topologically conjugate (TC) dynamic \( g \). Here the dynamic \( g: T \to T \) is TC to \( f \) if and only if a homeomorphism \( b: S \to T \) exists and \( b \circ f = g \circ b \). Clearly, \( g = b \circ f \circ b^{-1} \) is a continuous mapping, and there is a one-to-one correspondence between the orbit of \( x \) \( (x, f(x), f^2(x), \ldots) \) and the orbit of \( y = b(x) \) \( (y, g(y), g^2(y), \ldots) \), so that all the topologically interesting properties of \( f \) are inherited by \( g \), if \( g \) is TC to \( f \).

The methodology we want to call attention to in this paper is the use of what is called the topologically semiconjugate (TSC) dynamic. The notion of semiconjugacy is obtained by relaxing the definition and replacing “a homeomorphism \( b \)” with “a continuous \( b \)”. Formally,

**Definition 2.1.** A dynamic \( g: T \to T \) is TSC to \( f: S \to S \) if and only if there exists a continuous mapping \( b: S \to T \) such that \( b \circ f = g \circ b \).

Note that \( \dim T = \dim S \) (the dimension of \( T \) and \( S \), respectively), if \( g \) is TC to \( f \). A distinctive feature of TSC \( g \), if it exists, is that we can let \( \dim T \leq \dim S \), so that TSC \( g \) can have a lower dimensionality than that of TC \( g \).

Generally, a TSC \( g \) exists if and only if there is an equivalence relation \( \sim \) on \( S \) and \( \sim \) is compatible with \( f \) (i.e., the condition \( x \sim x' \Rightarrow f(x) \sim f(x') \) holds for any \( x, x' \in S \)). For then, endowing the quotient topology to the quotient set \( T = S/\sim \), the surjection \( b: S \to T \) naturally becomes continuous, and we have a mapping \( g: T \to T \), \( g(y) = b(f(b^{-1}(y))) \), which leads to \( g \circ b = b \circ f \). Necessity is clear if we let \( x \sim x' \Leftrightarrow b(x) = b(x') \) (note that \( b \) must be surjective).

In what amounts to the same thing, suppose we have a topological space \( S \), a dynamic \( f \) on \( S \), a topological (e.g., Euclidean) space \( T \), a dynamic \( g \) on \( T \), and a continuous function \( b: S \to T \) (i.e., continuous with respect to the given topologies of \( S \) and \( T \)) that is compatible with \( f \) (i.e., \( b \circ f = g \circ b \)). Then we can say that \( g \) is TSC to \( f \) under \( b \), identifying the set \( b(S) \) with the quotient set \( S/\sim \) and using the given topology of \( T \) for \( b(S) \).

For example, if we can choose a real-valued function \( b: S \to \mathbb{R} \) that makes the equivalence relation defined by \( x \sim_b x' \Leftrightarrow b(x) = b(x') \) compatible with \( f \), then we obtain a “one-dimensional” TSC dynamic.
We say that $g: T \to T$, identifying $T = b(S)$ with $S/\sim_b$. Figure 1 illustrates one such situation. The left-hand side shows a two-dimensional dynamic of “points” on a disc. Observe that the distance of each point from the center diminishes uniformly; hence, by classifying the points according to their distance from the center, we have a one-dimensional TSC dynamic of “radius” (the right-hand side). In the following, we will mainly be concerned with the cases in which $0 < \dim T < \dim S$, and especially in which $\dim T = 1$ for our example case.\(^1\)

As we stated in the introduction, we are looking at TSC $g$ as representing a macrostructure of $f$. Unfortunately, but naturally, though, TSC dynamics have less-appealing features than TC dynamics. For example, concerning the inheritance of the properties of $f$, we can at best say the following (in essence) at this state of generality. Suppose $g$ is TSC to $f$ under $b$. Then

(i) If $x$ is a periodic point of $f$ with prime period\(^2\) $k$, then $y = b(x)$ is a periodic point of $g$ with some prime period $k' \leq k$.

(ii) If $y$ is a periodic point of $g$ with prime period $k$, then $x \in b^{-1}(y)$ satisfies $f^{k'}(x) \sim_b x$.

Those assertions themselves are easily verified. For (i), assume that $f^k(x) = x$. By applying $b(f^k(x)) = g(b(f^{k-1}(x)))$ recursively, we have $b(f^k(x)) = g^k(b(x))$, and the periodic point of $g$, $y = b(x) = b(f^k(x))$. Generally, $k' \leq k$, since for $k \geq 2$, it is possible that $g^k(b(x)) = b(x)$ for some $k' < k$.

Assertion (ii) is immediate from $b(f^k(x)) = g^k(b(x)) = b(x)$.

