Evacuation Distributed Feedback Control and Abstraction

SABIHA A. WADOO

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Pushkin Kachroo
William T. Baumann
Daniel J. Stilwell
Ahmad Safaai-Jazi
Mahendra P. Singh

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(ABSTRACT)

In this dissertation, we develop feedback control strategies that can be used for evacuating people. Pedestrian models are based on macroscopic or microscopic behavior. We use the macroscopic modeling approach, where pedestrians are treated in an aggregate way and detailed interactions are overlooked. The models representing evacuation dynamics are based on the laws of conservation of mass and momentum and are described by nonlinear hyperbolic partial differential equations. As such the system is distributed in nature.

We address the design of feedback control for these models in a distributed setting where the problem of control and stability is formulated directly in the framework of partial differential equations. The control goal is to design feedback controllers to control the movement of people during evacuation and avoid jams and shocks. We design the feedback controllers for both diffusion and advection where the density of people diffuses as well as moves in a specified direction with time. In order to achieve this goal we are assuming that the control variables have no bounds. However, it is practically impossible to have unbounded controls so we modify the controllers in order to take the effect of control saturation into account. We also discuss the feedback control for these models in presence of uncertainties where the goal is to design controllers to minimize the effect of uncertainties on the movement of people during evacuation. The control design technique adopted in all these cases is feedback linearization which includes *backstepping* for higher order two-equation models, Lyapunov redesign for uncertain models and *robust backstepping* for two-equation uncertain models.

The work also focuses on the abstraction of an evacuation system for developing models with lesser number of partial differential equations than the original one. The feedback control design of a higher level two-equation model is more difficult than the lower order one-equation model. Therefore, it is desirable to perform control design for a simpler abstracted model and then transform control design back to the original model.

To my husband, Saajed

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CHAPTER 1

INTRODUCTION

In this dissertation, we are interested in developing feedback control strategies for models that can be used for evacuation of people. We want to design feedback controllers to control the movement of people during evacuation. In recent years, there has been an increasing interest in modeling crowd and evacuation dynamics. The development of pedestrian dynamic models to implement evacuation system strategies is an ongoing research area. Since the early 1990s a strong interest in this topic has shown the importance of this issue [1]. Nevertheless, as stated in [2], our knowledge of the flow of crowds is inadequate and behind that of other transportation modes. Studies like the ones in [3] and [4] are concerned with evacuation strategies for regional areas like cities and states. They are important in the decision making of an emergency evacuation such as in the case of a natural disaster. In smaller areas like airports, stadiums, theaters, buildings and ships an evacuation system is an important element in design safety [5]. A good evacuation system in the case of an emergency can prevent a catastrophic outcome as shown by [6].

Pedestrian models are based on macroscopic or microscopic behavior. We use the macroscopic modeling approach, where pedestrians are treated in an aggregate way and detailed interactions are overlooked. The models representing evacuation dynamics are based on the laws of conservation of mass and momentum and are described by nonlinear hyperbolic partial differential equations, as such the system is distributed in nature. In the analytical design of feedback controllers for distributed parameter systems, two approaches are adopted.

In the conventional approach the distributed mathematical model is approximated by a lumped parameter model having finite dimensions, using either finite difference or finite element methods. The controllers are then designed on the basis of resulting linear or nonlinear ordinary differential equation model using known techniques available for such systems. This approach however has certain disadvantages. By neglecting the infinite dimensional nature of the original system, design of controllers may result in instability even though the resulting finite dimensional system is stable using the same controllers. Thus in order to avoid errors introduced by spatial discretization it is desirable to formulate the control and stability problem directly in the framework of a distributed model of partial differential equations. Generally speaking these problems are very difficult. The design of feedback controllers in a distributed framework, although desirable is much harder than in the ordinary differential equation framework. In our work we address the design of distributed feedback control for two different models for an evacuation system. One model is described by only one partial differential equation while the other is described by two. The problem of control and stability is formulated directly in the framework of partial differential equations. We design a preliminary control to evacuate pedestrians based on feedback linearization applied directly on the PDE equation.

The first step in this study is to start investigating potential mathematical models governing crowd dynamics. Two factors need to be considered in deriving the models: the accurate representation of the pedestrian flow in an emergency situation (evacuation), and the complexity of the selected model. It has been suggested in [7] that pedestrian traffic flow can be treated similarly to vehicle traffic flow where we can divide the problem into two categories: the microscopic level and the macroscopic level. The former involves individual units with characteristics such as individual speed and individual interaction. Microscopic pedestrian analysis studies were presented by [8] and followed by many researchers to improve on the Car-Following model. In [9] the importance of a detailed design and pedestrian interactions is shown by implementing several case studies. On the other hand, the drawback to mathematical microscopic models is their difficult and expensive simulation.

To introduce feedback control as a strategy in evacuation plans where the objective is to control each pedestrian in the model area is difficult and not practical. The obvious reasons for the impracticality are the detailed design and pedestrian interactions which make it very difficult to control such models. Instead, the macroscopic modeling approach is well suited for understanding the rules governing the overall behavior of pedestrian flow for which individual differences are not that important. Macroscopic models aimed at studying pedestrian behavior use a continuum approach, where the movement of large crowds exhibits many of the attributes of fluid motion. As a result, pedestrian dynamics are treated as a fluid. This idea provides flexibility since detailed interactions are overlooked, and the model's characteristics are shifted toward parameters such as flow rate q , concentration ρ (also known as traffic density), and average speed v, all being functions of space and time. This class of models is classified under hyperbolic partial differential equations. This way of modeling the pedestrian behavior was first introduced by [10] and adopted by [11], where macroscopic models were developed using (a) fluid flow theory, (b) a continuum responding to influences (local or non-local). The drawback to this type of modeling is the assumption that pedestrians behave similarly to fluids. Pedestrians tend to interact among themselves and with obstacles in their model area, which is not captured by macroscopic models. In this work we first develop two models based on continuum theory and conservation laws such of continuity and momentum. The first model uses a single hyperbolic partial differential equation with a velocity-density relationship, while the other is a system of hyperbolic partial differential equations. The models are nonlinear and time-varying hyperbolic partial differential equations.

These models will be utilized for the feedback control design in the following chapters. The goal there is to design feedback controllers in order to control the movement of people during evacuation as dictated by these models. Since the models are hyperbolic PDEs movement control becomes a tough task. The hyperbolic PDEs are harder to analyze, simulate and control since there are shocks and jams associated with these. In the following chapters we will adopt the method of feedback linearization to convert these models into much simpler parabolic PDEs. We will design both diffusion and advection controllers where we make the density profiles of people decrease as well as move with time. In order to achieve this goal we are assuming that the control variables have no bounds on them. However, it is practically impossible to have unbounded controls so we modify the controllers in order to take the effect of control saturation into account. We also discuss the feedback control for these models in presence of uncertainties where the goal is to design the controllers to minimize the effect of uncertainties on the movement of people during evacuation. The control design technique adopted in all these cases is feedback linearization which includes *backstepping* for two-equation models, Lyapunov redesign for uncertain models and *robust backsteeping* for two-equation uncertain models.

The models representing evacuation dynamics are based on the laws of conservation of mass and momentum and are described by nonlinear partial differential equations. In order to accurately model evacuation dynamics it is desirable to have a more detailed model with a larger number of partial differential equations. The more the number of partial differential equations, more accurate is the model. However, this increase in the number of PDEs in the system increases the difficulty in control design. This motivates the need for doing an abstraction as it is desirable to perform control design for a simpler abstracted model and then transform control design back to the original model. The abstraction is done by abstracting certain number of partial differential equations from the original model in such a manner that certain properties of the original system are preserved. For our work we discuss abstraction of a two equation model to a one equation model that can be used for control design transfer. For the abstraction of infinite dimensional systems we need a framework which we first develop for finite dimensional case.

The second part of this dissertation therefore discusses the abstraction of finite and infinite dimensional systems. To reduce complexity of system analysis, simplified models that capture the behavior of interest in the original system can be obtained. These simplified models, called abstractions, can be analyzed more easily than the original complex model [12]. This hierarchical structure allows the systems located at various levels to operate without regard to the details of the other levels. Each layer of the hierarchy uses a less detailed model of the system than the lower levels and at each level the system functions according to its own model and control objectives. This structure clearly reduces complexity since every level of the hierarchy is required to know only the necessary information in order to perform its function successfully. This may also increase efficiency and performance of the overall system since each level of hierarchy has the minimal complexity. It is desirable that properties of interest, such as controllability, are propagated between levels. In the analysis of complex systems, abstraction plays an important role. Again, the abstracted system should reflect the qualitative behavior of the complex system. If the abstracted model truly captures the properties of interest from the complex system, then analyzing those properties in the simpler abstraction provides the same information as that of the original system. Such abstractions are called consistent abstractions. Abstraction can be categorized as discrete or continuous. Hierarchical abstractions for discrete event systems have been formally considered by the control community in [13] and [14]. Discrete abstractions for continuous systems have been studies in [15] and [16]. Continuous abstractions for continuous systems were considered in [17], where characterizations were obtained for constructing reachability preserving abstractions of linear control systems. In this work our focus is to discuss continuous abstractions.

