

Pricing, Variety, and Inventory Decisions
in Retail Operations Management

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(ABSTRACT)

This dissertation is concerned with decision making in retail operations management. Specifically, we focus on pricing, variety, and inventory decisions, which are at the interface of the marketing and operations functions of a retail firm. We consider two problems that relate to two major types of retail goods. First, we study joint pricing, variety, and inventory decisions for a set of “substitutable” items that serve the same need for the consumer (commonly referred to as a “retailer’s product line”). Second, we present a novel model of a selling strategy for “complementary” items that we refer to as “convenience tying,” and focus on analyzing the effect of this selling strategy on pricing and profitability. We also study inventory decisions under convenience tying and exogenous pricing.

For a product line of substitutable items, the retailer’s objective is to jointly determine the set of variants to include in her product line (assortment), together with their prices and inventory levels, so as to maximize her expected profit. We model the consumer choice process using a multinomial logit choice model and consider a newsvendor type inventory setting. We derive the structure of the optimal assortment for a special case where the non-ascending order of items in mean consumer valuation and the non-descending order of items in unit cost agree. For this special case, we find that an optimal assortment has a limited number of items with the largest values of the mean consumer valuation (equivalently, the items with the smallest values of the unit cost). For the general case, we propose a dominance rule that significantly reduces the number of different subsets to be considered

when searching for an optimal assortment. We also present bounds on the optimal prices that can be obtained by solving single variable equations. Finally, we combine several observations from our analytical and numerical study to develop an efficient heuristic procedure, which is shown to perform well on many numerical tests.

With the objective of gaining further insights into the structure of the retailer’s optimal decisions, we study a special case of the product line problem with “similar items” having equal unit costs and identical reservation price distributions. We also assume that all items in a product line are sold at the same price. We focus on two situations: (i) the assortment size is exogenously fixed, while the retailer jointly determines the pricing and inventory levels of items in her product line; and (ii) the pricing is exogenously set, while the retailer jointly determines the assortment size and inventory levels. We also briefly discuss the joint pricing/variety/inventory problem where the pricing, assortment size, and inventory levels are all decision variables.

In the first setting, we characterize the structure of the retailer’s optimal pricing and inventory decisions. We then study the effect of limited inventory on the optimal pricing by comparing our results (in the “risky case” with limited inventory) with the “riskless case,” which assumes infinite inventory levels. In addition, we gain insights on how the optimal price changes with product line variety as well as demand and cost parameters, and show that the behavior of the optimal price in the risky case can be quite different from that in the riskless case.

In the second setting, we characterize the retailer’s optimal assortment size considering the trade-off between sales revenue and inventory costs. Our stylized model allows us to obtain strong structural and monotonicity results. In particular, we find that the expected profit at optimal inventory levels is unimodal in the assortment size, which implies that the optimal assortment size is finite. By comparison to the riskless case, we find that this finite variety level is due to inventory costs. Finally,

for the joint pricing/variety/inventory problem, we find that even when the retailer has control over the price, finite inventories still restrict the variety level. We also propose several bounds that can be useful in solving the joint problem.

We then study a *convenience tying* strategy for two complementary items that we denote by “primary” and “secondary.” The retailer sells the primary item in an appropriate department of her store. In addition, to stimulate demand, the secondary item is offered in two locations: its appropriate department and the primary item’s department where it is displayed in very close proximity to the primary item. We analyze the profitability of this selling practice by comparing it to the traditional *independent components* strategy, where the two items are sold independently (each in its own department). We focus on understanding the effect of convenience tying on pricing. We also briefly discuss inventory considerations. First, assuming infinite inventory levels, we show that convenience tying decreases the price of the primary item and adjusts the price of the secondary item up or down depending on its popularity in the primary item’s department. We also derive several structural and monotonicity properties of the optimal prices, and provide sufficient conditions for the profitability of convenience tying. Then, under exogenous pricing, we find that convenience tying is profitable only if it generates enough demand to cover the increase in inventory costs due to decentralizing the sales of the secondary item.

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Chapter 1

Introduction

1.1 Motivations and Objectives

Integrating operations and marketing decisions is an important objective for firms in today's competitive environment. The interaction between operations management (OM) and marketing is clear. Marketing decisions drive the consumer demand, which is an input to the OM models that address issues such as capacity planning and inventory control. On the other hand, the Marketing Department of a firm relies on the OM cost estimates in making decisions concerning pricing, variety, promotions, etc. Therefore, developing joint operations and marketing models is a research objective that arises naturally. The interest in joint marketing/OM models is reflected in many works in the literature (see, for example, Eliashberg and Steinberg [23], Griffin and Hauser [35], Karmarkar [46], and Porteus and Whang [78]). In this dissertation, we study pricing, variety, and inventory decisions in retail operations management. Deciding on the prices and the breadth of items to be offered in a retail store is among the main functions of marketing. Moreover, inventory decisions that take into account the uncertainty in demand are the responsibility of OM. Our work thus contributes to the growing literature on joint marketing/OM models.

Within the spirit of an integrated marketing/OM approach, one of our main contributions is to

study the aforementioned pricing, assortment, and inventory decisions *jointly*. Under this integrative framework, the retailer sets two or more of the above decisions simultaneously. This seems to be a successful business practice for many retailers. For example, JCPenney received the “Fusion Award” in supply chain management for “its innovation in integrating upstream to merchandising and allocation systems and then downstream to suppliers and sourcing.” A JCPenney vice president attributes this success to the fact that, at JCPenney, “assortments, allocations, markdown pricing are all linked and optimized together” (Frantz [28]). Northern Group, the Canadian retailer, managed to get out of an unprofitable situation by implementing a merchandise optimization tool. Northern Group’s CFO credits this turnaround to “assortment planning” and the attempt to “sell out of every product in every quantity for full price” (Okun [71]).

More specifically, an important contribution of this research is to include inventory costs within pricing and assortment optimization models. Most of the previous literature along this avenue assumes infinite inventory levels and, therefore, excludes inventory considerations (see, for example, Aydin and Ryan [6], Dobson and Kalish [20], Green and Krieger [32], Kaul and Rao [48], and the references therein). We believe that this is due in part to the complexities introduced by inventory modeling. For example, the review paper by Petruzzi and Dada [76] indicates a high level of difficulty associated with joint pricing and inventory optimization even for the single item case. These difficulties do not, however, justify ignoring inventory effects in modeling. For example, in 2003 the average End-of-Month capital invested in inventory of food retailers (grocery and liquor stores) in the U.S. was approximately 34.5 Billion dollars, with an inventory/sales ratio of approximately 82% (U.S. Census Bureau [87]). On the other hand, the net 2003 profit margin in food retailing is estimated to be 0.95% (Food Marketing Institute [29]). With an inventory cost of capital commonly estimated at 20% (annually) or higher, these numbers indicate that food retailers can significantly

increase their profitability by reducing inventory costs.

We develop models that reflect the actual way consumer demand is generated in practice. For this purpose, we adopt state of the art demand models from the marketing and economics literature, which reflect the central role of pricing in consumer purchase decisions. We develop two models for a family of *substitutable* items that serve the same need for the consumer (commonly referred to as a “product line” or a “category”). In both models, consumer demand is generated based on the classical utility maximization principle. The consumer choice process is modeled by a Multinomial Logit Choice Model (MNL), which leads to a demand function where the demand for an item depends on its own price and consumer valuation as well as those of other items in the product line. This reflects the price/quality based substitution that the consumer is engaged in upon every visit to a retail store. We note that the MNL can be estimated from actual store sales data with relative ease, especially with the wide availability of business information software that dynamically tracks store operations (see, for example, Guadagni and Little [36]).

Finally, we present a third model for pricing and inventory decisions under convenience tying, where we perform a novel analysis of a selling strategy for *complementary items*. Complementary goods are on the other extreme of product lines of substitutable items. Thus, we intend to gain insights by comparing these extreme situations in future research. We note that the demand function for the convenience tying model is also developed in a realistic manner by aggregating consumer preferences with customers acting to maximize their surplus (utility).

In the remainder of this chapter, we provide details on the specific research problems that we consider in this dissertation.

1.2 Joint Pricing, Assortment, and Inventory Decisions for a Retailer's Product Line

Retailers display their goods in sets of items referred to as *product lines* or *categories*. The items in each product line serve the same basic need for the consumer (e.g., drinking coffee), but are different in some secondary features (e.g., flavor, aroma, color). These distinctions lead to different consumer valuation of each item in the product line. When faced with a purchase decision from a product line, a consumer selects her most preferred item, given the trade-off between price and quality. She may also choose not to buy any of the displayed items and postpone her purchase, or seek a different retailer. Pricing has a major impact on the consumer's choice among the available alternatives. However, other factors are also important. Such factors include the *assortment size* or *variety level*, in terms of which items are offered in the product line, and the shelf inventory levels of items in the product line.

We analyze the joint pricing, inventory, and assortment size decisions for a retailer's product line. Our objective is to characterize the structure of the retailer's optimal assortment and to gain insights into the combined effect of pricing, inventory, and variety on the profitability of a product line. We are also interested in developing easily implementable and effective methods for practical applications that generate profitable solutions for the retailer's product line problem. For this purpose, we develop a multi-item inventory model with stochastic and price-dependent demand. The randomness of the demand in our model is due to uncertainty in both the number of customers arriving and the valuations of those customers for the offered products.

Our model is developed with the following assumptions. Consumers arrive at the retailer's store during a selling period. A consumer chooses at most one item from the product line based on price and quality only, independently of the inventory status at the time of her arrival. This assumption

implies that if a consumer's most preferred item is out of stock, then the consumer leaves the store empty handed, without considering the purchase of other available items. That is, we consider a "static choice model" with no stockout based substitution. This assumption is mainly for analytical tractability and is common in the literature (see Mahajan and van Ryzin [57] for a discussion of this assumption and other aspects of the inventory and consumer choice problem).

Consumers act to maximize their surplus (utility) defined as the difference between the consumer reservation price (valuation) and the retail price of an item. The consumer surplus is determined based on the well-known Multinomial Logit Choice Model (MNL) (see, for example, Anderson and de Palma [2], Ben-Akiva and Lerman [10], Manski and McFadden [61], and McFadden [64]). The MNL is widely utilized as a consumer choice model due to the following reasons: (i) The MNL yields closed-form expressions of the purchase probabilities of items in the product line, leading to tractable analytical models; and (ii) it is easy to statistically estimate the parameters (and test the goodness of fit) of the MNL based on data from actual store transactions, especially with the wide spread of technology that tracks such transactions (see, for example, Guadagni and Little [36], Hauser [41], McFadden [64], and McFadden et al. [65]). We point out that these references indicate that the MNL predicts product line demand with high accuracy. The interest in the MNL is reflected in many recent works on variety, inventory, and pricing models. Examples of these works include Aydin and Ryan [6], Aydin and Porteus [7], Besanko et al. [11], Cattani et al. [15], Hanson and Martin [39], Hopp and Xu [43], and van Ryzin and Mahajan [80]. By utilizing the MNL, we derive the demand distribution for items in a product line in a realistic fashion by aggregating the individual consumer choices.

To model the inventory costs, we consider a newsvendor type setting where the items in a product line are to be sold within a single time period with leftover inventory not carried to subsequent

periods. This type of inventory models is utilized in many works similar to ours. Examples include Aydin and Porteus [7], Cattani et al. [15], Gaur and Honhon [31], Netessine and Rudi [70], Smith and Agrawal [85], and van Ryzin and Mahajan [80]. Moreover, Smith and Agrawal [85] cite several studies which indicate that the newsvendor model is suitable for many retail systems that utilize Electronic Data Interchange. We note finally that the newsvendor type model provides a basis that can be built on to extend our work to more sophisticated (multi-period) inventory models.

Under the above assumptions, we derive the structure of the optimal assortment for a special case where the non-ascending order of items in mean consumer valuation and the non-descending order of items in unit cost agree. For this special case, we find that an optimal assortment has a limited number of items with the largest values of the mean consumer valuation (equivalently, the items with the smallest values of the unit cost). For the general case, we propose a dominance rule that significantly reduces the number of different subsets to be analyzed when searching for an optimal assortment. We also present bounds on the optimal prices that can be obtained by solving single variable equations. Finally, we combine several observations from our analytical and numerical study to develop an efficient heuristic procedure, which is shown to perform well on many numerical tests.

Chapter 3 of this dissertation studies the above problem in detail.

1.3 Joint Pricing, Assortment, and Inventory Decisions for a Retailer’s Product Line: A Special Case

In the second part of this dissertation, we make certain simplifications to the model presented above to allow for the analysis of the complex effects of variety, pricing, and limited shelf inventory within a simplified framework. Our objective is to gain insights through the analysis of this stylized model. In particular, we consider a situation where all the items that may be included in the product line

have the equal unit costs and identical consumer reservation prices. In this stylized model, variety and profitability are determined only by the number of items in an assortment.

Although stylized, this model may nevertheless apply to certain situations where the items of a product line are distinguishable by a minor attribute. For example, in a product line of clothing items belonging to the same broad color group (such as the reds or the greens, etc.), it is likely that the items will have the same cost structure and similar consumer valuations.

All the assumptions made in Section 1.2 hold here. We make the following additional assumptions. We assume that all the items in the product line are to be sold at the same price. This assumption serves to simplify the problem further. We note that our extensive numerical experimentation suggests that such a pricing structure is optimal (although an analytical proof is lacking).

We focus on two situations: (i) the assortment size is exogenously fixed, while the retailer jointly determines the pricing and inventory levels of items in her product line; and (ii) the pricing is exogenously set, while the retailer jointly determines the assortment size and inventory levels. We also briefly discuss the joint pricing/variety/inventory problem where the pricing, assortment size, and inventory levels are all decision variables.

The first setting allows us to characterize the structure of the retailer's optimal pricing and inventory decisions for a given assortment. We then study the effect of limited inventory on the optimal pricing by comparing our results (the "risky case") with the "riskless case," which assumes infinite inventory levels. In addition, we gain insights on how the optimal price changes with product line variety as well as demand and cost parameters, and show that the behavior of the optimal price in the risky case can be quite different from that in the riskless case.

Considering the second setting, we characterize the retailer's optimal assortment size (variety level) considering the trade-off between sales revenue and inventory costs: While a high variety

increases the overall demand for the retailer’s product line, it also leads to thinning of demand for each individual item (due to cannibalization), resulting in possibly higher inventory costs (van Ryzin and Mahajan [80]). Our stylized model allows us to obtain strong results on the finiteness of the optimal assortment size and on how demand and cost parameters and the market price affect the retailer’s optimal variety level.

Finally, we briefly discuss the joint pricing/variety/inventory problem and find that even when the retailer has control over both the price and the variety level, finite inventories still restrict the variety level. We also propose several bounds that can be useful in solving the joint problem.

Chapter 4 of this dissertation studies the above problem in detail.

1.4 Pricing and Inventory Decisions under Convenience Tying

Retailers utilize various selling strategies to benefit from the complementarity in consumption of some items. Among the widely studied techniques are those involving “bundling” where one item is packaged with one or more complementary items and the whole package is sold for one price (see, for example, Eppen et al. [24]). However, retailers often “tie” two complementary items together by physically displaying them in near proximity in order to induce customers to buy the two items together (and hence expand the demand). More specifically, a “secondary” item (e.g., cakes) is “tied-in” to a primary item (e.g., berries) by displaying it next to the primary item in the appropriate location of the latter (e.g., the produce department). In addition, the secondary item is also sold in its own appropriate location (e.g., the bakery department) for customers who do not consume it in conjunction with the primary item. We refer to this selling strategy as *convenience tying*.¹

¹The economics literature defines tying as the situation where a firm “makes the sales (or price) of one of its products conditional upon the purchaser also buying some other product from it” (Whinston [89], see also, Burstein [14] and Bowman [13]). This is also known as *line forcing*. The selling strategy we consider does not require such line forcing since the customer freely chooses to buy the tied-in item at her own convenience (hence, the term “convenience tying”).

The convenience tying practice seems to be gaining a wide popularity among retailers. For example, while walking through the aisles of our local Wal-Mart or Kroger stores, we notice several items tied together. Examples include beer (primary) and lemon (secondary) displayed in overhanging baskets, chips (primary) and dippings (secondary) placed on small shelves encastrated in the chips shelves, milk (primary) and cereals (secondary) offered in separate shelves just next to the fridges where milk is sold, etc. Moreover, many retailers are showing interest in understanding and profitably implementing convenience tying. In fact, this part of the dissertation is motivated by the author's work on a berries and cakes pricing and demand forecasting problem for a large chain of grocery stores in New England.

Important questions facing a retailer engaging in the convenience tying practice include selecting which items to tie together and deciding on the prices and the inventory levels for these items. To the best of our knowledge, the above questions (and apparently the entire concept of convenience tying), have not been studied in the academic literature. In this dissertation, we take the first step in this research direction. Our objectives are (i) to gain insights into the convenience tying practice through an analytical model, and (ii) to provide possible answers to the foregoing questions.

Our model is developed with the following assumptions. Consumers arrive to the retailer store and choose to buy the primary item only, the secondary item only, both items, or neither, in a way as to maximize their surplus (utility) similar to the aforementioned models on retail product lines. Consumer reservation prices are random and, to simplify the analysis, are uniformly distributed. This allows us to develop the demand distribution by aggregating consumer preferences in a realistic manner. The secondary item is sold at the same price in both locations where it is offered. Moreover, customers who buy it in the primary item's location are those willing to buy the primary item first. Hence, the demand for the secondary item in the primary item's location is a fraction of the demand

of the latter (and therefore it depends on the price of the primary item in addition to its own price). This leads to “cross-price elasticity effects” between the primary and secondary items. As a result, their prices (that maximize the retailer’s profit) should be determined jointly.

We focus our analysis on the pricing implications of convenience tying under the assumption that inventory levels of both the primary and the secondary item are infinite.² This setting is common in the related literature on bundling of complementary items (see the references in Section 2.2). In addition, this assumption holds in certain practical situations, when, for example, the inventory levels are always sufficiently high due to the marketing requirements of a full shelf. To gain insights into the inventory implications of convenience tying, we briefly study another situation, where the prices of both the primary and secondary items are exogenous and the inventory levels are set optimally within a newsvendor type inventory model. In both situations, we compare convenience tying to the classical “independent components” strategy where the two items are sold independently each in its appropriate department.

In the first setting with ample inventories, we find that convenience tying leads to a lower price (than under independent components) of the primary item in order to increase the demand volume for the secondary item. On the other hand, the change in the price of the secondary item depends on how its consumer valuation shifts when it is tied-in to the primary item. This may be understood by thinking of an overall (system) consumer valuation that dictates the price of the secondary item (which is sold at the same price in the two echelons of the system). We also derive sufficient conditions for the profitability of convenience tying, and perform a detailed comparative statics analysis on the effect of changing demand and cost parameters on the optimal prices. Even though the problem of finding the optimal prices generally has no closed-form solution, we show that this problem is “well-

²Since, apparently, convenience tying was not analyzed in the previous literature, we believe that this is a necessary step to initiate its study.

behaved” in the sense that the optimal prices are the unique solutions to the first-order optimality conditions under some reasonable assumptions.

In the second situation with exogenous prices and limited inventory, we find that convenience tying is profitable only if it leads to a higher total demand volume relative to independent components. Specifically, the increase in total demand should be large enough to cover the additional inventory costs, which arise as a consequence of demand decentralization under convenience tying (see, for example, Eppen [25]). We also discuss the effect of the primary item stockouts on the demand function under convenience tying.

Chapter 5 of this dissertation studies the above problem in detail.

Chapter 2

Literature Review

In this chapter, we provide a brief overview of the literature that is most relevant to the problems of interest in this dissertation. Specifically, Section 2.1 presents a brief literature review on product line pricing, inventory, and variety decisions, while Section 2.2 surveys some of the well-known quantitative works on bundling (which relates to our convenience tying model).

2.1 Review of the Literature on Product Line Pricing, Variety, and Inventory Decisions

The literature on this area is at the interface of economics, marketing, and operations management (OM). Mahajan and van Ryzin [57] present a comprehensive review of this literature. The economics literature approaches this topic from the point of view of product differentiation (see Lancaster [50] for a review). The focus of this literature is on developing consumer choice models that reflect the way consumers make their purchase decisions from a set of differentiated products (see, for example, Hotelling [45], Lancaster [51], and McFadden [64]). The Multinomial Logit Choice Model (MNL) that we utilize is among the most popular consumer choice models (see, for example, Anderson and de Palma [2] and Ben-Akiva and Lerman [10]). Apparently, the MNL has its roots in Mathematical

Psychology (see, for example, Luce [55] and Luce and Suppes [56]). It has also been widely used to model travel demand in transportation systems (see, for example, Domencich and McFadden [22]). The economics literature also utilizes the MNL and other consumer choice models in modeling variety within a market-equilibrium framework in a market with many firms selling different products (see, for example, Anderson and de Palma [3] and [4]).

The marketing literature emphasizes data collection and model fitting issues (see, for example, Besanko et al. [11], Guadagni and Little [36], and Jain et al. [37]). The data is usually collected based on actual consumer behavior compiled from scanner data (log of all sales transactions in a store) and panel data (obtained by tracking the buying habits of a selected group of customers). A popular technique for measuring consumer utilities from store data is conjoint analysis (see, for example, Green and Krieger [34]). Several works in the marketing literature address the problem of product line design (in terms of what items to offer to consumers, i.e., variety decisions) and pricing utilizing data obtained from conjoint analysis (see Green and Krieger [32] and Kaul and Rao [48] for reviews). A typical approach is to utilize deterministic estimates of utilities of the consumer segments and formulate the resulting problem as a mixed integer program with the objective of maximizing the firm's profit subject to consumer utility maximization constraints (see, for example, Dobson and Kalish [20] and [21] and Green and Krieger [33]). Other works on product line design and pricing include Hanson and Martin [39], Moorthy [68], Mussa and Rosen [69], and Oren et al. [72].

In the following, we review with some detail the recent works on product line pricing, variety, and inventory decisions that are mostly related to our work. Van Ryzin and Mahajan [80] consider the problem of inventory and assortment decisions for a product line under an MNL consumer choice process, while assuming that the prices are exogenously determined. They demonstrate that the optimal assortment has a simple structure having items with the largest mean consumers valuations.

Van Ryzin and Mahajan find that a large assortment size (high variety level) is desired if either the price, or the no-purchase utility, or the sales volume are sufficiently high. Aydin and Ryan [6] consider the problem of pricing and assortment size decisions for a product line in the riskless case (i.e., assuming infinite inventory levels) under an MNL consumer choice process. They show that the optimal prices can be characterized by equal profit margins, with the expected profit being unimodal in the common margin. They further find that the optimal profit margin and the expected profit are increasing in the average margin of an item in the product line, where the average margin is defined as the difference between the mean consumer valuation of an item and its unit cost. Our work may be seen as an extension of van Ryzin and Mahajan [80] and Aydin and Ryan [6] in the sense that we study the problem of joint inventory, pricing, and assortment size decisions for a product line under MNL choice. Hanson and Martin [39] consider a model similar to that of Aydin and Ryan [6] under a fairly general form of the MNL choice model. They show that the expected profit function is not, in general, concave or even quasiconcave in the prices of items in an assortment, and they propose a numerical search technique to determine the optimal prices.

We are also aware of many very recent works that are closely related to ours. Aydin and Porteus [7] consider a model similar to ours (see Section 1.2), with MNL purchase probabilities and a newsvendor type inventory model. They show that the problem of finding the optimal prices is well-behaved in the sense that the first-order optimality conditions have a unique solution, and derive some monotonicity properties of the optimal prices and inventory levels. Aydin and Porteus also prove that the optimal price of an item is increasing in the item's own unit cost and decreasing in the unit costs of the other items in an assortment. The results in Aydin and Porteus [7] do not apply to our model because of differences in demand assumptions. Aydin and Porteus consider a multiplicative demand model where the total (stochastic) demand is deterministically split between the products according

to the MNL purchase probabilities. This implies that the coefficient of variation of the demand for an item is independent of the pricing. Our model is different in the sense that the total demand is split between the different products in a probabilistic way leading to a demand function with a coefficient of variation that depends on the prices of all items in the assortment in a complex form. Moreover, Aydin and Porteus do not address the optimal assortment problem that we consider in detail. Despite the complex nature of our demand function, we derive some important structural properties of the optimal assortment as detailed in Chapter 3.

Cattani et al. [15] consider two products (custom and standard) under an MNL consumer choice process with the objective of determining the optimal product prices and capacity levels for a dedicated and a flexible resource. Through a set of assumptions, the problem is reduced to the problem of finding the optimal prices and inventory levels for two products in a newsvendor setting under price dependent demand, similar to our model (see Section 1.2). They propose a heuristic solution procedure to determine the optimal prices and inventory levels for the two products. Their heuristic iterates between a marketing (riskless) model that sets the prices (assuming infinite inventory levels) and an operations model that sets the inventory levels (assuming prices are fixed). The approach of Cattani et al. [15] is mostly numerical. Moreover, they do not address the optimal assortment problem like we do.

Hopp and Xu [43] consider an MNL-based model for a product line with the expected profit consisting of sales revenues minus operations cost, where the latter is modeled as an increasing function of the number of items in an assortment. Under these assumptions, they derive several properties of the optimal prices and assortment size. They further discuss risk attitudes of the retailer and markets with multiple segments of consumers. They find that, in the case of a market with a single customer segment, both the optimal variety level and the optimal price increase if

either the fixed cost or variable cost of an item is reduced. Hopp and Xu also conclude that risk averse firms should not offer a high variety level. Gaur and Honhon [31] consider a problem similar to that of Van Ryzin and Mahajan [80], but utilize the Lancaster consumer choice model instead of the MNL.

Several authors study the problem of pricing and/or inventory decisions of a product line considering competition between retailers under consumer choice processes (see, for example, Anderson and de Palma [3], Besanko et al. [11], Hopp and Xu [44], and Mahajan and van Ryzin [59]). The problem of determining inventory levels considering stockout based substitution in a product line has also received considerable attention in the recent literature (see, for example, Agrawal and Smith [85], Netessine and Rudi [70], and Mahajan and van Ryzin [58]).

Finally, the works on single item inventory models with price dependent demand are also relevant to our research. Examples of these works include Chen and Simchi-Levi [17], Federgruen and Heching [27], Karlin and Carr [47], Mills [67], Petruzzi and Dada [76], Whitin [90], and Young [91].

2.2 Review of the Literature Related to Convenience Tying

The literature on bundling (i.e., the sales of items jointly as a bundle or a package) is broadly related to the convenience tying problem described in Section 1.4. Most of the quantitative literature deals with bundling as a price discrimination tool that allows the retailer to extract more consumer surplus (see, for example, Adams and Yellen [1] and Stigler [86]). Adams and Yellen (AY) consider the following selling strategies for two items which are independent in demand: (i) *pure components*, when the two items are sold separately; (ii) *pure bundling*, if the two items are sold as a package only; and (iii) *mixed bundling*, when the items are offered both separately and as a package. The convenience tying strategy we study in this dissertation may be seen as an additional selling alternative to the

these strategies. AY find that mixed bundling is more profitable than the two other forms of sales in most situations, and that negative correlation of reservation prices of the two items encourages some form of bundling.

Several papers extend the work of AY. In particular, Schmalensee [83] applies the AY framework to the case where one of the products is sold competitively, while the other is controlled by a monopolist. Dansby and Conard [18] and Lewbel [52] extend the AY model to handle items that are substitutable or complementary in demand. Paroush and Peles [75] consider a model with linear, price dependent demand, and compare the pure components and the pure bundling strategies. Schmalensee [82] considers the AY model with the reservation prices following a bivariate Normal distribution and derive conditions under which bundling is more profitable than pure components. Long [54] and McAfee et al. [62] derive conditions under which mixed bundling is more profitable than pure components for any reservation price distribution. Pierce and Winter [77] illustrate analytically and empirically that pure bundling may be more profitable than mixed bundling. Salinger [81] compares pure bundling and pure components under somewhat general settings.

More recently, Bakos and Brynjolfsson [8] consider the problem of bundling information goods, with zero marginal costs for each good, and prove that pure bundling is asymptotically optimal as the number of bundled goods increases. Hanson and Martin [40] analyze bundles of two or more components, while assuming that consumer reservation prices are well-known (deterministic). They formulate the problem as an integer program and propose an efficient solution procedure. Ansari and Weinberg [5] and Venkatesh and Mahajan [88] study the profitability of bundling in the entertainment sector, with season tickets and single-event tickets being sold for a series of performances.

In terms of cost side implications of bundling, very limited work has been done. In a recent paper, Ernst and Kouvelis [26] investigate the effect of bundling on inventory costs. They argue that

the benefits of bundling stem from its ability to allow demand substitution at stockout situations. In another recent work, McCardle et al. [63] propose a joint inventory and pricing model for pure bundling assuming uniformly distributed reservation prices. We also note that Hanson and Martin [40] and Salinger [81] briefly discuss some cost aspects of bundling.

Other works that are also related to convenience tying are on the “loss leader” selling practice (see, for example, Hess and Gerstner [42], Lal and Matutes [49], and the references therein). Loss leader pricing is a strategy where a retailer sells an item at or below cost in order to increase store traffic and, consequently, generate high profits from other items that can be sold at sufficiently large profit margins. Convenience tying can be seen as a loss leader strategy if the primary item is priced below its unit cost (in Chapter 5, we show that such situations may indeed be profitable). Hess and Gerstner [42] study a loss leader strategy with rain checks. In this strategy, a customer who finds an advertised loss leader item out of stock is given a rain check which entitles her to buy the item at the same reduced price in a future date. They utilize a game theoretic model with multiple firms in the market competing for the sales of one “shopping good” (candidate to be a loss leader) and a set of “impulse goods” (defined as items bought on the spot without price comparison with other stores). Hess and Gerstner show that retailers offering rain checks may deliberately run out of stock on loss leaders in order to have customers visit their stores a second time.

Lal and Matutes [49] consider a duopoly model with two competing firms selling two products one or both of which could be advertised (and the customers become aware of prices of the advertised items before visiting the store). They model the competition between the two firms as a multi-stage game that includes advertising and pricing decisions of the retailer as well as the customer’s rational for choosing one store over the other. Lal and Matutes show that a loss leader strategy could exist in market equilibrium.

Chapter 3

Joint Pricing, Inventory, and Assortment Decisions for a Retailer's Product Line

In this chapter we consider the problem of joint pricing, inventory and assortment decisions for a retailer's product line as introduced in Section 1.2. This chapter is organized as follows. Section 3.1 introduces the basic model and assumptions. Section 3.2 presents structural properties of the optimal assortment. Section 3.3 discusses structural properties and bounds on the optimal prices. Finally, Section 3.4 presents an efficient heuristic procedure that is suited for practical applications.