To add a few remarks, the prime period is by definition a positive integer, so that if $k = 1$ in (i), $k' = 1$ is asserted (i.e., a fixed point of $f$, if any, is mapped to a fixed point of $g$). But a periodic point of $f$ with prime period $k > 1$, if any, may also be mapped to a fixed point of $g$. As (ii) indicates, we cannot infer in general from the existence of a periodic (or a fixed) point of $g$ the number of periodicity, nor even the existence, of a periodic (or a fixed) point of $f$.

This may explain why TSC dynamics have received less attention than TC dynamics; TSC dynamics are generally “loose” and do not generally enable us to infer the nature of $f$. On this point, however, we can prove the following lemma:

**Lemma 2.1a.** Suppose $g$ is TSC to $f$ under $b$: $S \to T$. If the set $b^{-1}(y) \subseteq S$ has the fixed-point property (i.e., if every continuous mapping from $b^{-1}(y)$ to itself has a fixed point) for all $y \in T$, then

(iii) If $y$ is a periodic point of $g$ with prime period $k$, then, in the set $b^{-1}(y)$, there is a periodic point of $f$ with prime period $k$.

---

\(^1\)It is clear that every $f$ has a meaningless TSC $g$ with $\dim T = 0$, given the compatible relation $x \sim x'$, $\forall x, x' \in S$; a TSC $g$ with $\dim T = \dim S$ is nothing but a TC dynamic of $f$. For this reason the interesting cases are only those where $0 < \dim T < \dim S$.

\(^2\)We say that $k$ is the prime period of a periodic point $x$ if $x = f^k(x)$ and $x \neq f^v(x)$ for all positive $v < k$. 

Takuya Iimura 305
Proof. Let \( g^k(y) = y \). As we stated in (ii) above, \( x \in b^{-1}(y) \Rightarrow f^k(x) \sim_b x \) (i.e., after \( k \) iterations of \( f \), every \( x \in b^{-1}(y) \) returns to the set \( b^{-1}(y) \)). This means that there is a restriction \( f^k|_{b^{-1}(y)}: b^{-1}(y) \to b^{-1}(y) \). Since we are assuming that \( f \) is continuous, \( f^k|_{b^{-1}(y)} \) is also continuous, so that if \( b^{-1}(y) \) has the fixed-point property, then there exists \( x \in b^{-1}(y) \) that satisfies \( x = f^k|_{b^{-1}(y)}(x) = f^k(x) \). The number \( k \) is also the prime period of \( x \), since if it were less than \( k \), so would be the prime period of \( y \) by (i).

Lemma 2.1a allows the nature of the original \( f \) to be inferable to some extent, in the sense that it enables us to infer the existence and periodicity of a periodic point of \( f \) from an observed periodic point of the lower-dimensional \( g \). Of course, the inferable nature of \( f \) is not exhaustive, since \( f \) may also have a periodic point of higher prime period. But for example, if we find a periodic point with prime period 3 in a continuous one-dimensional TSC \( g \) under the condition of Lemma 2.1a, then by the famous Sarkovskii’s theorem, \( g \) has periodic points of all prime periods, and we can infer that \( f \) has periodic points of all prime periods, too.

Actually, the only thing we needed in Lemma 2.1a and its proof was the existence of a fixed point of \( f^k|_{b^{-1}(y)} \). We thus state the following lemma (which does not rely on the property of the set \( b^{-1}(y) \)) without proof:

**Lemma 2.1b.** Suppose \( g \) is TSC to \( f \) under \( b \): \( S \to T \). If there exists a periodic point \( y \) of \( g \) with prime period \( k \), and the restriction of \( f^k \) to \( b^{-1}(y) \), \( f^k|_{b^{-1}(y)}: b^{-1}(y) \to b^{-1}(y) \), has a fixed point \( x \), then \( x \) is a periodic point of \( f \) with prime period \( k \).

Perhaps the most difficult problem in our methodology is that we have no general rule to find such \( g \), given \( f \); it is generally difficult to find an equivalence relation that is compatible with \( f \). Nevertheless, we will see in the next section a concrete economic example of one-dimensional TSC income dynamics, from which the nature of the original price dynamics is inferable. The principle behind this existence result is stated in the following lemma:

**Lemma 2.2.** Let \( S \) be a convex subset of \( \mathbb{R}^n \), \( f \in C^1(S) \), and \( J_f(x) \) be the Jacobian of \( f \) at \( x \). If \( f \) satisfies the condition

\[
(*) \text{ there exist a constant row vector } w \neq 0 \text{ and a constant } \lambda \neq 0 \text{ such that } w J_f(x) = \lambda w \text{ for all } x \in S
\]

then there exists a one-dimensional linear TSC dynamic \( g \in C^1(b(S)) \) of \( f \), with gradient \( \lambda \), under the linear transformation function \( b: S \to \mathbb{R} \). The converse also holds.