Hierarchies of consistent abstractions can significantly reduce the complexity in determining the reachability properties of nonlinear systems. Such consistent hierarchies of reachability-preserving nonlinear abstractions are considered for finite as well as infinite dimensional systems. Not only can these abstractions be analyzed with respect to some behavior of interest, they can also be used to transfer control design for the complex model to the simplified model. This work is an initial step towards this study. Here, we study the abstractions of both finite and infinite dimensional cases.

In case of finite dimensions, abstraction would mean to obtain models with lesser dimensions than the original one that would preserve the property of interest. For our work we have considered abstraction of a robotic car as an application. The abstraction of a robotic car to a rolling disk that preserves controllability properties, in particular local accessibility is reviewed. In this framework, showing the local accessibility of the abstracted rolling disk is equivalent to showing local accessibility of the robotic car. Then, working towards the study of control design, it is seen that there are certain classes of trajectories that exist in the rolling disk system that cannot be achieved by the robotic car. In order to account for these cases, new concepts of *traceability*, *ε-traceability*, and *ε-consistency* are defined. With these notions, a framework has been provided that gives conditions for the existence of trajectories in the original system and allows controllability propagation between the two systems.

In case of infinite dimensional systems abstraction would mean to obtain models with lesser number of partial differential equations than the original one. In case of infinite cases, our application is the evacuation control system. We extend the framework for finite dimensional case to evacuation system and discuss its abstraction. As already stated it is desirable to abstract certain number of partial differential equations from the original model. This abstraction is performed in such a manner that certain properties of the original system are preserved and we can transform the control design from simpler to original model.

1.1 Objectives and Contributions

The contributions presented in this dissertation are:

- Modeling of evacuation dynamics in one and two dimensional spaces, presented in chapter 2
- Design of distributed feedback controllers in both original and abstract setting for evacuation problem in both one and two dimensions, which forms the discussion in chapter 3 and chapter 4
- Modification of feedback controllers of chapters 3 and 4 by adding motion to them and design of advection feedback controllers in both one and two dimensions, which forms the discussion in chapters 5 and 6 respectively
- Design of robust distributed feedback controllers for evacuation problem in both one and two dimensions, which is discussed in chapter 7

- Development of a framework for abstraction of evacuation system. The framework is first presented for a finite dimensional case which forms the material covered in chapter 8, the framework is then extended to evacuation system in chapter 9
- Study of abstraction first for finite dimensional systems with robotic car as an example and then for evacuation system, this part of research is covered in chapter 9

1.2 Organization

The organization of this dissertation is as follows:

Chapter 2: In this chapter we present the macroscopic models for both one-dimensional and two-dimensional evacuation systems. The models presented here are based on the laws of conservation of mass and momentum. The equations of motion in both cases are described by nonlinear hyperbolic partial differential equations. The first model is the classical one-equation model for a traffic flow based on conservation of mass with a prescribed relationship between density and velocity. In this model dynamics are represented by means of a single partial differential equation. The other model is a two-equation model in which the velocity is independent of the density. This model is based on conservation of mass and momentum. As such the dynamics are represented by a set of two partial differential equations.

Chapter 3: In this chapter the models developed in chapter 2 will be utilized for the feedback control design in one-dimension. The goal here is to design feedback controllers in order to control the movement of people during evacuation as dictated by these models. Since the models are hyperbolic PDEs movement control becomes a tough task because of shocks and jams. In this chapter we will adopt the method of feedback linearization to convert these models into much simpler parabolic PDEs. We will design diffusion controllers where we make the density profiles of people decrease with time. The control design technique adopted here is feedback linearization which includes *backstepping* for two-equation model. The problem of control and stability is formulated directly in the framework of partial differential equations. Sufficient conditions for Lyapunov stability for distributed control are derived.

Chapter 4: This chapter presents design of nonlinear feedback controllers for two different models representing evacuation dynamics in two-dimensions. The discussion in this chapter is similar to that of chapter 3 with the difference that evacuation dynamics are two dimensional. Here the dynamics are represented by means of a set of three partial differential equations.

Chapter 5: This chapter presents design of advection nonlinear feedback controllers for two different models representing evacuation dynamics in one-dimensions. The control design is done in such a way that now in our models not only do have diffusion (density decreases with time) but motion also where the density profile changes with time. In order to achieve this goal we are assuming that the control variables have no bounds on them. However, it is practically impossible to have unbounded controls so we modify the controllers in order to take the effect of control saturation into account.

Chapter 6: This chapter presents design of advection nonlinear feedback controllers for two different models representing evacuation dynamics in two-dimensions.

Chapter 7: This chapter presents design of robust nonlinear feedback controllers for the models representing evacuation dynamics in one-dimensions. We discuss the feedback control for these models in presence of uncertainties where the goal is to design the controllers to minimize the effect of uncertainties on the movement of people during evacuation. The robust controllers are designed in distributed setting using Lyapunov redesign and *robust backstepping* methods for PDEs.

Chapter 8: This chapter presents the mathematical background needed for finite dimensional abstractions which can be later on extended for infinite dimensional systems. Some basic differential geometric concepts that are used in later chapters are introduced in this chapter.

Chapter 9: This chapter discusses the abstraction of finite dimensional cases with robotic car as an application. Some important results on abstracted linear control systems from [17] are presented. Not only can the abstractions be analyzed with respect to some behavior of interest, but the transformation of the control design from the simplified model to the complex model can also be studied. This chapter presents an initial step towards that study. The chapter also discusses the abstraction for an evacuation system. The goal here is to abstract the system having higher number of partial differential equations by a system which is represented by lower number of partial differential equations. The abstraction is constructed in such a way that it enables to preserve controllability of the system. For that purpose the notions of consistent abstractions introduced in [17] need to be modified for infinite dimensional systems. We extend the framework presented for finite dimensional abstractions to evacuation problem.

Chapter 10: We summarize the main results of this work and present some future directions. We would like to study more detailed models for evacuation dynamics and address other types of distributed feedback control design problems. Another main objective of the future research is to study the abstraction of evacuation system. As future work we seek to study the transformation of feedback control from abstracted system to original system for an evacuation system and other infinite dimensional problems.

CHAPTER 2
MODELING OF ONE DIMENSIONAL EVACUATION SYSTEM

2.1 Introduction

The goal of this dissertation is on developing feedback control strategies for models that can be used for evacuation control. The first step towards this goal is to obtain mathematical models governing crowd dynamics. The development of pedestrian evacuation dynamic systems follows from the traffic flow theory in one-dimensional space [18],[19] and [20]. In many ways, the pedestrian evacuation system is similar to the vehicle traffic flow problem [7]. The main conservation equations used in modeling the vehicle traffic flow and the pedestrian evacuation flow are the same with the exception that vehicle traffic is a one-dimensional space problem and the evacuation system is a two-dimensional space problem.

It has been suggested in [7] that pedestrian traffic flow can be treated similarly to vehicle traffic flow and there are two main approaches to modeling pedestrian dynamics. One approach is microscopic [21] where each individual is taken into consideration as a unit and his behavior is expressed by characteristics such as individual speed and individual interaction. Microscopic pedestrian analysis studies are presented in [4], [5] and followed by many researchers. In [6], the importance of a detailed design and pedestrian interactions is shown by implementing several case studies. However, the drawback to mathematical microscopic models is their difficult and expensive simulation since each unit has an ODE to be solved at each time step.

The other approach is macroscopic [22]. Here the overall behavior of the flow of people is considered. The area is treated as a series of sections within each of which the density and average velocity of people can be measured for a given time. The changes in these variables may then be described using partial differential equations. The macroscopic models are computationally easier because they have fewer design details in terms of interaction among people and between people and their environment. While deriving the models for an evacuation system two factors need to be considered: the accurate representation of the pedestrian flow during evacuation and the complexity of the selected model. In order to introduce feedback control strategies in evacuation plans microscopic modeling approach is difficult and not practical. The obvious reasons for the impracticality are the detailed design and individual pedestrian interactions which makes control of each pedestrian very difficult. Therefore, the macroscopic modeling approach is well suited for understanding the rules governing the overall behavior of pedestrian flow. For this work we use macroscopic modeling approach, where pedestrians are treated in an aggregate way and detailed interactions are overlooked.