3.1 Model and Assumptions

Let $\Omega = \{1, 2, \dots, n\}$ be the set of possible variants from which the retailer can compose her product line. Let $S \subseteq \Omega$ denote the set of items stocked by the store. Demand for items in S is generated from customers arriving to the retailer's store during a single selling period. A customer chooses to purchase *at most* one item from set S so as to maximize her "surplus," which is the difference between her reservation price and the retail price of an item. We adopt a Multinomial Logit Choice Model (MNL) with the consumer surplus (utility) for item $i \in S$ given by $U_i = \alpha_i - p_i + \epsilon_i$, and

the utility of the no-purchase option given by $U_0 = u_0 + \epsilon_0$, where p_i and α_i respectively denote the retail price and the mean reservation price (consumer valuation) of item $i \in S$, u_0 is the mean utility for the no-purchase option, and $\epsilon_i, i \in S \cup \{0\}$, are independent and identically distributed Gumbel random variables with mean 0 and shape factor μ .¹ (See Johnson et al. [38] and Patel et al. [74] for details on the Gumbel distribution.)

The probability that a consumer buys item $i \in S$ is given by $q_i(S, \mathbf{p}) = Pr\{U_i = \max_{j \in S \cup \{0\}} U_j\}$, and the no-purchase probability is given by $q_0(S, \mathbf{p}) = 1 - \sum_{j \in S} q_j(S, \mathbf{p})$, where $\mathbf{p} = (p_1, \dots, p_{|S|})$ is the price vector corresponding to items in S , with $|S|$ denoting the cardinality of set S . Utilizing the expression for the Gumbel distribution function and simplifying (see, for example, Anderson et al. [2], pp. 29-42), it can be shown that $q_i(S, \mathbf{p}), i \in S \cup \{0\}$, can be expressed as

$$q_i(S, \mathbf{p}) = \frac{e^{(\alpha_i - p_i)/\mu}}{v_0 + \sum_{j \in S} e^{(\alpha_j - p_j)/\mu}}, \quad i \in S, \quad q_0(S, \mathbf{p}) = \frac{v_0}{v_0 + \sum_{j \in S} e^{(\alpha_j - p_j)/\mu}}, \quad (3.1)$$

where $v_0 = e^{u_0/\mu}$.

We adopt a demand model similar to that in Cattani et al. [15] and van Ryzin and Mahajan [80]. Denote by λ the mean number of customers arriving during the selling period. Under the assumptions that (i) consumers make their purchasing decisions independently of the inventory status at the moment of their arrival, and (ii) they will leave the store empty-handed if their preferred item (in S) is out of stock (i.e., there is no stockout based substitution), the expected demand for item $i \in S$ is $\lambda q_i(S, \mathbf{p})$. Then, the demand for item $i \in S$, X_i , is assumed to be a Normal

¹The model parameters are commonly estimated by assuming $\alpha_i = \sum_{j \in T_i} \beta_{ji} x_{ji}$, $i \in \Omega$, where T_i is the set of “attributes” corresponding to item i , x_{ji} is the observed value of attribute j for item i (as measured from actual consumer behavior data), and β_{ji} is the “utility weight” of attribute j for item i . The coefficients β_{ji} are evaluated using maximum likelihood estimation. See Guadagni and Little [36] and McFadden [64] for details. Guadagni and Little [36] further indicate that parameters such as u_0 and μ can be “absorbed” into estimates of the α_i by adequately scaling the model. In this dissertation, we assume that the MNL parameters have been estimated accurately, and address the retailer’s problem of making operational decisions based on these estimates.

random variable with mean $\lambda q_i(S, \mathbf{p})$ and standard deviation $\sqrt{\lambda q_i(S, \mathbf{p})}$, which represents a Normal approximation to demand generated from customers arriving according to a Poisson process with rate λ per selling period. We note that all the subsequent results would also extend to the case where X_i is a Normal random variable with mean $\lambda q_i(S, \mathbf{p})$ and standard deviation $\sigma(\lambda q_i(S, \mathbf{p}))^\beta$, where $\sigma > 0$ and $0 \leq \beta < 1$, with the coefficient of variation of X_i being decreasing in λ (as in van Ryzin and Mahajan [80]).

In our model, the coefficient of variation and standard deviation of X_i are given by $1/\sqrt{\lambda q_i(S, \mathbf{p})}$ and $\sqrt{\lambda q_i(S, \mathbf{p})}$, respectively. That is, both the demand standard deviation and coefficient of variation are functions of the price vector of the product line. Most of the literature on joint inventory and pricing models assumes that the demand is either “additive,” with the demand standard deviation being independent of the price, or “multiplicative,” with the demand coefficient of variation not depending on the price (see, for example, Petruzzi and Dada [76]). In that sense, our demand model may be seen as “mixed multiplicative/additive.” This follows from the fact that $X_i = \lambda q_i(S, \mathbf{p}) + \sqrt{\lambda q_i(S, \mathbf{p})} Z_i$, where Z_i are independent and identically distributed random variables with a standard Normal distribution. Young [91] considers a demand with a similar multiplicative/additive structure for the case of a single item. However, even for the single item case, none of Young’s results are applicable to our model because Young makes restrictive assumptions on the range of cost and demand parameters.

Our model may be seen as a multi-item newsvendor model with items having Normal demands, under the additional complexities of pricing and assortment decisions. On the cost side, we assume that items of the product line do not have a salvage value and no additional holding or shortage costs apply (as in Aydin and Porteus [7] and van Ryzin and Mahajan [80]). We note that the essence of inventory costs in terms of overage and underage costs are captured here. In addition, these

assumptions can be easily relaxed to include holding and shortage costs and salvage values without changing the structure of our results. By utilizing the well-known results for the newsvendor model under Normal demand (see, for example, Silver et al. [84], pp. 404-408), we can write the optimal inventory level for item $i \in S$, $y_i^*(S, \mathbf{p})$, and the expected profit from S at optimal inventory levels, $\Pi(S, \mathbf{p})$, as:

$$y_i^*(S, \mathbf{p}) = \lambda q_i(S, \mathbf{p}) + \Phi^{-1}(1 - c_i/p_i) \sqrt{\lambda q_i(S, \mathbf{p})}, \quad i \in S, \quad (3.2)$$

$$\Pi(S, \mathbf{p}) = \sum_{j \in S} \left[\lambda q_j(S, \mathbf{p})(p_j - c_j) - p_j \sqrt{\lambda q_j(S, \mathbf{p})} \phi(\Phi^{-1}(1 - c_j/p_j)) \right], \quad (3.3)$$

where $c_i < p_i$ is the unit cost of item $i \in S$, and $\phi(\cdot)$ and $\Phi(\cdot)$ are the probability density function and the cumulative distribution function of the standard Normal distribution, respectively. Observe that in (3.3), the first term is the “riskless” expected profit (assuming an infinite supply of items), while the second term involving the demand standard deviation represents the inventory costs.

The optimal prices that maximize the expected profit from S can be obtained via a multiple variable search on $\Pi(S, \mathbf{p})$ in (3.3). The optimal inventory levels can then be obtained using (3.2). However, no closed-form expression exists for the term $\Phi^{-1}(1 - c_i/p_i)$ which complicates the analysis of $\Pi(S, \mathbf{p})$. Furthermore, approximate expressions for $\Phi^{-1}(\cdot)$ are quite cumbersome and are not promising in obtaining a simplified expression for $\Pi(S, \mathbf{p})$ (see, for example, Patel and Read [73], pp. 66-70). Consequently, we propose the following simple approximation for $\phi(\Phi^{-1}(1 - x))$, $0 \leq x \leq 1$:

$$\phi(\Phi^{-1}(1 - x)) \approx -ax(x - 1), \quad (3.4)$$

where $a > 0^2$; see Appendix A. With this approximation, $\Pi(S, \mathbf{p})$ in (3.3) simplifies to the following:

$$\Pi(S, \mathbf{p}) = \sum_{j \in S} (p_j - c_j) \left[\lambda q_j(S, \mathbf{p}) - a \frac{c_j}{p_j} \sqrt{\lambda q_j(S, \mathbf{p})} \right]. \quad (3.5)$$

That is, $\Pi(S, \mathbf{p}) = \sum_{j \in S} \Pi_j(S, \mathbf{p})$, where $\Pi_j(S, \mathbf{p}) = (p_j - c_j) \left[\lambda q_j(S, \mathbf{p}) - a \frac{c_j}{p_j} \sqrt{\lambda q_j(S, \mathbf{p})} \right]$ is the expected profit from item $j \in S$.

Our objective is to find the assortment yielding the maximum profit, Π^* :

$$\Pi^* = \Pi(S^*, \mathbf{p}^*) = \max_{S \subseteq \Omega} \max_{\mathbf{p} \in \Gamma_S} \{\Pi(S, \mathbf{p})\}, \quad (3.6)$$

where S^* is an optimal assortment, \mathbf{p}^* is the corresponding optimal price vector, and

$$\Gamma_S = \{(p_1, \dots, p_{|S|}) \mid p_1 > c_1, \dots, p_{|S|} > c_{|S|}\}.$$

We use the expression of $\Pi(S, \mathbf{p})$ in (3.5) in the remainder of this chapter. We observe, through an extensive numerical study, that the approximate expected profit in (3.5) behaves in a similar fashion to the exact expected profit in (3.3). We also observe that, for a given assortment, the optimal prices under the approximate expected profit are very close to their counterparts under the exact expected profit. In summary, our numerical study indicates that our approximation does not change the main structural properties of the optimal solution (see Appendix A for details of the numerical study).

The following assumption guarantees that the retailer will not be better off not selling anything (hence, the optimal expected profit is positive).

(A1): Let \bar{i} be the item in Ω such that $\alpha_{\bar{i}} - c_{\bar{i}} = \max_{j \in \Omega} \{\alpha_j - c_j\}$. The expected profit from assortment $\{\bar{i}\}$ is increasing in $p_{\bar{i}}$ at $p_{\bar{i}} = c_{\bar{i}}$. That is, $\frac{\partial \Pi(\{\bar{i}\}, p_{\bar{i}})}{\partial p_{\bar{i}}} \Big|_{p_{\bar{i}}=c_{\bar{i}}} > 0$, or equivalently,

²We find that with $a = 1.66$, the approximation is reasonably accurate with an average error of 8.6%. However, our subsequent analytic results hold for any positive constant a .

$$\lambda > a^2 \left(1 + v_0 e^{-(\alpha_i - c_i)/\mu}\right).^3$$

Assumption **(A1)** implies that in an optimal assortment, S^* , the contribution of each item in S^* to the expected profit is positive, and that the optimal price vector, \mathbf{p}^* , for S^* is an internal point solution. The following lemma states these results formally.

Lemma 3.1.1 *Assume that **(A1)** holds. Then, the contribution of each item in an optimal assortment to the expected profit is positive, i.e., $\Pi_i(S^*, \mathbf{p}^*) > 0, i \in S^*$. In addition, the optimal price vector, \mathbf{p}^* , satisfies $c_i < p_i^* < \infty$, with $\left. \frac{\partial \Pi(S^*, \mathbf{p})}{\partial p_i} \right|_{\mathbf{p}=\mathbf{p}^*} = 0, i \in S^*$.*

Proof. See Appendix B. ■

In the remainder of this chapter, we assume that **(A1)** holds. When a result involves changing the values of the model parameters, we assume that the change is restricted to the range where **(A1)** holds.

3.2 Structure of the Optimal Assortment

Our main structural result allows us to determine whether a given item “dominates” another item. The dominance relationship that we consider is quite intuitive as it requires the dominating item to have a lower or equal unit cost and a higher or equal mean reservation price than the dominated item (with one of the two inequalities being strict). We show that an optimal assortment cannot contain the dominated item and not contain the dominating item. While this type of a dominance relationship can considerably reduce the computational effort needed to determine the optimal assortment, it also

³In fact, our results hold under a weaker assumption than **(A1)**. However, we choose to report **(A1)** here because (i) it is easy to verify whether **(A1)** holds or not; (ii) **(A1)** guarantees that an item’s mean demand is reasonably large, which leads to a low probability of negative demand under the Normal demand distribution; and (iii) **(A1)** is not too restrictive, e.g., if $(\alpha_i - c_i) \gg 0$, then **(A1)** holds if $\lambda > a^2 \approx 3$ when $a = 1.66$.

allows us to derive the structure of the optimal assortment for a special case that may apply to many practical situations. We present below some important lemmas that are utilized to derive our main result. Other supporting lemmas are presented in Appendix C.

Lemma 3.2.1 *Consider an assortment $S \subseteq \Omega$. Assume that prices of items in S are fixed at some price vector \mathbf{p} . Then, the expected profit from S , $\Pi(S, \mathbf{p}, \alpha_i)$, is strictly pseudoconvex in α_i , the mean reservation price of item $i \in S$.*

Proof. See Appendix C. ■

Lemma 3.2.1 extends a similar result in van Ryzin and Mahajan [80]. The intuition behind Lemma 3.2.1 is as follows. Recall that the demand coefficient of variation and standard deviation of item $i \in S$ are $1/\sqrt{\lambda q_i(S, \mathbf{p})}$ and $\sqrt{\lambda q_i(S, \mathbf{p})}$, respectively. Then, it can be easily shown that (i) the demand coefficient of variation of item $j \neq i, j \in S$, is increasing in α_i , while that of item i is decreasing in α_i ; (ii) the demand standard deviation of item $j \in S, j \neq i$, is decreasing in α_i , while that of item i is increasing in α_i ;⁴ and (iii) the total expected demand for the product line, $\lambda(1 - q_0(S, \mathbf{p}))$, is increasing in α_i . With the inventory costs being increasing in demand variability and sales revenues being increasing in the total expected demand (at fixed prices), these conflicting effects of α_i explain the pseudoconvexity result in Lemma 3.2.1.

The following lemma studies the effect of changing the mean reservation price of an item in an optimal assortment.

Lemma 3.2.2 *Consider an optimal assortment $S^* \subseteq \Omega$. Let \mathbf{p}^* be the optimal price vector when $\alpha_i = \alpha'_i$, for some $i \in S^*$. Assume that prices of items in S^* are fixed at \mathbf{p}^* . Then, the expected profit from S^* , $\Pi(S^*, \mathbf{p}^*, \alpha_i)$, is increasing in α_i for $\alpha_i \geq \alpha'_i$.*

⁴Here we are measuring demand variability by both the standard deviation and the coefficient of variation of the demand, similar to Petruzzi and Dada [76].

Proof. See Appendix C. ■

Lemma 3.2.2 states that increasing the mean reservation price, α_i , of an item in an optimal assortment increases the expected profit from the assortment when the prices of items in the assortment are unchanged. Clearly, this implies that the optimal profit (with prices adjusted optimally and possibly a new optimal assortment obtained as α_i changes) also increases in α_i , $i \in S^*$.

Another key parameter that drives the profitability of an item is its unit cost. The following lemma studies the effect of changing the unit cost of an item on the profitability of a product line.

Lemma 3.2.3 *Consider an assortment $S \subseteq \Omega$. Assume that prices of items in S are fixed at some price vector \mathbf{p} and that $\Pi_i(S, \mathbf{p}, c_i) > 0$ when $c_i = c'_i$, for some $i \in S$. Then, the expected profit from S , $\Pi(S, \mathbf{p}, c_i)$, increases if c_i is decreased below c'_i .*

Proof. See Appendix C. ■

Lemma 3.2.3 states that decreasing the unit cost of an item returning a positive expected profit will increase the expected profit from an assortment. It can be easily seen that this results also extends to the case of an optimal assortment. That is, decreasing the unit cost of an item in the optimal assortment increases the optimal profit. A question that arises naturally is whether increasing the mean reservation price of an item (by perhaps an advertisement campaign) is more profitable than decreasing the item's unit cost, or vice versa. We numerically observe, in Section 3.4, that decreasing the unit cost is slightly more profitable.

Next, we present our main dominance result.

Lemma 3.2.4 *Consider two items $i, k \in \Omega$ such that $\alpha_i \leq \alpha_k$ and $c_i \geq c_k$, with at least one of the two inequalities being strict, that is, item k “dominates” item i . Then, an optimal assortment cannot contain item i and not contain item k .*

Proof. See Appendix C. ■

Observe that the number of assortments to be considered in the search for an optimal assortment can be significantly reduced if there are few dominance relations like the one described in Lemma 3.2.4. For example, with exactly one pair of items satisfying the dominance relation in the lemma, the number of assortments to be considered is reduced by $\sum_{r=1}^{n-1} \binom{n-2}{r-1} = 2^{n-2}$, which leads to more than 25% reduction in computational effort since the total number of subsets of Ω to be considered in an exhaustive search is $2^n - 1$. (Under **(A1)**, the empty set, i.e., the option of selling nothing, cannot be optimal.) In addition, Lemma 3.2.4 allows the development of the structure of an optimal assortment in a special case which may be commonly encountered in practice. This is stated in the following theorem.

Theorem 3.2.1 *Assume that the items in Ω are such that $\alpha_1 \geq \alpha_2 \geq \dots \alpha_n$, and $c_1 \leq c_2 \leq \dots c_n$. Then, an optimal assortment is $S^* = \{1, 2, \dots, k\}$, for some $k \leq n$.*

Proof. The proof follows directly from Lemma 3.2.4. Lemma 3.2.4 implies that if the optimal assortment is of cardinality 1, then $\{1\}$ is an optimal assortment. Similarly, if the optimal assortment is of cardinality 2, then $\{1, 2\}$ is an optimal assortment, and so on. ■

In the special case of a product line with all items having the same unit cost, Theorem 3.2.1 implies that an optimal assortment has the $k, k \leq n$, items with the largest values of α_i . Van Ryzin and Mahajan [80] prove a similar result when, in addition to having the same unit cost, all items have the same price (or same price to unit cost ratio) which is exogenously determined. Thus, Theorem 3.2.1 extends the result of van Ryzin and Mahajan to a product line with items having distinct endogenous prices. Another special case of Theorem 3.2.1 is a product line with items having the same mean reservation prices; in this case, an optimal assortment has the $k, k \leq n$, items with the

smallest values of c_i .

Theorem 3.2.1 is quite intuitive: An optimal assortment simply contains either the most popular item (with the least possible cost), or the two most popular items (with the least costs), and so on. Furthermore, Theorem 3.2.1 greatly simplifies the search for an optimal assortment in the special case where it applies, as it suffices to consider only n assortments out of $(2^n - 1)$ possible assortments. We point out that in many cases, the economies of scale in the supply of popular items may lead to a structure of reservation prices/unit costs that is similar to that in the theorem. That is, high customer demand for popular items implies larger order size for these items, which tends to decrease the unit cost per item with business practices such as quantity discounts.

For cases where Theorem 3.2.1 does not apply, one may expect an optimal assortment to have the k , $k \leq n$, items with the largest “average margins,” $\alpha_i - c_i$, similar to the structure of the optimal assortment in Theorem 3.2.1. In fact, a somewhat similar result holds in the riskless case (which assumes infinite inventory levels), where it can be shown that while the optimal assortment is Ω (i.e., there is no limit on variety), the optimal assortment of size k has the k items with the largest average margins; see, for example, Aydin and Ryan [6]. However, in Section 3.4, we find several counterexamples of optimal assortments not having items with the largest average margin, indicating that a result similar to Theorem 3.2.1 does not hold in general. Nevertheless, we observe that assortments consisting of items with the largest average margins return expected profits that are very close to the optimal profit. This last observation is one of the main motivations for our heuristic procedure, discussed in Section 3.4, which can be utilized in cases where Theorem 3.2.1 does not hold.

3.3 Properties and Bounds on the Optimal Prices

We first exploit the optimality conditions to gain insight into the structure of the optimal prices.

The following lemma is a consequence of the first-order optimality conditions.

Lemma 3.3.1 *Consider an optimal assortment $S^* \subseteq \Omega$. Then, the optimal prices of any two items $i, j \in S^*$ satisfy the following equation:*

$$\frac{1}{\mu}(p_i^* - c_i) \left(1 - \frac{a}{2} \frac{c_i}{p_i^*} \frac{1}{\sqrt{\lambda q_i(S^*, \mathbf{p}^*)}} \right) + a \frac{c_i^2}{p_i^{*2}} \frac{1}{\sqrt{\lambda q_i(S^*, \mathbf{p}^*)}} = \frac{1}{\mu}(p_j^* - c_j) \left(1 - \frac{a}{2} \frac{c_j}{p_j^*} \frac{1}{\sqrt{\lambda q_j(S^*, \mathbf{p}^*)}} \right) + a \frac{c_j^2}{p_j^{*2}} \frac{1}{\sqrt{\lambda q_j(S^*, \mathbf{p}^*)}}.$$

Proof. See Appendix C. ■

Lemma 3.3.1 has several important implications. First, note that for λ large enough ($\lambda \rightarrow \infty$), and with the optimal prices being finite (as shown in Lemma 3.1.1), the optimal profit margins for all items in S^* will be exactly equal, i.e., $p_i^* - c_i = p_j^* - c_j, i, j \in S^*$. This is due to the fact that for large λ our problem converges to the “riskless case,” as it can be easily shown that for finite prices $\Pi(S^*, \mathbf{p})$ in (3.5) converges to $\sum_{j \in S} (p_j - c_j) \lambda q_j(S^*, \mathbf{p})$, the expected profit for the riskless case. For the riskless case, many authors indeed show that the optimal prices are characterized by “equal profit margins” (see, for example, Anderson et al. [2], Aydin and Ryan [6], and Cattani et al. [15]).

Second, Lemma 3.3.1 shows that with finite inventory levels (i.e., in the “risky” case), the “equal margins” property is no longer guaranteed to hold at optimality. However, Lemma 3.3.1 suggests that for large mean demands, $\lambda q_i(S^*, \mathbf{p}^*)$ and $\lambda q_j(S^*, \mathbf{p}^*)$, items i and j will have *approximately* equal profit margins, i.e., $p_i^* - c_i \approx p_j^* - c_j, i, j \in S^*$. We observe, through an extensive numerical study, that the optimal profit margins are indeed quite close (see Section 3.4 for details). Finally, Lemma 3.3.1 may be useful in numerically solving for the optimal prices.

Next, in Lemma 3.3.2 and Corollary 3.3.1, we develop bounds on the optimal prices that can be obtained by solving single variable equations. These bounds are useful in the numerical search for

the optimal prices of an assortment. Moreover, in practice, product lines may have a handful of “fast movers” that attract most of the demand and numerous “slow movers,” each attracting only a thin fraction of the demand. Lemma 3.3.2 can identify highly unprofitable slow movers and eliminate these items from consideration. Define $\tilde{\pi}_i(p_i) \equiv \lambda p_i^2 q_i(\{i\}, p_i) - a^2 c_i^2$ and $h_i^{max} \equiv \max_{p_i > 0} \{p_i^2 q_i(\{i\}, p_i)\}$.

Lemma 3.3.2 *Consider item $i \in \Omega$. Then,*

(i) *If $h_i^{max} > a^2 c_i^2 / \lambda$ and $\bar{p}_i > c_i$ then $p_i^* \in (\max\{c, \underline{p}_i, \bar{p}_i\})$ in any assortment containing i ,*

where \underline{p}_i and \bar{p}_i are such that $\tilde{\pi}_i(\underline{p}_i) = \tilde{\pi}_i(\bar{p}_i) = 0$ with $\underline{p}_i < \bar{p}_i$.⁵

(ii) *Otherwise, item i must not be included in an optimal assortment.*

Proof. See Appendix D. ■

Note that the bounds in Lemma 3.3.2 apply to the optimal price of an item $i \in \Omega$ in *any* assortment $S \subseteq \Omega$ containing i . Finally, bounds on the optimal prices of all items (in all possible assortments) can be obtained by solving one simple equation, as indicated in the following result.

Corollary 3.3.1 *Consider item $i \in \Omega$. Then, $p_i^* \in (\max\{c_i, \underline{p}, \bar{p}\})$ in any assortment containing i , where \bar{p} and \underline{p} are such that $\tilde{\pi}_{\bar{i}}(\underline{p}) = \tilde{\pi}_{\bar{i}}(\bar{p}) = 0$ with $\underline{p} < \bar{p}$, and item \bar{i} is characterized by $c_{\bar{i}} = \min_{j \in \Omega} \{c_j\}$, and $\alpha_{\bar{i}} = \max_{j \in \Omega} \{\alpha_j\}$.*

Proof. See Appendix D. ■

3.4 Numerical Results and a Heuristic Procedure

The objectives of our numerical study in this section are twofold: (i) To study the properties of an optimal solution for the general case (when Theorem 3.2.1 does not hold); and (ii) to develop a

⁵The condition that $h_i^{max} > a^2 c_i^2 / \lambda$ guarantees that there are exactly two solutions of $\tilde{\pi}_i(p_i) = 0$; see Appendix D.

simple effective heuristic for the problem at hand, motivated mainly by our analytical results and numerical observations.

Table 3.1 presents the numerical results for a three-item case. The optimal assortment, S^* , is obtained by enumerating over all subsets of $\Omega = \{1, 2, 3\}$ and determining the corresponding optimal profit (together with optimal prices and inventory levels) for each subset. In addition to the optimal assortment, S^* , and its expected profit, Π^* , Table 3.1 reports the optimal profit margins, $p_i^* - c_i, i \in \Omega$, and the no-purchase probability, $q_0(S^*, \mathbf{p}^*)$ (the fraction of customers who leave the store empty-handed). Note that an infinite profit margin indicates that the item is not included in S^* . The second column of Table 3.1 shows the modification from the “base case,” described in the table heading. Each modification involves changing the parameters given in the second column of the table only, while keeping other parameters at their base values.

Table 3.1 reveals three important insights. First, items in S^* have *approximately* equal profit margins, $p_i^* - c_i, i \in S^*$. This finding is not surprising given our discussion of Lemma 3.3.1. Second, an optimal assortment need not have the items with the largest values of $\alpha_i - c_i$ (which shows that a result similar to Theorem 3.2.1 does not hold in general). For example, in Case 8 the two items with the largest values of $\alpha_i - c_i$ are items 1 and 2, while the optimal assortment contains items 1 and 3 only. A similar observation holds for Case 9.⁶ We observe, however, (on Cases 8 and 9, and many other cases) that assortments containing items with the largest values of $\alpha_i - c_i$ yield expected profits that are very close to the optimal expected profits. (This will be further discussed within the context of our proposed heuristic procedure.) Third, we observe that reducing an item’s unit cost by a certain amount is slightly more profitable than increasing its mean reservation price by the same

⁶We have verified that similar counter-examples exist under the exact expected profit in (3.3) as well as under the exact expected profit obtained based on a Poisson arrival process, which indicates that the counter-examples are not due to our approximations.

amount. For example, while item 1 has the same value of $\alpha_1 - c_1 = 3$ in Cases 2 and 3 (with items 2 and 3 having the same parameters in both cases), Case 3 (where c_i is smaller) yields a slightly higher expected profit than Case 2. The same observation is valid when comparing Cases 4 and 5, or Cases 6 and 7.

Table 3.2 presents the results for a four-item case, which confirm the three main insights observed in Table 3.1. In particular, a similar observation on decreasing c_i being more profitable than increasing α_i can be made by comparing Cases 2 and 3, or 4 and 5, or 6 and 7 in Table 3.2. In addition, Cases 8 and 9 in Table 3.2 furnish more examples on optimal assortments not having the items with the largest values of $\alpha_i - c_i$. Moreover, it can be observed throughout Table 3.2 that optimal profit margins for items in S^* are approximately equal.

Table 3.1. Optimal solution ($n = 3$)

Base case: $\lambda = 100$, $\alpha_1 = 11$, $\alpha_2 = 10$, $\alpha_3 = 9$, $c_1 = 9$, $c_2 = 8$, $c_3 = 7$, $v_0 = 1$, $\mu = 1$.

Case	Modification	$p_1^* - c_1$	$p_2^* - c_2$	$p_3^* - c_3$	$q_0(S^*, \mathbf{p}^*)$	S^*	Π^*
1	None	2.531	2.534	2.536	0.362	{1, 2, 3}	117.453
2	$\alpha_1 = 12$	2.790	2.904	2.904	0.329	{1, 2, 3}	142.528
3	$c_1 = 8$	2.795	2.909	2.908	0.330	{1, 2, 3}	143.175
4	$\alpha_1 = 12.75$	3.066	∞	3.331	0.308	{1, 3}	173.950
5	$c_1 = 7.25$	3.075	∞	3.345	0.310	{1, 3}	175.667
6	$\alpha_1 = 15$	4.673	∞	∞	0.210	{1}	323.935
7	$c_1 = 5$	4.692	∞	∞	0.213	{1}	333.694
8	$\alpha_1 = 12.8, c_2 = 7.98$	3.096	∞	3.379	0.305	{1, 3}	176.660
9	$c_1 = 7.25, c_2 = 7.98$	3.075	∞	3.343	0.310	{1, 3}	175.667

Table 3.2. Optimal solution ($n = 4$)**Base case:** $\lambda = 150, \alpha_1 = 20, \alpha_2 = 22, \alpha_3 = 24, \alpha_4 = 26, c_1 = 18, c_2 = 20, c_3 = 22, c_4 = 24, v_0 = 1, \mu = 1.$

Case	Modification	$p_1^* - c_1$	$p_2^* - c_2$	$p_3^* - c_3$	$p_4^* - c_4$	$q_0(S^*, \mathbf{p}^*)$	S^*	Π^*
1	None	2.663	2.661	2.659	2.658	0.326	{1, 2, 3, 4}	190.200
2	$\alpha_1 = 21.6$	3.027	3.215	3.214	∞	0.297	{1, 2, 3}	252.286
3	$c_1 = 16.4$	3.029	3.217	3.216	∞	0.297	{1, 2, 3}	252.816
4	$\alpha_1 = 21.8$	3.078	3.299	∞	∞	0.300	{1, 2}	267.338
5	$c_1 = 16.2$	3.081	3.302	∞	∞	0.300	{1, 2}	268.009
6	$\alpha_1 = 22$	3.155	∞	∞	∞	0.300	{1}	285.400
7	$c_1 = 16$	3.158	∞	∞	∞	0.301	{1}	286.236
8	$\alpha_1 = 21.6, \alpha_4 = 26.015$	3.027	3.215	3.214	∞	0.297	{1, 2, 3}	252.286
9	$\alpha_1 = 21.6, c_4 = 23.98$	3.027	3.215	3.214	∞	0.297	{1, 2, 3}	252.286

Table 3.3. Optimal solution ($n = 3$), small λ **Base case:** $\lambda = 100, \alpha_1 = 11, \alpha_2 = 10, \alpha_3 = 9, c_1 = 9, c_2 = 8, c_3 = 7, v_0 = 1, \mu = 1.$

Case	Modification	$p_1^* - c_1$	$p_2^* - c_2$	$p_3^* - c_3$	$q_0(S^*, \mathbf{p}^*)$	S^*	Π^*
1	$\lambda = 30$	2.425	2.534	2.536	0.339	{1, 2, 3}	24.379
2	$\lambda = 30, \alpha_1 = 11.75$	2.467	∞	2.598	0.348	{1, 3}	29.548
3	$\lambda = 30, \alpha_1 = 11.75, \alpha_2 = 10.07$	2.467	∞	2.598	0.348	{1, 3}	29.548
4	$\lambda = 10$	∞	∞	1.751	0.438	{3}	4.328
5	$\lambda = 10, \alpha_1 = 11.08$	∞	∞	1.751	0.438	{3}	4.328
6	$\lambda = 4$	∞	∞	1.400	0.354	{3}	0.503

As discussed in Section 3.3, the optimal profit margins are exactly equal for λ large enough. However, how the optimal margins compare for small λ is not clear. This is studied in Table 3.3, which indicates that the optimal margins remain approximately equal even for small λ , with the number of items in S^* decreasing as λ decreases. Note also that Cases 3 and 5 in Table 3.3 provide more examples of optimal assortments with items not having the largest value of $\alpha_i - c_i$.