**Remarks.** For \( x \in S \) on the boundary of \( S \), if any, use the left or right derivatives. Apparently, the condition refers to a very special situation, but it seems to be a possible one in economic applications; see section 3.1 for the economic meaning of the condition \((*)\). Since \( g \) is linear, the prime period of any periodic point we can find in this \( g \) is at most 2; the more general case (in which \( \lambda \) is variable and the TSC dynamic is nonlinear) will be discussed in the paper’s final section.

**Proof.** Let \( w \) and \( \lambda \) be such that \( w J_f(x) = \lambda w , \forall x \in S \), and define \( b: S \to \mathbb{R} \) by \( b(x) = w \cdot x \). We first show that \( b(x') = b(x) \Rightarrow b(f(x')) = b(f(x)) \). To exclude the obvious case, let \( x' \neq x \) and define \( x(\theta) = \theta(x' - x) + x \), with \( \theta \in \mathbb{R} \) such that \( x(\theta) \in S \). Since

\[
\frac{d b(f(x(\theta)))}{d\theta} = w J_f(x(\theta))(x' - x) = \lambda w(x' - x) = 0
\]

the function \( b(f(x(\theta))) \) is constant over \( \theta \). Hence, letting \( \theta = 1 \), we have \( b(f(x')) = b(f(x)) \). The one-dimensional dynamic \( g \in C^1(b(S)) \) is well-defined by \( g(y) = b(f(x)) \), \( x \in b^{-1}(y) \), and satisfies \( b(f(x)) = g(b(x)) \) for all \( x \in S \). Differentiating this, we have \( w J_f(x) = dg(y)/dy \cdot w \) for all \( x \), so that \( dg(y)/dy = \lambda \). Letting \( b(x) = w \cdot x \), this also shows the necessity of the condition \((*)\).
3 Example: A Dynamic of Temporary Equilibrium with Loans and Consumption

3.1 General considerations

In this section, we will present a model of the dynamics of temporary equilibrium with loan and consumption markets. The purpose of this model is to show a concrete example of TSC dynamics in economic terms, as simply as possible. Hence, some of the assumptions may be rather restrictive (or in some cases ad hoc).

Within this limitation, however, we will see a one-dimensional linear TSC income dynamic that is associated with an $n$-dimensional, nonlinear, nondegenerate price dynamic (the case of nonlinear TSC dynamics will be discussed in the final section); the existence of a stationary temporary equilibrium (STE) will be inferred from this linear TSC dynamic. But before stating the specifications of our model, we will give a few general considerations that may also apply to other constructions than ours.

Suppose some dynamic of prices is given by an implicit form

$$z(q^{-1}, q') = 0, \quad \forall t \quad (3.1)$$

where $z = (z_1, \ldots, z_n)$, and $q^{-1}$ and $q'$ are the vectors of $\mathbb{R}^n_+$ representing the prices at $t - 1$ and $t$, respectively. Here we assume that $|J_q z| \neq 0$ and that the dynamic has an explicit form $q' = f(q^{-1})$. We also assume that the prices are normalized by using a numeraire (good 0).

In many cases, we can choose the excess demand function of the nonnumeraire goods as $z$, imposing the condition $z(q^{-1}, q') = 0$ (the excess demand at $t$ is zero). If Walras’s law is effective at $t$, and if $z_0(q^{-1}, q')$ is the excess demand for the numeraire, we have, for any $q^{-1}$ and for any $q'$,

$$q' \cdot z(q^{-1}, q') = -z_0(q^{-1}, q') \quad (3.2)$$

Differentiating both sides at $(q^{-1}, f(q^{-1}))$, with $q^{-1}$ and $q'$, respectively, we have

$$q' J_q z = -J_q z_0, \quad \text{and} \quad q' J_{q'} z = -J_{q'} z_0 \quad (3.3)$$

From the latter we have $q' = -J_q z_0[J_q z]^{-1}$, so that substituting it into the former,

$$- J_q z_0[J_q z]^{-1} J_{q'} z = -J_{q'} z_0 \quad (3.4)$$

or

$$J_q z_0 J_{q'} f(q^{-1}) = -J_{q'} z_0 \quad (3.5)$$

since $J_{q'} f(q^{-1}) = -[J_q z]^{-1} J_{q'} z$ given $z(q^{-1}, f(q^{-1})) = 0$.