Macroscopic models aimed at studying pedestrian behavior use a continuum approach, where the movement of large crowds exhibits many of the attributes of fluid motion. As a result, pedestrian dynamics are treated as a fluid. This idea provides flexibility since detailed interactions are overlooked, and the model's characteristics are shifted toward parameters such as flow rate q, traffic density ρ and average speed v, all of which are functions of space and time. This class of models is classified under hyperbolic partial differential equations. This way of modeling the pedestrian behavior was first introduced by [10] and [11] where macroscopic models were developed using fluid flow theory. The drawback is the assumption that pedestrians behave similarly to fluids. However, pedestrians tend to interact among themselves and with obstacles in their model area, which is not captured by macroscopic models.

In this chapter we discuss models for a one-dimensional evacuation system based on traffic flow theory and the existing macroscopic mathematical models. These models will be modified for crowd flow in two-dimensional space in section 2.3. The resulting models for each case (one and two dimensions) are divided into two types which are based on continuum theory and conservation laws such as the continuity and momentum equations. These models start from conventional theory for ordinary fluids and the equations of motion in both cases are described by nonlinear partial differential equations. The first model uses a single hyperbolic partial differential equation with a velocity-density relationship, while the other is a system of two hyperbolic partial differential equations. The first we call the one-equation model and the other we refer to as the two-equation model.

2.2 Modeling of One-Dimensional Evacuation System

In this section we will present two one-dimensional models which describe the behavior of people using a single partial differential equation in first case and two partial differential equations in the other. The first one is a basic one-equation model which is based on the equation of continuity or conservation of mass. This model also has a fundamental relationship between density of people and speed of flow. According to the law of conservation of mass total flow of people exiting from any section cannot be higher than the total flow of the people that are entering which means that the "total number of people is conserved in the system". The number of people moving in and out accounts for the change in density in that area. To represent the flows, Greenshields model [23] is used to show the dependence of speed on the density of people.

The second model is a two-equation model based on conservation of mass and momentum where the velocity is independent of density. To increase resolution and accuracy of the model we add the equation of conservation of momentum. We will discuss the evacuation model of a one dimensional corridor of length L. The model is similar to the one dimensional traffic flow model and is described by a nonlinear hyperbolic partial differential equation. The models presented here are macroscopic with the dynamics being represented in terms of density, flow and speed. As a result the system is distributed with all the parameters as functions of space and time.

2.2.1 Continuity One-Equation Model

This model is based on equation of conservation of mass. The conservation law of mass in case of an evacuation system means that the number of people is conserved in the system. The derivation of the conservation law is given in [24]. Let us consider the case of a one dimensional corridor of length L with exits at both ends. Let $\rho(x,t)$ denote the density of people as a function of position vector x and time t, $q(x,t)$ the flow at a given x and t, and $v(x,t)$ the velocity vector field associated with the flow. The one-dimensional conservation law of mass is given by

$$\frac{\partial\rho(x,t)}{\partial t}+\frac{\partial(q(x,t))}{\partial x}=0 \tag{2.1}$$

with initial condition and a boundary condition given by

$$\rho(x,t_0)=\rho_0(x) \tag{2.2}$$

$$\rho(0,t)=0 \text{ and } \rho(L,t)=0 \quad \forall \quad t\in[0,\infty) \tag{2.3}$$

Here $\rho(x,t)\in H^2[(0,L),\Re]$ with $H^2[(0,L),\Re]$ being the infinite dimensional Hilbert space of one dimensional vector function defined for an interval $[0,L]$ whose spatial derivatives upto second order are square integrable with a specified L_2 norm. $q(x,t)\in H^2[(0,L),\Re]$ and $\rho_0(x)\in H[(0,L),\Re]$. The vectors $x\in[0,L]\subset\Re$ and $t\in[0,\infty)$ denote position and time respectively. For the rest of the chapter it will be assumed that the vector spaces are Sobolev spaces or Banach spaces [25].

The relationship between the three variables: density, velocity and flow is given by the following equation

$$q(x,t)=\rho(x,t)v(x,t) \tag{2.4}$$

The one-dimensional conservation equation is therefore given by

$$\frac{\partial\rho(x,t)}{\partial t}+\frac{\partial(\rho(x,t)v(x,t))}{\partial x}=0 \tag{2.5}$$

subject to the conditions given by (2.2) and (2.3). For simplicity we will omit the arguments of the functions for the rest of this chapter.

2.2.2 Velocity-Density Relationship

To describe the relationship between velocity vector field v and density ρ we need one more equation. Crowd density and velocity are related through the conservation law (2.5). The velocity function for the pedestrian flow is dependent on density. The choice of such function depends on the behavior the model is trying to mimic. There are a number of representations for this velocity function used throughout the literature some of which have been discussed in [26]. In our case we use the Greenshield's model.

Greenshield's Model

The Greenshield model [23] is one of the simplest and widely used models for velocity-density relationship. This model assumes velocity as a linearly decreasing function of the flow density, and is given by

$$v = v_f (1 - \rho / \rho_{\max}) \tag{2.6}$$

where $v_f = v_f(x,t)$ is the free flow speed and $\rho_{\max}$ is the maximum or jam density. Jam density is the maximum number of people that could possibly fit a single cell. From (2.6) it is clear that for zero density the model allows the people to move with free flow velocity and for jam density there is no flow at all. Using (2.6) we get modified one-equation conservation model as

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho v_f (1 - \rho / \rho_{\max})\right) = 0 \tag{2.7}$$

subject to conditions (2.1) and (2.2). The model given by (2.7) is also known as LWR model named after the authors in Lighthill, Whitham and Richards in [22]. The LWR model is a scalar, time-varying, non-linear, hyperbolic partial differential equation. This is a simple model and is thus unable to capture all the complex interactions for a realistic crowd flow. In order to make this model more accurate we need some modifications. One such modification is discussed in the next section.

2.2.3 Two-Equation System Model

The model presented earlier is one of the original models representing the traffic flow dynamics. Since its appearance, a number of other models have emerged in literature where velocity is independent of density. One such model is being presented here. In this section we consider a higher order model or more precisely a system model of two partial differential equations for a one dimensional corridor. This model consists of conservation of mass equation (2.1) coupled with a second equation based on the principle of conservation of momentum. The first equation is the conservation of mass equation (2.5). The second equation is derived from conservation of momentum for one dimensional flow [27]. This equation tries to mimic the flow motion instead of the velocity-density models and is given by

$$\frac{\partial(\rho v)}{\partial t}+\frac{\partial(\rho v^2)}{\partial x}=-\frac{\partial p}{\partial x} \tag{2.8}$$

where $p=p(\rho,t)\in H^1[(0,L),\Re]$ is pressure and is a function of density ρ . The second equation is derived from the Navier-Stokes equation of motion of a one-dimensional compressible flow with the pressure term as a function of density given as $p=C\rho^2$. The constant C is known as the anticipation factor and describes the response of macroscopic driver to crowd density, i.e. space concentration. The relationship between density and flow is again given by (2.4). Thus we have the dynamics of an evacuation system given by following two-equation system model

$$\frac{\partial \rho}{\partial t}+\frac{\partial q}{\partial x}=0 \tag{2.9}$$

$$\frac{\partial q}{\partial t}+\frac{\partial}{\partial x}(q^2/\rho)=-\frac{\partial p}{\partial x} \tag{2.10}$$

with initial conditions and subject to boundary conditions

$$\rho(x,t_0)=\rho_0(x), \quad q(x,t_0)=q_0(x)$$

$$\rho(0,t)=0 \quad q(0,t)=0 \quad \forall \quad t\in[0,\infty) \tag{2.11}$$

2.3 Modeling of Two Dimensional Evacuation System

In this section we present two macroscopic crowd dynamic models that can be used to study crowd behavior in a two-dimensional space. The first is a one-equation model similar to the one given in subsection 2.1.1. The other is the two-equation model similar to the one given in subsection 2.1.3.

In this section we will discuss the evacuation model of a two dimensional space of dimensions $L \times L$ with exits at both ends. There are two partial differential equations that we use to model the control problem. The first is the equation of conservation of mass and the second is the conservation of momentum.