The analytical and numerical results presented thus far suggest that a solution with equal profit margins and having some assortment of items with the largest values of $\alpha_i - c_i$ is expected to yield an expected profit which is quite close to the optimal profit. This is the main motivation behind our proposed heuristic solution procedure, referred to as the ‘‘Equal Margins Heuristic (EMH).’’ Details of the EMH procedure are as follows.

Equal Margins Heuristic (EMH)

Step 0: Eliminate any item $i \in \Omega$ that cannot return a positive expected profit (assuming equal profit margins, with a common margin m). That is, eliminate any item such that $\lambda(c_i + m)^2 \frac{e^{(\alpha_i - c_i - m)/\mu}}{v_0 + e^{(\alpha_i - c_i - m)/\mu}} - a^2 c_i^2 < 0$, for all $m > 0$. Let $\tilde{\Omega} \subseteq \Omega$, with cardinality $\tilde{n} \leq n$, be the set of items which are not eliminated in this step.

Step 1: Sort items in $\tilde{\Omega}$ in nonincreasing order of $\alpha_i - c_i$. Break ties according to the smaller value of c_i . Number items in $\tilde{\Omega}$ such that item 1 is the item with the largest $\alpha_i - c_i$, item 2 is the item with the second largest $\alpha_i - c_i$, and so on.

Step 2: Assuming equal profit margins, find the common margin, m_k , that would yield the highest profit from $S_k = \{1, 2, \dots, k\}$, $k = 1, 2, \dots, \tilde{n}$. That is, find

$$\Pi_k^H(S_k, m_k) = \max_{m>0} m \sum_{i \in S_k} \left[\lambda q_i(S_k, m) - a \frac{c_i}{c_i + m} \sqrt{\lambda q_i(S_k, m)} \right],$$

$$\text{where } q_i(S_k, m) = \frac{e^{(\alpha_i - c_i - m)/\mu}}{v_0 + \sum_{j \in S_k} e^{(\alpha_j - c_j - m)/\mu}}.$$

Step 3: Find k^H such that $k^H = \arg \max_{k=1, \dots, \tilde{n}} \Pi_k^H(S_k, m_k)$. Set $S^H = S_{k^H}$, $m^H = m_{k^H}$, and $\Pi^H = \Pi_{k^H}^H(S_{k^H}, m_{k^H})$. Set prices and inventory levels of items in S^H to

$$\begin{aligned} p_i^H &= c_i + m^H, \\ y_i^H &= \lambda q_i(S^H, m^H) + \Phi^{-1}(1 - c_i/(c_i + m^H)) \sqrt{\lambda q_i(S^H, m^H)}. \end{aligned}$$

In the EMH Algorithm, Step 0 is a direct consequence of Lemma 3.3.2. The tie breaking rule in Step 1 is motivated by the observation that decreasing c_i is more profitable than increasing α_i , as discussed above. The remaining steps then find the most profitable assortment consisting of the $k \leq n$ items with the largest values of $\alpha_i - c_i$, and under the restriction of equal profit margins.

The advantage of EMH is that it requires little computational effort relative to the effort required to find the optimal solution: The EMH generates at most n assortments, each requiring a single variable search (over the common margin), while determining the optimal solution requires generating up to $2^n - 1$ assortments (when Theorem 3.2.1 does not hold), with a multiple variable search (over the price vector) for each assortment. Moreover, our numerical study suggests that EMH generates

solutions that are very close to the optimal solution, with the ratio of heuristic expected profit to the optimal expected profit, Π^H/Π^* , being larger than 99.5% in all tested cases; see the last column in Tables 3.4, 3.5, and 3.6, which report this ratio for the examples in Tables 3.1, 3.2, and 3.3, respectively. These results indicate an excellent performance for the EMH, which is not surprising given the analytical motivations upon which it is founded. Tables 3.4, 3.5, and 3.6 also report the best heuristic assortment, S^H , together with the corresponding common margin, m^H , the no-purchase probability, $q_0(S^H, m^H)$, and the expected profit, Π^H . Observe that the no-purchase probabilities for the heuristic and the optimal solution are quite close (see Tables 3.1-3.6). This suggests that the heuristic solution exhibits a similar performance to the optimal solution in satisfying secondary objectives such as the service level.

Table 3.4. EMH solution ($n = 3$)

Base case: $\lambda = 100$, $\alpha_1 = 11$, $\alpha_2 = 10$, $\alpha_3 = 9$, $c_1 = 9$, $c_2 = 8$, $c_3 = 7$, $v_0 = 1$, $\mu = 1$.

Case	Modification	m^H	$q_0(S^H, m^H)$	S^H	Π^H	Π^H/Π^*
1	None	2.534	0.362	{1, 2, 3}	117.453	100.00 %
2	$\alpha_1 = 12$	2.839	0.328	{1, 2, 3}	142.446	99.94%
3	$c_1 = 8$	2.839	0.329	{1, 2, 3}	143.094	99.94%
4	$\alpha_1 = 12.75$	3.184	0.317	{1, 3}	173.760	99.98%
5	$c_1 = 7.25$	3.192	0.319	{1, 3}	175.472	99.89%
6	$\alpha_1 = 15$	4.673	0.210	{1}	323.935	100.00%
7	$c_1 = 5$	4.692	0.213	{1}	333.694	100.00%
8	$\alpha_1 = 12.8, c_2 = 7.98$	3.125	0.294	{1, 2}	176.00	99.63%
9	$c_1 = 7.25, c_2 = 7.98$	3.194	0.300	{1, 2}	175.144	99.70%

Table 3.5. EMH solution ($n = 4$)

Base case: $\lambda = 150$, $\alpha_1 = 20$, $\alpha_2 = 22$, $\alpha_3 = 24$, $\alpha_4 = 26$, $c_1 = 18$, $c_2 = 20$, $c_3 = 22$,
 $c_4 = 24$, $v_0 = 1$, $\mu = 1$.

Case	Modification	m^H	$q_0(S^H, m^H)$	S^H	Π^H	Π^H/Π^*
1	None	2.66	0.326	{1, 2, 3, 4}	190.200	100.00%
2	$\alpha_1 = 21.6$	3.027	0.297	{1, 2, 3}	252.021	99.89%
3	$c_1 = 16.4$	3.074	0.296	{1, 2, 3}	252.551	99.90%
4	$\alpha_1 = 21.8$	3.103	0.299	{1, 2}	267.138	99.92%
5	$c_1 = 16.2$	3.106	0.300	{1, 2}	267.809	99.92%
6	$\alpha_1 = 22$	3.155	0.300	{1}	285.400	100.00%
7	$c_1 = 16$	3.158	0.301	{1}	286.236	100%
8	$\alpha_1 = 21.6, \alpha_4 = 26.015$	3.073	0.295	{1, 2, 4}	251.972	99.87%
9	$\alpha_1 = 21.6, c_4 = 23.98$	3.073	0.295	{1, 2, 4}	252.018	99.89%

Table 3.6. EMH solution ($n = 3$) small λ

Base case: $\lambda = 100$, $\alpha_1 = 11$, $\alpha_2 = 10$, $\alpha_3 = 9$, $c_1 = 9$, $c_2 = 8$, $c_3 = 7$, $v_0 = 1$, $\mu = 1$.

Case	Modification	m^H	$q_0(S^H, m^H)$	S^H	Π^H	Π^H/Π^*
1	$\lambda = 30$	2.432	0.339	{1, 2, 3}	24.378	100.00%
2	$\lambda = 30, \alpha_1 = 11.75$	2.504	0.347	{1, 3}	29.525	99.92%
3	$\lambda = 30, \alpha_1 = 11.75, \alpha_2 = 10.07$	2.504	0.343	{1, 2}	29.522	99.91%
4	$\lambda = 10$	1.751	0.438	{3}	4.328	100.00%
5	$\lambda = 10, \alpha_1 = 11.08$	1.744	0.417	{1}	4.315	99.70%
6	$\lambda = 4$	1.400	0.354	{3}	0.503	100.00%

Chapter 4

Joint Pricing, Assortment, and Inventory Decisions for a Retailer's Product Line: A Stylized Model with Similar Items

In this chapter, we consider a special case of the model presented in Chapter 3 by requiring that all items that may be included in the product line have the same unit cost and identical consumer reservation price distributions. The resulting stylized model is amenable to analytical study, and the optimal pricing, inventory, and assortment decisions can be characterized with stronger results than in Chapter 3. These results allow us to gain more insight into the product line problem that we introduced in Chapter 3.

This chapter is organized as follows. In Section 4.1, we introduce our model and assumptions. In Section 4.2, we study the optimal pricing problem when the assortment size is fixed. Then in Section 4.3, we analyze the optimal assortment size problem when the price is exogenously determined. Finally, in Section 4.4, we present bounds on the optimal price and variety level, and discuss the joint pricing/variety/inventory problem.

4.1 Model and Assumptions

We study the joint pricing, inventory, and assortment size decisions of a retailer's product line, considering the framework of Section 3.1. However, we make the simplifying assumption that all items in Ω have equal mean reservation prices (consumer valuations) and unit costs, denoted by α and c , respectively. Let $S_k \subseteq \Omega$ denote a subset of any $k \in \mathbf{Z}^+$ items in Ω , stocked by the store. Under the above assumption, all subsets of Ω with cardinality k (S_k) will be equally profitable. We also assume that all items in an assortment S_k are to be sold at the same price $p > c$, as discussed in Section 1.3.

Considering the MNL consumer choice model described in Section 3.1, the probability that a consumer buys an item $i \in S_k$ reduces to the following in our stylized model

$$q(p, k) = \frac{e^{(\alpha-p)/\mu}}{v_0 + ke^{(\alpha-p)/\mu}}, \quad (4.1)$$

where $v_0 = e^{u_0/\mu}$.

In addition, the optimal inventory level, $y^*(p, k)$, and the expected profit at optimal inventory levels, $\Pi(p, k)$, for item $i \in S_k$ are given by

$$y^*(p, k) = \lambda q(p, k) + \Phi^{-1}(1 - c/p) \sqrt{\lambda q(p, k)}, \quad \text{and} \quad (4.2)$$

$$\Pi(p, k) = k(p - c) \left[\lambda q(p, k) - a \frac{c}{p} \sqrt{\lambda q(p, k)} \right]. \quad (4.3)$$

We use the expression of $\Pi(p, k)$ in (4.3) in the remainder of this chapter. Note that (4.3) indicates that the profitability of the product line depends on the price of the items and the assortment size, k , only, while in the general model of Section 3.1 the profitability of an assortment depends in a complex

way on the individual characteristics of each item in the assortment. The simplified objective function in (4.3) allows us to provide a more explicit characterization of the optimal solution as shown below.

4.2 On the Optimal Price when the Assortment Size is Fixed

In this section, we assume that the assortment size, k , is fixed, and analyze properties of the optimal price, $p_k^* = \arg \max_{p>c} \Pi(p, k)$, where $\Pi(p, k)$ is as given in (4.3). The analysis here is useful when the assortment size is imposed by space constraints or by other, possibly, marketing considerations. In addition, the analysis in this section allows the determination of an optimal solution for the joint variety/pricing/inventory problem by enumerating over all possible values of k . We discuss the joint variety/pricing/inventory problem in Section 4.4.

Similar to Assumption **(A1)** in the previous chapter, we make the following assumption throughout this section, which ensures that the retailer will not be better off by not selling anything.

(A2): The expected profit, $\Pi(p, k)$, is increasing in p at $p = c$, that is, $\left. \frac{\partial \Pi(p, k)}{\partial p} \right|_{p=c} > 0$, or equivalently, $\lambda > a^2(k + v_0 e^{-(\alpha-c)/\mu})$.¹

It can be easily shown that $\lim_{p \rightarrow \infty} \Pi(p, k) \rightarrow 0^-$. This, together with **(A2)**, ensures that the optimal price, p_k^* , is an internal point solution satisfying the first- and second-order optimality conditions. In addition, our extensive numerical study suggests that $\Pi(p, k) > 0$ is pseudoconcave in p . In fact, we have failed to find a counter-example. However, this result does not seem to lend itself to an analytical proof easily. The following lemma indicates that under a somewhat weak condition, $\Pi(p, k) > 0$ is pseudoconcave in p .

¹Note that **(A2)** is equivalent to **(A1)** when all items in Ω have equal mean reservation prices and unit costs. However, we include **(A2)** here for completeness.

Lemma 4.2.1 *The expected profit $\Pi(p, k)$ is pseudoconcave in $p \geq c$ in the region where $\Pi(p, k) > 0$ and $q(p, k) \geq 1/3$.*

Proof. See Appendix E. ■

Note that the item purchase probability, $q(p, k)$, is decreasing in p , with a maximum value of $q(c, k)$. Thus, if $q(c, k) > 1/3$, then Lemma 4.2.1 will determine an interval to the right of c where $\Pi(p, k) > 0$ is pseudoconcave in p . (We show that under **(A2)**, $\Pi(p, k) > 0$ on exactly one interval to the right of c ; see Appendix J.)

4.2.1 Comparative Statics on the Optimal Price

Next we analyze the behavior of p_k^* as a function of problem parameters. The following theorem summarizes our comparative statics analysis on p_k^* .

Theorem 4.2.1 *For a fixed assortment size k , the optimal price p_k^* is:*

- (i) *Increasing in the assortment size, k , if $p_k^* \geq \frac{3}{2}c$;*
- (ii) *Increasing in the expected store volume, λ , if $p_k^* < 2c$; and decreasing in λ , otherwise;*
- (iii) *Increasing in the unit cost per item, c , if $p_k^* > 2c$.*

Proof. See Appendix F. ■

Theorem 4.2.1 has important consequences as indicated in the two corollaries below as well as in some results in the next section.

Corollary 4.2.1 *If $p_{\underline{k}}^* \geq \frac{3}{2}c$ for some $\underline{k} \in \mathbf{Z}^+$, then p_k^* is increasing in k for all $k \geq \underline{k}$. In particular, if $p_1^* \geq \frac{3}{2}c$, then p_k^* is increasing in k for all $k \in \mathbf{Z}^+$.*

Proof. Follows directly from Theorem 4.2.1. ■

Corollary 4.2.2 *If $p_k^* > 2c$ ($p_k^* < 2c$) at some $\lambda = \lambda_0$, then p_k^* is almost everywhere decreasing (increasing) in λ for all λ with $\lim_{\lambda \rightarrow \infty} p_k^*(\lambda) \geq 2c$ ($\lim_{\lambda \rightarrow \infty} p_k^*(\lambda) \leq 2c$).*

Proof. Consider that $p_k^* > 2c$ at some $\lambda = \lambda_0$. Then by Theorem 4.2.1 p_k^* is decreasing in λ for all $\lambda \leq \lambda_0$. Next we analyze the behavior of p_k^* in $\lambda \geq \lambda_0$. Theorem 4.2.1 also implies that as λ increases from λ_0 , p_k^* decreases to the extent that p_k^* may approach $2c$ from above. In this case, if λ increases further by an infinitesimal amount and p_k^* drops below the $2c$ threshold, then Theorem 4.2.1 implies that p_k^* becomes increasing in λ , and eventually approaches $2c$ from below, and may exceed $2c$ to decrease again, and so on. Thus, we conclude that p_k^* is almost everywhere decreasing in λ for all λ . A similar argument holds if $p_k^* < 2c$ at some λ_0 . ■

We note that in Corollary 4.2.2 we use the term “almost everywhere decreasing (increasing),” because if p_k^* approaches $2c$ from above (below), then it might be increasing (decreasing) in λ on some interval of λ with an infinitely small length (see Rudin [79], p. 317, for a precise definition of the term “almost everywhere”).

Corollary 4.2.1 states that if the optimal price is relatively high ($p_k^* \geq 3c/2$) at a given variety level, then increasing variety will also increase the optimal price. The condition, $p_k^* \geq 3c/2$, can be seen as an indicator that consumers tolerate high prices well so that the retailer is induced to increase the price if the breadth of the assortment is enlarged. This could be the case of a store located in an upscale neighborhood. Our numerical study suggests that in environments where consumers do not tolerate high prices well (with $p_k^* < 3c/2$), the retailer may expand the breadth of the assortment, while decreasing the price (see Appendix K). In such cases, the retailer may achieve a higher profit due to the increase in sales volume as a result of the higher variety and the price drop. This could

be the case of a large discount store where a wide array of brands are sold at low prices.

Corollary 4.2.2 asserts that the optimal price as a function of the expected store volume moves in one direction only, all else held constant (see Appendix K for some numerical examples). This might be the case of an expensive store with a low volume where the price decreases as a result of an increase in volume, or the case of a low-price high-volume store where the price increases with volume. The condition, $p_k^* > 2c$, may be seen as an indicator of the nature of the marketplace and the store. It would be interesting to test the validity of the threshold-based results of Corollaries 4.2.1 and 4.2.2 in more complex situations (such as a product line with items having different unit costs and reservation price distributions).

Unfortunately, we have little to say on p_k^* as a function of c , except for the result in Theorem 4.2.1 (iii). However, we have found interesting counter-intuitive examples where p_k^* is decreasing in c for cases not covered by Theorem 4.2.1 (iii) (see Appendix K).

We note, finally, that we have verified that our findings that p_k^* can be decreasing in k or non-increasing in c continue to hold for the optimal price obtained from the exact expected profit function in (3.3) as well as from the exact profit function generated from a Poisson arrival process (see Appendix K). Thus, we believe that these findings are not due to our approximation in (4.3).

4.2.2 Comparison to the Riskless Case

Our objective, in this section, is to better understand the implications of limited inventory on the optimal pricing of a product line. For this purpose, we relax the limited inventory assumption and assume that there is an ample supply of inventory. Since the expected profit function under this assumption becomes independent of demand variability (see (4.4) below), we refer to this case as the “riskless case” (like many researches do, e.g., Petruzzi and Dada [76]). This case is extensively

studied in the economics and marketing literature. In the remainder of this section, we assume that the assortment size (k) is given and analyze the optimal pricing problem.

Formally, in the riskless case the expected profit in (3.3) reduces to:

$$\Pi^0(p, k) = k(p - c)\lambda q(p, k). \quad (4.4)$$

The following result indicates that $\Pi^0(p, k)$ attains a unique maximum that we denote by p_k^0 .

Corollary 4.2.3 *The expected riskless profit $\Pi^0(p, k)$ is unimodal in p .*

Proof. Follows as a special case of the results in Anderson et al. [2] or Cattani et al. [15]. ■

In the following theorem, we present a comparative statics analysis on p_k^0 , similar to that on p_k^* in Section 4.2.1.

Theorem 4.2.2 *For a fixed assortment size k , the optimal riskless price p_k^0 is:*

- (i) *Increasing in the assortment size, k ;*
- (ii) *Increasing in the unit cost per item, c ;*
- (iii) *Increasing in the mean reservation price, α .*

Proof. See Appendix G. ■

The intuitions behind Theorem 4.2.2 are clear: (i) Higher variety increases the overall demand for the product line, so the retailer can afford increasing the price and losing some of the demand to achieve higher profits; (ii) a higher unit cost requires a higher price to maintain a reasonable profit margin; and (iii) if consumers are willing to pay more, on average, then the price should be raised to increase the profit. More importantly, by comparing Theorem 4.2.2 to Theorem 4.2.1, we get another

interpretation of the effect of limited inventory on pricing. Recall that we were able to show that the risky price, p_k^* , is increasing in k or c above certain threshold values only (see Theorem 4.2.1). Indeed, we observe, numerically, that p_k^* can be decreasing in k or c at advanced stages of thinning of demand (characterized by a large value of k ; see Appendix K), when the inventory costs are high. In these cases, p_k^* decreases with k or c to sustain a level of demand yielding a sales revenue that covers the inventory cost. On the other hand, when inventory is not a factor, the riskless price, p_k^0 , is always increasing in k and c , as stated in Theorem 4.2.2.

An important question in the literature on the single item joint pricing and inventory problem is how the risky price, p^* , compares with the riskless price, p^0 . For an additive demand function, Mills [67] finds that $p^* \leq p^0$. On the other hand, for the multiplicative demand case, Karlin and Carr [47] prove that $p^* \geq p^0$. Young [91] generalizes these two results by proving that (i) $p^* \leq p^0$, if the demand variance is nondecreasing in the price, p , and the demand coefficient of variation is increasing in p ; and (ii) $p^* \geq p^0$, if the demand variance is decreasing in p and the demand coefficient of variation is nonincreasing in p . In our case, the demand variance ($\lambda q(p, k)$) is decreasing in p , while the demand coefficient of variation ($1/\sqrt{\lambda q(p, k)}$) is increasing in p : a case which does not fit into the framework of Young's results. In fact, for cases like ours, Petruzzi and Dada [76] conjecture, on the relationship between p^* and p^0 , that "either the price dependency of demand variance or of demand coefficient of variation will take precedence, thereby ensuring a determinable direction for the relationship." The following result confirms Petruzzi and Dada's conjecture and suggests the criterion, $p_k^0 < 2c$, with which to determine the direction of the relationship.

Lemma 4.2.2 *If $p_k^0 < 2c$, then $p_k^* \leq p_k^0$. Otherwise, $p_k^* \geq p_k^0$.*

Proof. Observe that $\lim_{\lambda \rightarrow \infty} \Pi(p, k) = \Pi^0(p, k)$ and hence $\lim_{\lambda \rightarrow \infty} p_k^* = p_k^0$. Then, applying Corollary 4.2.2 with $\lambda_0 = \infty$ completes the proof. ■

We observe that there exists cases where $p_k^0 < 2c$ as well as cases where $p_k^0 > 2c$, depending on the value of α (see the examples in Appendix K). In particular, by Theorem 5 (iii) we expect to have $p_k^* \leq p_k^0$ for small values of α and $p_k^* \geq p_k^0$ for large values of α . The interpretation of these findings follows the reasoning by Petruzzi and Dada [76] detailed above. For small α , the coefficient of variation of the demand $\left(1/\sqrt{\lambda q(p, k)}\right)$ is large, while the variance $(\lambda q(p, k))$ is small. Therefore, p_k^* is decreased below p_k^0 to decrease the demand coefficient of variation (which takes precedence over the demand variance). A similar interpretation applies to the case of large α , where p_k^* is increased over p_k^0 to reduce the somewhat large variance in this case.

4.3 On the Optimal Assortment Size when the Price is Fixed

In this section, we consider a retailer who is a “price-taker” in the market. With the market price of $p > c$, the retailer determines the optimal assortment size, $k_p^* = \arg \max_{k>0} \Pi(p, k)$, for her product line, where $\Pi(p, k)$ is as given in (4.3). (Note that we assume here that the cardinality of Ω is infinite; that is, there is no upper bound on the variety level, k .) This case applies to items for which the retailer’s pricing flexibility is quite limited due to factors such as high market competition. We note that our model in this section can be seen as a special case of the model in van Ryzin and Mahajan [80]. Our simplified model, with all items having identical reservation price distributions, allows us to obtain somewhat stronger results than those in [80], especially when analyzing the effect of changing model parameters on the optimal assortment size.

We make the following assumption throughout this section, which guarantees that the optimal assortment set is not empty, i.e., the retailer will not be better off by not selling anything.

(A3): The expected profit, $\Pi(p, k)$, is increasing in k at $k = 1$, that is, $\left.\frac{\partial \Pi(p, k)}{\partial k}\right|_{k=1} > 0$, or equivalently,

$$\lambda > \frac{a^2 c^2}{p^2} (1 + v_0 e^{-(\alpha-p)/\mu}) \left(1 + \frac{e^{(\alpha-p)/\mu}}{2v_0}\right)^2.$$

(A side benefit of **(A3)** is that it reduces the probability of negative demand by requiring the mean of the Normal demand to be above a certain value.)

As discussed above, a large assortment size, k , leads to a higher expected total demand (and, therefore, to a higher sales revenue when the price is exogenously determined as in this section). On the other hand, a large k may also lead to higher inventory costs due to the thinning of demand per item. Even though we consider no direct costs for adding items to the optimal assortment, as a result of this trade-off one would expect the optimal assortment size, k_p^* , to be neither too small nor too large. The following result confirms this intuition.

Theorem 4.3.1 *The expected profit $\Pi(p, k)$ is strictly pseudoconcave and unimodal in k .*

Proof. See Appendix H. ■

Theorem 4.3.1 states that the expected profit increases with variety (k) up to $k = k_p^*$. For $k > k_p^*$, adding more items to the product line will only diminish the expected profit. Thus, Theorem 4.3.1 implies that $k_p^* < \infty$ (i.e., there exists an upper limit on the variety level). On the other hand, in the riskless case (which assumes infinite inventory levels), the expected profit, $k(p - c)\lambda q(p, k)$, is increasing in k , and there is no upper bound on variety in the product line (Aydin and Ryan [6] prove a similar result for the riskless case). That is, inventory costs limit the variety level of the product line.

Theorem 4.3.1 leads to the following optimality condition for determining k_p^* when the assortment size, k , is treated as a continuous variable.

Corollary 4.3.1 *Assume that **(A3)** holds. Then, the first-order optimality condition, $\left. \frac{\partial \Pi(p, k)}{\partial k} \right|_{k=k_p^*} = 0$, is necessary and sufficient to determine k_p^* .*

Proof. Follows directly from Theorem 4.3.1. ■

Note that the optimal assortment size with integral value is given by $\tilde{k}_p^* = \arg \max_{k \in \{\lfloor k_p^* \rfloor, \lceil k_p^* \rceil\}} \Pi(p, k)$, where $\lfloor x \rfloor$ ($\lceil x \rceil$) is the largest (smallest) integer \leq (\geq) x .

4.3.1 Comparative Statics on the Optimal Assortment Size

We next analyze the effect of changing various model parameters on k_p^* . In all the subsequent results, we assume that the parameters change within the range where **(A3)** holds. The following theorem states that k_p^* is monotone in the unit cost per item, c , and the total mean demand, λ .

Theorem 4.3.2 *For a fixed price p , the optimal assortment size k_p^* is:*

- (i) *Decreasing in the unit cost per item, c ;*
- (ii) *Increasing in the expected store volume (arrival rate), λ .*

Proof. See Appendix I. ■

Theorem 4.3.2 states that the higher the unit cost per item, the lower the optimal variety level. That is, retailers selling expensive items should not offer a wide variety. On the other hand, retailers with low cost items should diversify their assortments. This kind of practice is adopted by many retailers. For example, a grocery store offers several types of the relatively low cost ordinary produce, while offering only a few types of the more expensive organic produce. A popular clothing retailer offers relatively few expensive models (possibly from reputable fashion brands) along with a wide array of the low quality/price brands. Theorem 4.3.2 also indicates that a higher store volume allows the retailer to offer a wide variety. Van Ryzin and Mahajan [80] prove that a similar result holds asymptotically (for λ large) in a more general case where items in the product line do not need to have the same mean consumer valuation. Our result is stronger (as it holds for any λ) for the special case that we consider. Van Ryzin and Mahajan [80] offer an excellent interpretation of this finding

and utilize it to justify the fact that “Super Stores” carry a much higher level of variety than small stores. The interested reader is referred to van Ryzin and Mahajan [80] for details.

In general, similar monotonicity properties do not seem to hold for the behavior of k_p^* as a function of other model parameters. However, the following theorem establishes monotonicity results under a fairly mild condition.

Theorem 4.3.3 *For a fixed price p , if $q(p, 1) > \frac{1}{2}$ (equivalently, $u_0 < \alpha - p$), then the optimal assortment size k_p^* is:*

- (i) *Increasing in the price, p (in the range where $p < \alpha - u_0$);*
- (ii) *Decreasing in the mean reservation price, α (in the range where $\alpha > u_0 + p$);*
- (iii) *Increasing in the utility of the no-purchase option, u_0 (in the range where $u_0 < \alpha - p$).*

Proof. See Appendix I. ■

The condition in Theorem 4.3.3 simply states that, on average, the no-purchase option is less appealing than buying from the retailer’s product line even when it consists of a single item (observe that $kq(p, k) \geq q(p, 1)$, for $k \geq 1$). Clearly, for relatively high quality and/or low-priced items, this condition is expected to hold (since $\alpha - p$ would be large). Theorem 4.3.3 (i) indicates that a higher price increases the cost of underage, so variety is increased to reduce the risk of losing a customer. We note that van Ryzin and Mahajan [80] show that such a result holds for a price that is high enough. Thus, Theorem 4.3.3 (i) complements van Ryzin and Mahajan’s Result since it holds for a relatively low price ($p < \alpha - u_0$). Theorem 4.3.3 (ii) indicates that as the quality of the items increases, there will be less need for variety. This is, for example, the case of a sandwich shop that is recognized by customers for offering a few good sandwiches. Theorem 4.3.3 states that the shop will not gain much by offering a more diversified menu. Finally, Theorem 4.3.3 (iii) also states that

a high no-purchase utility (possibly indicating a fierce competitive environment) forces the retailer to increase the breadth of her product line in order to reduce the number of unsatisfied customers. Van Ryzin and Mahajan [80] also present a weaker asymptotic version of Theorem 4.3.3 (iii) in a more general case and provide a detailed discussion. Their discussion also applies to our case.

4.4 Bounds and the Joint Variety/Pricing/Inventory Problem

In this section, we first present bounds on the optimal price, p_k^* , when the variety level is fixed. We then briefly discuss the joint variety/pricing/inventory problem. The following lemma presents an upper bound on p_k^* , which can be obtained by solving a simple equation.