From this observation, we can conclude that if $J_q z_0$ and $J_{q'} z_0$ are constant and proportional with a constant factor $-\lambda$ on any equilibrium path, then they are related as $w J_{q'} f(q^{-1}) = \lambda w$, and Lemma 2.2 implies the existence of a one-dimensional linear TSC dynamic. In addition, if $w$ is a strictly positive vector, then any class of $q \in \mathbb{R}^n_+$ given by $q \sim q' \Leftrightarrow w \cdot q = w \cdot q'$ is a compact and convex subset of $\mathbb{R}^n_+$, so that Lemma 2.1a implies the inferability of the original price dynamic.

The economic meaning of the above considerations is straightforward; if the current and past prices’ effects on the excess demand for the numeraire are constant and proportional at any equilibrium (a natural case may involve $\partial z_0/\partial q'_i > 0 > \partial z_0/\partial q_i^{-1}$ and $\partial z_0/\partial q'_i = -\lambda \partial z_0/\partial q_i^{-1}$ for all $i$), then we have the desired property. We think that this is not an unrealistic situation, especially if the numeraire is a kind of “money.”

---

1Although it is customary in the literature to study such problems in the framework of the “overlapping-generation” model (Samuelson 1958), we do not follow suit here (though some similarity exists). The reason is, in essence, that we need a situation in which the agents anticipate their future endowments (their incomes in real terms), in order for the TSC income dynamics to arise.
3.2 Description of the model

There are \( k \) consumers \((i = 1, \ldots, k)\), one central bank, and \( n \) markets for nonstorable goods \((l = 1, \ldots, n)\) in our model. The consumers live forever and have stocks of savings at the central bank. At the beginning of each period \( t \), \( i \) has a stock of savings \( s_{i,t-1} \in \mathbb{R} \) (or dissavings if it is negative), which he set in the previous period. He receives an endowment of the nonstorable goods \( w^i \in \mathbb{R}^n_{++} \), the flow of which is assumed to be constant over time. We assume that the one-period consumption set is \( X = \mathbb{R}^n_+ \) for every \( i \) and denote the total endowment \( \sum_{i=1}^{k} w^i \) by \( w \). No production is assumed.

Suppose we are now in period \( t \) and the prices of the goods are given by \( q^i = (q^i_1, \ldots, q^i_n) \in Q = \mathbb{R}^n_+ \), measured in the unit of savings. We assume that every consumer’s planning horizon is only one period ahead and that the intertemporal utility function of \( i \), \( u^i: X \times X \rightarrow \mathbb{R} \), is given by

\[
u^i(x^i_t, x^{i,t+1}) = u^i(x^i_t)u^i(x^{i,t+1})
\]

where \( x^i_t \) and \( x^{i,t+1} \) are the consumption at \( t \) and \( t+1 \), respectively, and

\[
u^i(x) = \prod_{l=1}^{n} x_l^i, \quad 0 < \alpha^i_l, \quad \sum_{l=1}^{n} \alpha^i_l = 1
\]

(Cobb-Douglas), which is homogeneous of degree one.

Now, letting \( s_{i,t} \in \mathbb{R} \) be the stock of savings (or dissavings) to be decided at \( t \), the decision problem of \( i \) is to choose \( x^i_t, x^{i,t+1}, \) and \( s_{i,t} \), to

\[
\text{maximize } u^i(x^i_t, x^{i,t+1}), \\
\text{s. t. } s_{i,t} + x^i_t - q^i \leq s_{i,t-1} + w^i - q^i, \\
s_{i,t} + x^{i,t+1} - q^{i,t+1} \leq s_{i,t} + w^i - q^{i,t+1}
\]

where \( q^{i,t+1} \) is the price expectation of \( i \). To simplify the matter, we assume, as a behavioral assumption, that

\[
\forall i, s_{i,t+1} = s_{i,t-1}
\]

That is, every consumer plans to recover the given stock of savings in the next period. Note that, by choosing \( s^i_t = s^i_{t-1} \), the consumer can always select \( x^i_t \in X \) (even when \( s^i_{t-1} + w^i - q^i < 0 \); this excludes the possibility of bankruptcy at \( t \)) and will be able to plan \( x^{i,t+1} \in X \) if \( s^{i,t+1} = s^{i,t-1} \). We assume that the price expectation of \( i \) is given by a \( C^1 \) function \( \psi^i: Q \times Q \rightarrow Q, \psi^i(q_{i-1}, q^i) = q^{i,t+1} \). We also assume that the consumers are sufficiently heterogeneous in their tastes \( v^i \) and endowments \( w^i \); this will be made more precise in section 3.4.