2.3.1 Continuity One-Equation Crowd Model

In this section, we present a 2-D scalar, nonlinear, time-varying, hyperbolic PDE crowd model. This is a simple extension to the LWR one-dimension conservation of continuity model given in subsection 2.2.1. This model is based on equation of conservation of mass, i.e. the number of people is conserved in the system. Let us consider the case of a two- dimensional corridor of dimension $L \times L$. Let $\Omega = (0, L) \times (0, L)$ be a bounded, open subset of a two-dimensional Euclidean space R^2 and $\partial\Omega$ be its boundary. Let $\rho(x,t)$ denote the density of people as a function of position vector x and time t. The vector $x \in \Omega \subset \Re^2$ is expressed in terms of its coordinates as $x = [x_1, x_2]^T$. Let $q(x,t)$ be the flow at a given x and t with $q_1(x,t)$ and $q_2(x,t)$ as flow in x_1 and x_2 directions. $v(x,t)$ is the velocity vector field associated with the flow with $v_1(x,t)$ and $v_2(x,t)$ as x_1 and x_2 components respectively. We use the fundamental velocity-density relation given by (2.6) to describe $v_1(x,t)$ and $v_2(x,t)$. The conservation one-equation model is given by

$$\frac{\partial \rho(x,t)}{\partial t} + div(q(x,t)) = 0 \qquad \forall x \in \Omega \tag{2.12}$$

with initial condition and a boundary condition

$$\rho(x,t_0) = \rho_0(x) \tag{2.13}$$

$$\rho(x,t) = 0 \quad \forall \ t \in [0,\infty), x \in \partial\Omega \tag{2.14}$$

Here $\rho(x,t) \in H^2[\Omega, \Re]$ with $H^2[\Omega, \Re]$ being the infinite dimensional Hilbert space of one dimensional vector function defined on domain Ω, whose spatial derivatives upto second order are square integrable with a specified L_2 norm $q(x,t) \in H^2[\Omega, \Re]$ and $\rho_0(x) \in H[\Omega, \Re]$ .The vectors $x \in \Omega \subset \Re^2$ and $t \in [0, \infty)$ denote position and time respectively. For the rest of the chapter it will be assumed that the vector spaces are Sobolev spaces (Banach spaces) [25]. For simplicity again we drop the arguments. The flows q_i's are obtained as a product of density and velocity as

$$q_i = \rho v_i \ , \ i = 1,2 \tag{2.15}$$

The dynamics for two dimensions are therefore given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_1)}{\partial x_1} + \frac{\partial(\rho v_2)}{\partial x_2} = 0 \tag{2.16}$$

subject to the initial conditions and boundary conditions given by (2.13) and (2.14) respectively. To describe the relationship between velocity vector field $v(x,t)$ and density $\rho(x,t)$ we use the fundamental velocity-density relation of (2.6) to give $v_1(x,t)$ and $v_2(x,t)$ as

$$v_i = v_{if}(1 - \rho/\rho_{\max}) \tag{2.17}$$

where $v_{if}(x,t)$ is the free flow speed in x_i direction, $i = 1,2$ and $\rho_{\max}$ is the jam density. Using (2.17) in (2.16) we get the modified one-equation model given by

$$\frac{\partial \rho(x,t)}{\partial t} + \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(\rho v_{if}(1 - \rho/\rho_{\max})\right) = 0 \tag{2.18}$$

subject to conditions (2.12) and (2.13). The free flow speeds $v_{1f}(x,t)$ and $v_{2f}(x,t)$ are the velocities in the x-axis and y-axis, and they are used as control parameters to direct crowd flow to any desired direction and are both a function of time and space.

2.3.2 Two-Equation System Crowd Dynamic Model

In this section we consider a higher order model or more precisely a system of two partial differential equations for a two dimensional corridor. This model consists of conservation of mass equation coupled with a second set of equations based on the principle of conservation of momentum. The first equation is the conservation of mass equation (2.16) for two-dimensions

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho v_1)}{\partial x_1}+\frac{\partial(\rho v_2)}{\partial x_2}=0$$

The second equation is derived from conservation of momentum by taking a 2-D form of equation (2.8) and is found to be

$$\frac{\partial(\rho v_i)}{\partial t}+div(\rho v_i v)=-\frac{\partial p}{\partial x_i}, \quad i=1,2$$

The x_1 and x_2 components of momentum equation can be written as

$$\frac{\partial(\rho v_1)}{\partial t}+\frac{\partial(\rho v_1{}^2)}{\partial x_1}+\frac{\partial(\rho v_1 v_2)}{\partial x_2}=-\frac{\partial p}{\partial x_1} \tag{2.19}$$

$$\frac{\partial(\rho v_2)}{\partial t}+\frac{\partial(\rho v_1 v_2)}{\partial x_1}+\frac{\partial(\rho v_2{}^2)}{\partial x_2}=-\frac{\partial p}{\partial x_2} \tag{2.20}$$

where $p=p(x,t)\in H^1[\Omega,\Re]$ is pressure. The pressure term is the same as in 1-D case and is equal to $p=C_0^2\rho$ with C_0 being the anticipation term. The relationship between density and flow is given by $q_i=\rho v_i$. Thus we have the dynamics of an evacuation system given by following two-equation system model

$$\frac{\partial \rho}{\partial t}+\sum_{i=1}^{2}\frac{\partial q_i}{\partial x_i}=0 \tag{2.21}$$

and

$$\frac{\partial q_1}{\partial t}+\frac{\partial}{\partial x_1}(q_1{}^2/\rho)+\frac{\partial}{\partial x_2}(q_1 q_2/\rho)=-\frac{\partial p}{\partial x_1} \tag{2.22}$$

$$\frac{\partial q_2}{\partial t}+\frac{\partial}{\partial x_1}(q_1 q_2/\rho)+\frac{\partial}{\partial x_2}(q_2{}^2/\rho)=-\frac{\partial p}{\partial x_2} \tag{2.23}$$

with initial conditions and subject to boundary conditions given by

$$\rho(x,t_0)=\rho_0(x), \quad q(x,t_0)=q_0(x) \qquad \forall\ x\in\Omega \tag{2.24}$$

$$\rho(x,t)=0,\ q(x,t)=0,\ \forall\ \ t\in[0,\infty),\ x\in\partial\Omega \tag{2.25}$$

This concludes the modeling section. In this chapter we presented two macroscopic models for both one and two dimensional spaces. The models are macroscopic and are represented by nonlinear hyperbolic PDEs. We divided the models for each case into two types: one is based on continuum theory and is referred to as one-equation model and the other is based on conservation laws of continuity and momentum and will be referred to as two-equation model. These models will be utilized for the feedback control design in following chapters. The goal there is to design feedback controllers in order to control the movement of people during evacuation as dictated by these models. Since the models are hyperbolic PDEs movement control becomes a tough task because of shocks and jams. In the following chapters we will adopt the method of feedback linearization to convert these models into much simpler parabolic PDEs. We will design both diffusion and advection controllers where we make the density profiles of people decrease as well as move with time. In order to achieve this goal we are assuming that the control variables have no bounds on them. However, it is practically impossible to have unbounded controls so we modify the controllers in order to take the effect of control saturation into account. We also discuss the feedback control for these models in presence of uncertainties where the goal is to design the controllers to minimize the effect of uncertainties on the movement of people during evacuation. The control design technique adopted in all these cases is feedback linearization which includes *backstepping* for two-equation models, Lyapunov redesign for uncertain models and *robust backsteeping* for two-equation uncertain models.

CHAPTER 3

FEEDBACK CONTROL DESIGN AND STABILITY ANALYSIS OF ONE DIMENSIONAL CROWD MODELS

3.1 Introduction

This chapter presents design of nonlinear feedback controllers for two models that are given in chapter 2, representing crowd dynamics in one dimension. The models are based on the laws of conservation of mass and momentum. The first model is the classical one-equation model for a traffic flow based on conservation of mass with a prescribed relationship between density and velocity given in subsection 2.2.1. The model dynamics are represented by a single partial differential equation. The other model is a two equation model given in subsection 2.2.3 in which the velocity is independent of the density. This model is based on conservation of mass and momentum. As such, the model dynamics are represented by means of a system of two partial differential equations. The equations of motion in both cases are described by nonlinear partial differential equations. The system is distributed, i.e. both the state and control variables are distributed in time and space.

The control objectives are to design feedback controllers for removing people from the evacuation area effectively by generating distributed control commands. We address the feedback control problem for both models. The objective is to synthesize a nonlinear distributed feedback controller that guarantees stability of a closed loop system. The problem of control and stability is formulated directly in the framework of partial differential equations. Sufficient conditions for Lyapunov stability for distributed control are derived.

There are two approaches to the design of feedback controllers for distributed systems. In the conventional approach, the distributed mathematical model is approximated by a lumped parameter model having finite dimensions. The spatial discretization of the system is performed using either the finite difference or the finite element method. The controllers are designed on the basis of the resulting linear or nonlinear ordinary differential equation model using known techniques available for such systems [28], [29]. This approach however has certain disadvantages. By neglecting the infinite dimensional nature of original system, design of controllers may result in instability even though the resulting finite dimensional system is stable using the same controllers. Moreover, properties like controllability and observability depend on the method of discretization used [30]. Thus, in order to avoid errors introduced by spatial discretization, it is desirable to formulate the control and stability problem directly in the framework of a distributed model of partial differential equations [25]. In this chapter the latter approach is used.