Lemma 4.4.1 *Assume (A2) holds. Then, for a fixed assortment size k , an upper bound on p_k^* is given by \overline{p}_k , which is the largest solution to the following equation:*

$$\lambda p^2 q(p, k) - a^2 c^2 = 0. \quad (4.5)$$

Proof. See Appendix J. ■

We note here that under (A2), (4.5) has exactly one solution in (c, ∞) and $\Pi(p, k) > 0$ for $p \in (c, \overline{p}_k)$; see Appendix J. Lemma 4.4.1 also allows the development of an upper bound on p_k^* for any assortment size k , as given in the following result.

Corollary 4.4.1 *Assume (A2) holds. Then, for any assortment size $k \in \mathbf{Z}^+$, an upper bound on p_k^* is given by \overline{p}_1 , which is the largest solution to (4.5) with $k = 1$.*

Proof. See Appendix J. ■

Other bounds on p_k^* may be obtained by applying the first-order optimality conditions as indicated in the following lemma.

Lemma 4.4.2 *Assume (A2) holds. Then, for a fixed assortment size k , let $p_i^k, i = 1, 2$, denote the (unique) solution to $w_i(p, k) = 0$, where $w_1(p, k) = 1 - \frac{(p-c)}{\mu}(1 - kq(p, k))$ and $w_2(p, k) = \frac{2c}{p} - \frac{(p-c)}{\mu}(1 - kq(p, k))$. Let $p_{(1)}^k = \min(p_1^k, p_2^k)$ and $p_{(2)}^k = \max(p_1^k, p_2^k)$. Then, $p_k^* \notin (p_{(1)}^k, p_{(2)}^k)$.*

Proof. See Appendix J. ■

Next we discuss the joint variety/pricing/inventory problem in which the retailer has the flexibility to jointly determine the price, assortment size, and inventory level in a way as to maximize her expected profit. Formally, the objective of the joint problem is as follows:

$$\Pi(p^{**}, k^{**}) = \max_{k \in \mathbf{Z}^+} \max_{p \geq c} \{\Pi(p, k)\}. \quad (4.6)$$

The joint optimal price and assortment size may be determined by enumerating over all possible values of the assortment size $k \in \mathbf{Z}^+$ and performing a single variable search on p for each value of k . (The bounds on p_k^* given in Lemmas 4.4.1 and 4.4.2 and Corollary 4.4.1 can be useful in this search.) Then, the value k^{**} yielding the maximum expected profit is the optimal assortment size and the corresponding $p^{**} \equiv p_{k^{**}}^*$ is the optimal price.

Note that \bar{p}_1 in Corollary 4.4.1 determines an upper bound on p^{**} . The following lemma determines an upper bound on k^{**} .

Lemma 4.4.3 *The set $\bar{K} = \{\bar{k} \mid \Pi(p, k) < 0 \text{ for } p \in (c, \infty), k \geq \bar{k}\}$ is not empty. Moreover, let $\underline{k} \equiv \inf\{\bar{K}\}$. Then $k^{**} < \underline{k} < \infty$.*

Proof. See Appendix J. ■

Therefore, the total enumeration on k in the joint variety/pricing/inventory problem can be done on finite values of $k = 1, 2, \dots, \bar{k}$. More importantly, Lemma 4.4.3 implies that in the joint problem the optimal assortment size is finite, while the optimal assortment size for the riskless case is infinite. That is, even when the retailer has control over the price, inventory costs impose an upper limit on variety beyond which adding more items to the product line is not profitable.

Chapter 5

Pricing and Inventory Decisions under Convenience Tying

In this chapter, we study the pricing and inventory decisions for the convenience tying strategy described in Section 1.4. The remainder of this chapter is organized as follows. Section 5.1 introduces our model and assumptions. Section 5.2 studies the optimal pricing problem (assuming infinite inventory levels) and analyzes the effect of convenience tying on pricing and profitability (by comparing it to the independent components strategy). Finally, Section 5.3 analyzes the profitability of convenience tying when the inventory levels are set optimally, while the prices are exogenously determined.

5.1 Model and Assumptions

Consider two complementary items that we denote by the “primary” (or tying) item, P , and the “secondary” (or tied-in) item, S , that can be sold according to two different strategies in a retail store. Under the “independent components strategy (IC),” items P and S are sold independently in two different locations of the store, denoted by E_P and E_S , respectively. Under the “convenience

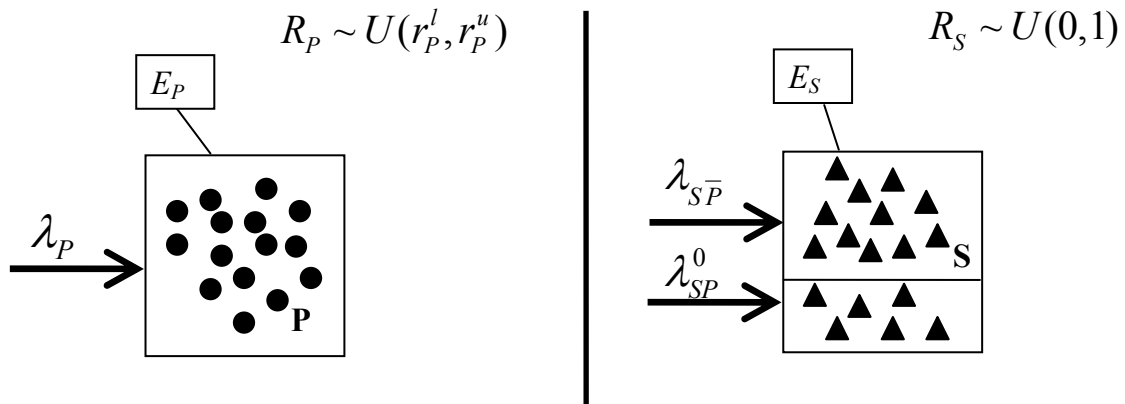
tying strategy (CT),” P is sold in location E_P , while S is sold in both locations E_S and E_P , where we refer to the latter selling practice of S as “tying S to P ” or making “ S tied-in to P ,” see Figure 4.1 for a representation of these strategies, where λ_i^J and R_i^J respectively denote the arrival rate per selling period and the corresponding consumer reservation price for item $i \in \{P, S, S\bar{P}, SP\}$ under selling strategy $J \in \{0, T\}$. The subscripts “ SP ” and “ $S\bar{P}$ ” refer to customers who buy both S and P and customers who buy S and do not buy P , respectively, while the subscripts “ P ” and “ S ” respectively refer to customers who buy P and who buy S . The superscripts “ 0 ” and “ T ” respectively denote the IC and CT strategies. We omit these superscripts when the corresponding parameter is the same under both IC and CT. In addition, let p_i , $i \in \{P, S\}$, denote the selling price of item i , and let $q_P(p_P)$ denote the purchase probability of the primary item P .

We consider the following demand model.

- All demand is generated from customers arriving in a single selling period.
- The arrival process for customers who purchase P has a rate λ_P (per selling period) under both IC and CT.
- The arrival process to E_S for customers who buy S and not buy P has a rate $\lambda_{S\bar{P}}$ under both IC and CT.
- Under IC, the arrival process to E_S for customers who buy both P and S has a rate λ_{SP}^0 , and this arrival process will vanish under CT.¹
- Under CT, a customer will buy S in E_P only if she is willing to buy P first, which implies that the “effective” arrival rate for customers who buy S in E_P is given by $\lambda_P q_P(p_P)$.

¹This is a reasonable assumption, since customers who buy S to consume it with P will most likely buy S in E_P under CT. We believe that from a psychological point of view a consumer will buy the primary item, P , first, and then consider buying the secondary item, S , that will be consumed with P .

a) IC Strategy



b) CT Strategy

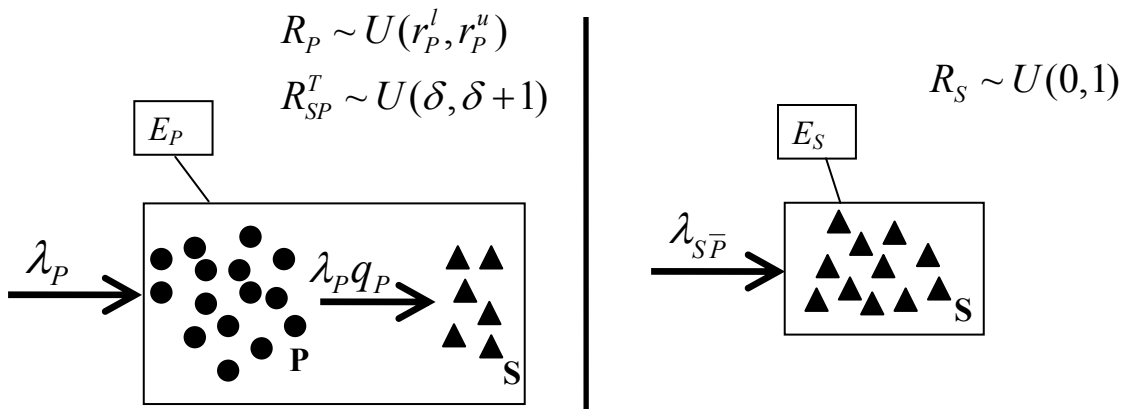


Figure 4.1. Independent components (IC) and convenience tying (CT) strategies

In practice, arrival rates $\lambda_{S\bar{P}}$ and λ_{SP}^0 can be estimated from a “basket analysis,” which tracks the items bought by each customer on every shopping occasion.

Purchase probabilities for items P and S are determined as follows. Consumers act to maximize their surplus, with a no-purchase utility equal to zero (this is a common assumption in the bundling literature; see, for example, Schmalensee [82]). The reservation prices (consumer valuations) for P and S (in locations E_S and E_P) have the following uniform distributions.

- The consumer reservation price for P , R_P , is uniformly distributed on $[r_P^l, r_P^u]$, i.e.,
 $R_P \sim U(r_P^l, r_P^u)$, under both IC and CT.
- The reservation price for S among customers who visit E_S is $R_S \sim U(0, 1)$ (without loss of generality) under both IC and CT.
- The reservation price for S under CT among customers who visit E_P is $R_{SP}^T \sim U(\delta, \delta + 1)$.

The parameter δ represents the shift in the consumer valuation for S as a result of tying. If $\delta \geq 0$, then customers who visit E_P are willing to pay for S at least as much as those customers who visit E_S . This could be the case of a “strong complementarity” between P and S , e.g., salads and dressings, business suits and shirts or ties, etc. Otherwise, if $\delta < 0$, then customers who visit E_P value S less than those who visit E_S . This may be the case where only a small fraction of the consumers in E_P values highly the consumption of S in conjunction with P , e.g., a mobile phone and expensive accessories such as a wireless headset. Then the purchase probabilities for P in E_P (under both IC and CT), S in E_S (under both IC and CT), and S in E_P under CT are given by

$$q_P(p_P) = \frac{\left(r_P^u - p_P + (p_P - r_P^l)^-\right)^+}{r_P^u - r_P^l}, \quad (5.1)$$

$$q_{SP}^T(p_S) = (\delta + 1 - p_S + (p_S - \delta)^-)^+, \quad (5.2)$$

$$q_S(p_S) = \left(1 - p_S + p_S^-\right)^+, \quad (5.3)$$

where $x^+ \equiv \max(0, x)$ and $x^- \equiv \min(0, x)$.

We assume infinite inventory levels of items throughout the system, which allows us to focus on the pricing aspect of CT. Moreover, this assumption applies in certain practical situations as discussed in Section 1.4. We note that in Section 5.3 we relax this assumption and study a variation of our model under limited inventories.

Define $\Pi_S^0(p_S)$ and $\Pi_P(p_P)$ as the expected profit from S under IC, and that from P under both IC and CT. Define also $\Pi_{S\bar{P}}^T(p_S)$ and $\Pi_{SP}^T(p_S, p_P)$ as the expected profits from S under CT in E_S and E_P , respectively. Then, the expected profits under IC and CT can be written as

$$\begin{aligned} \Pi^0(p_S, p_P) &= \Pi_S^0(p_S) + \Pi_P(p_P) = (\lambda_{S\bar{P}} + \lambda_{SP}^0)(p_S - c_S)q_S(p_S) + \lambda_P(p_P - c_P)q_P(p_P), \\ &= (\lambda_{S\bar{P}} + \lambda_{SP}^0)(p_S - c_S) \left(1 - p_S + p_S^-\right)^+ + \lambda_P(p_P - c_P) \frac{\left(r_P^u - p_P + (p_P - r_P^l)^-\right)^+}{r_P^u - r_P^l}, \end{aligned} \quad (5.4)$$

$$\begin{aligned} \Pi^T(p_S, p_P) &= \Pi_{S\bar{P}}^T(p_S) + \Pi_{SP}^T(p_S, p_P) + \Pi_P(p_P), \\ &= \lambda_{S\bar{P}}(p_S - c_S)q_S(p_S) + \lambda_P q_P(p_P)(p_S - c_S)q_{SP}^T(p_S) + \lambda_P(p_P - c_P)q_P(p_P), \\ &= \lambda_{S\bar{P}}(p_S - c_S) \left(1 - p_S + p_S^-\right)^+ + \lambda_P q_P(p_P)(p_S - c_S) (\delta + 1 - p_S + (p_S - \delta)^-)^+ \\ &\quad + \lambda_P(p_P - c_P) \frac{\left(r_P^u - p_P + (p_P - r_P^l)^-\right)^+}{r_P^u - r_P^l}. \end{aligned} \quad (5.5)$$

Then, the retailer's objective of maximizing the expected profit is given by

$$\Pi^* = \max_{J \in \{0, T\}} \max_{p_S, p_P} \left\{ \Pi^J(p_S, p_P) \right\}.$$

5.2 Pricing under Convenience Tying and Independent Components Strategies

In this section we study the problem of optimal pricing under both IC and CT. This study is within the same spirit of the conventional analysis of bundling in the economics literature. We make the following assumptions throughout this section.

Assumption **(A4)**: $\delta^+ < c_S < 1 + \delta^-$, and $r_P^l < c_P < r_P^u$.

Assumption **(A5)**: $\delta^+ \leq 1 - c_S$, and $\delta^- \geq -(1 - c_S)/2$.

Assumption **(A4)** imposes realistic ranges on the unit costs of S and P , while Assumption **(A5)** states that the shift in the reservation price of S , $|\delta|$, should not be too large, ensuring that S is demanded at both E_S and E_P , which seems to be the case in practice.²

Under IC, the solution to the pricing problem is straightforward as indicated by the following theorem.

Theorem 5.2.1 *The optimal prices under IC, $(p_S^0, p_P^0) \equiv \arg \max_{p_S, p_P} \Pi^0(p_S, p_P)$, are given by*

$$p_S^0 = \frac{1 + c_S}{2}, \quad p_P^0 = \frac{r_P^u + c_P}{2}.$$

Proof. It can be easily shown from (5.4) that an optimal price, (p_S^0, p_P^0) , must satisfy $p_S^0 \in (c_S, 1)$ and $p_P^0 \in (c_P, r_P^u)$. Under these restrictions, the expected profit under IC reduces to

$$\Pi^0(p_S, p_P) = (\lambda_{S\bar{P}} + \lambda_{SP}^0)(p_S - c_S)(1 - p_S) + \lambda_P(p_P - c_P) \frac{(r_P^u - p_P)}{r_P^u - r_P^l},$$

which is differentiable everywhere in this range. The proof then follows from the first- and second-

²Specifically, **(A5)** implies that when $\delta < 0$ ($\delta > 0$), $|\delta|$ is not too large leading to a high optimal price of S relative to customers who demand it in E_P (E_S), to the extent that none of these customers could buy it.

order optimality conditions. ■

Next we characterize the structure of the optimal prices under CT.

5.2.1 Structure of the Optimal Prices under CT

This section focuses on the problem of determining the optimal prices under CT given by $(p_S^T, p_P^T) \equiv \arg \max \Pi^T(p_S, p_P)$, where $\Pi^T(p_S, p_P)$ is given in (5.5). The following lemma shows that under assumption **(A4)** the optimal prices under CT are restricted to a bounded domain.

Lemma 5.2.1 *Under CT, the optimal prices $(p_S^T, p_P^T) \in D$, where*

$$D = \{(p_S, p_P) | c_S < p_S < 1 + \delta^+ \text{ and } r_P^l \leq p_P < r_P^u\}.$$

Proof. It can be easily shown, by enumerating the different cases for $\delta > 0$ and $\delta \leq 0$, that for every point $(p'_S, p'_P) \notin D$, there exists an improving direction that leads to a point $(p''_S, p''_P) \in D$ such that $\Pi^T(p''_S, p''_P) > \Pi^T(p'_S, p'_P)$. This implies that $(p_S^T, p_P^T) \in D$. ■

Lemma 5.2.1 is intuitive. If S is priced below its unit cost, then the profit could be improved by increasing p_S , while if S is priced too high (i.e., $p_S \geq 1 + \delta^+$), then no consumer buys S , and the profit can be improved by decreasing p_S . The same reasoning applies when the price of P is too high (i.e., $p_P \geq r_P^u$). However, the price of P may be set below its unit cost (when $r_P^l \leq p_P < c_P$) if this generates enough revenue from the sales of S that could cover the loss from P ; see Example 1 below for such a case.

The following theorem characterizes the structure of the optimal prices under CT.

Theorem 5.2.2 *The optimal prices under CT, (p_S^T, p_P^T) , are the unique local maximum of $\Pi^T(p_S, p_P)$ on $G = \{(p_S, p_P) | c_S < p_S < 1 + \delta^- \text{ and } r_P^l \leq p_P < r_P^u\}$, and are determined as follows:*

(i) *If the first-order optimality conditions have a solution on G , then this solution is unique, and the optimal prices can be determined by solving the following equations:*

$$p_S^T = \frac{\lambda_P(\delta + 1 + c_S)(r_P^u - p_P^T) + \lambda_S \bar{p}(1 + c_S)(r_P^u - r_P^l)}{2(\lambda_S \bar{p}(r_P^u - r_P^l) + \lambda_P(r_P^u - p_P^T))}, \quad (5.6)$$

$$p_P^T = \frac{-(p_S^T - c_S)(\delta + 1 - p_S^T) + r_P^u + c_P}{2}. \quad (5.7)$$

(ii) *Otherwise, $p_P^T = r_P^l$ and $p_S^T = \frac{(1+c_S)}{2} + \delta \frac{\lambda_P}{2(\lambda_S \bar{p} + \lambda_P)}$.*

Proof. See Appendix L.

As stated above, we can have interesting situations where P is sold below its unit cost. Such a situation is illustrated in the following example.

Example 1.

If $\delta = 0$, then (5.6) and (5.7) have closed-form solutions given by

$$\hat{p}_S = \frac{1 + c_S}{2}, \quad \hat{p}_P = \frac{r_P^u + c_P}{2} - \frac{(1 - c_S)^2}{8}.$$

If $\hat{p}_P > r_P^l$, i.e., $4(r_P^u + c_P - 2r_P^l) - (1 - c_S)^2 > 0$, then $(p_S^T, p_P^T) = (\hat{p}_S, \hat{p}_P)$. Otherwise, if $\hat{p}_P < r_P^l$, i.e., $4(r_P^u + c_P - 2r_P^l) - (1 - c_S)^2 < 0$, then $(p_S^T, p_P^T) = (\frac{1+c_S}{2}, r_P^l)$.

In Example 1, P is priced below its unit cost, at r_P^l , if $4(r_P^u + c_P - 2r_P^l) - (1 - c_S)^2 < 0$, which is more likely to hold if the range of the reservation price of P ($r_P^u - r_P^l$) is small (leading to a low profit margin for P), or the unit cost of S (c_S) is low (allowing a higher profit margin on S). Example 1

also reveals another situation where P is priced below its unit cost. Specifically, this case happens when $4(r_P^u + c_P - 2r_P^l) - (1 - c_S)^2 > 0$, and $\hat{p}_P = \frac{r_P^u + c_P}{2} - \frac{(1 - c_S)^2}{8} < c_P$. This is the case of a relatively high unit cost of P . In these cases, P may be seen as a “loss leader” priced below its unit cost in order to increase the demand for S (see, for example, Hess and Gerstner [42] and Lal and Matutes [49]).

5.2.2 Comparison of the Optimal Prices under IC and CT

The following lemma investigates the effect of convenience tying on pricing.

Lemma 5.2.2 *The optimal prices under CT, (p_S^T, p_P^T) , compare to the optimal prices under IC, (p_S^0, p_P^0) , as follows:*

(i) $p_S^T > p_S^0$ if and only if $\delta > 0$;

(ii) $p_S^T < p_S^0$ if and only if $\delta < 0$;

(iii) $p_S^T = p_S^0$ if and only if $\delta = 0$;

(iv) $p_P^T < p_P^0$.

Proof. See Appendix M.

Lemma 5.2.2 states that convenience tying will increase the price of S (the tied-in item) if S is more “popular” among consumers who visit E_P than among those who visit E_S . This can be explained by considering a “system-wide” reservation price of S which dictates its price. This system reservation price is similar to a weighted average of the reservation prices of S in E_S and in E_P , with the weights being the demand volumes for S in E_S ($\lambda_{S\bar{P}}$) and in E_P ($\lambda_{Pq_P}(p_P)$). If $\delta > 0$, then the system reservation price of S under CT is higher than its reservation price under IC, and hence

the price of S is increased. The converse happens if $\delta < 0$ (and if $\delta = 0$ the price of S is the same under both IC and CT). Lemma 5.2.2 also states that convenience tying always decreases the price of P (the tying item). This can be interpreted by the fact that under convenience tying P generates more demand to the “system,” which allows the reduction of its profit margin in exchange for an additional profit from S .

5.2.3 Comparative Statics on the Optimal Prices under CT

The optimal prices under CT do not have closed-form expressions in most cases as shown above. Moreover, these cases (with $\delta \neq 0$ indicating a change of consumer taste in E_P , and $p_P > r_P^l$, implying that some customers will not buy P) seem to be the most reasonable in practice. The following lemmas perform comparative statics analysis on the effect of changing demand and cost parameters on the optimal prices under CT in such cases.

Lemma 5.2.3 *Assume that $\delta \neq 0$ and $p_P^T > r_P^l$. Then, p_S^T is:*

(i) *decreasing (increasing) in $\lambda_{S\bar{P}}$, the mean demand of S in E_S , if $\delta > 0$ ($\delta < 0$);*

(ii) *increasing (decreasing) in λ_P , the mean demand of P in E_P , if $\delta > 0$ ($\delta < 0$);*

(iii) *increasing (decreasing) in r_P^l if $\delta > 0$ ($\delta < 0$);*

(iv) *increasing in δ if $\delta > 0$;*

(v) *decreasing (increasing) in c_P , the unit cost of P , if $\delta > 0$ ($\delta < 0$).*

Proof. See Appendix N.

The results in Lemma 5.2.3 can be explained by utilizing the system reservation price concept for S explained above. For example, if $\delta > 0$, then increasing $\lambda_{S\bar{P}}$ will decrease the system reservation

price of S , and the converse happens if $\delta < 0$. The opposite happens if λ_P or r_P^l increases (which will increase the volume for S in E_P , $\lambda_P q_P(p_P)$). Increasing $\delta > 0$ increases the valuation of S in E_S , and hence leads to a higher system reservation price for S . Finally, increasing the unit cost of P leads to a higher optimal price of P (see Lemma 5.2.4 below), and hence results in a lower demand volume for S in E_P (since $q_P(p_P)$ is decreasing in p_P), which results in similar effects as the above.

Lemma 5.2.4 *Assume that $\delta \neq 0$, and $p_P^T > r_P^l$. Then, p_P^T is:*

- (i) *increasing in $\lambda_{S\bar{P}}$, the mean demand of S in E_S ;*
- (ii) *decreasing in λ_P , the mean demand of P in E_P ;*
- (iii) *decreasing in δ if $\delta > 0$;*
- (iv) *decreasing in r_P^l ;*
- (v) *increasing in c_P , the unit cost of P .*

Proof. See Appendix N.

Lemma 5.2.4 indicates that any change in the model parameters that could increase the revenue from S in E_P (when the price of P is fixed) is “amplified” by a further decrease in the price of P . That is due to the fact that a higher volume of S in E_P allows the retailer to forfeit some of her revenues from P in exchange for higher returns from S . For example, decreasing $\lambda_{S\bar{P}}$ increases the system reservation price of S as explained above, and hence leads to a higher price for S in E_P (if the price of P is unchanged), which leads to a lower price of P . Similarly, increasing λ_P , $\delta > 0$, or r_P^l works to increase the volume of S , and, hence, decreases the price of P . We note here that the result on r_P^l is interesting and may appear counter-intuitive, since it indicates that increasing the consumer valuation for P decreases its price. Finally, changing the unit cost of P does not have

a “direct” effect on the sales of S (but rather an indirect one as explained above), and hence the corresponding change in the price of P confirms with the conventional intuition and varies in the same direction as its unit cost.

5.2.4 Comparison of the Optimal Profits under IC and CT

The following lemma studies the profitability of CT when the price of one of the items is exogenous (that is, the price of one of the items is fixed at the same value under both IC and CT). This applies in cases where the retailer has control over the price of only one of the items. Define Π^{T^*} and Π^{0^*} as the optimal expected profits under CT and IC respectively.

Lemma 5.2.5 *CT is more profitable than IC, i.e., $\Pi^{T^*} > \Pi^{0^*}$, if*

(i) *The price of P is fixed at $p_P \in (r_P^l, r_P^u)$ and*

$$-\gamma(p_P)^2 \lambda_{S\bar{P}} + (\lambda_{PQP}(p_P) - \lambda_{SP}^0) \left(\frac{1 - c_S}{2} \right)^2 + \lambda_{PQP}(p_P) \left[\delta \frac{(1 - c_S)}{2} + \gamma(p_P)^2 \alpha(p_P) (1 + 1/\alpha(p_P)) \right] > 0,$$

where $\gamma(p_P) \equiv \delta \frac{\alpha(p_P)}{1 + 2\alpha(p_P)}$ and $\alpha(p_P) \equiv \frac{\lambda_{PQP}(p_P)}{2\lambda_{S\bar{P}}}$.

(ii) *The price of S is fixed at $p_S \in (c_S, 1 + \delta^-)$, $p_P^T = p_P^0 - \frac{\beta(p_S)}{2} > r_P^l$, and*

$$\lambda_P \frac{\beta(p_S)}{2(r_P^u - r_P^l)} \left(r_P^u - c_P + \frac{\beta(p_S)}{2} \right) - \lambda_{SP}^0 (1 - p_S)(p_S - c_S) > 0,$$

where $\beta(p_S) \equiv (\delta + 1 - p_S)(p_S - c_S)$.

Proof. See Appendix O. ■

Lemma 5.2.5 has the following corollary which provides sufficient profitability conditions that are easy to verify.

Corollary 5.2.1 *CT is more profitable than IC, i.e., $\Pi^{T^*} > \Pi^{0^*}$, if*

(i) *The price of P is fixed at $p_P \in (r_P^l, r_P^u)$, $\delta > 0$, and $\lambda_P q_P(p_P) > \lambda_{S\bar{P}} + \lambda_{SP}^0$.*

(ii) *The price of P is fixed at $p_P \in (r_P^l, r_P^u)$, $\delta < 0$, and $\lambda_P q_P(p_P)(1+2\delta/(1-c_S)) > \lambda_{S\bar{P}}/4 + \lambda_{SP}^0$.*

Proof. Follows by noting that, under **(A5)**, the first term of the condition in Lemma 5.2.5 (i) is bounded below by $-\left(\frac{1-c_S}{2}\right)^2 \lambda_{S\bar{P}}$ if $\delta > 0$, and by $-\left(\frac{1-c_S}{4}\right)^2 \lambda_{S\bar{P}}$ if $\delta < 0$. ■

Corollary 5.2.1 (i) states that CT is profitable if the demand volume for S in E_P under CT is larger than its demand volume in E_S under IC. This is intuitive. Corollary 5.2.1 (ii) presents a similar result and it also indicates that a smaller unit cost of S improves the profitability of CT (relative to that of IC).

The following corollary presents a necessary condition for the profitability of CT when the price of S is fixed.

Corollary 5.2.2 *When the price of S is fixed at $p_S \in (c_S, 1 + \delta^-)$ and $p_P^T = p_P^0 - \frac{\beta(p_S)}{2} > r_P^l$, CT is more profitable than IC, i.e., $\Pi^{T^*} > \Pi^{0^*}$, only if $\lambda_P q_P(p_P^T) q_{SP}^T(p_S) > 2\lambda_{SP}^0 q_S(p_S)$.*

Proof. Follows by noting that the condition in Lemma 5.2.5 (ii) may be rewritten as

$$\lambda_P q_P(p_P^T) q_{SP}^T(p_S) - \frac{\lambda_P}{2} q_{SP}^T(p_S)^2 \frac{(p_S - c_S)}{2(r_P^u - r_P^l)} - 2\lambda_{SP}^0 q_S(p_S) > 0. \blacksquare$$

Corollary 5.2.2 states that CT is profitable only if it increases the mean demand for S by at least two fold. This indicates that when the price of S is fixed, CT is unlikely to be profitable.

Finally, the following corollary utilizes Lemma 5.2.5 to provide sufficient conditions for the profitability of CT when the prices of S and P are both decision variables.