3.3 Existence and uniqueness of temporary equilibrium

We must first prove the existence of a temporary equilibrium in this model. This is done by making a rather standard assumption:

**Assumption 3.1.** For all \( i \), \( \psi^i \) satisfies the condition

\[
\forall q \in Q, \text{ the set } \psi^i(q, Q) \text{ is bounded}
\]

(This means that price expectations do not depend too much on the current prices, a condition that dates back to Grandmont 1974.)
A simple calculation\(^4\) shows that the optimum stock of savings \(s^{t\ast}\) and the optimum demand for goods \(x^{t\ast}\) for each \(i\) are given by

\[
s^{t\ast} = \frac{1}{2}(w^t \cdot q^t - w^t \cdot q^{t+1}) + s^{t-1}
\]

\[
x^{t\ast}_i = (w^t \cdot q^t + w^t \cdot q^{t+1}) \frac{a^t_i}{2q^t_i}, \quad l = 1, \ldots, n
\]  

(3.11)  

(3.12)

Hence, the individual excess-demand functions are given by

\[
z^t_i(q^{t-1}, q^t) = \frac{1}{2}(w^t \cdot q^t - w^t \cdot \psi_i(q^{t-1}, q^t))
\]

\[
z^t_i(q^{t-1}, q^t) = (w^t \cdot q^t + w^t \cdot \psi_i(q^{t-1}, q^t)) \frac{a^t_i}{2q^t_i} - w^t_i, \quad l = 1, \ldots, n
\]

(3.13)  

(3.14)

The excess-demand functions of the system are given by

\[
z_0(q^{t-1}, q^t) = \sum_{i=1}^k \frac{1}{2}(w^t \cdot q^t - w^t \cdot \psi_i(q^{t-1}, q^t))
\]

\[
z_l(q^{t-1}, q^t) = \sum_{i=1}^k (w^t \cdot q^t + w^t \cdot \psi_i(q^{t-1}, q^t)) \frac{a^t_i}{2q^t_i} - w^t_i, \quad l = 1, \ldots, n
\]

(3.15)  

(3.16)

Note that the system satisfies Walras’s law.

The temporary equilibrium price is a system of prices \(q^{\ast}\) that, for some \(q^{t-1} \in Q\), satisfies

\[
z(q^{t-1}, q^{\ast}) = 0
\]

(3.17)

(If \(z(q^{t-1}, q^{\ast}) = 0\), then \(z_0(q^{t-1}, q^{\ast}) = 0\) by Walras’s law.)

**Proposition 3.1.** Under Assumption 3.1, there exists a temporary equilibrium price \(q^{\ast}\) for any \(q^{t-1} \in Q\).

*Proof.* Use the standard argument (see, e.g., Grandmont 1977) on the set

\[
P = \{p \in \mathbb{R}^{n+1}_{++}, \|p\| = 1 \text{ (the sum norm)}\}
\]

(3.18)

---

\(^4\)Fix \(s^t \in \mathbb{R}\) temporarily and define \(\xi^t\): \(Q \to X\) by

\[
\xi^t(q) = \arg\max_{x^t} v^t(x) \text{ s. t. } x^t, q^t \leq 1.
\]

Here \(\xi^t(q) \neq q\) is well defined (\(\xi^t(q) = a^t_i/q^t_i, i = 1, \ldots, n\)). The optimum demand for goods \(x^{t\ast}\) and the optimum plan \(x^{t+1\ast}\) are then

\[
x^{t\ast} = (w^t \cdot q^t + q^{t-1} - s^t)\xi^t(q^t)
\]

\[
x^{t+1\ast} = (w^t \cdot q^{t+1} + s^t - s^{t-1})\xi^t(q^{t+1})
\]

Hence, the optimum stock of savings \(s^{t\ast}\) is uniquely determined as

\[
\arg\max_{s^t} v^t((w^t \cdot q^t + s^{t-1} - s^t)\xi^t(q^t)) = \arg\max_{s^t}(w^t \cdot q^t + s^{t-1} - s^t)(w^t \cdot q^{t+1} + s^t - s^{t-1})
\]

(we used the homogeneity of \(v^t\), which amounts to

\[
s^{t\ast} = \frac{1}{2}(w^t \cdot q^t - w^t \cdot q^{t+1}) + s^{t-1}
\]

The optimum demand for goods \(x^{t\ast}\) is also derived from here.
which is homeomorphic to $Q$ via $\Phi$: $Q \to P$, where $\Phi_0(q) = 1/(1 + q||)$ and $\Phi_i(q) = q_i/(1 + q||)$, $l = 1, \ldots, n$. It is easy to see from the construction and Assumption 3.1 that the excess demand for the savings and goods points inward as $\Phi(q')$ goes to the boundary of $P$. Indeed, for every $i$, the variable parts of the individual excess demand functions behave as follows:

$$\Phi_0(q') \to 0(||q'|| \to \infty) \Rightarrow u^i \cdot q' - u^i \cdot \psi^i(q'^{-1}, q') \to \infty$$

$$\Phi_i(q') \to 0(q'_{-i} \to 0) \Rightarrow (u^i \cdot q' + u^i \cdot \psi^i(q'^{-1}, q'))q'_i \to \infty, \quad \forall l$$

Hence, there exists $q^{*i}$ such that $z(q'^{-1}, q^{*i}) = 0$.