Here we are interested in designing a feedback controller for evacuating pedestrians from a one-dimensional area (e.g., corridor). We adopt the method of feedback linearization for control design. The method works by canceling the nonlinearities in the system and is well known in nonlinear control for ODEs [31]. This method is introduced to quasi-linear hyperbolic PDEs in [25]. This chapter presents the feedback linearization control design for the LWR model. First we discuss the design of nonlinear feedback control for the model based on continuity equation alone and next we will discuss the feedback control design for the system described by both the equations where *backstepping* approach is used for the control design. In both cases the objective of control design is to synthesize a nonlinear distributed feedback controller that stabilizes the system and guarantees stability in the closed loop system.

The organization of this chapter is as follows. In Section 3.2 we formulate the control model and present feedback control design for the one-equation model. This section also studies Lyapunov stability and finally presents some simulation results. Section 3.3 presents the feedback control design and stability analysis for the two-equation system Simulation results for closed loop dynamics are also presented.

3.2 Feedback Control for Continuity One-Equation Model

In this section we formulate the control model and present feedback control design for the one-equation model. This section also studies Lyapunov stability and presents some simulation results.

3.2.1 Continuity One-Equation Model

In this section we present the mathematical model of crowd dynamics for a one-dimensional area. The one dimensional area can be a corridor of length L with exits at both ends. The model is given in subsection 2.2.1 by (2.7) as

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial (q(x,t))}{\partial x} = 0 \tag{3.1}$$

with initial and a boundary condition given by

$$\rho(x,t_0) = \rho_0(x) \quad \forall \; x \in [0,L] \tag{3.2}$$

$$\rho(0,t) = 0 \quad \forall \; t \in [0,\infty) \tag{3.3}$$

Here $\rho(x,t)$ is the variable we want to control. For pedestrian evacuation the final density should be equal to zero; i.e. $\rho(x,t_f) = 0$. The length of the corridor is $[0,L]$ and the boundary condition given by (3.3) is that the corridor is open at both ends. The flow $q(x,t)$ is obtained as a product of density and velocity as

$$q(x,t) = \rho(x,t)v(x,t) \tag{3.4}$$

The velocity-density relationship as given by Greenshield's model is

$$v = v_f (1 - \rho/\rho_{\max}) \tag{3.5}$$

where $v_f = v_f(x,t)$ is the free flow speed and $\rho_{\max}$ is the maximum or jam density. The one-equation model is therefore given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho v_f (1 - \rho/\rho_{\max})\right) = 0 \tag{3.6}$$

3.2.2 Continuity One- Equation Control Model

To formulate the control problem based on the continuity equation model we need to choose a control variable. For this model we take free flow velocity vector field $v_f = v_f(x,t)$ as the distributed control variable denoted by u. If the density at a location is zero then the speed at that location will be the free flow speed. However, with the actuation system implemented, we can inform people to change their speed. Also the crowd density affects the achievable speed. Therefore we choose v_f as the control variable, giving us the following representation of the control system

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho u (1 - \rho/\rho_{\max})\right) = 0 \tag{3.7}$$

where $u(x,t) \in H[(0,L),\Re]$ is the control variable.

3.2.3 State Feedback Control

Here we address the problem of synthesizing a distributed state feedback controller $u(x,t)$ that stabilizes origin ($\rho(x,t) = 0$) of system (3.7) or in other words control the evacuation of pedestrians from a one-dimensional area. We shall use the method of feedback linearization for PDEs. The method of feedback linearization works by canceling the nonlinearities in the system and is discussed for nonlinear ODEs in [31]. More specifically we consider control law of the form

$$u = F(\rho(x,t))$$

which makes origin of the closed loop dynamics exponentially stable. Here $F(\rho)$ is a nonlinear operator mapping $H^2[(0,L),\Re]$ into $H^1[(0,L),\Re]$. Designing the operator $F(\rho)$ we get our state feedback controller as

$$F(\rho(x,t)) = -[(1-\rho/\rho_{\max})\rho]^{-1} D\frac{\partial \rho}{\partial x} \tag{3.8}$$

The closed loop dynamics can be found by substituting (3.8) in (3.7) to get

$$\frac{\partial \rho}{\partial t} - D\frac{\partial^2 \rho}{\partial x^2} = 0 \tag{3.9}$$

where with boundary conditions is given by (3.3). The closed loop dynamics represent the heat equation with D being the diffusion constant. (3.9) suggests that the people are diffusing or in other words the motion of people is because of diffusion only. There is no direction to the motion of people. The rate of diffusion is determined by the diffusion constant D. The direction of motion is important in order to have an effective evacuation. In chapter 5 we will add this component to the control law (3.9) so that there is diffusion as well as direction to the motion. The diffusion term enhances motion and becomes important due to the limitation on the control variable. The controller in (3.8) has saturation issues which will be a topic in chapter 5.

3.2.4 Lyapunov Stability Analysis

In order to check the stability of the closed loop system (3.9) we use the Lyapunov function analysis. The stability problem is to establish sufficient conditions for which the origin of the closed loop dynamics (3.9) is exponentially or asymptotically stable. Within the framework of our system the definition of Lyapunov stability can be established by writing system dynamics (3.9) as an abstract differential equation in terms of operator theory. Towards that end, we introduce a differential operator A on $H^2[(0,L),\Re]$ defined by

$$A\varphi = k\frac{\partial^2 \varphi}{\partial x^2}, \quad \forall \ \varphi \in D(A) \tag{3.10}$$

Here $D(A) \subset H^2[(0,L),\Re]$ is the domain of operator A defined as

$$D(A) = \{\varphi \in H^2 : \varphi,\ \varphi' \in H^2[0,L];\ \varphi(0) = \varphi(L) = 0\}$$

Using operator (3.10), the system dynamics (3.9) can be written as an abstract differential equation making the dynamics look like an ordinary differential equation in Sobolev (Banach) space [32], [33]. The abstract version or state space representation of (3.9) can be put into the form

$$\frac{d}{dt}\rho(t) = A\rho(t),\ t > 0\ ;\ \ \rho(0) = \rho_0 \tag{3.11}$$

The operator A is known to generate a strongly continuous semigroup $U(t)$ of bounded linear operators on a normed linear space $H^2[(0,L),\Re]$. The system motion starting from any initial state $\rho(t_0)$ at time t_0 is defined by $U(t)\rho(t_0)$. Thus a partial differential equation can be regarded as an evolution system where $U(t)$ evolves ρ_0 forward in time. Within this abstract framework the definition of Lyapunov stability is given as follows.

Definition 3.2.1: An equilibrium state ρ_{eq} of a dynamical system (3.11) is stable with respect to a specified L_2 norm $\|\rho(x,t)\|_2$ if for every real number $\varepsilon > 0$ there exists a real number $\delta(\varepsilon, t_0) > 0$ such that

$$\|\rho(t_0) - \rho_{eq}\|_2 < \delta \Rightarrow \|U(t)\rho(t_0) - \rho_{eq}\|_2 < \varepsilon \qquad \forall t > t_0$$

If in addition $\|U(t)\rho(t_0) - \rho_{eq}\|_2 \to 0$ as $t \to \infty$ then the equilibrium is said to be asymptotically stable. Furthermore, if there exist two positive constants a and b such that

$$\|U(t)\rho(t_0) - \rho_{eq}\|_2 \le a\|\rho(t_0) - \rho_{eq}\|_2 e^{-b(t-t_0)} \qquad \forall t > t_0$$

is satisfied, then ρ_{eq} is said to be exponentially asymptotically stable. More precisely the problem is to find a condition under which operator A generates an exponentially stable semigroup $U(t)$ that satisfies the following growth property with respect to a specified L_2 norm

$$\|U(t)\|_2 \leq ae^{-bt} \quad t \geq 0 \tag{3.12}$$

where $b > 0$. In other words we say that A generates an exponentially stable semigroup $U(t)$ [34]. The determination of conditions for which estimate (3.12) is satisfied amounts to establishing conditions for which the null state of the linear system (3.9) is exponentially stable with respect to the specified L_2 norm; that is $\|\rho(t)\|_2 \to 0$ as $t \to \infty$.