Corollary 5.2.3 *CT is more profitable than IC, i.e., $\Pi^{T*} > \Pi^{0*}$, if*

(i) *There exists $p_P \in (r_P^l, r_P^u)$ such that*

$$-\gamma(p_P)^2 \lambda_{S\bar{P}} + (\lambda_P q_P(p_P) - \lambda_{SP}^0) \left(\frac{1-c_S}{2} \right)^2 + \lambda_P q_P(p_P) \left[\delta \frac{(1-c_S)}{2} + \gamma(p_P)^2 \alpha(p_P) (1 + 1/\alpha(p_P)) \right] + \lambda_P \left[q_P(p_P)(p_P - c_P) - \left(\frac{r_P^u - c_P}{2(r_P^u - r_P^l)} \right)^2 \right] > 0.$$

(ii) *There exists $p_S \in (c_S, 1 + \delta^-)$ such that $p_P^T = p_P^0 - \frac{\beta(p_S)}{2} > r_P^l$ and*

$$\lambda_P \frac{\beta(p_S)}{2} \left(\frac{r_P^u - c_P}{(r_P^u - r_P^l)} + \frac{\beta(p_S)}{2(r_P^u - r_P^l)} \right) + (\lambda_{S\bar{P}} - \lambda_{SP}^0)(1 - p_S)(p_S - c_S) - \lambda_{S\bar{P}} \left(\frac{1 - c_S}{2} \right)^2 > 0.$$

Proof. Follows from Lemma 5.2.5 by noting that

$$\Pi^{T*} - \Pi^{0*} = \Pi_S^T(p_S^T, p_P^T) - \Pi_S^0(p_S^0, p_P^0) \geq \Pi_S^T(p_S, p_P) - \Pi_S^0(p_S^0, p_P^0),$$

for all $(p_S, p_P) \in G$. ■

5.3 A Special Case with Limited Inventory and Exogenous Pricing

In this section, we assume that prices of P and S are fixed (under both IC and CT), and study a cost side aspect of convenience tying. Specifically, we relax the assumption of infinite inventory levels of Section 5.1, and assume that inventory levels are finite and set optimally in a way so as to maximize the retailer's profit (equivalently, so as to reduce the inventory costs). We assume, similar to the models in Chapters 3 and 4, that all the inventory is to be sold in a single selling period with leftover inventory not being carried to subsequent periods within a newsvendor type inventory setting. We adopt the following demand model:

- The demand for P in E_P , X_P , has a Normal distribution with mean $\mu_P \equiv \lambda_P q_P(p_P)$ and standard deviation $\sqrt{\mu_P}$, i.e., $X_P \sim N(\mu_P, \sqrt{\mu_P})$, under both IC and CT.
- The demand in E_S of customers who buy S and not buy P is $X_{S\bar{P}} \sim N(\mu_{S\bar{P}}, \sqrt{\mu_{S\bar{P}}})$ under both IC and CT, where $\mu_{S\bar{P}} \equiv \lambda_{S\bar{P}} q_S(p_S)$.
- Under IC, the demand in E_S of customers who buy both S and P is $X_{SP}^0 \sim N(\mu_{SP}^0, \sqrt{\mu_{SP}^0})$, where $\mu_{SP}^0 \equiv \lambda_{SP}^0 q_S(p_S)$.
- Under CT, the demand for S in E_P is $X_{SP}^T \sim N(\mu_{SP}^T, \sqrt{\mu_{SP}^T})$, where $\mu_{SP}^T \equiv \lambda_P q_P(p_P) q_{SP}^T(p_S)$.

This demand model is based on assuming that all arrival processes to the system are Poisson processes and then utilizing the well-known Normal approximation to the Poisson distribution similar to the approach in Chapters 3 and 4. Note that the total demand for S in E_S under IC is $X_S^0 \equiv X_{S\bar{P}} + X_{SP}^0$. Therefore, $X_S^0 \sim N(\mu_{S\bar{P}} + \mu_{SP}^0, \sqrt{\mu_{S\bar{P}} + \mu_{SP}^0})$.

Our model may be seen as a two-location two-item newsvendor model with items having Normal demands, under the additional complexity of tying decisions. Adopting a cost structure similar to that used in Chapters 3 and 4, and assuming $p_i > c_i$, $i = S, P$, the expected profits at optimal inventory levels under IC and CT in (5.4) and (5.5) can be written as $\Pi^0 = \Pi_S^0 + \Pi_P$ and $\Pi^T = \Pi_{S\bar{P}}^T + \Pi_{SP}^T + \Pi_P$, or equivalently

$$\Pi^0 = (p_S - c_S)(\mu_{S\bar{P}} + \mu_{SP}^0) - p_S \theta_S \sqrt{\mu_{S\bar{P}} + \mu_{SP}^0} + (p_P - c_P)\mu_P - p_P \theta_P \sqrt{\mu_P}, \quad (5.8)$$

$$\Pi^T = (p_S - c_S)(\mu_{S\bar{P}} + \mu_{SP}^T) - p_S \theta_S \left(\sqrt{\mu_{S\bar{P}}} + \sqrt{\mu_{SP}^T} \right) + (p_P - c_P)\mu_P - p_P \theta_P \sqrt{\mu_P}, \quad (5.9)$$

where $\theta_i = \phi(\Phi^{-1}(1 - c_i/p_i))$, $i = S, P$, and $\phi(\cdot)$ and $\Phi(\cdot)$ respectively denote the probability density function and the cumulative distribution function of the standard Normal distribution.

Note first that the problem of finding the optimal inventory levels (under both IC and CT) is straightforward within our newsvendor setting. Specifically, the optimal inventory level for an item i with demand $X_i \sim N(\mu_{X_i}, \sigma_{X_i})$ is given by

$$y_i^* = \mu_{X_i} + \Phi^{-1}(1 - c_i/p_i)\sigma_{X_i}, \quad (5.10)$$

where c_i and p_i respectively denote the unit cost and the price of item i . We therefore focus on comparing IC and CT in the remainder of this section. Note also that since the demand for P is the same under both IC and CT, the inventory level and the expected profit from P will not change under IC and CT. Therefore, in the remaining analysis we focus on S . We make the following assumption that guarantees that the retailer will not be better off not selling any of item S at either location under both IC and CT.

Assumption (A6): $\Pi_S^0 > 0$, $\Pi_{S\bar{P}}^T > 0$, and $\Pi_{SP}^T > 0$, or equivalently, $\min(\sqrt{\mu_{S\bar{P}}}, \sqrt{\mu_{SP}^T}) > \theta_S \frac{p_S}{p_S - c_S}$.

The following lemma compares the optimal profits under IC and CT.

Lemma 5.3.1 *CT is more profitable than IC if and only if*

$$\mu_{SP}^T - \mu_{SP}^0 > \theta_S \frac{p_S}{p_S - c_S} \left(\sqrt{\mu_{SP}^T} + \sqrt{\mu_{S\bar{P}}} - \sqrt{\mu_{SP}^0 + \mu_{S\bar{P}}} \right) > 0.$$

Proof. See Appendix O. ■

Lemma 5.3.1 states that in order for convenience tying to be profitable, the mean demand of S should increase to an extent that generates enough revenue to cover the additional inventory costs incurred due to decentralizing the sales of S . This is due to the well-known result that demand variability (and hence inventory costs) are lower in centralized systems due to risk pooling; see, for

example, Eppen [25].

Next, we compare the optimal inventory levels under IC and CT.

Lemma 5.3.2 *Let y_S^T and y_S^0 respectively denote the optimal inventory level of S under CT and IC.*

Then, we have

$$y_S^T - y_S^0 = \mu_{SP}^T - \mu_{SP}^0 + \Phi^{-1}(1 - c_S/p_S) \left(\sqrt{\mu_{SP}^T} + \sqrt{\mu_{SP}^0} - \sqrt{\mu_{SP}^0 + \mu_{SP}^T} \right).$$

Proof. Follows from (5.10). ■

Lemmas 5.3.1 and 5.3.2 indicate that, when CT is profitable, the optimal inventory level under CT is higher than that under IC for a wide range of cases. Specifically, a conservative condition is that the inventory level of S will increase for any “critical ratio” $(1 - c_S/p_S)$ larger than 0.5 (corresponding to $\Phi^{-1}(1 - c_S/p_S) > 0$).

5.3.1 The Effect of Stockouts of P on CT

The above analysis is under the approximate assumption that a customer who is willing to buy P may buy S even if P is out of stock. This assumption is justified if the stockouts of P occur sporadically. In reality, a customer may not buy S in E_P if P is stocked out (even if she has the willingness to do so when P is in-stock). The above analysis, therefore, provides an *upper bound* on the profitability of CT. In the following, we discuss the development of a lower bound on this profitability which allows the retailer to gauge the effect of stockouts of P .

The lower bound is developed based on the assumption that if P is stocked out then no customer buys S . This case differs from the above in the distribution of X_{SP}^T , the demand for S in E_P . For example, assume that the demand for P , X_P , has a discrete distribution with mean $\lambda_P q_P(p_P)$. By

conditioning on the demand for P , the probability mass function of X_{SP}^T is given by

$$\begin{aligned} Pr\{X_{SP}^T = i\} &= \sum_{j=i}^{y_P} Pr\{X_P = j\} \binom{j}{i} q_{SP}^T(p_S)^i (1 - q_{SP}^T(p_S))^{j-i} \\ &\quad + Pr\{X_P > y_P\} \binom{y_P}{i} q_{SP}^T(p_S)^i (1 - q_{SP}^T(p_S))^{y_P-i}, \\ &\quad i = 0, \dots, y_P, \end{aligned} \quad (5.11)$$

where y_P is the inventory level of P . With this demand model, the optimal inventory levels for P and S in E_P can be determined jointly via discrete optimization on expected newsvendor type profits under integral demands.

An approximate approach, which is expected to perform well if X_P has a Poisson distribution, is to utilize Normal approximations to X_P and X_{SP}^T . This approximation significantly reduces the computational effort. We approximate X_P by a Normal random variable \hat{X}_P with mean $\lambda_P q_P(p_P)$ and standard deviation $\sqrt{\lambda_P q_P(p_P)}$. We then approximate X_{SP}^T by another Normal random variable with mean

$$\hat{\mu}_{SP}^T = \left[\int_0^{y_P} x_P f_{\hat{X}_P}(x_P) dx_P + y_P \bar{F}_{\hat{X}_P}(y_P) \right] q_{SP}^T(p_S), \quad (5.12)$$

and standard deviation $\hat{\sigma}_{SP}^T = \sqrt{\hat{\mu}_{SP}^T}$, where $f_{\hat{X}_P}(\cdot)$ is the density function of \hat{X}_P and $\bar{F}_{\hat{X}_P}(y_P) = \int_{y_P}^{\infty} f_{\hat{X}_P}(x_P) dx_P$. Our numerical study suggests that this approximation is reasonably accurate, but it tends to slightly overestimate the standard deviation, of X_{SP}^T (see Table 5.1 below). This will lead to a conservative lower bound on the profitability of CT and a minor overstock of S in E_P in most cases.

Table 5.1 below tests the accuracy of the Normal approximation to X_{SP}^T . In this table, the exact mean and standard deviation of X_{SP}^T , μ_{SP}^T and σ_{SP}^T , are evaluated from (5.11) with X_P having a

Poisson distribution with mean $\lambda_P q_P(p_P)$, while the approximate ones, $\hat{\mu}_{SP}^T$ and $\hat{\sigma}_{SP}^T$, are estimated from (5.12). The inventory level of P , y_P , is set optimally based on the Poisson distribution of X_P .

Table 5.1. Testing the Normal approximation to X_{SP}^T ($q_{SP}^T(p_S) = 0.400$)

$\lambda_P q_P(p_P)$	$c_p/p_P = 0.300$					$c_p/p_P = 0.450$				
	y_P	μ_{SP}^T	σ_{SP}^T	$\hat{\mu}_{SP}^T$	$\hat{\sigma}_{SP}^T$	y_P	μ_{SP}^T	σ_{SP}^T	μ_{SP}^T	$\hat{\sigma}_{SP}^T$
30.000	33.000	11.588	3.094	11.598	3.406	31.000	11.309	2.943	11.311	3.363
40.000	44.000	15.584	3.619	15.595	3.949	41.000	15.176	3.401	15.178	3.896
50.000	54.000	19.485	4.021	19.496	4.415	51.000	19.058	3.804	19.060	4.366

Chapter 6

Conclusions and Suggestions for Further Research

In this dissertation, we study the retailer's pricing, variety, and inventory decisions considering substitutable as well as complementary items. Specifically, we first study the joint pricing, assortment, and inventory decisions for a retailer's product line comprised of a set of substitutable items. Then, we study the effectiveness of a selling strategy for complementary items. In this chapter, we present conclusions and suggestions for future research on these two types of problems.

6.1 The Retailer's Product Line Problem

For the retailer's product line problem we utilize the multinomial logit choice model for consumer behavior, which allows a realistic and analytically tractable framework for analyzing pricing and assortment decisions. We also consider a newsvendor type setting, which allows us to capture the essence of inventory costs in terms of the risks of understocking and overstocking, and, in addition, provides a building block for more complex multi-period (stochastic) inventory models.

Our work reveals several important insights and provides decision tools that can be applied in

practice as well. In particular, we find that an optimal assortment consists of items with the largest values of the mean reservation price, α_i (equivalently, the smallest unit cost, c_i , or the largest average margin, $\alpha_i - c_i$), for the special case where the non-ascending order of α_i 's and the non-descending order of c_i 's are the same. We also propose a dominance relationship for the general case, which could significantly reduce the number of assortments to be analyzed in the search for an optimal assortment. We show, through counter-examples, that the optimal assortment does not, in general, consist of the items with the largest average margins (which indicates that a related result in the riskless case, which assumes infinite inventory levels, does not extend to the finite inventory case). However, we observe, via many numerical tests, that assortments having items with the largest average margins yield expected profits that are quite close to the optimal profits.

Furthermore, we show that the “equal profit margins” property of the riskless case no longer holds in our model with finite inventory. (In fact, the equal profit margins result in the riskless case can be deduced asymptotically from our model.) Nonetheless, we argue, based on the optimality conditions, that the profit margins in our model would be *approximately* equal at optimality. Our numerical study confirms this argument. In addition, we propose “easy-to-compute” bounds on the optimal prices. Finally, based on our analytical results and observations from our numerical study, we propose a simple heuristic that performs quite well on many numerical tests.

Next, we study a “stylized” version of this problem with similar items having equal unit costs and identical reservation prices, which leads to many interesting insights. We consider two variations of this stylized model: (i) the retailer is a “price-taker,” while the assortment size and inventory levels are decision variables; and (ii) the assortment size is exogenously determined, while the retailer controls the price and inventory levels. We also briefly discuss the joint problem where the price, assortment size, and inventory levels are decision variables. We demonstrate that the expected

profit function at optimal inventory levels is well-behaved in the sense that many unimodularity and monotonicity results hold. However, we also show, via numerical examples, that for a given variety level, the optimal price can behave in a somewhat counter-intuitive way (for example, we find that the optimal price can be decreasing in the variety level or in the unit cost). In addition, we analyze the effect of limited inventory on variety and pricing. We conclude that finite inventories restrict the assortment size (variety level) even when the retailer has control over the price. We also find that with finite inventory, the optimal price is adjusted up or down from the riskless price (which assumes infinite inventory levels) in such a way as to reduce demand uncertainty (measured in terms of both demand variance and coefficient of variation).

Several extensions to our work deserve further analysis. In particular, one may be able to utilize the conceptual results for the stylized similar items case to develop insights and bounds on the optimal solution in the general case. For example, the insight that the optimal assortment size is finite for the similar items case seems to be also valid for the general items case. The proof of this observation appears to be possible but definitely not trivial. Similarly, it may be possible to develop a tighter upper bound on the optimal assortment size for the general items case based on results from the similar items case.

While our work has been developed with retail applications in mind, it might have applications in other areas such as the choice of manufacturing technology. For example, Bish and Wang [12] study the technology choice decision of a two-product firm: The firm can invest in a *flexible* resource that can manufacture both products as well as two *dedicated* cheaper resources, each of which can manufacture a single product. Considering that demand of each product is a function of its own price only, they characterize the structure of the firm's optimal "resource investment portfolio" (i.e., resources mix and capacities of flexible and dedicated resources). Incorporating realistic MNL-based

demand models (similar to what we consider in this dissertation) into the firm’s technology choice decision can be a promising avenue for future research.

Important extensions of our work are those that will broaden the scope of practical problems covered by our analysis. For example, our customer choice model is based on a subtle “Independence from Irrelevant Alternatives” (IIA) property, which implies that the ratio of the purchase probabilities of two items is independent of the assortment containing them (see, for example, Debreu [19] and Luce [55]). That is, adding a new item, k , to an assortment containing items i and j will not affect the ratio of the purchase probabilities of i and j . While this assumption is valid in many situations and there exists statistical tests to validate it (see, for example, McFadden et al. [65]), there can also be situations where certain items in a product line share more similarities with each other than with other items. For example, introducing a new flavor of chocolate ice-cream to a diversified line of ice-cream (e.g., a line of Ben & Jerry’s brands) will most likely affect other chocolate-based flavors more than strawberry-based flavors. Fortunately, the consumer choice theory offers an elegant solution to the violation of the IIA property through an enhanced choice model referred to as the “nested logit model” (see, for example, Anderson et al. [2] and McFadden [66]). We believe that the nested logit will not only rectify such shortcomings of the MNL but can also capture important inter-store dynamics such as the cannibalization between the different categories in a store (e.g., between ordinary produce and organic produce, ready baked cakes and cakes mix, fresh seafood and frozen seafood).

Another problem that deserves further study is to consider “dynamic” stockout based substitution between items in the product line. The exact solution to this problem is quite difficult (especially with the additional complexities of pricing and variety decisions) as indicated in the recent papers by Mahajan and van Ryzin [58], Netessine and Rudi [70], and Smith and Aggrawal [85]. However,

certain heuristic procedures may perform very well on this problem. For example, when the profit margins are equal, Mahajan and van Ryzin [58] report the excellent performance of a simple “pulled newsboy” heuristic. With the profit margins being approximately equal in our current model, such a heuristic may work very well.

Finally, considering a multi-period inventory setting, where leftover inventory is carried from one period to the next, and new inventory is introduced as needed is also worthy of further investigation. This will lead our research to the widely studied area of “dynamic pricing,” which is crucial for many industries such as airlines and sports organizations (see, for example, Gallego and van Ryzin [30]). While most work on dynamic pricing involves a single commodity, this extension of our work involves dynamic pricing of multiple substitutable items and “dynamic assortment” decisions (see, for example, Caro and Gallien [16] and Lin and Li [53]). Although this is an analytically challenging problem, there have been some recent advances on joint pricing and inventory models within a multi-period setting. These works may prove to be useful to us. In particular, Chen and Simchi-Levi [17] and Federgruen and Heching [27] utilize a similar mixed multiplicative/additive demand structure and develop interesting results for single-item dynamic pricing problems.

6.2 The Convenience Tying Problem

In this dissertation, we present an original model and analysis for the convenience tying strategy (CT). Our study focuses on the pricing and inventory implications of this strategy. When compared with the conventional strategy in which items are sold separately in different locations (IC), CT decreases the price of the tying item and adjusts up or down the price of the tied-in item based on its popularity in the tying item’s department. Moreover, when the prices are fixed, CT must generate a sufficiently higher demand than IC to be profitable due to decentralizing the sales of S (which leads

to higher stock levels in most cases).

Our analysis exploits two facets of CT. First, we incorporate the cross-price elasticity effects into our model, reflecting the dependence of the demand of one item on the price of the other item (in addition to its own price). This appears to be a realistic phenomenon that we observed in a data set we obtained from a retailer adopting CT. Second, our model considers the concept of risk pooling or demand consolidation, which allows us to assess the inventory costs and demand variability effects of CT. In that sense, our work introduces a new dimension to risk pooling modeling by exploiting a situation where decentralization could lead to a larger demand volume that covers additional inventory costs and improves profitability.

As aforementioned, our work is a first step in studying CT. It can be extended (and enhanced) in several directions. Different reservation price distributions may be utilized in the pricing analysis. Perhaps the Normal or another smooth unimodal distribution may be more applicable than the Uniform distribution that we utilize here. However, we expect that the analysis will be more complicated with these distributions. In addition, our modeling of the change in the reservation price of the tied-in item (when it is sold with the primary item) by a simple shift in the distribution support could be enhanced by a more elaborate formulation that takes into account various measurable attributes such as search costs and visual effects. The same is true for our second model that considers a single-period inventory cost structure. A worthwhile extension would be to consider the possibility of replenishing the tied-in item in the tied item's department from its original department, which seems to be the case in some practical situations, especially with limited display space. The effect of stockouts of the tying item that we discuss briefly may also merit further study. Finally, the problem of joint pricing and inventory decisions under CT, where the retailer decides on both the prices and inventory levels simultaneously, is an important area for further research. For example,

it is of interest to investigate how the pricing effects of CT (such as decreasing the price of the tying item) would change if the inventory levels were finite.

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Appendix

Appendix A. Approximation for $\phi(\Phi^{-1}(1-x)), 0 \leq x \leq 1$.

This approximation is developed by fitting a quadratic function to $\phi(\Phi^{-1}(1-x))$, as shown in Figure A.1. To simplify the expression for $\Pi(S, \mathbf{p})$, we let the quadratic pass through points $(0,0)$ and $(0,1)$. This leads to a quadratic of the form $u(x) = -ax(x-1)$. The value of a is chosen so as to minimize the sum of squared deviations from the actual function, i.e., $\min_a \int_0^1 [\phi(\Phi^{-1}(1-x)) - u(x)]^2 dx$. Solving for a numerically, we find that $a \approx 1.66$, for which the average relative error, $\int_0^1 |\phi(\Phi^{-1}(1-x)) - u(x)| / \phi(\Phi^{-1}(1-x)) dx$, is approximately 8.6% and the maximum absolute error is 0.027 at $x = 0.101$ and $x = 0.890$.

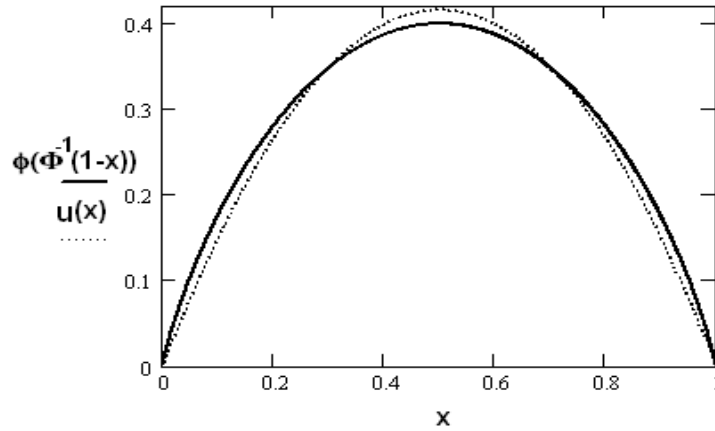


Figure A.1. Approximation for $\phi(\Phi^{-1}(1-x))$.

In the following we present a graphical comparison between the exact expected profit function in (3.3) and the expected profit with the above approximation in (3.5) for the case of a single item.

Testing the standard Normal approximation - Single item case ($\mu = v_0 = 1$)

$$q(p, \alpha) := \frac{e^{\alpha-p}}{1 + e^{\alpha-p}}$$

Expected profit with Normal demand

$$\Pi_e(p, \lambda, \alpha, c) := (p - c) \cdot \lambda \cdot q(p, \alpha) - p \cdot \sqrt{\lambda \cdot q(p, \alpha)} \cdot \text{dnorm}\left(\text{qnorm}\left(1 - \frac{c}{p}, 0, 1\right), 0, 1\right)$$

Expected profit with our approximated Normal demand

$$\Pi_a(p, \lambda, \alpha, c) := (p - c) \cdot \left(\lambda \cdot q(p, \alpha) - 1.66 \cdot \frac{c}{p} \cdot \sqrt{\lambda \cdot q(p, \alpha)} \right)$$

Minimum value of λ for which **(A1)** holds

$$\lambda_{\min}(\alpha, c) := 1.66^2 \cdot \left[1 + e^{-(\alpha-c)} \right]$$

Observations (see graphs on pages 89-90)

1. The approximation is most accurate when λ or $\alpha-c$ are large.
2. Cases where the approximation is not accurate are when the expected profit is close to zero (which are not so important cases).
3. The exact and approximated profits follow similar trends.
4. The optimal prices under exact and approximated expected profits are close.
5. Assumption **(A1)** contributes to approximation accuracy (compare λ and λ_{\min} on pages 89-90).

Note. For the multiple items case, the observations are expected to hold since the expected profit in that case is the sum of several single item expected profit functions.

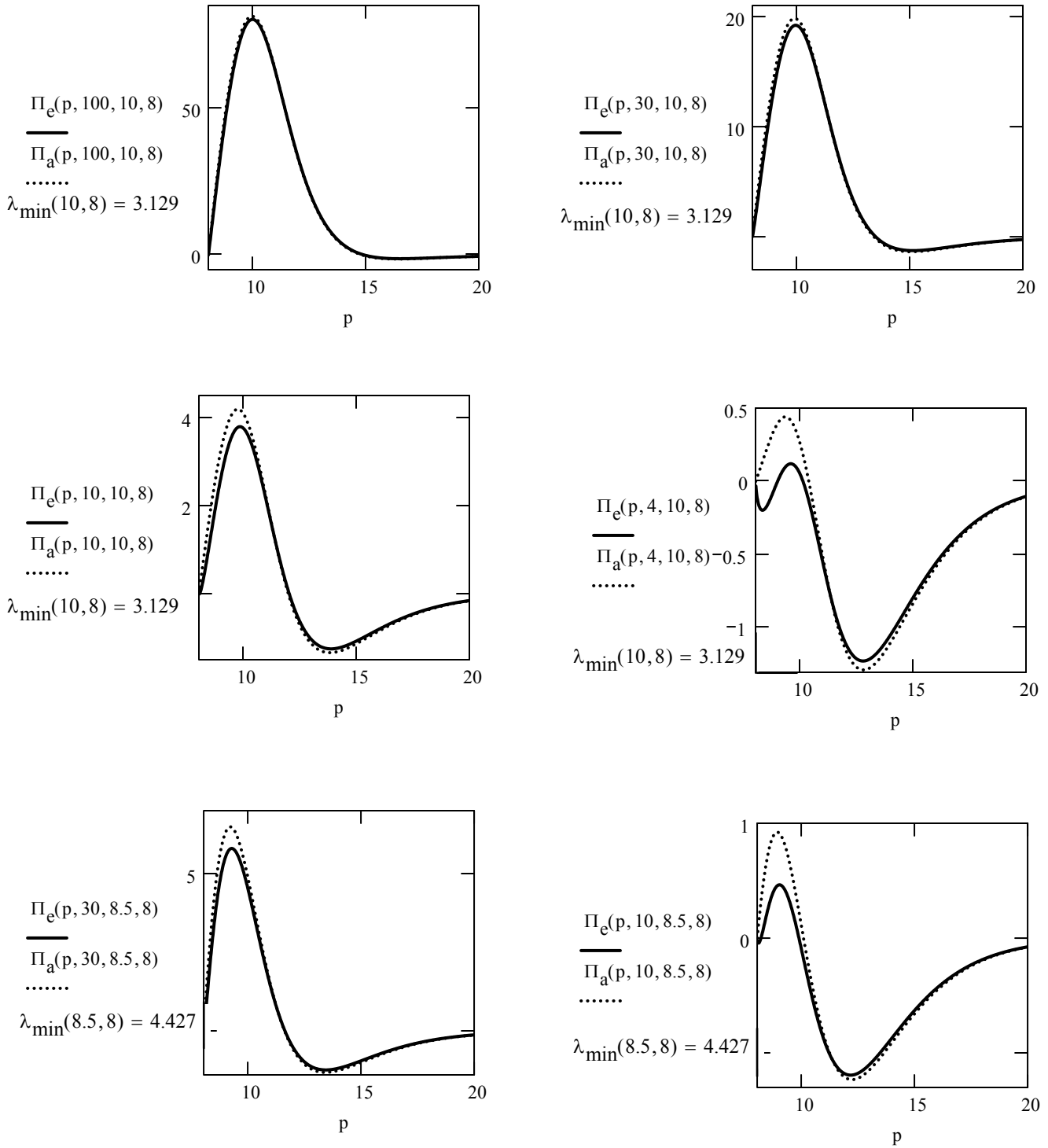


Figure A.2. Graphical Comparison of the exact and the approximate expected profits.

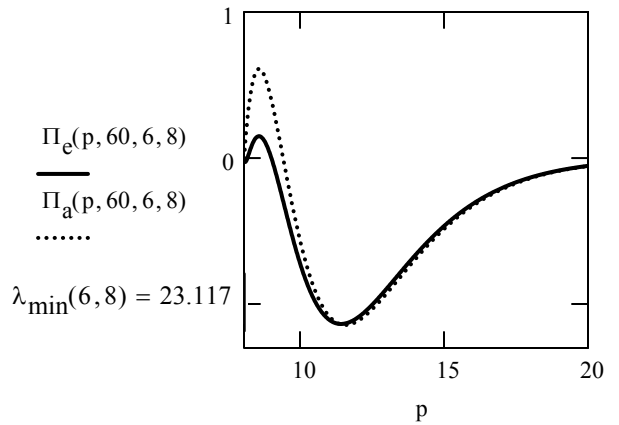
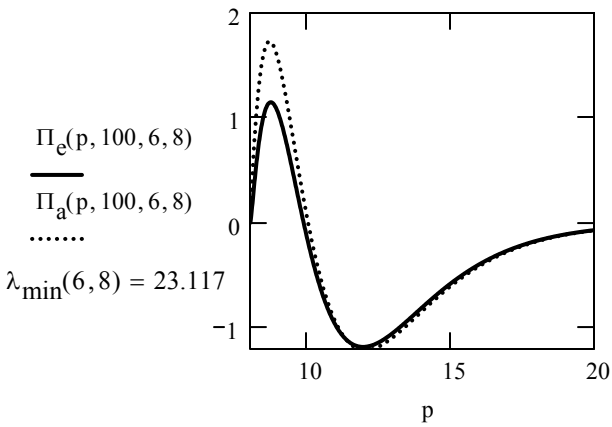
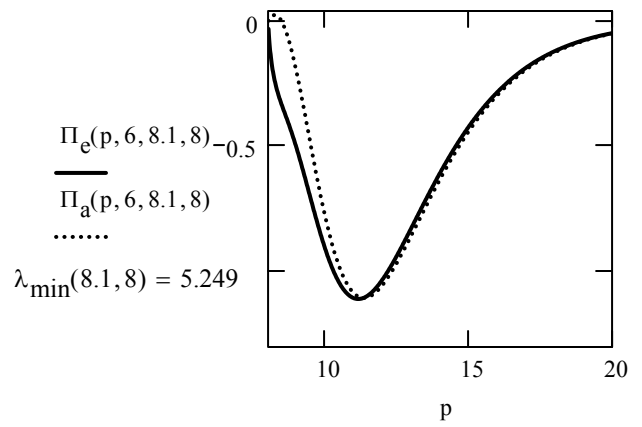
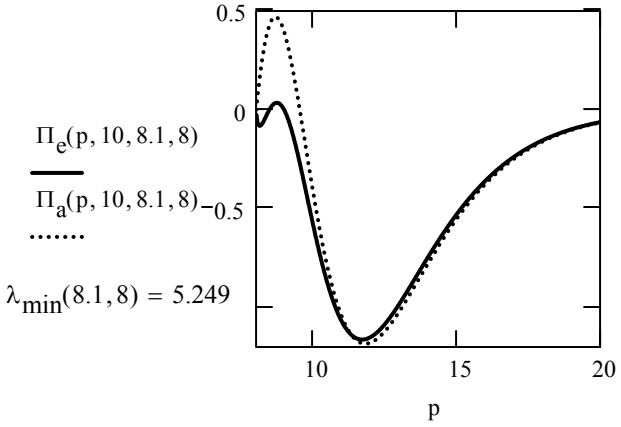
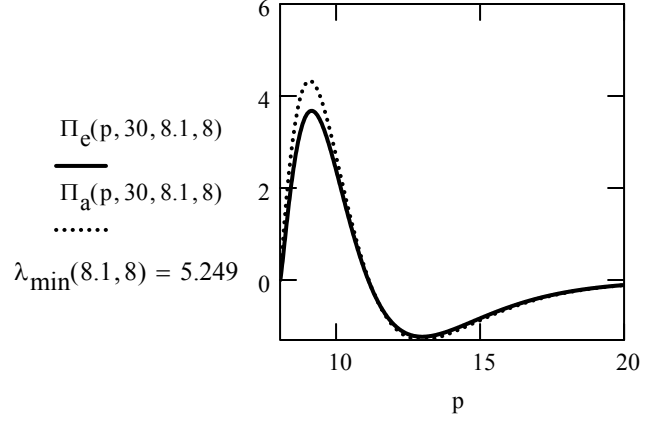
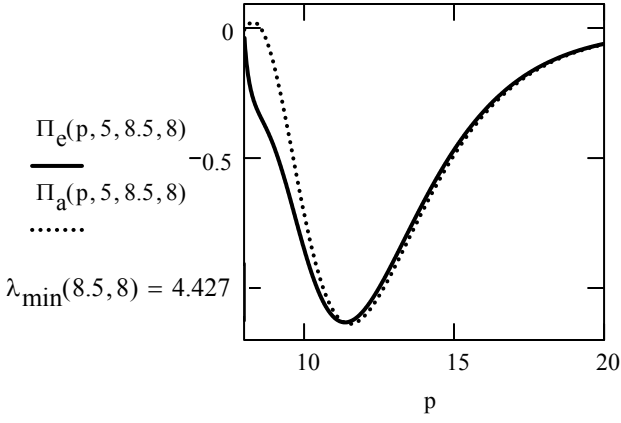


Figure A.3. Graphical Comparison of the exact and the approximate expected profits.