By inspecting the functional form of $z^i$ (Equation (3.14)), we can say that if every $i$’s income expectation $u^i \cdot \psi^i(q'^{-1}, q')$ does not depend too much on $q'$ (an obvious case is where every $i$’s $u^i \cdot \psi^i(q'^{-1}, q')$ is independent of $q'$), then $z^i$’s possess gross substitutability and $z$ also possesses gross substitutability, which entails the uniqueness of equilibrium price $q^{*}$. In the following, we will assume, in addition to Assumption 3.1, this insensitivity of the expected income for all $i$, guaranteeing the uniqueness of the temporary equilibrium price.

### 3.4 Price dynamics and TSC income dynamics

The above suggests that there is a price dynamic $f$: $Q \to Q$, $f(q'^{-1}) = q'$, which is implicitly given by the excess-demand function as

$$z(q'^{-1}, f(q'^{-1})) = 0$$

Since $|J_q z| \neq 0$, $f$ is $C^1$ and $Jf = -[J_q z]^{-1}J_q \cdot z$, by the implicit function theorem. Consider, then, the rank of $f$ (or the dimensionality of $f(Q)$). It is not necessary that $|J_q z| \neq 0$, since the Jacobian of individual excess demand with respect to $q'^{-1}$ is given by the $n \times n$ matrix

$$\frac{1}{2} \begin{pmatrix} \alpha^i_1/q^i_1 \\ \vdots \\ \alpha^i_n/q^i_n \end{pmatrix} \left( \frac{\partial (u^i \cdot \psi^i)}{\partial q'^{-1}} \ldots \frac{\partial (u^i \cdot \psi^i)}{\partial q'^{-1}} \right)$$

which is at most rank 1. However, if the consumers’ $\alpha^i = (\alpha^i_1, \ldots, \alpha^i_n)$ are sufficiently different from one another, and their $u^i$’s are also sufficiently different from one another to invoke sufficiently different income change prospects as $q'^{-1}$ varies, then the sum of these rank-1 matrices can be of rank $n$, the price dynamic then becomes nondegenerate. This is what we assumed above concerning the heterogeneity of the agents.\(^5\)

Now let us make an assumption on $\psi^i$ that induces a TSC income dynamic:

**Assumption 3.2.** Every consumer $i$ has a belief that the lower bound of his per period income is $z^i > 0$, and his $\psi^i$ is such that

$$\psi^i(q'^{-1}, q') = \lambda q'^{-1} + \frac{r^i}{u^i \cdot q'} q', \quad 0 < \lambda < 1$$

The weight $\lambda$ is identical for every $i$.

---

\(^5\)Generally, rank $(A + B) \leq$ rank $A$ + rank $B$. An example in $2 \times 2$ matrices is

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 4 & 5 \end{pmatrix}$$

The first two matrices are rank 1 and singular, but their sum is nonsingular.
This is certainly an ad hoc assumption, but it will not damage the point we want to make clear. The functional form is chosen so that the expected income \( w' \cdot q^{t+1} \) is equal to \( \lambda w' \cdot q^{t-1} + r' \), which is above \( r' \) and is positively correlated with the past income \( w' \cdot q^{t-1} \); the price expectation \( q^{t+1} \) is a positive combination of \( q^{t-1} \) and \( q' \). This satisfies Assumption 3.1 (recall that \( w' \in \mathbb{R}^* \)), and the expected income \( w' \cdot q^{t+1} \) is independent of \( q' \). Since \( J_{q^{-1}}\psi = \lambda I \), it is easy to see that the heterogeneity amounts to that of \( q' \)'s and \( w' \)'s.\(^6\)

Now let us examine the \( n \)-dimensional price dynamic under Assumption 3.2. Substituting the functional form of \( \psi' \) into the excess demand for the savings, we have

\[
  z_0(q^{t-1}, q') = \frac{1}{2}(w \cdot q' - \lambda w \cdot q^{t-1} - L) 
\]

where \( L = \sum_{i=1}^k L^i \). If we impose the condition \( z_0(q^{t-1}, q') = 0 \), we have an implicit expression of \( f \), though \( f \), is not determined from it, of course. But the fact that \( J_{q'} z_0 = (1/2) w \) and \( J_{q^{-1}} z_0 = -\lambda (1/2) w \) enables us to know, as we have already noted in the general considerations, that \( f \) has a one-dimensional linear TSC dynamic defined on the quotient set of \( Q \), where the classification is given by \( q \sim q' \iff w \cdot q = w \cdot q' \) (i.e., by the value of income). We thus have

**Proposition 3.2.** Let \( b : Q \to \mathbb{R}^+ \) be defined by \( b(q) = w \cdot q \). Under Assumption 3.2, there exists a one-dimensional linear income dynamic \( g : Q' \sim_b Q' \sim_b \) that is TSC to \( f : Q \to Q \). We can also infer from the existence of a fixed point of \( g \) the existence of a fixed point of \( f \), the stationary temporary equilibrium price.