It should be noted here that for infinite dimensional systems stability with respect to one norm does not necessarily imply stability with respect to others unlike finite dimensional systems where all norms are equivalent. For our system we have chosen the following L_2 norm defined by

$$\|\rho(t,x)\|_2 \to \left[\int_0^L |\rho(t,x)|^2 dx\right]^{1/2} \tag{3.13}$$

This norm represents an "energy like" function of the system at any time. Let us now consider a Lyapunov functional $V(t)$ for the system (3.9). Here $V : H^2[0,L] \to \Re_+$ is a smooth functional of the form

$$V(t) = \tfrac{1}{2}\|\rho\|_2^2 = \tfrac{1}{2}\int_0^L |\rho|^2 dx \tag{3.14}$$

Using the norm properties we can easily see that $V(t)$ is a positive definite function. The time rate of change of $V(t)$ using Leibniz rule in (3.14) is given as

$$\frac{dV(t)}{dt} = \int_0^L \rho \frac{\partial \rho}{\partial t} dx \tag{3.15}$$

Substituting the system dynamics from (3.12) in (3.15) and integrating it we get

$$\frac{dV(t)}{dt} = \left[D\rho \frac{\partial \rho}{\partial x}\bigg|_0^L \right] - D\int_0^L \left(\frac{\partial \rho}{\partial x}\right)^2 dx$$

The first term vanishes by boundary condition (3.4). For the second integral we make use of Gagliardo-Nirenberg-Sobolev Inequality [35], [36]. The inequality as applied to our case states

$$\|\rho\|_2 \le C\|\nabla\rho\|_2 = C\left\|\frac{\partial\rho}{\partial x}\right\|_2 \tag{3.16}$$

where C is a positive real number. Using (3.16) we have the following inequality

$$\int_0^L (\frac{\partial\rho}{\partial x})^2 \ge C^{-1}\int_0^L \rho^2 dx$$

The time rate of change of Lyapunov functional $V(t)$ can thus be bounded as

$$\frac{dV(t)}{dt} \le -DC^{-1}\int_0^L \rho^2 dx = -2DC^{-1}V(t) = -\beta V(t)$$

with $\beta = 2DC^{-1}$ a constant. Therefore the Lyapunov functional satisfies following property

$$V(t) \le V(t_0)e^{-\beta(t-t_0)}$$

which can also be written as

$$\|\rho(t,x)\|_2 \le \|\rho(t_0,x)\|_2 e^{-\beta(t-t_0)}$$

As long as $\beta > 0$ the null state of (3.9) is exponentially stable and condition (3.12) $\|U(t)\|_2 \le e^{-b(t-t_0)}$ is satisfied with $b = \beta$. Thus equilibrium of closed loop system (3.9) using feedback control (3.8) is exponentially stable.

3.2.5 Simulation Results

This section shows simulation results for closed loop system (3.9) using the control law (3.8). The numerical method used is the Lax-Friedrichs scheme [37]. For simulation the initial distribution for density is considered to be Gaussian. The initial density distribution is given by

$$\rho(x,0) = G\exp(-(x-a)^2)$$

with a being the centre of the Gaussian distribution and G is the highest magnitude of the distribution.

We have three simulation results for the one-equation model. The control action response for a corridor with exit at the right is shown in Fig.3.1. The simulation shows the density response as a mesh plot. After some finite time the density has decreased to zero. The second simulation is shown as a contour plot in Fig. 3.2 where the contour lines vary from 0.6 to 0.1 and as seen from plots density at every point in space is decreasing exponentially with time. The third simulation shows the density response at different time instants in Fig. 3.3, where the density flattens out with time. The plots indicate the diffusion of people throughout the length of the corridor.

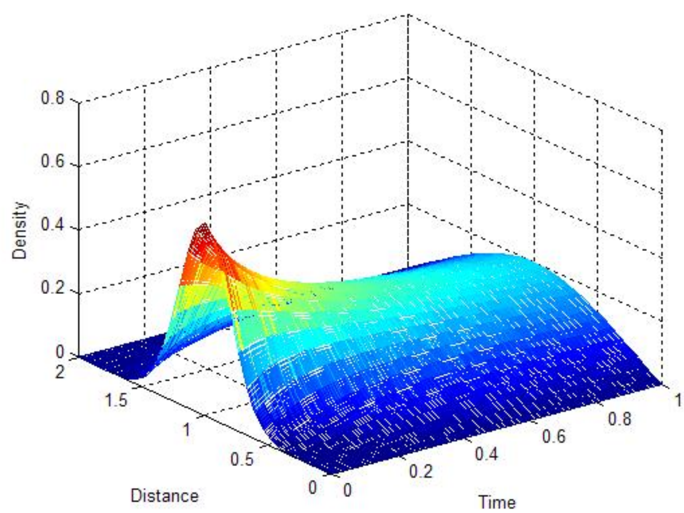


Figure 3. 1: Density response for one-equation model.

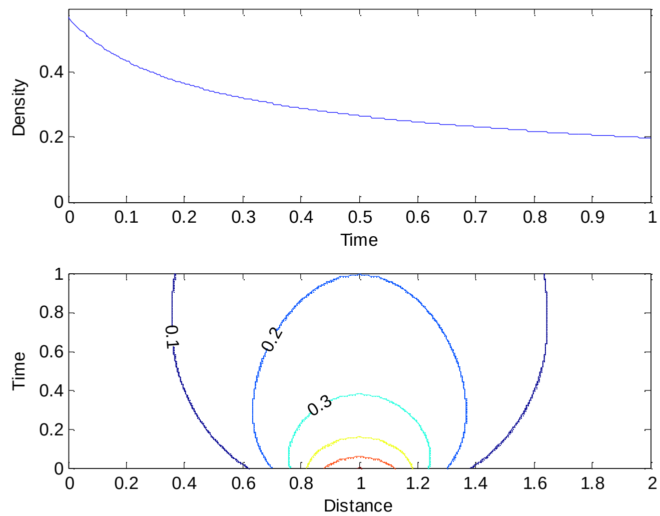


Figure 3. 2: Contours of the density response for one equation model.

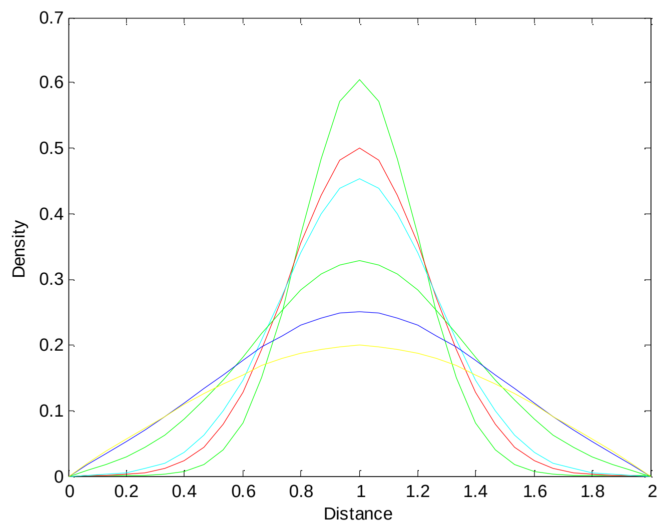


Figure 3. 3: Density response at different time instants for one-equation model.

3.3 Feedback Control for Two-Equation Model

In this section we formulate the control model and present feedback control design for the two-equation model. This section also studies Lyapunov stability for this model and presents some simulation results. In this section we design a feedback control for the two-equation model of the evacuation system given by (2.9) and (2.10) subject to boundary conditions (2.11) using the *backstepping* approach. The Lyapunov functional (3.14) which was used as a stability analysis tool for the one-equation model will be used as a feedback control design tool for this system. The design of feedback control is done in such a way that Lyapunov functional for the system or its derivative has certain properties that guarantee boundedness or convergence to an equilibrium point.

3.3.1 Two Equation System Model

We consider a higher order model or more precisely a system of two partial differential equations for a one dimensional corridor. This model consists of conservation of mass equation coupled with a second equation based on the principle of conservation of momentum. The model is given in subsection 2.2.3 by (2.9) and (2.10) as

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{3.17}$$

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}(q^2/\rho) = -\frac{\partial p}{\partial x} \tag{3.18}$$

with initial conditions and subject to boundary conditions

$$\rho(x,t_0) = \rho_0(x), \quad q(x,t_0) = q_0(x) \ \forall \ x \in [0,L]$$

$$\rho(0,t) = 0 \ \ q(0,t) = 0 \quad \forall \ \ t \in [0,\infty) \tag{3.19}$$

where $p = p(x,t) \in H^1[(0,L),\Re]$ is pressure. Here ρ and q are the variables we want to control. For pedestrian evacuation the final density and flow should be equal to zero; that is $\rho(x,t_f) = 0$ and $q(x,t_f) = 0$. The length of the corridor is $[0,L]$ and the boundary condition given by (3.19) is that the corridor is open at both the ends.

3.3.2 Two Equation Control Model

To formulate the control problem we need to choose a control variable. For the two-equation model (3.17) and (3.18) we choose divergence of pressure $\partial p/\partial x$ as distributed control variable u giving us the following representation of the control system

$$\frac{\partial \rho}{\partial t} = -\frac{\partial q}{\partial x}$$

$$\frac{\partial q}{\partial t} = -\frac{\partial}{\partial x}(q^2/\rho) + u$$

The above system can be rewritten in the following form

$$\frac{\partial \rho}{\partial t} = -\frac{\partial q}{\partial x} \qquad (3.20)$$

$$\frac{\partial q}{\partial t} = \bar{u} \qquad (3.21)$$

where $\bar{u} = -\frac{\partial}{\partial x}(q^2/\rho) + u$ is the new control variable.