Appendix B. Proof of Lemma 3.1.1

The proof is by contradiction. Consider an optimal assortment S^* , and let $S^+ \subset S^*$ and $S^- \subset S^*$ denote the subsets of items having positive contribution and nonpositive contribution to the expected profit, respectively. Under Assumption **(A1)**, $\Pi(\{\bar{i}\}, c_i^\pm) > 0$ and, consequently, S^+ is not empty. By contradiction, assume that S^- is also not empty. Noting that $\Pi_i(S^*, \mathbf{p}^*, \alpha_j) = (p_i^* - c_i) \left[\lambda q_i(S^*, \mathbf{p}^*, \alpha_j) - a \frac{c_i}{p_i^*} \sqrt{\lambda q_i(S^*, \mathbf{p}^*, \alpha_j)} \right]$ and $\frac{\partial q_i(S^*, \mathbf{p}^*, \alpha_j)}{\partial \alpha_j} = -(1/\mu) q_i(S^*, \mathbf{p}^*, \alpha_j) q_j(S^*, \mathbf{p}^*, \alpha_j) < 0$ for $i, j \in S^*, i \neq j$, we derive

$$\begin{aligned} \frac{\partial \Pi_i(S^*, \mathbf{p}^*, \alpha_j)}{\partial \alpha_j} &= (p_i^* - c_i) \frac{\partial q_i(S^*, \mathbf{p}^*, \alpha_j)}{\partial \alpha_j} (1/q_i(S^*, \mathbf{p}^*, \alpha_j) \left[\lambda q_i(S^*, \mathbf{p}^*, \alpha_j) - \frac{a}{2} \frac{c_i}{p_i^*} \sqrt{\lambda q_i(S^*, \mathbf{p}^*, \alpha_j)} \right]) \\ &< (1/q_i(S^*, \mathbf{p}^*, \alpha_j)) \frac{\partial q_i(S^*, \mathbf{p}^*, \alpha_j)}{\partial \alpha_j} \Pi_i(S^*, \mathbf{p}^*, \alpha_j), \text{ by definition of } \Pi_i(S^*, \mathbf{p}^*, \alpha_j), \\ &< 0, \text{ in the range where } \Pi_i(S^*, \mathbf{p}^*, \alpha_j) > 0. \end{aligned} \quad (6.1)$$

Since, by definition, $\Pi_i(S^*, \mathbf{p}^*, \alpha_j) > 0$ for $i \in S^+$, (6.1) implies that $\frac{\partial \Pi_i(S^*, \mathbf{p}^*, \alpha_j)}{\partial \alpha_j} < 0$ in the decreasing direction of α_j . Hence, $\Pi_i(S^*, \mathbf{p}^*, \alpha_j)$ remains positive and increases as α_j decreases. Next consider an assortment $S' = S^* \setminus \{j\}$, where $j \in S^-$. Let $\mathbf{p}_{-j}^* = (p_1^*, \dots, p_{j-1}^*, p_{j+1}^*, \dots, p_{|S^*|}^*)$, denote the optimal price vector for S^* , corresponding to the items in S' . Noting that S' is equivalent to S^* with $\alpha_j = -\infty$ (since $\lim_{\alpha_j \rightarrow -\infty} q_j(S^*, \mathbf{p}^*, \alpha_j) \rightarrow 0$), it follows that

$$\Pi_i(S', \mathbf{p}_{-j}^*) > \Pi_i(S^*, \mathbf{p}^*), i, j \in S^*, i \neq j, \text{ (by (6.1))},$$

which implies that $\Pi(S', \mathbf{p}_{-j}^*) > \Pi(S^*, \mathbf{p}^*)$. This contradicts with the optimality of S^* . Therefore, S^- must be empty. This proves the first part of the lemma.

Now we can show that \mathbf{p}^* is an internal point solution. Denote by $\Pi_i(S^*, p_i, \mathbf{p}_{-i}^*)$, the expected

contribution of $i \in S^*$ when the prices of other items are fixed at their optimal values in S^* . It can be easily shown that $\Pi_i(S^*, c_i, \mathbf{p}_{-i}^*) = 0$ and $\lim_{p_i \rightarrow \infty} \Pi_i(S^*, p_i, \mathbf{p}_{-i}^*) \rightarrow 0^-$. Moreover, as proved above, under **(A1)** $\Pi_i(S^*, p_i^*, \mathbf{p}_{-i}^*) = \Pi_i(S^*, \mathbf{p}^*) > 0$, for all $i \in S^*$. Thus, it must be true that $c_i < p_i^* < \infty, i \in S^*$, and, therefore, $\left. \frac{\partial \Pi(S^*, \mathbf{p}^*)}{\partial p_i} \right|_{\mathbf{p}=\mathbf{p}^*} = 0$. ■

Appendix C. Structure of the optimal assortment

The following lemmas are utilized to support the proofs of the results in Section 3.2.

Lemma 6.2.1 below is presented as an exercise in Mangasarian [60] (p. 148) and Bazaraa et al. [9] (p. 123).

In the following, we provide a proof for it for the sake of completeness.

Lemma 6.2.1 *Consider the function $\pi(x) = \theta(x)/\delta(x)$ defined over an open set $\Gamma \subset \Re$. If $\theta(x)$ is a strictly convex function and $\delta(x)$ is a positive linear function, then $\pi(x)$ is strictly pseudoconvex.*

Proof. We need to show that for all $x_1, x_2 \in \Gamma$, if $\nabla\pi(x_1)(x_2 - x_1) \geq 0$, then $\pi(x_2) > \pi(x_1)$, where

$\nabla\pi(x_1) = \left. \frac{\partial\pi(x)}{\partial x} \right|_{x=x_1}$, (see, for example, Bazaraa et al. [9], pp. 113-114). Note that

$$\nabla\pi(x_1) = \frac{\nabla\theta(x_1)\delta(x_1) - \theta(x_1)\nabla\delta(x_1)}{(\delta(x_1))^2}.$$

Note also that since $\theta(x)$ is a convex function, then $\theta(x_2) - \theta(x_1) > \nabla\theta(x_1)(x_2 - x_1)$. Furthermore, since $\delta(x)$ is linear, then $\nabla\delta(x_1)(x_2 - x_1) = \delta(x_2) - \delta(x_1)$. Upon replacement and simplification, it follows that if $\nabla\pi(x_1)(x_2 - x_1) \geq 0$, then

$$\frac{\delta(x_2)}{\delta(x_1)} \left(\frac{\theta(x_2)}{\delta(x_2)} - \frac{\theta(x_1)}{\delta(x_1)} \right) > 0,$$

which implies that $\pi(x_2) > \pi(x_1)$. ■

Lemma 6.2.2 Consider an assortment $S \subseteq \Omega$. Assume that prices of items in S are fixed at some price vector \mathbf{p} . Then, the expected profit from S , $\Pi(S, \mathbf{p}, v_i)$, is strictly pseudoconvex in $v_i \equiv e^{(\alpha_i - p_i)/\mu}$, $i \in S$.

Proof. Note that $q_i(S, \mathbf{p}) = v_i / (v_0 + \sum_{j \in S} v_j)$, $i \in S$. Then, $\Pi(S, \mathbf{p})$ can be written as $\Pi(v_i) = \theta(v_i) / \delta(v_i)$, $i \in S$, where $\theta(v_i) = (K_1 + K_2 v_i) - (K_3 + K_4 v_i^{1/2})(K_5 + v_i)^{1/2}$, with $K_1 = \sum_{j \neq i, j \in S} (p_j - c_j) \lambda v_j > 0$, $K_2 = (p_i - c_i) \lambda > 0$, $K_3 = \sum_{j \neq i, j \in S} (p_j - c_j) a_{p_j}^{c_j} \sqrt{\lambda} \sqrt{v_j} > 0$, $K_4 = (p_i - c_i) a_{p_i}^{c_i} \sqrt{\lambda} > 0$, $K_5 = \sum_{j \neq i, j \in S} v_j + v_0 > 0$, and $\delta(v_i) = \sum_{j \neq i, j \in S} v_j + v_0 + v_i$. Upon differentiation and simplification, it follows that $\frac{\partial^2 \theta(v_i)}{\partial v_i^2} = \frac{1}{4} (K_5 + v_i)^{-3/2} (K_4 K_5^2 v_i^{-3/2} + K_3) > 0$. Therefore, $\theta(v_i)$ is strictly convex in v_i . On the other hand, $\delta(v_i)$ is linear in v_i . Applying Lemma 6.2.1, it follows that $\Pi(v_i)$ is strictly pseudoconvex in v_i , $i \in S$. ■

Lemma 6.2.3 Consider two real functions $v(x)$ and $\pi(x)$ defined on \mathfrak{R} . If $v(x)$ is increasing in x and $\pi(v(x))$ is strictly pseudoconvex in $v(x)$, then $\pi(v(x))$ is strictly pseudoconvex in x .

We need to show that for all x_1, x_2 , if $\nabla \pi(v(x_1))(x_2 - x_1) \geq 0$, then $\pi(v(x_2)) > \pi(v(x_1))$. Since v is increasing in x , then $x_2 - x_1$ is of the same sign as $v(x_2) - v(x_1)$. Therefore, if $\nabla \pi(v(x_1))(x_2 - x_1) \geq 0$, then $\nabla \pi(v(x_1))(v(x_2) - v(x_1)) \geq 0$. Since π is strictly pseudoconvex in v , the last inequality implies that $\pi(v(x_2)) > \pi(v(x_1))$. ■

Lemma 6.2.4 Consider an assortment $S \subseteq \Omega$. Assume that the prices of items in S are fixed at some price vector \mathbf{p} and $\Pi_i(S, \mathbf{p}, \alpha_i) > 0$ when $\alpha_i = \alpha'_i$, for some $i \in S$. Then, $\Pi_i(S, \mathbf{p}, \alpha_i)$ is increasing in α_i for $\alpha_i \geq \alpha'_i$.

Proof. Similar to the proof of Lemma 3.1.1, the derivative of $\Pi_i(S, \mathbf{p}, \alpha_i)$ with respect to α_i , $i \in S$,

is given by

$$\begin{aligned} \frac{\partial \Pi_i(S, \mathbf{p}, \alpha_i)}{\partial \alpha_i} &= (p_i - c_i) \frac{\partial q_i(S, \mathbf{p}, \alpha_i)}{\partial \alpha_i} (1/q_i(S, \mathbf{p}, \alpha_i)) \left[\lambda q_i(S, \mathbf{p}, \alpha_i) - \frac{a c_i}{2 p_i} \sqrt{\lambda q_i(S, \mathbf{p}, \alpha_i)} \right] \\ &> (1/q_i(S, \mathbf{p}, \alpha_i)) \frac{\partial q_i(S, \mathbf{p}, \alpha_i)}{\partial \alpha_i} \Pi_i(S, \mathbf{p}, \alpha_i), \end{aligned}$$

where the inequality follows by definition of $\Pi_i(S, \mathbf{p}, \alpha_i)$ and since $\frac{\partial q_i(S, \mathbf{p}, \alpha_i)}{\partial \alpha_i} = (1/\mu) q_i(S, \mathbf{p}, \alpha_i) (1 - q_i(S, \mathbf{p}, \alpha_i)) > 0$. Using an argument similar to that in the proof of Lemma 3.1.1, it follows that $\frac{\partial \Pi_i(S, \mathbf{p}, \alpha_i)}{\partial \alpha_i} > 0$, in the increasing direction of α_i . That is, $\Pi_i(S, \mathbf{p}, \alpha_i)$ remains positive and increases as α_i increases. ■

Next, we present proofs for the results in Section 3.2.

Proof of Lemma 3.2.1. For any fixed price vector \mathbf{p} , $\Pi(S, \mathbf{p}, v_i)$ is strictly pseudoconvex in v_i by Lemma 6.2.2, and v_i is increasing in α_i . Then, Lemma 6.2.3 implies that $\Pi(S, \mathbf{p}, \alpha_i)$ is strictly pseudoconvex in $\alpha_i, i \in S$. ■

Proof of Lemma 3.2.2. Lemma 3.2.1 implies that $\Pi(S^*, \mathbf{p}^*, \alpha_i)$ is strictly pseudoconvex in $\alpha_i, i \in S^*$, when prices of items in S^* are fixed at \mathbf{p}^* . Therefore, it is sufficient to show that $\Pi(S^*, \mathbf{p}^*, \alpha_i)$ is increasing in α_i at $\alpha_i = \alpha'_i$. By contradiction, assume that $\Pi(S^*, \mathbf{p}^*, \alpha_i)$ is nonincreasing in α_i at $\alpha_i = \alpha'_i$. Then, the strict pseudoconvexity of $\Pi(S^*, \mathbf{p}^*, \alpha_i)$ in α_i implies that $\Pi(S^*, \mathbf{p}^*, \alpha_i)$ is decreasing in α_i , for $\alpha_i < \alpha'_i$, and hence $\Pi(S^*, \mathbf{p}^*, -\infty) > \Pi(S^*, \mathbf{p}^*, \alpha'_i)$, or equivalently $\Pi(S^* \setminus \{i\}, \mathbf{p}^*_{-i}) > \Pi(S^*, \mathbf{p}^*)$ (since setting $\alpha_i = -\infty$ is equivalent to removing i from S^*), which contradicts with the optimality of S^* . ■

Proof of Lemma 3.2.3. Note that $\frac{\partial \Pi(S, \mathbf{p}, c_i)}{\partial c_i} = \frac{\partial \Pi_i(S, \mathbf{p}, c_i)}{\partial c_i} = -\frac{\Pi_i(S, \mathbf{p}, c_i)}{p_i - c_i} + a \sqrt{\lambda q_i(S, \mathbf{p})} \left(\frac{c_i}{p_i} - 1 \right)$. Therefore, if $\Pi_i(S, \mathbf{p}, c'_i) > 0$, then it can be shown, in a similar way to the proof of Lemma 3.1.1,

that $\Pi_i(S, \mathbf{p}, c_i)$ remains positive and increases as c_i decreases from c'_i . ■

Proof of Lemma 3.2.4. The proof is by contradiction. Assume that item k dominates item i and there exists an optimal assortment S' such that $i \in S'$ and $k \notin S'$. Let $\Pi(S', \mathbf{p}^*, \alpha_i, c_i)$ and $\Pi_i(S', \mathbf{p}^*, \alpha_i, c_i)$ denote the optimal profit from S' and the expected contribution of item i , respectively, with \mathbf{p}^* being the optimal price vector for items in S' . If α_i is changed to α_k in S' with the prices being held fixed at \mathbf{p}^* , then it follows, by Lemma 3.2.2, that $\Pi(S', \mathbf{p}^*, \alpha_k, c_i) \geq \Pi(S', \mathbf{p}^*, \alpha_i, c_i)$. Then by Lemma 6.2.4, $\Pi_i(S', \mathbf{p}^*, \alpha_k, c_i) \geq \Pi_i(S', \mathbf{p}^*, \alpha_i, c_i) > 0$, where the last inequality follows by Lemma 3.1.1. Similarly, further changing c_i to c_k in S' implies, by Lemma 3.2.3, that $\Pi(S', \mathbf{p}^*, \alpha_k, c_k) \geq \Pi(S', \mathbf{p}^*, \alpha_i, c_i)$. With one of the two inequalities being strict as in the statement of the lemma, it follows that $\Pi(S', \mathbf{p}^*, \alpha_k, c_k) > \Pi(S', \mathbf{p}^*, \alpha_i, c_i)$, or equivalently, $\Pi(S' \cup \{k\} \setminus \{i\}, \mathbf{p}^*) > \Pi(S', \mathbf{p}^*)$, which contradicts with the optimality of S' . ■

Appendix D. Properties of the optimal prices 3.3

Proof of Lemma 3.3.1. By Lemma 3.1.1, the optimal prices satisfy the first-order optimality conditions given below for any $i \in S^*$.

$$\begin{aligned} \frac{\partial \Pi(S^*, \mathbf{p})}{\partial p_i} = e^{(\alpha_i - p_i)/\mu} & \left\{ \frac{1}{\mu \psi(S^*, \mathbf{p})} \left[\sum_{j \in S^*} (p_j - c_j) \left(\lambda q_j(S^*, \mathbf{p}) - \frac{a c_j}{2 p_j} \sqrt{\lambda q_j(S^*, \mathbf{p})} \right) \right] \right. \\ & + \frac{\lambda}{\psi(S^*, \mathbf{p})} \left(-\frac{1}{\mu} (p_i - c_i) + 1 \right) - a \frac{c_i^2}{p_i^2} \sqrt{\lambda} \frac{e^{-(\alpha_i - p_i)/(2\mu)}}{\sqrt{\psi(S^*, \mathbf{p})}} \\ & \left. + (p_i - c_i) \frac{a}{2\mu} \sqrt{\lambda} \frac{c_i}{p_i} \frac{e^{-(\alpha_i - p_i)/(2\mu)}}{\sqrt{\psi(S^*, \mathbf{p})}} \right\}, \end{aligned}$$

where $\psi(S^*, \mathbf{p}) = v_0 + \sum_{j \in S^*} e^{(\alpha_j - p_j)/\mu}$. Since $e^{(\alpha_i - p_i^*)/\mu} \neq 0$ because $p_i^* < \infty$ (by Lemma 3.1.1), then

setting $\frac{\partial \Pi(S^*, \mathbf{p})}{\partial p_i} = 0$ and dividing by $(\lambda/\psi(S^*, \mathbf{p}))$ yields

$$\frac{1}{\mu}(p_i^* - c_i) \left(1 - \frac{a c_i}{2 p_i^*} \frac{1}{\sqrt{\lambda q_i(S^*, \mathbf{p}^*)}} \right) + a \frac{c_i^2}{p_i^{*2}} \frac{1}{\sqrt{\lambda q_i(S^*, \mathbf{p}^*)}} = \Upsilon(S^*, \mathbf{p}^*),$$

where $\Upsilon(S^*, \mathbf{p}^*) = 1 + \frac{1}{\mu\lambda} \sum_{j \in S^*} (p_j^* - c_j) \left(\lambda q_j(S^*, \mathbf{p}^*) - \frac{a c_j}{2 p_j^*} \sqrt{\lambda q_j(S^*, \mathbf{p}^*)} \right)$. Noting that $\Upsilon(S^*, \mathbf{p}^*)$ is the same for all $i \in S^*$ yields the result in the lemma. ■

The following result is utilized in the proof of Lemma 3.3.2.

Lemma 6.2.5 *Consider item $i \in \Omega$. Then, either $\Pi_i(\{i\}, p_i) > 0$ on exactly one interval of $p_i \in (c_i, \infty)$, or $\Pi_i(\{i\}, p_i) \leq 0$ for all $p_i \in (c_i, \infty)$.*

Proof. Setting $\Pi_i(\{i\}, p_i) = 0$ for $p_i \in (c_i, \infty)$, implies that

$$h_i(p_i) = \frac{a^2 c_i^2}{\lambda}, \tag{6.2}$$

where $h_i(p_i) \equiv p_i^2 q_i(\{i\}, p_i)$ (or, equivalently, $\tilde{\pi}_i(p_i) = 0$). In the following, we prove that $h_i(p_i)$ is unimodal in $p_i \geq 0$. Note that $h_i(0) = 0$, $h_i(p_i) > 0$ for $0 < p_i < \infty$, and $\lim_{p_i \rightarrow +\infty} h_i(p_i) \rightarrow 0^+$. Therefore, $h_i(p_i)$ attains a maximum in $(0, \infty)$. Next we show that this maximum is unique by showing that $\frac{\partial h_i(p_i)}{\partial p_i}$ has a single zero in $(0, \infty)$. Note that $\frac{\partial h_i(p_i)}{\partial p_i} = 2p_i q_i(\{i\}, p_i) - \frac{p_i^2}{\mu} q_i(\{i\}, p_i)(1 - q_i(\{i\}, p_i))$. Setting $\frac{\partial h_i(p_i)}{\partial p_i} = 0$ yields $p_i(1 - q_i(\{i\}, p_i)) = 2\mu$. It can be easily shown that the left hand side of this equation is increasing in p_i . Therefore, the equation has a unique solution. Hence, it follows that $h_i(p_i)$ is unimodal in $p_i \geq 0$. This, together with the fact that the right hand side of (6.2) is constant in p_i , implies that $\Pi_i(\{i\}, p_i)$ has at most two zeroes on in (c_i, ∞) . (Observe that a unimodal function can take on the same value at most twice.)

Define $h_i^{max} \equiv \max_{p_i > 0} h_i(p_i)$. Note that it can be easily shown that for p_i large enough ($p_i \rightarrow \infty$),

$\Pi_i(\{i\}, p_i) < 0$. Therefore, (1) if $h_i^{max} \leq a^2 c_i^2 / \lambda$, then (6.2) has at most one solution (equivalently, $\Pi_i(\{i\}, p_i)$ has at most one zero in $p_i \in (c_i, \infty)$). Then, $\Pi_i(\{i\}, p_i) \leq 0$ for all $p_i \in (c_i, \infty)$; (2) if $h_i^{max} > a^2 c_i^2 / \lambda$, then (6.2) has two solutions, \underline{p}_i and \overline{p}_i with $\underline{p}_i < \overline{p}_i$. Then, (2.a) if $\underline{p}_i < \overline{p}_i \leq c_i$, then $\Pi_i(\{i\}, p_i) \leq 0$ for all $p_i \in (c_i, \infty)$; (2.b) if $c_i < \underline{p}_i < \overline{p}_i$, then $\Pi_i(\{i\}, p_i) > 0$ for $p_i \in (\underline{p}_i, \overline{p}_i)$, and $\Pi_i(\{i\}, p_i) \leq 0$ otherwise (within the (c_i, ∞) region); (2.c) if $\underline{p}_i \leq c_i < \overline{p}_i$, then $\Pi_i(\{i\}, p_i) > 0$ for $p_i \in (c_i, \overline{p}_i)$, and $\Pi_i(\{i\}, p_i) \leq 0$ otherwise (within the (c_i, ∞) region). ■

Proof of Lemma 3.3.2. Recall that (6.2) is equivalent to $\Pi_i(\{i\}, p_i) = 0$ for $p_i \in (c_i, \infty)$. Note also that $\Pi_i(\{i\}, p_i) > \Pi_i(S, p_i, \mathbf{p}_{-i})$, for all $p_i > c_i$ in the region where $\Pi_i(S, p_i, \mathbf{p}_{-i}) > 0$, since $\Pi_i(S, p_i, \mathbf{p}_{-i}) > 0$ increases as α_j decreases, $j \neq i, j \in S$, as shown in the proof of Lemma 3.1.1, and $\Pi_i(\{i\}, p_i)$ is equivalent to $\Pi_i(S, p_i, \mathbf{p}_{-i})$ with $\alpha_j = -\infty, j \neq i, j \in S$. This, together with Lemma 6.2.5, yields the desired result. ■

Proof of Corollary 3.3.1. Lemmas 3.2.3 and 6.2.4 imply that $\Pi_{\overline{i}}(\{\overline{i}\}, p_i) \geq \Pi_i(\{i\}, p_i)$ for all $p_i > c_i$ in the region where $\Pi_i(\{i\}, p_i) > 0$. This, together with Lemma 3.3.2, yields the desired result. Note that by Assumption (A1), $\Pi_{\overline{i}}(\{\overline{i}\}, p_i) > 0$ for $p_i \in (\max\{c_{\overline{i}}, \underline{p}\}, \overline{p})$. ■

Appendix E. Structural results on p_k^*

Proof of Lemma 4.2.1. Let $\gamma_1(p, k) = \sqrt{\lambda q(p, k)} - a \frac{c}{p}$. It can be shown that $\gamma_1(p)$ is concave when $q(p, k) \geq 1/3$ as

$$\frac{\partial^2 \gamma_1(p, k)}{\partial p^2} = \frac{1}{4\mu^2} \sqrt{\lambda q(p, k)} (1 - kq(p, k))(1 - 3kq(p, k)) - 2a \frac{c}{p^3} < 0.$$

Let $\gamma_2(p, k) = k(p - c)\sqrt{\lambda q(p, k)}$. Then for $q(p, k) \geq 1/3$,

$$\frac{\partial^2 \gamma_2(p, k)}{\partial p^2} = -\frac{k}{\mu} \sqrt{\lambda q(p, k)}(1 - kq(p, k)) + k(p - c) \frac{1}{4\mu^2} \sqrt{\lambda q(p, k)}(1 - kq(p, k))(1 - 3kq(p, k)) < 0.$$

Thus, $\gamma_2(p, k)$ is also concave for $q(p, k) \geq 1/3$. A result in Mangasarian [60] (p. 149), which can be proven in a similar way to Lemma 6.2.1, states that the product of two positive concave functions is pseudoconcave. Therefore, $\Pi(p, k) = \gamma_2(p, k)\gamma_1(p, k)$ is pseudoconcave in p in the range $q(p, k) \geq 1/3$ and $\Pi(p, k) > 0$. (Note that if $\Pi(p, k) > 0$, then $\gamma_1(p, k) > 0$.) ■

Appendix F. Monotonicity results on p_k^*

Proof of Theorem 4.2.1. Since under **(A2)** p_k^* is an internal point solution, $\frac{\partial \Pi(p, k)}{\partial p} \Big|_{p=p_k^*} = 0$ and $\frac{\partial^2 \Pi(p, k)}{\partial p^2} \Big|_{p=p_k^*} < 0$. To prove part (i), differentiate $\frac{\partial \Pi(p, k)}{\partial p} \Big|_{p=p_k^*}$ with respect to k . Then by implicit differentiation, we can obtain

$$\frac{\partial}{\partial k} \left(\frac{\partial \Pi(p, k)}{\partial p} \Big|_{p=p_k^*} \right) = \frac{\partial^2 \Pi(k, p)}{\partial p^2} \Big|_{p=p_k^*} \frac{\partial p_k^*}{\partial k} + \frac{\partial^2 \Pi(k, p)}{\partial k \partial p} \Big|_{p=p_k^*} = 0.$$

Therefore, $\frac{\partial p_k^*}{\partial k} = - \left(\frac{\partial^2 \Pi(k, p)}{\partial k \partial p} \Big|_{p=p_k^*} \right) / \left(\frac{\partial^2 \Pi(k, p)}{\partial p^2} \Big|_{p=p_k^*} \right)$, and $\frac{\partial p_k^*}{\partial k}$ is of the same sign as $\frac{\partial^2 \Pi(k, p)}{\partial k \partial p} \Big|_{p=p_k^*}$.

In the following we show that $\frac{\partial^2 \Pi(k, p)}{\partial k \partial p} \Big|_{p=p_k^*} > 0$ if $p_k^* \geq \frac{3c}{2}$, which completes the proof of (i).

We have

$$\frac{\partial \Pi(p, k)}{\partial p} = \frac{\Pi(p, k)}{p - c} + k(p - c) \left(\lambda \frac{\partial q(p, k)}{\partial p} + a \frac{c}{p^2} \sqrt{\lambda q(p, k)} - a \frac{c}{2p} \sqrt{\frac{\lambda}{q(p, k)}} \frac{\partial q(p, k)}{\partial p} \right). \quad (6.3)$$

Setting $\frac{\partial \Pi(p, k)}{\partial p} = 0$ implies that

$$\lambda \frac{\partial q(p, k)}{\partial p} - a \frac{c}{p^2} \sqrt{\lambda} \left(\frac{1}{2} q(p, k)^{-\frac{1}{2}} p \frac{\partial q(p, k)}{\partial p} - q(p, k)^{\frac{1}{2}} \right) = -\frac{\Pi(p, k)}{k(p-c)^2}. \quad (6.4)$$

Differentiating (6.3) with respect to k , we have

$$\begin{aligned} \frac{\partial^2 \Pi(p, k)}{\partial k \partial p} &= \frac{1}{k} \left(\frac{\Pi(p, k)}{p-c} + k(p-c) \left(\lambda \frac{\partial q(p, k)}{\partial p} + a \frac{c}{p^2} \sqrt{\lambda q(p, k)} - a \frac{c}{2p} \sqrt{\frac{\lambda}{q(p, k)}} \frac{\partial q(p, k)}{\partial p} \right) \right) \\ &\quad + k \left(\lambda \frac{\partial q(p, k)}{\partial k} - a \sqrt{\lambda} \frac{c}{2p} q(p, k)^{-\frac{1}{2}} \frac{\partial q(p, k)}{\partial k} \right) + k(p-c) \lambda \frac{\partial^2 q(p, k)}{\partial p \partial k} \\ &\quad - k(p-c) \frac{ac}{p^2} \sqrt{\lambda} \left(-\frac{p}{4} q(p, k)^{-\frac{3}{2}} \frac{\partial q(p, k)}{\partial k} \frac{\partial q(p, k)}{\partial p} + \frac{p}{2} q(p, k)^{-\frac{1}{2}} \frac{\partial^2 q(p, k)}{\partial p \partial k} \right. \\ &\quad \left. - \frac{1}{2} q(p, k)^{-\frac{1}{2}} \frac{\partial q(p, k)}{\partial k} \right). \end{aligned}$$

Recall that $\frac{\partial \Pi(p, k)}{\partial p} \Big|_{p=p_k^*} = 0$. Note also that $\frac{\partial q(p, k)}{\partial k} = -q(p, k)^2$ and $\frac{\partial^2 q(p, k)}{\partial p \partial k} = -2q(p, k) \frac{\partial q(p, k)}{\partial p}$.