Clearly, the explicit form of \( g \) is given from the excess-saving function by

\[
  r' = \lambda r^{t-1} + L
\]

where \( r' = w \cdot q' \) and \( r^{t-1} = w \cdot q^{t-1} \). It has a fixed point \( r^* = (1 - \lambda)^{-1} L \) and no other periodic point (see Figure 2). The inferability of an STE is verified as follows: Let \( q \in b^{-1}(r^*) \). Similarly to the proof of Proposition 3.1, this time the vector \( f|_{b^{-1}(r^*)}(q) - q \) points inward near the boundary of \( b^{-1}(r^*) \) (if \( q^t \to 0 \) and \( f(q^t) \to 0, q^{t-1}, f(q^{t-1}) \in b^{-1}(r^*) \), then the variable part of individual excess demand \( (w' \cdot q' + \lambda w' \cdot q^{t-1} + r')a_0/q_0 \to \infty \), a contradiction), so that the continuous function \( f|_{b^{-1}(r^*)} \) has a fixed point \( q^* \in b^{-1}(r^*) \).\(^7\) By Lemma 2.1b, this is an STE. Note that other periodic points of \( f \) (with prime period greater than 1), if any, may also be contained in the set of stationary income class.

### 4 Extensions of the Results

This section notes some possible extensions of our results. First, we presented a condition for the existence of one-dimensional linear TSC dynamics in Lemma 2.2. This lemma should be extended in two directions: that of higher-dimensional TSC dynamics and that of nonlinear TSC dynamics.

The first line of extension may not seem so attractive, since the higher the dimensionality of TSC dynamics, the smaller their significance as a device to summarize the original dynamics. As a device to analyze the originals, however, it might also be required. We note just a simple case that should also be considered in a generalization of Lemma 2.2 in this direction. Suppose, for example, that the excess demand for goods \( z \) is separable into the real goods \( z_R \) and the financial goods \( z_F \), \( z = (z_R, z_F) \), and that each group has its own proper vector, \( w_R, w_F \), and proper value, \( \lambda_R, \lambda_F \), representing the proportional effects of its past prices and

---

\(^6\)This type of price expectation implies a static expectation about income in the aggregate level, since \( \sum w' \cdot q^{t+1} = \lambda w' \cdot q^{t-1} + r \) by definition, which equals \( w' \cdot q' \) under the income dynamics to be derived below. The same holds for the function in Assumption 4.1.

\(^7\)See Dierker 1982, p. 799, for more on this type of existence argument, which also appeared in our proof of Lemma 2.2 (in the form of the market equilibrium lemma).
current prices on the current excess demand for the numeraire. That is,

\[ w_R f_R = \lambda_R w_R \quad \text{from} \quad -J_{q_R} z_0 J_{z_R}^{-1} J_{z_R}^{-1} z_R = -J_{z_R}^{-1} z_0 \]  
\[ w_F f_F = \lambda_F w_F \quad \text{from} \quad -J_{q_F} z_0 J_{z_F}^{-1} J_{z_F}^{-1} z_F = -J_{z_F}^{-1} z_0 \]

Then we have a two-dimensional linear dynamic \( g = (g_R, g_F) \),

\[ r'_R = g_F (r''_R), \quad g_R = \lambda_R \]  
\[ r'_F = g_F (r''_F), \quad g_F = \lambda_F \]

Now, consider the second line of extension of Lemma 2.2, returning to our temporary-equilibrium model. Let us make the following assumption:

**Assumption 4.1.** In addition to Assumption 3.2, every consumer has a belief that the upper bound of total income is \( \tilde{r} > 0 \). Consumers’ \( \lambda \) is a function of \( q^{-1} \):

\[ \lambda(q^{-1}) = \beta \tilde{r} - w \cdot q^{-1} > 0, \quad \beta > 0 \]  

Under this assumption, consumer \( i \)'s price expectation \( \psi^i \) becomes

\[ \psi^i(q^{-1}, q') = \beta \tilde{r} - w \cdot q^{-1} + \frac{r^i}{w^i \cdot q} q' \]
which leads to the expected income of \( i \)

\[
\bar{r} - w \cdot q_{t}^{i-1} w' \cdot q_{t}^{i-1} + l'
\]  

(4.7)

and the income dynamic

\[
r' = \frac{\beta}{\bar{r}} (r' - r(t-1)) r(t-1) + l
\]