3.3.3 State Feedback Control by Backstepping

Here we address the problem of synthesizing a distributed state feedback controller $\bar{u}(x,t)$ that stabilizes the origin ($\rho(x,t) = 0, q(x,t) = 0$) of the two-equation control system given by (3.20) and (3.21). More specifically we consider the control law

$$\bar{u} = \bar{F}(\rho(x,t), q(x,t))$$

such that origin of the closed loop dynamics are exponentially stable. $\bar{F}$ is a nonlinear operator mapping $H^2[(0,L),\Re]$ into $H^1[(0,L),\Re]$. The control strategy adopted here is similar in principle to feedback control by backstepping for ordinary differential equations [31] and is extended to PDEs. Backstepping is a Lyapunov-based control method of feedback linearization. It is a recursive method that designs the feedback control law based on the choice of the Lyapunov function. It breaks the design problem for a system of equations into a sequence of design problems for scalar systems. We proceed with the control design as follows:

1. First we design control law for (3.20) where $q(x,t)$ can be assumed as an input. We proceed to design a conceptual control law $q = G(\rho)$ for this control input to stabilize the origin $\rho(x,t) = 0$. G is a nonlinear operator mapping $H^2[(0,L),\Re]$ into $H^2[(0,L),\Re]$. With the control law

$$q(x,t) = G(\rho) = -D\int_0^x \frac{\partial^2 \rho}{\partial m^2}\, dm \qquad (3.22)$$

we get the conceptual closed loop dynamics for (3.20) as

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial x^2} \qquad (3.23)$$

which is similar to (3.9). As already shown this system is exponentially stable with a conceptual Lyapunov functional (3.14)

$$V(t) = \tfrac{1}{2}\|\rho\|_2^2 = \tfrac{1}{2}\int_0^L |\rho|^2\, dx$$

which satisfies $\frac{dV(t)}{dt} \le -\beta V(t)$ and ensures stability.

2. We have used the term “conceptual” with the control law, closed loop system and the Lyapunov function. This is done in order to stress the fact that the control law (3.22) cannot be implemented in practice as $q(x,t)$ is not a control variable. However, this conceptual design helps us to recognize the benefit of the input $q(x,t)$ being close to $G(\rho)$. From the knowledge of the conceptual Lyapunov function $V(t)$ we want to design a smooth feedback control for stabilizing the origin of the overall system. We therefore add to the conceptual Lyapunov function (3.14) a term penalizing the difference between q and $G(\rho)$. For this purpose we rewrite the dynamics (3.20) as

$$\frac{\partial \rho}{\partial t} = -\frac{\partial G(\rho)}{\partial x} - \frac{\partial}{\partial x}\left(q - G(\rho)\right)$$

Defining the difference between q and $G(\rho)$ by an error variable $z = q - G(\rho)$, we get the following modified dynamics.

$$\frac{\partial \rho}{\partial t} = -\frac{\partial G(\rho)}{\partial x} - \frac{\partial z}{\partial x} \tag{3.24}$$

$$\frac{\partial z}{\partial t} = u_n \tag{3.25}$$

where $u_n = \bar{u} - \frac{\partial G(\rho)}{\partial t}$ is the new control variable and $z(x,t) \in H^2[(0,L),\Re]$.

3 Now let us modify the Lyapunov functional (3.14) by adding an error term to it thus resulting in the Lyapunov function for the overall system as

$$V_a(t) = V(t) + \tfrac{1}{2}\|z\|_2^2 = \tfrac{1}{2}\int_0^L |\rho|^2 dx + \tfrac{1}{2}\int_0^L |z|^2 dx \tag{3.26}$$

The time rate of change of this functional using (3.24) and (3.25) is given as

$$\frac{dV_a(t)}{dt} = \int_0^L \rho \frac{\partial^2 \rho}{\partial x^2} dx + \int_0^L \rho \frac{\partial z}{\partial x} dx + \int_0^L z u_n dx$$

Using (3.16) we know that the first term is bounded by $-\beta V(t)$. Therefore

$$\frac{dV_a(t)}{dt} \le -\beta V(t) + \int_0^L \rho(x,t)\frac{\partial z}{\partial x} dx + \int_0^L z u_n dx$$

We have to choose a new control law $u_n(x,t)$ in such a manner that the time derivative of the new functional or the sum of second and third terms is also bounded by a negative definite function. By choosing the following control law

$$u_n(x,t) = -Kz - z^{-1}\rho\frac{\partial z}{\partial x} \tag{3.27}$$

we get the following result

$$\frac{dV_a(t)}{dt} \le -\beta V(t) - K\|z\|_2^2 = -2\beta V_a(t)$$

with $K = 2\beta > 0$. This shows that the origin ($\rho(x,t) = 0, z(x,t) = 0$) of the system (3.24) and (3.25) is exponentially stable.

4. With the feedback controller (3.27) we have proved that the origin ($\rho(x,t) = 0, z(x,t) = 0$) of (3.24) and (3.25) is asymptotically stable. But we need to prove the asymptotic stability of the origin ($\rho(x,t) = 0, q(x,t) = 0$) of the system

(3.20) and (3.21) or we need to prove the exponential stability of $q(x,t)=0$. The Proposition 3.3.1 establishes the asymptotic stability of the original closed-loop system.

Proposition 3.3.1 *The origin ($\rho(x,t)=0, q(x,t)=0$) of the system (3.20) and (3.21) is exponentially stable under the control law* $\bar{u}=u_n+\partial G(\rho)/\partial t$ *with* u_n *given by (3.27).*

Proof: The control laws (3.22) and (3.27) make the origin $\rho(x,t)=0, z(x,t)=0$ for equation (3.24) and (3.25) asymptotically go to zero. From the equation $z=q-G(\rho)$ we want to prove the asymptotic stability of $q(x,t)=0$. We will show that as $t\to\infty$ the control law $G(\rho)\to 0$. The control law $G(\rho)$ is given as

$$G(\rho)=-D\int_0^x \frac{\partial^2 \rho}{\partial m^2}\,dm \tag{3.28}$$

Therefore using (3.28) we need to prove the following

$$\text{limit}_{t\to\infty} G(\rho(x,t))=\text{limit}_{t\to\infty}\left(-D\int_0^x \frac{\partial^2\rho}{\partial s^2}\,ds\right)=0 \tag{3.29}$$

The proof is done in the following three steps.

Step1: Solution of heat equation. In order to prove (3.29) we consider the solution of heat equation (3.9) given by

$$\frac{\partial \rho}{\partial t}=D\frac{\partial^2 \rho}{\partial x^2} \tag{3.30}$$

The solution of the equation (3.30) as given in [24] is

$$\rho(x,t)=\sum_{n=1}^{\infty} A_n e^{-D\lambda_n^2 t}\sin(\lambda_n x) \tag{3.31}$$

where $\lambda_n = n\pi/L$. For our case let us take $L=1$ and the coefficients in (3.31) are given by

$$A_n=\frac{2}{\pi}\int_0^L \phi(x)\sin(n\pi x)dx \tag{3.32}$$

where the function $\phi(x) = \rho(x,0)$ is the initial condition. The rate of change of density in (3.31) is given as

$$\frac{\partial \rho}{\partial t} = -(\pi)^2 \sum_{n=1}^{\infty} n^2 A_n e^{-D\lambda_n^2 t} \sin(\lambda_n x) \tag{3.33}$$

Step 2: Convergence of $\partial\rho/\partial t$. The next step is to show the convergence of rate of change of density as given by (3.32). Let us construct a sequence of time denoted by t_k with $k = 1,2,......\infty$. Now for a fixed time sample t_k we have the estimate of absolute value $\partial\rho/\partial t$ given by

$$\left|\partial\rho/\partial t\right|_{t=t_k} = \pi^2 \sum_{n=1}^{\infty} \left| n^2 A_n e^{-\lambda_n^2 t_k} \sin(\lambda_n x) \right| \le \pi^2 \sum_{n=1}^{\infty} \left| n^2 A_n e^{-\lambda_n^2 t_k} \right| \tag{3.34}$$

since $\left|\sin(\lambda_n x)\right| \le 1$ for $(x,t) \in [0,L] \times [0,\infty)$. Thus we have

$$\left|\partial\rho/\partial t\right|_{t=t_k} \le \pi^2 \sum_{n=1}^{\infty} \left| n^2 A_n e^{-\lambda_n^2 t_k} \right| \le \pi^2 \sum_{n=1}^{\infty} \left| \frac{n^2 A_n}{(n^2 \pi^2 t_k)^2} \right|$$