Applying (6.3) with the fact that $\frac{\partial \Pi(p, k)}{\partial p} \Big|_{p=p_k^*} = 0$ and simplifying leads to

$$\begin{aligned} \frac{\partial^2 \Pi(p, k)}{\partial k \partial p} \Big|_{p=p_k^*} &= k \left(-\lambda q(p_k^*, k)^2 + a \frac{c}{2p_k^*} \sqrt{\lambda} q(p_k^*, k)^{\frac{3}{2}} \right) \\ &\quad - k(p_k^* - c) q(p_k^*, k) \left\{ 2\lambda \frac{\partial q(p_k^*, k)}{\partial p} - a \frac{c}{p_k^{*2}} \sqrt{\lambda} \left(\frac{3}{4} p_k^* q(p_k^*, k)^{-\frac{1}{2}} \frac{\partial q(p_k^*, k)}{\partial p} - \frac{1}{2} q(p_k^*, k)^{\frac{1}{2}} \right) \right\}. \end{aligned}$$

Applying (6.4) to the term in the curly brackets and simplifying yields

$$\begin{aligned} \frac{\partial^2 \Pi(p, k)}{\partial k \partial p} \Big|_{p=p_k^*} &= k \left(-\lambda q(p_k^*, k)^2 + a \frac{c}{2p_k^*} \sqrt{\lambda} q(p_k^*, k)^{\frac{3}{2}} \right) + 2 \frac{\Pi(p_k^*, k)}{p_k^* - c} \\ &\quad + k(p_k^* - c) q(p_k^*, k) a \frac{c}{p_k^{*2}} \sqrt{\lambda} \left(-\frac{1}{4} p_k^* q(p_k^*, k)^{-\frac{1}{2}} \frac{\partial q(p_k^*, k)}{\partial p} + \frac{3}{2} q(p_k^*, k)^{\frac{1}{2}} \right). \end{aligned}$$

Replacing $\Pi(p_k^*, k)$ by its expression in (4.3) and simplifying yields

$$\left. \frac{\partial^2 \Pi(p, k)}{\partial k \partial p} \right|_{p=p_k^*} = k\lambda q(p_k^*, k)^2 + k\sqrt{\lambda}a \frac{c}{p_k^*} \left(-\frac{(p_k^* - c)}{4} q(p_k^*, k)^{\frac{1}{2}} \frac{\partial q(p_k^*, k)}{\partial p} - \frac{3}{2} \frac{c}{p_k^*} q(p_k^*, k)^{\frac{3}{2}} \right).$$

Utilizing the fact that $\frac{\partial q(p_k^*, k)}{\partial p} = -\frac{1}{\mu} q(p_k^*, k)(1 - kq(p_k^*, k))$ and further simplifying gives

$$\left. \frac{\partial^2 \Pi(p, k)}{\partial k \partial p} \right|_{p=p_k^*} = kq(p_k^*, k) \left(\lambda q(p_k^*, k) - a \frac{c}{4p_k^*} \sqrt{\lambda q(p_k^*, k)} \left(6 \frac{c}{p_k^*} - \frac{1}{\mu} (p_k^* - c)(1 - kq(p_k^*, k)) \right) \right).$$

Thus, it follows that $\frac{\partial p_k^*}{\partial k}$ is of the same sign as

$$W(p, k) = \lambda q(p_k^*, k) - a \frac{c}{4p_k^*} t(p_k^*, k) \sqrt{\lambda q(p, k)},$$

where $t(p_k^*, k) = 6 \frac{c}{p_k^*} - \frac{1}{\mu} (p_k^* - c)(1 - kq(p_k^*, k))$. Observe that since $\Pi(p_k^*, k) > 0$, we have $\lambda q(p_k^*, k) - a \frac{c}{p_k^*} \sqrt{\lambda q(p, k)} > 0$. Then, $W(p, k) > 0$ if $t(p_k^*, k) \leq 4$. Observe also that $t(p, k)$ is decreasing in $p \geq c$ with a maximizer $p = c$ and a maximum value of $t(c, k) = 6$. Then, there exists p_k^c such that for $p_k^* > p_k^c$, $t(p_k^*, k) < 4$ (i.e., $t(p_k^c, k) = 4$). Note finally that $t(\frac{3c}{2}, k) = 4 - \frac{1}{\mu} \frac{c}{2} (1 - kq(\frac{3c}{2}, k)) < S(p_k^c, k) = 4$, which implies that $p_k^c < \frac{3c}{2}$. Therefore, if $p_k^* \geq \frac{3c}{2}$, then $p_k^* > p_k^c$ and $W(p, k) > 0$, which also imply that $\frac{\partial p_k^*}{\partial k} > 0$.

The proof of (ii) is more straightforward. Denote $\Pi(p, k)$ as $\Pi(p, \lambda)$. Similar to the above, $\frac{\partial p_k^*}{\partial \lambda}$ is of the same sign as $\left. \frac{\partial^2 \Pi(p, \lambda)}{\partial p \partial \lambda} \right|_{p=p_k^*}$. Setting $\frac{\partial \Pi(p, \lambda)}{\partial p} = 0$ implies that

$$\left. \frac{\partial q(p)}{\partial p} \right|_{p=p_k^*} = -\frac{\lambda q(p_k^*) - a \frac{c^2}{p_k^{*2}} \sqrt{\lambda q(p_k^*)}}{(p_k^* - c) \left(\lambda - a \frac{c}{2p_k^*} \sqrt{\frac{\lambda}{q(p_k^*)}} \right)}. \quad (6.5)$$

Differentiating $\Pi(p, \lambda)$ with respect to λ , we have

$$\frac{\partial \Pi(p, \lambda)}{\partial \lambda} = (p - c) \left(q(p) - \frac{ac}{2p\sqrt{\lambda}} \sqrt{q(p)} \right) = \frac{\Pi(p, \lambda)}{\lambda} + \frac{(p - c)ac}{\lambda} \frac{1}{2p} \sqrt{\lambda q(p)}.$$

Differentiating the last equation with respect to p , we have

$$\frac{\partial^2 \Pi(p, \lambda)}{\partial p \partial \lambda} = \frac{1}{\lambda} \left(\frac{\partial \Pi(p, \lambda)}{\partial p} + \frac{ac^2}{2p^2} \sqrt{\lambda q(p)} + (p - c) \frac{ac}{4p} \sqrt{\frac{\lambda}{q(p)}} \frac{\partial q(p)}{\partial p} \right).$$

Therefore, since $\frac{\partial \Pi(p, \lambda)}{\partial p} \Big|_{p=p_k^*} = 0$, then $\frac{\partial^2 \Pi(p, \lambda)}{\partial p \partial \lambda} \Big|_{p=p_k^*}$ is of the same sign as

$$W_1(p_k^*, \lambda) = \frac{ac^2}{2p_k^{*2}} \sqrt{\lambda q(p_k^*)} + (p_k^* - c) \frac{ac}{4p_k^*} \sqrt{\frac{\lambda}{q(p_k^*)}} \frac{\partial q(p)}{\partial p} \Big|_{p=p_k^*}.$$

Utilizing (6.5), we have

$$W_1(p_k^*, k) = \frac{\lambda \frac{ac}{2p} \sqrt{\lambda q(p_k^*)} \left(\frac{2c - p_k^*}{2p_k^*} \right)}{\left(\lambda - a \frac{c}{2p_k^*} \sqrt{\frac{\lambda}{q(p_k^*)}} \right)},$$

where $\left(\lambda - a \frac{c}{2p_k^*} \sqrt{\frac{\lambda}{q(p_k^*)}} \right) > 0$ since $\Pi(p_k^*, \lambda) > 0$. Therefore, $W_1(p_k^*, k) > 0$ if $p_k^* < 2c$ and $W_1(p_k^*, k) < 0$ if $p_k^* > 2c$. This completes the proof of (ii).

The proof of (iii) is similar to that of (ii). Denote $\Pi(p, k)$ as $\Pi(p, c)$. Similar to the above, $\frac{\partial p_k^*}{\partial c}$ is of the same sign as $\frac{\partial^2 \Pi(p, c)}{\partial p \partial c} \Big|_{p=p_k^*}$. Differentiation implies that

$$\begin{aligned} \frac{\partial \Pi(p, c)}{\partial c} &= -\frac{\Pi(p, c)}{p - c} - k(p - c) \frac{a}{p} \sqrt{\lambda q(p)}, \\ \frac{\partial^2 \Pi(p, c)}{\partial p \partial c} &= -\frac{1}{p - c} \frac{\partial \Pi(p, c)}{\partial p} + \frac{\Pi(p, c)}{(p - c)^2} - k \frac{a}{p} \sqrt{\lambda q(p)} + k(p - c) \frac{a}{p^2} \sqrt{\lambda q(p)} \\ &\quad - k(p - c) \frac{a}{2p} \sqrt{\frac{\lambda}{q(p)}} \frac{\partial q(p)}{\partial p}. \end{aligned}$$

The facts that $\Pi(p_k^*, c) > 0$ and $\left. \frac{\partial \Pi(p, c)}{\partial p} \right|_{p=p_k^*} = 0$ imply that $\left. \frac{\partial \Pi(p, c)}{\partial p \partial c} \right|_{p=p_k^*} > 0$ if $W_2(p_k^*, c) > 0$, where

$$W_2(p_k^*, c) = -ka \frac{c}{p_k^{*2}} \sqrt{\lambda q(p_k^*)} - k(p_k^* - c) \frac{a}{2p_k^*} \sqrt{\frac{\lambda}{q(p_k^*)}} \left. \frac{\partial q(p)}{\partial p} \right|_{p=p_k^*}.$$

Utilizing (6.5), we have

$$W_2(p_k^*, c) = k\lambda \frac{a}{2p_k^{*2}} \sqrt{\lambda q(p_k^*)} \left(\frac{p_k^* - 2c}{\lambda - a \frac{c}{2p_k^*} \sqrt{\frac{\lambda}{q(p_k^*)}}} \right).$$

Therefore, $W_2(p_k^*, c) > 0$ if $p_k^* > 2c$, which completes the proof of (iii). ■

Appendix G. Riskless Case

Proof of Theorem 4.2.2. Parts (i) and (ii) follow directly by setting $a = 0$ in the proof of Theorem 4.2.1. To prove (iii), we adopt a more direct approach. Note that $\frac{\partial p_k^0}{\partial \alpha}$ is of the same sign as $\left. \frac{\partial^2 \Pi^0(p, \alpha)}{\partial \alpha \partial p} \right|_{p=p_k^0}$. The fact that $\left. \frac{\partial \Pi^0(p, \alpha)}{\partial p} \right|_{p=p_k^0} = 0$ (from Corollary 4.2.3) implies that

$$q(p_k^0) = -(p_k^0 - c) \left. \frac{\partial q(p, \alpha)}{\partial p} \right|_{p=p_k^0}.$$

In addition, it can be shown that

$$\frac{\partial^2 \Pi^0(p, \alpha)}{\partial \alpha \partial p} = k\lambda \frac{\partial q(p, \alpha)}{\partial \alpha} + k\lambda(p - c) \frac{\partial^2 q(p, \alpha)}{\partial \alpha \partial p},$$

where $\frac{\partial q(p, \alpha)}{\partial \alpha} = \frac{1}{\mu} q(p, \alpha)(1 - kq(p, \alpha))$ and $\frac{\partial^2 q(p, \alpha)}{\partial \alpha \partial p} = \frac{1}{\mu} \frac{\partial q(p, \alpha)}{\partial p} (1 - 2kq(p, \alpha))$. Then,

$$\left. \frac{\partial^2 \Pi^0(p, \alpha)}{\partial \alpha \partial p} \right|_{p=p_k^0} = \frac{k\lambda}{\mu} \left(q(p_k^0, \alpha) \right)^2 > 0,$$

which completes the proof. ■

Appendix H. Structural results on k_p^*

Proof of Theorem 4.3.1. Assume k is a continuous variable. Rewrite (3.5) as $\Pi(p, k) = \theta(p, k)/\delta(p, k)$,

where

$$\theta(p, k) = (p - c) \left(\lambda e^{(\alpha-p)/\mu} k - a \frac{c}{p} k \sqrt{\lambda e^{(\alpha-p)/\mu}} \left(1 + k e^{(\alpha-p)/\mu} \right)^{1/2} \right),$$

and $\delta(p, k) = 1 + k e^{(\alpha-p)/\mu}$. Clearly, $\delta(p, k)$ is linear in k . Moreover, $\theta(p, k)$ is strictly concave in k

as

$$\frac{\partial^2 \theta(p, k)}{\partial k^2} = -(p - c) a \frac{c}{p} \lambda^{\frac{1}{2}} \left(e^{(\alpha-p)/\mu} \right)^{\frac{3}{2}} \left(1 + k e^{(\alpha-p)/\mu} \right)^{-\frac{3}{2}} \left(1 + \frac{3}{4} k e^{(\alpha-p)/\mu} \right) < 0.$$

A result in Mangasarian [60] (p. 149) implies that the function obtained by dividing a concave function by a linear positive function is pseudoconcave. (It can be easily verified, utilizing arguments like those in the proof of Lemma 6.2.1, that a similar result holds under strict concavity.) Therefore, $\Pi(p, k) = \theta(p, k)/\delta(p, k)$ is strictly pseudoconcave in k . To prove that $\Pi(p, k)$ is unimodal in k (since a pseudoconcave function can be either unimodal or monotone), it is sufficient to show that $\Pi(p, k)$ attains a maximum in $k \in (1, \infty)$; see, for example, Bazaraa et al., p. 116 and 123. Note that $\Pi(p, k)$ is increasing in k at $k = 1$ under **(A3)** with $\Pi(p, 1) > 0$. In addition, it can be easily shown that $\lim_{k \rightarrow \infty} \Pi(p, k) \rightarrow 0^-$. Thus, it follows that $\Pi(p, k)$ starts as an increasing function at $k = 1$ and becomes decreasing at some $k > 1$. Therefore, $\Pi(p, k)$ attains a local maximum in $k \in (1, \infty)$, which is the unique global maximum due to the strict pseudoconcavity of $\Pi(p, k)$. This completes the proof.

■

Appendix I. Monotonicity results on k_p^*

Proof of Theorem 4.3.2. Theorem 4.3.1 implies that $\frac{\partial \Pi(p,k)}{\partial k} \Big|_{k=k_p^*} = 0$ and $\frac{\partial^2 \Pi(p,k)}{\partial k^2} \Big|_{k=k_p^*} < 0$. To prove part (i), denote $\Pi(p,k)$ as $\Pi(k,c)$ and note that $\frac{\partial k_p^*}{\partial c}$ is of the same sign as $\frac{\partial^2 \Pi(k,c)}{\partial c \partial k} \Big|_{k=k_p^*}$ (similar to the proof of Theorem 4.2.1). We have

$$\frac{\partial \Pi(k,c)}{\partial k} = (p-c) \left\{ \lambda q(k) - a \frac{c}{p} \sqrt{\lambda q(k)} + k \frac{\partial q(k)}{\partial k} \left(\lambda - a \frac{c}{2p} \sqrt{\frac{\lambda}{q(k)}} \right) \right\}. \quad (6.6)$$

Then,

$$\lambda q(k) - a \frac{c}{p} \sqrt{\lambda q(k)} + k \frac{\partial q(k)}{\partial k} \left(\lambda - a \frac{c}{2p} \sqrt{\frac{\lambda}{q(k)}} \right) \Big|_{k=k_p^*} = 0, \quad (6.7)$$

which also implies that

$$\frac{\partial q(k)}{\partial k} \Big|_{k=k_p^*} = - \frac{\left(\lambda q(k) - a \frac{c}{p} \sqrt{\lambda q(k)} \right) \Big|_{k=k_p^*}}{k \left(\lambda - a \frac{c}{2p} \sqrt{\frac{\lambda}{q(k)}} \right) \Big|_{k=k_p^*}}. \quad (6.8)$$

Differentiating (6.6) with respect to c at $k = k_p^*$ yields

$$\begin{aligned} \frac{\partial^2 \Pi(k,c)}{\partial c \partial k} \Big|_{k=k_p^*} &= - \left\{ \left(\lambda q(k) - a \frac{c}{p} \sqrt{\lambda q(k)} \right) + k \frac{\partial q(k)}{\partial k} \left(\lambda - a \frac{c}{2p} \sqrt{\frac{\lambda}{q(k)}} \right) \right\} \Big|_{k=k_p^*} \\ &\quad + (p-c) \left(-\frac{a}{p} \sqrt{\lambda q(k)} - \frac{a}{2p} k \frac{\partial q(k)}{\partial k} \sqrt{\frac{\lambda}{q(k)}} \right) \Big|_{k=k_p^*}. \end{aligned}$$

Applying (6.7) and (6.8),

$$\begin{aligned} \frac{\partial^2 \Pi(k,c)}{\partial c \partial k} \Big|_{k=k_p^*} &= -\frac{a}{p} (p-c) \left(\sqrt{\lambda q(k_p^*)} - \frac{(\lambda q(k_p^*) - a \frac{c}{p} \sqrt{\lambda q(k_p^*)}) \sqrt{\frac{\lambda}{q(k_p^*)}}}{2 \left(\lambda - a \frac{c}{2p} \sqrt{\frac{\lambda}{q(k_p^*)}} \right)} \right) \\ &= -\frac{a}{p} (p-c) \left(\frac{\lambda \sqrt{\lambda q(k_p^*)}}{2 \left(\lambda - a \frac{c}{2p} \sqrt{\frac{\lambda}{q(k_p^*)}} \right)} \right) < 0, \end{aligned}$$

where the last inequality follows since $\lambda - a \frac{c}{2p} \sqrt{\frac{\lambda}{q(k_p^*)}} > \lambda - a \frac{c}{p} \sqrt{\frac{\lambda}{q(k_p^*)}} = \frac{\Pi(k_p^*, c)}{k_p^* (p-c) q(k_p^*)}$, and $\Pi(k_p^*, c) > 0$

under **(A2)**. This completes the proof of (i).

The proof of (ii) is similar. $\frac{\partial k_p^*}{\partial \lambda}$ is of the same sign as $\frac{\partial^2 \Pi(k, \lambda)}{\partial \lambda \partial k} \Big|_{k=k_p^*}$. Differentiating (6.6) with respect to λ at $k = k_p^*$ and simplifying yields

$$\begin{aligned} \frac{\partial \Pi(k, \lambda)}{\partial \lambda \partial k} \Big|_{k=k_p^*} &= \frac{(p-c)}{2\lambda} \left\{ \left(\lambda q(k) - a \frac{c}{p} \sqrt{\lambda q(k)} \right) + k \frac{\partial q(k)}{\partial k} \left(\lambda - a \frac{c}{2p} \sqrt{\frac{\lambda}{q(k)}} \right) \right\} \Big|_{k=k_p^*} \\ &+ \frac{(p-c)}{2} \left(q(k) + k \frac{\partial q(k)}{\partial k} \right) \Big|_{k=k_p^*}. \end{aligned}$$

Applying (6.7) and (6.8),

$$\frac{\partial \Pi(k, \lambda)}{\partial \lambda \partial k} \Big|_{k=k_p^*} = \frac{(p-c)}{2} \left(q(k_p^*) - \frac{\left(\lambda q(k_p^*) - a \frac{c}{p} \sqrt{\lambda q(k_p^*)} \right)}{\left(\lambda - a \frac{c}{2p} \sqrt{\frac{\lambda}{q(k_p^*)}} \right)} \right) = \frac{(p-c) a \frac{c}{p} \sqrt{\lambda q(k_p^*)}}{4 \left(\lambda - a \frac{c}{2p} \sqrt{\frac{\lambda}{q(k_p^*)}} \right)} > 0,$$

which completes the proof. ■

Proof of Theorem 4.3.3. To prove (i), note that $\frac{\partial k_p^*}{\partial p}$ is of the same sign as $\frac{\partial^2 \Pi(p, k)}{\partial k \partial p} \Big|_{k=k_p^*}$. Note also that (6.8) implies that

$$\frac{1}{k_p^*} \left(\lambda q(p, k_p^*) - \frac{ac}{p} \sqrt{\lambda q(p, k_p^*)} \right) = \lambda (q(p, k_p^*))^2 - \frac{ac}{2p} \sqrt{\lambda} (q(p, k_p^*))^{3/2}. \quad (6.9)$$

By differentiating $\Pi(p, k)$ with respect to p then with respect to k and simplifying, it follows that

$$\begin{aligned} \frac{\partial^2 \Pi(p, k)}{\partial k \partial p} &= \frac{1}{p-c} \frac{\partial \Pi(p, k)}{\partial k} + (p-c) \left(\lambda \frac{\partial q(p, k)}{\partial p} + \frac{ac}{p^2} \sqrt{\lambda q(p, k)} - \frac{ac}{2p} \sqrt{\frac{\lambda}{q(p, k)}} \frac{\partial q(p, k)}{\partial p} \right) \\ &+ k(p-c) \left[-2\lambda q(p, k) \frac{\partial q(p, k)}{\partial p} - \frac{ac}{2p^2} \sqrt{\lambda} (q(p, k))^{3/2} + \frac{3}{4} \frac{ac}{p} \sqrt{\lambda q(p, k)} \frac{\partial q(p, k)}{\partial p} \right]. \end{aligned}$$

Then, $\left. \frac{\partial^2 \Pi(p, k)}{\partial k \partial p} \right|_{k=k_p^*}$ is of the same sign as

$$\begin{aligned} \Delta_1(p, k_p^*) &= \left\{ \lambda \frac{\partial q(p, k)}{\partial p} + \frac{ac}{p^2} \sqrt{\lambda q(p, k)} - \frac{ac}{2p} \sqrt{\frac{\lambda}{q(p, k)}} \frac{\partial q(p, k)}{\partial p} \right. \\ &\quad \left. + k \left[-2\lambda q(p, k) \frac{\partial q(p, k)}{\partial p} - \frac{ac}{2p^2} \sqrt{\lambda} (q(p, k))^{3/2} + \frac{3}{4} \frac{ac}{p} \sqrt{\lambda q(p, k)} \frac{\partial q(p, k)}{\partial p} \right] \right\} \Big|_{k=k_p^*} \\ &= \left\{ \frac{\partial q(p, k)}{\partial p} \left[\left(\lambda (q(p, k))^2 - \frac{ac}{2p} \sqrt{\lambda} (q(p, k))^{3/2} \right) \left(\frac{1}{(q(p, k))^2} - 2 \frac{k}{q(p, k)} \right) - k \frac{ac}{4p} \sqrt{\lambda q(p, k)} \right] \right. \\ &\quad \left. + \frac{ac}{p^2} \sqrt{\lambda q(p, k)} \left(1 - \frac{kq(p, k)}{2} \right) \right\} \Big|_{k=k_p^*}. \end{aligned}$$

Utilizing (6.9) and the fact that $kq(p, k) \leq 1$, it follows that

$$\begin{aligned} \Delta_1(p, k_p^*) &\geq \frac{\partial q(p, k)}{\partial p} \left[\frac{1}{k} \left(\lambda q(p, k) - \frac{ac}{p} \sqrt{\lambda q(p, k)} \right) \left(\frac{1}{(q(p, k))^2} - 2 \frac{k}{q(p, k)} \right) - k \frac{ac}{4p} \sqrt{\lambda q(p, k)} \right] \Big|_{k=k_p^*} \\ &= \frac{\partial q(p, k)}{\partial p} \left(\frac{1 - 2kq(p, k)}{k(q(p, k))^2} \right) \left[\lambda q(p, k) - \frac{ac}{p} \sqrt{\lambda q(p, k)} \left(1 + \frac{k^2 (q(p, k))^2}{4(1 - 2kq(p, k))} \right) \right] \Big|_{k=k_p^*}. \end{aligned}$$

Observe that $kq(p, k)$ is increasing in k . Then, $k_p^*(p, k_p^*) > q(p, 1)$. Therefore, if $q(p, 1) > 1/2$, then $k_p^*q(p, k_p^*) > 1/2$. It follows that $\Delta_1(p, k_p^*)$ is of the same sign as

$$\lambda q(p, k_p^*) - \frac{ac}{p} \sqrt{\lambda q(p, k_p^*)} \left(1 + \frac{k_p^{*2} (q(p, k_p^*))^2}{4(1 - 2k_p^*q(p, k_p^*))} \right) > \lambda q(p, k_p^*) - \frac{ac}{p} \sqrt{\lambda q(p, k_p^*)} > 0,$$

where the last inequality follows since $\Pi(p, k_p^*) > 0$ under **(A3)**. This proves (i).

To prove (ii), it can be shown, similar to the proof of Theorem 4.3.2, that $\frac{\partial k^*}{\partial \alpha}$ is of the same sign as $\left. \frac{\partial^2 \Pi(k, \alpha)}{\partial k \partial \alpha} \right|_{k=k_p^*}$, where

$$\frac{\partial \Pi(k, \alpha)}{\partial \alpha} = k(p - c) \frac{\partial q(k, \alpha)}{\partial \alpha} \left(\lambda - a \frac{c}{2p} \sqrt{\frac{\lambda}{q(k, \alpha)}} \right),$$

$$\begin{aligned} \frac{\partial^2 \Pi(k, \alpha)}{\partial k \partial \alpha} &= \left((p-c) \frac{\partial q(k, \alpha)}{\partial \alpha} + k(p-c) \frac{\partial^2 q(k, \alpha)}{\partial k \partial \alpha} \right) \left(\lambda - a \frac{c}{2p} \sqrt{\frac{\lambda}{q(k, \alpha)}} \right) \\ &\quad + k(p-c) \frac{\partial q(k, \alpha)}{\partial \alpha} a \frac{c}{4p} \sqrt{\lambda} (q(k, \alpha))^{-\frac{3}{2}} \frac{\partial q(k, \alpha)}{\partial k}, \end{aligned}$$

$\frac{\partial q(k, \alpha)}{\partial k} = -(q(k, \alpha))^2$, $\frac{\partial q(k, \alpha)}{\partial \alpha} = \frac{1}{\mu} q(k, \alpha) (1 - kq(k, \alpha)) > 0$, and $\frac{\partial^2 q(k, \alpha)}{\partial \alpha \partial k} = -2q(k, \alpha) \frac{\partial q(k, \alpha)}{\partial \alpha}$. It follows that $\frac{\partial^2 \Pi(k, \alpha)}{\partial k \partial \alpha}$ is of the same sign as

$$\Delta(k, \alpha) = (1 - 2kq(k, \alpha)) \left(\lambda - a \frac{c}{2p} \sqrt{\frac{\lambda}{q(k, \alpha)}} \right) - ka \frac{c}{4p} \sqrt{\lambda q(k, \alpha)}.$$

Recall, by **(A3)**, that $k_p^* \geq 1$. In addition, $k_p^* q(p, k_p^*) > 1/2$ (as argued above) and, hence, $(1 - 2k_p^* q(k_p^*, \alpha)) < 0$. Moreover, $\lambda - a \frac{c}{2p} \sqrt{\frac{\lambda}{q(k_p^*, \alpha)}} > 0$ as shown in the proof of Theorem 4.3.2. Then, it follows that if $q(1, \alpha) > 1/2$, then $\Delta(k_p^*, \alpha) < 0$ and therefore $\frac{\partial k_p^*}{\partial \alpha} < 0$, which completes the proof of (ii).

The proof of (iii) is similar. Note first that $v_0 = e^{u_0/\mu}$ is increasing in u_0 . We, therefore, establish the proof in terms of v_0 . Similar to the above, $\frac{\partial k_p^*}{\partial v_0}$ is of the same sign as $\left. \frac{\partial^2 \Pi(k, v_0)}{\partial v_0 \partial k} \right|_{k=k_p^*}$, where

$$\begin{aligned} \frac{\partial^2 \Pi(k, v_0)}{\partial k \partial v_0} &= \left((p-c) \frac{\partial q(k, v_0)}{\partial v_0} + k(p-c) \frac{\partial^2 q(k, v_0)}{\partial k \partial v_0} \right) \left(\lambda - a \frac{c}{2p} \sqrt{\frac{\lambda}{q(k, v_0)}} \right) \\ &\quad + k(p-c) \frac{\partial q(k, v_0)}{\partial v_0} a \frac{c}{4p} \sqrt{\lambda} (q(k, v_0))^{-\frac{3}{2}} \frac{\partial q(k, v_0)}{\partial k}. \end{aligned}$$

Therefore, substituting $\frac{\partial q(k, v_0)}{\partial k} = -(q(k, v_0))^2$, $\frac{\partial q(k, v_0)}{\partial v_0} = -\frac{e^{(\alpha-p)/\mu}}{(v_0 + e^{(\alpha-p)/\mu})^2} < 0$, and

$\frac{\partial^2 q(k, v_0)}{\partial k \partial v_0} = -2q(k, v_0) \frac{\partial q(k, v_0)}{\partial v_0}$, it follows that $\frac{\partial k_p^*}{\partial v_0}$ is of the opposite sign (because $\frac{\partial q(k, v_0)}{\partial v_0} < 0$) as

$$\Delta(k_p^*, v_0) = (1 - 2k_p^* q(k_p^*, v_0)) \left(\lambda - a \frac{c}{2p} \sqrt{\frac{\lambda}{q(k_p^*, v_0)}} \right) - k_p^* a \frac{c}{4p} \sqrt{\lambda q(k_p^*, v_0)},$$

where $\Delta(k_p^*, v_0) < 0$ under **(A3)** and the condition that $q(1, v_0) > 1/2$, as discussed above. ■

Appendix J. Bounds on p_k^*

Proof of Lemma 4.4.1. Setting $\Pi(p, k) = 0$ for $p \in (c, \infty)$ implies that

$$p^2 q(p, k) = \frac{a^2 c^2}{\lambda}. \quad (6.10)$$

Let $h(p, k) \equiv p^2 q(p, k)$. In the following we first prove that $h(p, k)$ is unimodal in p for $p \geq 0$. Note that $h(0, k) = 0$, $h(p, k) > 0$ for $0 < p < \infty$, and $\lim_{p \rightarrow +\infty} h(p, k) \rightarrow 0^+$. Therefore, $h(p, k)$ attains a maximum on $(0, \infty)$. Next we prove that this maximum is unique by showing that the first derivative of $h(p, k)$ has a single zero on $(0, \infty)$. Note that

$$\frac{\partial h(p, k)}{\partial p} = 2pq(p, k) + p^2 \frac{\partial q(p, k)}{\partial p} = 2pq(p, k) - \frac{p^2}{\mu} q(p, k)(1 - kq(p, k)).$$

Setting $\frac{\partial h(p, k)}{\partial p} = 0$ yields

$$p(1 - kq(p, k)) = 2\mu.$$

It can be easily shown that the left hand side of this equation is increasing in p . Therefore, the equation has a unique solution and hence it follows that $h(p, k)$ is unimodal in p for $p \in (0, \infty)$. Thus, the function $\gamma_1(p, k) = \sqrt{\lambda q(p, k)} - a \frac{c}{p}$ has at most two zeroes in $p \in (0, \infty)$. (Observe that a unimodal function can take on the same value at most twice.)