(4.8)

This is a form of logistic mapping (shifted upward by \( l \)), and if \( \beta > 1 \), it generates many possibilities on its orbits (see Figure 2; the gradient equals \( \beta \) near 0). This dynamic (call \( g \)) is defined on the same quotient set as before, and the infeasibility of \( f \) is still valid: Let \( r^* \) be a periodic point of \( f \) with prime period \( k \) and \( q \in \bar{h}^{-1}(r^*) \); again \( f_{k} \mid_{\bar{h}^{-1}(r^*)}(q) - q \) points inward near the boundary of \( h^{-1}(r^*) \) (if \( q_{t}^{k} \rightarrow 0 \) and \( f_{k}^{i}(q_{t}^{k}) \rightarrow 0 \), then the variable part of individual excess demand \( (w' \cdot q' + (\beta/\bar{r})(\bar{r} - w \cdot q_{t}^{i-1}) w' \cdot q_{t}^{i-1} + l') \alpha_{j}/q_{t}^{j} \rightarrow \infty \), irrespective of the movement of \( q_{t}^{i-1} = f_{k}^{i}(q_{t}^{i-k}) \) in its class, a contradiction). Hence, for example, if we find a periodic point of prime period \( \beta \), for some values of \( \bar{r} \), \( l \), and \( \beta > 1 \), we can say that there are periodic points of all prime periods (chaos?) in the original temporary-equilibrium price dynamic.

Note that in Lemma 2.2, \( \lambda \) was a constant. However, if such a \( \lambda \) is continuously chosen as being invariant on each class (like \( \lambda(q_{t}^{i-1}) = \beta(\bar{r} - w \cdot q_{t}^{i-1})/\bar{r} \), then a one-dimensional TSC dynamic exists (under a linear-transformation function). This is verified by examining the proof of Lemma 2.2; again, the converse also holds.

We have derived the excess-demand functions satisfying the condition of Lemma 2.2 by means of exact aggregation. Alternatively, and more generally, we may be able to obtain such a demand system by means of a distributional approach, as exemplified by, for example, Hildenbrand 1983, Grandmont 1992, and Quah 1997. We may then be interested in the following question: Under what distribution of the characteristics of agents (their preferences, endowments, and expectations) could we expect to find a kind of “law of consumption and/or savings”?

Finally, we would like to note that our macrostructure emerges also as a consequence of certain maximising (or minimizing) processes. Consider an exchange economy whose aggregate demand is given by \( x(q_{t}^{i-1}, q^{*}) \). We can say, for example, that if any pair of successive temporary-equilibrium prices \( (q_{t}^{i-1}, q^{*}) \) is an interior solution for the problem

\[
\max_{q_{t}^{i-1}, q^{*}} x(q_{t}^{i-1}, q^{*}) \cdot q^{*} \quad \text{s.t. } w \cdot q_{t}^{i-1} = \text{constant}
\]

(4.9)

then there is a TSC income dynamic.\(^8\) Although this is not relevant to the model in this paper, and we do not know what kind of economic dynamic could be seen as such a maximizing (or minimizing, or saddle) process, it suggests that our viewpoint has an intrinsic importance (and a wider applicability) in the study of dynamics in general.

---

\(^8\)Proof: Let \( q^{*} \cdot x(q_{t}^{i-1}, q^{*}) + \lambda(c - w \cdot q_{t}^{i-1}) \) be the Lagrangian. Differentiating, we have (1) \( q^{*} \cdot J_{q} \cdot x(q_{t}^{i-1}, q^{*}) - \lambda^{*} w = 0 \) (row vector) and (2) \( x(q_{t}^{i-1}, q^{*}) + \lambda^{*} J_{q_{t}} \cdot x(q_{t}^{i-1}, q^{*}) = 0 \) (row vector). By (2),

\[
J_{q_{t}} \cdot x(q_{t}^{i-1}, q^{*}) = \lambda^{*} w
\]

(4.9)

(4.10)

(4.11)

The value of \( w \cdot f(q_{t}^{i-1}) = x(q_{t}^{i-1}, f(q_{t}^{i-1})) \cdot f(q_{t}^{i-1}) \) is determined by the value of \( w \cdot q_{t}^{i-1} \), and the value \( \lambda^{*} \) (now the marginal propensity to consume) is determined by the value of \( w \cdot q_{t}^{i-1} \) alone. Therefore, there exists a TSC income dynamic \( r' = g(r^{*}) \), where

\[
r' = w \cdot f(q_{t}^{i-1}), \quad r(t-1) = w \cdot q_{t}^{i-1}, \quad \text{and } dq(r^{*}) \text{d}r^{*} = \lambda^{*}(r(t-1)).
\]
References