Note that $\Pi(c, k) = 0$, and **(A2)** implies that $\Pi(p, k) > 0$ for p to the right of c . In addition, it can be easily shown that for p large enough ($p \rightarrow \infty$), $\Pi(p, k) < 0$. Therefore, under **(A2)** $\Pi(p, k) = k(p-c)\sqrt{\lambda q(p, k)}\gamma_1(p, k)$ has two zeroes at points \underline{p}_k and \overline{p}_k , with $\underline{p}_k \in (0, c]$ and $\overline{p}_k \in (c, \infty)$. Then, it follows that $\Pi(p, k)$ is positive for $p \in (c, \overline{p}_k)$, and is negative otherwise (within the range

$p > c$), which implies that \overline{p}_k constitutes an upper bound on p_k^* . ■

Proof of Corollary 4.4.1. Follows since it can be shown that $\Pi(p, 1) > \Pi(p, k)/k$ for all $p \in (c, \infty)$ and $k = 2, 3, \dots$. Then, $\overline{p}_1 > \overline{p}_k$ for all $k = 2, 3, \dots$

Proof of Lemma 4.4.2. The first derivative of $\Pi(p, k)$ is given by

$$\begin{aligned} \frac{\partial \Pi(p, k)}{\partial p} &= k \left(\lambda q(p, k) - a \frac{c}{p} \sqrt{\lambda q(p, k)} \right) \\ &\quad + k(p - c) \left(\lambda \frac{\partial q(p, k)}{\partial p} + a \frac{c}{p^2} \sqrt{\lambda q(p, k)} - a \frac{c}{2p} \sqrt{\frac{\lambda}{q(p, k)}} \frac{\partial q(p, k)}{\partial p} \right), \end{aligned}$$

where $\frac{\partial q(p, k)}{\partial p} = -\frac{1}{\mu} q(p, k) (1 - kq(p, k))$. Setting $\frac{\partial \Pi(p, k)}{\partial p} = 0$ and simplifying leads to

$$\sqrt{\lambda q(p, k)} \left(1 - \frac{(p-c)}{\mu} (1 - kq(p, k)) \right) = a \frac{c}{2p} \left(\frac{2c}{p} - \frac{(p-c)}{\mu} (1 - kq(p, k)) \right).$$

Letting $w_1(p, k) \equiv 1 - \frac{(p-c)}{\mu} (1 - kq(p, k))$ and $w_2(p, k) \equiv \frac{2c}{p} - \frac{(p-c)}{\mu} (1 - kq(p, k))$, it follows that $\sqrt{\lambda q(p_k^*, k)} w_1(p_k^*, k) = a \frac{c}{2p} w_2(p_k^*, k)$. Therefore, $w_1(p_k^*, k)$ and $w_2(p_k^*, k)$ must be of the same sign. Note that both $w_1(p, k)$ and $w_2(p, k)$ are decreasing in p . In addition, $w_1(c, k) = 1 > 0$, $w_2(c, k) = 2 > 0$, $\lim_{p \rightarrow \infty} w_1(p, k) \rightarrow -\infty$, and $\lim_{p \rightarrow \infty} w_2(p, k) \rightarrow -\infty$. Therefore, the equation $w_i(p, k) = 0$, $i = 1, 2$, has a single solution for $p \in (c, \infty)$, denoted by p_i^k . The fact that $w_1(p_k^*, k)$ and $w_2(p_k^*, k)$ are of the same sign then implies the result in the lemma. ■

Proof of Lemma 4.4.3. The proof follows by noting that there exists $\overline{k} < \infty$ such that $p^2 q(p, \overline{k}) < a^2 c^2$ for all p (since $p^2 q(p, k)$ is decreasing in k with $\lim_{k \rightarrow \infty} p^2 q(p, k) \rightarrow 0$), which implies that $\Pi(p, k) < 0$ for all $p > c$ and $k \geq \overline{k}$ based on the results above. This proves that \overline{K} is not empty. Since, $\Pi(p^{**}, k^{**})$ should be positive (otherwise, the retailer makes at least the same expected profit

by selling nothing), it follows that $k^{**} < \inf\{\overline{K}\}$. ■

Appendix K. Numerical Examples

Example of p_k^* being decreasing in k

Table K.1. $\lambda = 100$, $\alpha = 5$, $c = 7.5$, $\mu = 1$, $v_0 = 1$

k	p_k^*
8	8.031
10	8.017
12	7.997
14	7.969
16	7.931

Example of p_k^* being decreasing in c

Table K.2. $\lambda = 100$, $\alpha = 7.5$, $k = 25$, $\mu = 1$, $v_0 = 1$

c	$p_k^*(c)$
9.00	10.015
9.35	10.014
9.40	10.013
9.45	10.010
9.50	10.007

Example of p_k^* being increasing in λ

Table K.3. $c = 8$, $\alpha = 10$, $k = 1$, $\mu = 1$, $v_0 = 1$

(Note: $p_k^0 = 10.000 < 2c$)

λ	$p_k^*(\lambda)$
100	9.936
120	9.943
140	9.947
160	9.951
180	9.954

Example of p_k^* being decreasing in λ

Table K.4. $c = 8$, $\alpha = 50$, $k = 1$, $\mu = 1$, $v_0 = 1$
(Note: $p_k^0 = 46.379 > 2c$)

λ	$p_k^*(\lambda)$
100	46.388
150	46.387
200	46.386
250	46.385
300	46.384

To show that the fact that p_k^* may be decreasing in k and nondecreasing c is not due to our approximation in (3.4), we present numerical examples on the behavior of the optimal price based on the exact expected profit in (3.3), p_k^E , and the optimal price based on the exact expected profit from Poisson demands, p_k^P . The expected profit under Poisson demands is based on the assumption that the arrival process is a Poisson process with rate λ , then it follows that the demand for item $i \in S_k$, X_i , has a Poisson distribution with mean $\lambda q(p, k)$, and the expected profit from Poisson demands is given by

$$\Pi^P(p, k) = k \left\{ p \left[\sum_{r=0}^{y^*} r f_{X_i}(r) + y^*(1 - F_{X_i}(r)) \right] - cy^* \right\},$$

where $f_{X_i}(r) = \frac{(\lambda q(p, k))^r}{r!} e^{-\lambda q(p, k)}$ is the density function of X_i , $F_{X_i}(r) = \sum_{l=0}^r g(l)$ is the corresponding cumulative density function, and $y^* = F_{X_i}^{-1}(1 - c/p)$ is the optimal inventory level for $i \in S_k$. Some Numerical examples on the behavior of p_k^E and p_k^P follow.

Example of p_k^E and p_k^P being decreasing in k

Table K.5. $\lambda = 105$, $\alpha = 5$, $c = 7.5$, $\mu = 1$, $v_0 = 1$

k	p_k^E	p_k^P
5	8.082	8.344
6	8.079	8.327
7	8.074	8.310

Example of p_k^E and p_k^P being nonincreasing in c

Table K.6. $\lambda = 115$, $\alpha = 8.09$, $k = 30$, $\mu = 1$, $v_0 = 1$

c	$p_k^E(c)$	$p_k^P(c)$
9.3015	10.8908297674	10.9888815078
9.3045	10.8908297674	10.9888815078
9.3075	10.8908297674	10.9888815078

Appendix L. Structure of the optimal prices under CT

Proof of Theorem 5.2.2. For $(p_S, p_P) \in D$, the expected profit is given by

$$\Pi^T(p_S, p_P) = \begin{cases} \Pi_1^T(p_S, p_P), & \text{if } c_S < p_S < 1 + \delta^- \quad ((p_S, p_P) \in G), \\ \Pi_2^T(p_S, p_P), & \text{if } \delta > 0 \text{ and } 1 \leq p_S < 1 + \delta \quad ((p_S, p_P) \in D \setminus G), \\ \Pi_3^T(p_S, p_P), & \text{if } \delta \leq 0 \text{ and } 1 + \delta \leq p_S < 1 \quad ((p_S, p_P) \in D \setminus G), \end{cases}$$

$$\begin{aligned} \text{where } \Pi_1^T(p_S, p_P) &= \lambda_S \bar{p} (p_S - c_S)(1 - p_S) + \lambda_P \frac{r_P^u - p_P}{r_P^u - r_P^l} (p_S - c_S)(\delta + 1 - p_S) + \lambda_P \frac{r_P^u - p_P}{r_P^u - r_P^l} (p_P - c_P), \\ \Pi_2^T(p_S, p_P) &= \lambda_P \frac{r_P^u - p_P}{r_P^u - r_P^l} (p_S - c_S)(\delta + 1 - p_S) + \lambda_P \frac{r_P^u - p_P}{r_P^u - r_P^l} (p_P - c_P), \\ \Pi_3^T(p_S, p_P) &= \lambda_S \bar{p} (p_S - c_S)(1 - p_S) + \lambda_P \frac{r_P^u - p_P}{r_P^u - r_P^l} (p_P - c_P). \end{aligned}$$

Observe that the function $\Pi^T(p_S, p_P)$ is everywhere continuous, and everywhere differentiable except for the lines defined by $p_P = r_P^l$ ($c_S < p_S < 1 + \delta^+$) and $p_S = 1 + \delta^-$ ($r_P^l \leq p_P < r_P^u$) (these are the boundaries of D and G not eliminated by Lemma 5.2.1).

In the following, we show that an optimal price cannot be in $D \setminus G$, i.e., $(p_S^T, p_P^T) \notin D \setminus G$.

Case 1: $\delta > 0$. In this case, setting $\frac{\partial \Pi_2^T(p_S, p_P)}{\partial p_P} = 0$ and solving for p_P yields

$p_P^T(p_S) \equiv \frac{-(p_S - c_S)(\delta + 1 - p_S) + r_P^u + c_P}{2}$. Now define $\tilde{\Pi}_2^T(p_S) \equiv \Pi_2^T(p_S, p_P^T(p_S))$. Then,

$$\tilde{\Pi}_2^T(p_S) = \frac{\lambda_P}{4(r_P^u - r_P^l)} [(p_S - c_S)(\delta + 1 - p_S) + r_P^u - c_P]^2.$$

Observe that a price of S corresponding to a local minimum/maximum of $\Pi_2^T(p_S, p_P)$ is a local minimum/maximum of $\tilde{\Pi}_2^T(p_S)$. Note that under **(A4)** and **(A5)**,

- (i) $\lim_{p_S \rightarrow \pm\infty} \Pi_2^T(p_S) = \infty$,
- (ii) $\left. \frac{\partial \Pi_2^T(p_S)}{\partial p_S} \right|_{p_S=c_S} = \frac{\lambda_P}{2(r_P^u - r_P^l)} (\delta + 1 - c_S)(r_P^u - c_P) > 0$,
- (iii) $\left. \frac{\partial \Pi_2^T(p_S)}{\partial p_S} \right|_{p_S=1} = \frac{\lambda_P}{2(r_P^u - r_P^l)} (\delta - 1 + c_S)[\delta(1 - c_S) + r_P^u - c_P] < 0$, and
- (iv) $\left. \frac{\partial \Pi_2^T(p_S)}{\partial p_S} \right|_{p_S=\delta+1} = -\frac{\lambda_P}{2(r_P^u - r_P^l)} (\delta + 1 - c_S)(r_P^u - c_P) < 0$.

These imply, since $\tilde{\Pi}_2^T(p_S)$ is a fourth degree polynomial, that $\Pi_2^T(p_S)$ admits one local minimum, for $p_S \in (-\infty, c_S)$, one local maximum for $p_S \in (c_S, 1)$, another local minimum for $p_S \in (\delta + 1, \infty)$ and no local minima/maxima elsewhere. In particular, $\tilde{\Pi}_2^T(p_S)$ has no local minimum/maximum in $(1, \delta + 1)$, and, therefore, $\Pi^T(p_S, p_P)$ admits no internal point local maxima on $D \setminus G$. Then, $\Pi^T(p_S, p_P)$ achieves its maximum within the boundaries of $D \setminus G$ (either on the line $p_S = 1$ or on the line $p_P = r_P^l$) or within the interior of G . However, note that $\left. \frac{\partial \Pi^T(p_S^T, p_P^T)}{\partial p_S} \right|_{p_S=1} = \lambda_S \bar{p}(-1 + c_S) + \lambda_P \frac{r_P^u - p_P}{r_P^u - r_P^l} (\delta - 1 + c_S) < 0$, which implies that there exists an improving direction (along the direction of p_S) from any point in $D \setminus G$ on the line $p_S = 1$ leading to the interior of G (because $\Pi(p_S, p_P)$ is continuous). Then, it follows that (p_S^T, p_P^T) cannot be on the line $p_S = 1$. Similarly, if $p_P = r_P^l$, then it can be easily shown that $\Pi_2^T(p_S, r_P^l)$ is decreasing in p_S for all $p_S > (\delta + 1 + c_S)/2 < 1$, which implies that $\Pi_2^T(p_S, r_P^l)$ is decreasing for $p_S > 1$, and, consequently, there exists an improving direction from any point in $D \setminus G$ on the line $p_P = r_P^l$ leading to the point $(1, r_P^l)$ (and from there to the interior of G based on the argument above). This proves that $(p_S^T, p_P^T) \notin D \setminus G$ if $\delta > 0$.

Case 2: $\delta < 0$. In this case, it can be easily shown that $\Pi_3^T(p_S, p_P)$ admits a unique local maximum $((1+c_S)/2, (r_P^u+c_P)/2) \notin D \setminus G$. Therefore, an optimal price (p_S^T, p_P^T) can only be on the boundaries of $D \setminus G$, and similar to the above case with $\delta > 0$, on the boundary lines $p_S = \delta + 1$ or $p_P = r_P^l$. An argument similar to the above shows that (p_S^T, p_P^T) cannot be on the line $p_S = \delta + 1$ (since $\frac{\partial \Pi_1^T(p_S^T, p_P^T)}{\partial p_S} \Big|_{p_S=\delta+1} = \lambda_{S\bar{P}}(-2\delta - 1 + c_S) + \lambda_P \frac{r_P^u - p_P}{r_P^u - r_P^l}(-\delta - 1 + c_S) < 0$). It can be also shown that the function $\Pi^T(p_S, r_P^l)$ is decreasing for $p_S > \delta + 1$ (since $\Pi^T(p_S, r_P^l)$ is decreasing for $p_S > (1+c_S)/2 = 1 - (1-c_S)/2 < 1 + \delta$), and then similar to the above, (p_S^T, p_P^T) cannot be on the line $p_P = r_P^l$ in $D \setminus G$. Therefore, $(p_S^T, p_P^T) \notin D \setminus G$ if $\delta \leq 0$.

Next we show that $\Pi_1^T(p_S, p_P)$ has at most one local maximum on G . Similar to the above, define $\tilde{\Pi}_1^T(p_S) \equiv \Pi_1^T(p_S, p_P^T(p_S))$. Then,

$$\tilde{\Pi}_1^T(p_S) = \lambda_{S\bar{P}}(1-p_S)(p_S - c_S) + \frac{\lambda_P}{4(r_P^u - r_P^l)} [(p_S - c_S)(\delta + 1 - p_S) + r_P^u - c_P]^2.$$

Then, it can be shown that

- (i) $\lim_{p_S \rightarrow \pm\infty} \Pi_1^T(p_S) = \infty$,
- (ii) $\frac{\partial \tilde{\Pi}_1^T(p_S)}{\partial p_S} \Big|_{p_S=c_S} = \lambda_{S\bar{P}}(1-c_S) + \frac{\lambda_P}{2(r_P^u - r_P^l)}(\delta + 1 - c_S)(r_P^u - c_P) > 0$.
- (iii) If $\delta > 0$, then $\frac{\partial \tilde{\Pi}_1^T(p_S)}{\partial p_S} \Big|_{p_S=1} = -\lambda_{S\bar{P}}(1-c_S) + \frac{\lambda_P}{2(r_P^u - r_P^l)}(\delta - 1 + c_S)[\delta(1-c_S) + r_P^u - c_P] < 0$.
- (iv) Otherwise, if $\delta \leq 0$, then $\frac{d\tilde{\Pi}_1^T(p_S)}{dp_S} \Big|_{p_S=\delta+1} = \lambda_{S\bar{P}}(-2\delta - 1 + c_S) - \frac{\lambda_P}{2(r_P^u - r_P^l)}(\delta + 1 - c_S)(r_P^u - c_P) < 0$.

It follows that $\tilde{\Pi}_1^T(p_S)$ admits one local minimum for $p_S \in (-\infty, c_S)$ and another one for $p_S \in (1+\delta^-, \infty)$. The fact that $\tilde{\Pi}_1^T(p_S)$ is a fourth degree polynomial then implies that it will admit exactly one local maximum for $p_S \in (c_S, 1+\delta^-)$. There are two cases to consider. Either this local maximum of $\tilde{\Pi}_1^T(p_S)$ will correspond to a unique internal point local maximum of $\Pi^T(p_S, p_P)$ within G (given by the solution to the first order optimality conditions in (5.6) and (5.7)), or, otherwise, $\Pi^T(p_S, p_P)$ will

admit a local maximum on its boundaries (because Lemma 5.2.1 guarantees that $\Pi^T(p_S, p_P)$ admits a local maximum in D). In the latter case, Lemma 5.2.1 and the above imply that a boundary point local maximum of $\Pi^T(p_S, p_P)$ within G will only be on the line $p_P = r_P^l$. Then, it can be seen by maximizing the function $\Pi_1(p_S, r_P^l)$ that this local maximum is $(\frac{1+c_S}{2} + \delta \frac{\lambda_P}{2(\lambda_{S\bar{P}} + \lambda_P)}, r_P^l) \in G$. (Observe that this solution is in G because under **(A4)** and **(A5)**, if $\delta > 0$, then $c_S < \frac{1+c_S}{2} < \frac{1+c_S}{2} + \delta \frac{\lambda_P}{2(\lambda_{S\bar{P}} + \lambda_P)} < \frac{1+c_S}{2} + \frac{1-c_S}{2} = 1$. If $\delta \leq 0$, then $\frac{1+c_S}{2} + \delta \frac{\lambda_P}{2(\lambda_{S\bar{P}} + \lambda_P)} \geq \frac{1+c_S}{2} + \delta/2 \geq \frac{1+c_S}{2} - \frac{(1-c_S)}{4} = \frac{1}{4} + \frac{3c_S}{4} > \frac{c_S}{4} + \frac{3c_S}{4} = c_S$, and $\frac{1+c_S}{2} + \delta \frac{\lambda_P}{2(\lambda_{S\bar{P}} + \lambda_P)} < \frac{1+c_S}{2} = 1 - \frac{(1-c_S)}{2} \leq 1 + \delta$.) ■

Appendix M. Comparison of the optimal prices under IC and CT

Proof of Lemma 5.2.2. Recall that $p_S^0 = \frac{1+c_S}{2}$. Then, Theorem 5.2.2 implies that either $p_S^T = p_S^0 + \delta \frac{\lambda_P}{2\lambda_{S\bar{P}} + \lambda_P}$, or $p_S^T = p_S^0 + \frac{\alpha}{1+2\alpha}\delta$, where $\alpha = \frac{\lambda_P}{2\lambda_{S\bar{P}}} \frac{(r_P^u - p_P^T)}{r_P^u - r_P^l} > 0$. This proves (i).

To prove (ii), recall that $p_P^0 = \frac{r_P^u + c_P}{2}$. Then, Theorem 5.2.2 implies that either $p_P^T = r_P^l < p_P^0$, or $p_P^T = p_P^0 - \frac{\beta}{2}$, where $\beta = (\delta + 1 - p_S^T)(p_S^T - c_S) > 0$ (by Theorem 5.2.2). This completes the proof. ■

Appendix N. Monotonicity results on the optimal prices under CT

Proof of Lemma 5.2.3. To prove (i), note that it can be shown, by implicit differentiation (in the light of Theorem 5.2.2), that $\frac{\partial p_S^T(\lambda_{S\bar{P}})}{\partial \lambda_{S\bar{P}}}$ is of the same sign as $\frac{\partial^2 \tilde{\Pi}_1^T(p_S, \lambda_{S\bar{P}})}{\partial \lambda_{S\bar{P}} \partial p_S} \Big|_{p_S=p_S^T}$. We derive

$$\begin{aligned} \frac{\partial \tilde{\Pi}_1^T(p_S)}{\partial p_S} &= -\lambda_{S\bar{P}}(p_S - c_S) + \lambda_{S\bar{P}}(1 - p_S) \\ &\quad + \frac{\lambda_P}{2(r_P^u - r_P^l)} [(\delta + 1 - p_S) - (p_S - c_S)] [(\delta + 1 - p_S)(p_S - c_S) + r_P^u - c_P]. \end{aligned}$$

Then, $\frac{\partial^2 \tilde{\Pi}_1^T(p_S, \lambda_{S\bar{P}})}{\partial \lambda_{S\bar{P}} \partial p_S} \Big|_{p_S=p_S^T} = -(p_S^T - c_S) + (1 - p_S^T) = -2(p_S^T - p_S^0)$. Then the result follows by Lemma 5.2.2.

Similarly, $\frac{\partial p_S^T(\lambda_P)}{\partial \lambda_P}$ is of the same sign as

$$\left. \frac{\partial^2 \tilde{\Pi}_1^T(p_S, \lambda_P)}{\partial \lambda_P \partial p_S} \right|_{p_S=p_S^T} = \frac{1}{2(r_P^u - r_P^l)} \left[(\delta + 1 - p_S^T) - (p_S^T - c_S) \right] \left[(\delta + 1 - p_S^T)(p_S^T - c_S) + r_P^u - c_P \right],$$

where $(\delta + 1 - p_S^T) > 0$ since $p_S^T \in G$, and therefore $\frac{\partial p_S^T(\lambda_P)}{\partial \lambda_P}$ is of the same sign as

$$(\delta + 1 - p_S^T) - (p_S^T - c_S) = -2(p_S^T - p_S^0 - \delta/2). \text{ Observe, from the proof of Lemma 5.2.2, that}$$

$$p_S^T - p_S^0 = \frac{\alpha}{(1+2\alpha)}\delta < \delta/2 \text{ if } \delta > 0 \text{ (and } > \delta/2 \text{ if } \delta < 0). \text{ This proves (ii).}$$

By the same reasoning, $\frac{\partial p_S^T(r_P^l)}{\partial r_P^l}$ is of the same sign as

$$\left. \frac{\partial^2 \tilde{\Pi}_1^T(p_S, r_P^l)}{\partial r_P^l \partial p_S} \right|_{p_S=p_S^T} = \frac{\lambda_P}{2(r_P^u - r_P^l)^2} \left[(\delta + 1 - p_S^T) - (p_S^T - c_S) \right] \left[(\delta + 1 - p_S^T)(p_S^T - c_S) + r_P^u - c_P \right] > 0,$$

if $\delta > 0$ (and < 0 if $\delta < 0$), which proves (iii).

Similarly, $\frac{\partial p_S^T(\delta)}{\partial \delta}$ is of the same sign as

$$\left. \frac{\partial^2 \tilde{\Pi}_1^T(p_S, \delta)}{\partial \delta \partial p_S} \right|_{p_S=p_S^T} = \frac{\lambda_P}{2(r_P^u - r_P^l)} \left\{ (\delta + 1 - p_S^T) - (p_S^T - c_S) + (\delta + 1 - p_S^T)(p_S^T - c_S) + r_P^u - c_P \right\} > 0,$$

if $\delta > 0$, which proves (iv).

Finally, $\frac{\partial p_S^T(c_P)}{\partial c_P}$ is of the same sign as

$$\left. \frac{\partial^2 \tilde{\Pi}_1^T(p_S, c_P)}{\partial c_P \partial p_S} \right|_{p_S=p_S^T} = -\frac{\lambda_P}{2(r_P^u - r_P^l)} \left[(\delta + 1 - p_S^T) - (p_S^T - c_S) \right] < 0,$$

if $\delta > 0$ (and > 0 if $\delta < 0$). ■

Proof of Lemma 5.2.4. By Theorem 5.2.2, we have that $p_P^T = \frac{-(p_S^T - c_S)(\delta + 1 - p_S^T) + r_P^u + c_P}{2}$. Therefore,

p_P^T is decreasing in p_S^T if $p_S^T < (\delta + c_S + 1)/2$ (or equivalently, $p_S^T - p_S^0 < \delta/2$), and otherwise increasing

in p_S^T . Moreover, as shown in Lemma 5.2.3, $p_S^T - p_S^0 < \delta/2$, if $\delta > 0$ (and $< \delta/2$ if $\delta < 0$). The proof then follows from Lemma 5.2.3. ■

Appendix O. Comparison of optimal profits under IC and CT

Proof of Lemma 5.2.5. To prove (i), note that when p_P is fixed, $\Pi^{T*} - \Pi^{0*} = \Pi_S^T(p_S^T, p_P) - \Pi_S^0(p_S^0)$, $\Pi_S^T(p_S^T, p_P)$ is the expected profit from S under CT. Note also that the first-order optimality conditions imply that $p_S^T = p_S^0 + \gamma(p_P)$ where $\gamma(p_P)$ is as defined above (similar to equation (5.6) in Theorem 5.2.2). Then, it can be shown that

$$\begin{aligned} \Pi^{T*} - \Pi^{0*} &= \lambda_{S\bar{P}}[(1 - p_S^0 - \gamma(p_P))(p_S^0 + \gamma(p_P) - c_S) - (1 - p_S^0)(p_S^0 - c_S)] \\ &\quad + \lambda_{Pq_P}(p_P)(\delta + 1 - p_S^0 - \gamma(p_P))(p_S^0 + \gamma(p_P) - c_S) - \lambda_{SP}^0(1 - p_S^0)(p_S^0 - c_S). \end{aligned}$$

Utilizing the identity that $p_S^0 = \frac{1+c_S}{2}$ and simplifying yields the desired result.

Similarly, to prove (ii) note that with p_S fixed,

$$\Pi^{T*} - \Pi^{0*} = \Pi_P(p_P^T) + \Pi_{SP}^T(p_S, p_P^T) - \Pi_P(p_P^0) - \Pi_{SP}^0(p_S),$$

where $\Pi_{SP}^0(p_S) = \lambda_{SP}^0(p_S - c_S)q_S(p_S)$, $p_P^T = p_P^0 - \frac{\beta(p_S)}{2}$ and $\beta(p_S) \equiv (\delta + 1 - p_S)(p_S - c_S)$. Then, it can be shown that

$$\Pi^{T*} - \Pi^{0*} = \lambda_{Pq_P}(p_P^0)\frac{\beta(p_S)}{2} + \lambda_P\frac{\beta(p_S)}{2(r_P^u - r_P^l)}\left(\frac{\beta(p_S)}{2} + p_P^0 - c_P\right) - \lambda_{SP}^0(1 - p_S)(p_S - c_S),$$

and the result follows by utilizing the identity that $p_P^0 = \frac{r_P^u + c_P}{2}$ and simplifying. ■

Appendix P. Comparison of CT and IC under exogenous pricing and finite inventories

Proof of Lemma 5.3.1. CT is more profitable than IC if and only if $\Pi^T - \Pi^0 > 0$, where Π^T and

Π^0 are as given in (5.8) and (5.9). Then

$$\Pi^T - \Pi^0 = (p_S - c_S) \left\{ \mu_{SP}^T - \mu_{SP}^0 - \theta_S \frac{p_S}{p_S - c_S} \left(\sqrt{\mu_{SP}^T} + \sqrt{\mu_{SP}^0} - \sqrt{\mu_{SP}^T + \mu_{SP}^0} \right) \right\}.$$

Next we show that $\Pi^T - \Pi^0 > 0$ only if $\mu_{SP}^T - \mu_{SP}^0 > 0$. By contradiction, assume that $\mu_{SP}^T - \mu_{SP}^0 \leq 0$ and $\Pi^T - \Pi^0 > 0$. Then the fact that $\sqrt{\mu_{SP}^T + \mu_{SP}^0} < \sqrt{\mu_{SP}^T} + \sqrt{\mu_{SP}^0}$ implies that

$$\Pi^T - \Pi^0 < \left(\sqrt{\mu_{SP}^T} - \sqrt{\mu_{SP}^0} \right) \left(\sqrt{\mu_{SP}^T} + \sqrt{\mu_{SP}^0} - \theta_S \frac{p_S}{p_S - c_S} \right).$$

Therefore, $\Pi^T - \Pi^0 > 0$ only if $\sqrt{\mu_{SP}^T} + \sqrt{\mu_{SP}^0} - \theta_S \frac{p_S}{p_S - c_S} < 0$. However, $\sqrt{\mu_{SP}^T} > \theta_S \frac{p_S}{p_S - c_S}$ from

(A6). This completes the contradiction proof. Note finally that when $\mu_{SP}^T - \mu_{SP}^0 > 0$,

$$\sqrt{\mu_{SP}^T} + \sqrt{\mu_{SP}^0} - \sqrt{\mu_{SP}^T + \mu_{SP}^0} > 0. \blacksquare$$

Vita

Bacel Maddah was born in Miness, South Lebanon on December 7, 1976. He obtained a Bachelor degree in Civil Engineering from the Lebanese University and a Masters degree in Engineering Management from the American University of Beirut. He moved from Lebanon to the U.S. in August 2000 and joined Virginia Tech in August 2001. Bacel research interests are in retail operations management, queueing theory, and inventory control. He has authored several research articles on these topics. Bacel has also been active in developing software tools for both practical and academic purposes, and has had several internships with the industry. He will work as a senior analyst with United Airlines upon graduating from Virginia Tech. His long-term goals are to continue research work and to someday go back to Lebanon and contribute to ongoing efforts of social reform.