The coefficients given by (3.32) are bounded as

$$|A_n| \le \frac{2}{\pi} \int_0^L |\phi(x)| dx = \frac{2}{\pi} \int_0^L |\rho(x,0)| dx \le \overline{A} \tag{3.35}$$

where $\overline{A}$ is a positive constant and $\overline{A}$ exists because $\phi(x) = \rho(x,0)$ is continuous on a compact (i.e. closed and bounded) interval. Therefore we have

$$\left|\partial\rho/\partial t\right|_{t=t_k} \le \frac{\overline{A}}{\pi^2 t_k^{\ 2}} \sum_{n=1}^{\infty} \frac{1}{n^2} \tag{3.36}$$

Since the series $\sum_{n=1}^{\infty} 1/n^2$ converges [38] therefore we conclude that $|\partial\rho/\partial t|$ in (3.36) converges for every $t_k > 0$. Next we need to show $|\partial\rho/\partial t| \to 0$ as $t \to \infty$ or as $k \to \infty$, which means $\forall \varepsilon > 0, \exists$ some $t_N > 0$ such that for all $t \ge t_N$, $|\partial\rho/\partial t| < \varepsilon$. From (3.34) and (3.35) we have

$$\left|\partial\rho/\partial t\right| \le \pi^2 \overline{A} \sum_{n=1}^{\infty} \left| n^2 e^{-(n\pi)^2 t} \right| \tag{3.36}$$

Now let us construct a series in $\varepsilon > 0$ given as

$$\varepsilon/2+\varepsilon/4+\varepsilon/8+........=\varepsilon(\sum_{n=1}^{\infty}1/2^n)=\varepsilon \tag{3.37}$$

For an arbitrary $\varepsilon > 0$ we estimate the *n*-th term of series (3.36) for some time t as

$$\pi^2\overline{A}\left|n^2e^{-(n\pi)^2t}\right|\le\varepsilon/2^n$$

which yields the following

$$t\ge\frac{\ln 2^n+\ln n^2+\ln\left(\pi^2\overline{A}\right)-\ln\varepsilon}{n^2\pi^2}=\frac{1}{\pi^2}\left(\frac{\ln(2)}{n}+\frac{2\ln(n)}{n^2}+\frac{\ln(\pi^2\overline{A})}{n^2}-\frac{\ln\varepsilon}{n^2}\right)$$

For $n=1$ let us denote the time $t_N=\frac{1}{\pi^2}\left(\ln(2)+\ln(\pi^2\overline{A})\right)$. Consequently for each n and $t\ge t_N$, we have $\pi^2\overline{A}\left|n^2e^{-(n\pi)^2t}\right|\le\varepsilon/2^n$ and the later in conjunction with (3.36) and (3.37) implies the following

$$\left|\partial\rho/\partial t\right|=\sum_{n=}^{\infty}\left|n^2\pi^2\overline{A}e^{-(n\pi)^2t}\right|\le\sum_{n=1}^{\infty}\varepsilon/2^n=\varepsilon\ .$$

Thus for $t\ge t_N$ we $\left|\partial\rho/\partial t\right|<\varepsilon$ .Therefore as $t\to\infty$, $\partial\rho/\partial t\to 0$ or from (3.30) we have $\partial^2\rho/\partial x^2\to 0$ as $t\to\infty$.

Step 3: Convergence of $G(\rho(x,t))$. So far we have shown the convergence of $\partial^2\rho/\partial x^2$ to zero. The last step is to establish the convergence of $G(\rho(x,t))$ from (3.28). In other words we need to show the following

$$\text{limit}_{t\to\infty}G(\rho(x,t))=\text{limit}_{t\to\infty}\left(-D\int_0^x\frac{\partial^2\rho}{\partial s^2}ds\right)=0 \tag{3.37}$$

Using the equations (3.33) and (3.30) we can rewrite (3.37) as

$$\text{limit}_{k\to\infty}\left(\int_0^x(\sum_{n=1}^{\infty}A_n\pi^2n^2e^{-(n\pi)^2t_k}\sin(n\pi s))ds\right)=0 \tag{3.38}$$

In order to prove (3.38) let us denote a sequence of functions $f_k(x)$ by

$$f_k(x)=\left|\partial\rho/\partial t\right|_{t=t_k}=-(\pi)^2\sum_{n=1}^{\infty}n^2A_ne^{-\lambda_n{}^2t_k}\sin(\lambda_nx)$$

We know from (3.36) that $f_k(x)$ converges for every $k>1$ (or $t_k>0$) and that $\text{limit}_{k\to\infty} f_k(x)$ is zero. Now we need to show that $\text{limit}_{k\to\infty}\int_0^x f_k(s)ds=0$. In order to prove this we make use of dominated convergence theorem [39]. From (3.36) we estimate of the absolute value of $f_k(x)$ given as

$$|f_k(x)|\le \frac{\overline{A}}{\pi^2 t_k{}^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Since the series $\sum_{n=1}^{\infty} 1/n^2$ converges to some positive constant M therefore we have

$$|f_k(x)|\le \frac{\overline{A}M}{\pi^2 t_k{}^2}=\frac{P}{t_k{}^2}$$

$P=\overline{A}M/\pi^2$. Let us define for some $t_1>0$, a function $\Phi(x)=P/t_1^2$. Now for all k we have $|f_k(x)|\le\Phi(x)$ and

$$\int_0^x \Phi(s)ds=\int_0^x \frac{P}{t_1{}^2}ds<\infty .$$

Thus we have a sequence of functions $f_k(x)$ for which there exists an integrable function $\Phi(x)$ such that $|f_k(x)|\le\Phi(x)$ and $\int_0^x\Phi(s)ds<\infty$. Also $\text{limit}_{k\to\infty} f_k(x)$ is zero. Therefore by dominated convergence theorem [40] this limit is integrable and we have the following

$$\text{limit}_{k\to\infty}\int_0^x f_k(s)ds=\int_0^x \text{limit}_{k\to\infty} f_k(s)ds=0 .$$

It follows from (3.28) and (3.30) that

$$\text{limit}_{t\to\infty} G(\rho(x,t))=\text{limit}_{t\to\infty}\left(-D\int_0^x\frac{\partial^2\rho}{\partial s^2}ds\right)=0$$

This completes the proof. ◊

Thus we conclude that the origin ($\rho(x,t)=0, q(x,t)=0$) for actual closed-loop system (3.22) and (3.23) is asymptotically stable. The feedback control law is given by the following partial differential-integral equation

$$u = u_n + \frac{\partial G(\rho)}{\partial t} + \frac{\partial}{\partial x}(q^2/\rho) \tag{3.39}$$

with u_n given by (3.27) as $u_n = -Kz - z^{-1}\rho\frac{\partial z}{\partial x}$, where $z = q - G(\rho)$. Hence the closed-loop dynamics (3.20) and (3.21) for the two equation model are exponentially stable with this feedback control.

3.3.4 Simulation

In this section we show simulation results for the closed loop system (3.20) and (3.21) using controller (3.27) designed in previous section. We have used the same Lax-Friedrichs numerical technique as before. For simulation the initial distribution for both density and flow is considered to be Gaussian.

The simulation results are shown in the following figures. The density plots are shown in Fig.3.4-Fig. 3.6 and are similar to those in Fig. 3.1 and Fig. 3.2. The flow plots are shown in Fig. 3.7 and Fig. 3.8. As is seen from the plots flow of people at every point in space is decreasing exponentially with time.

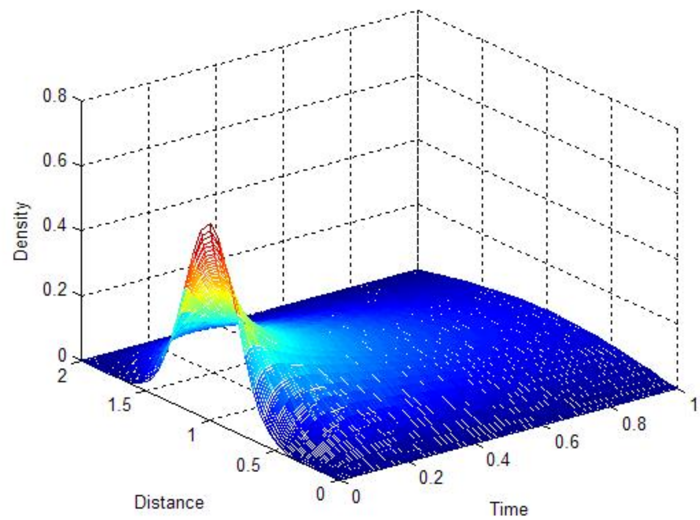


Figure 3. 4: Density response for the two-equation model.

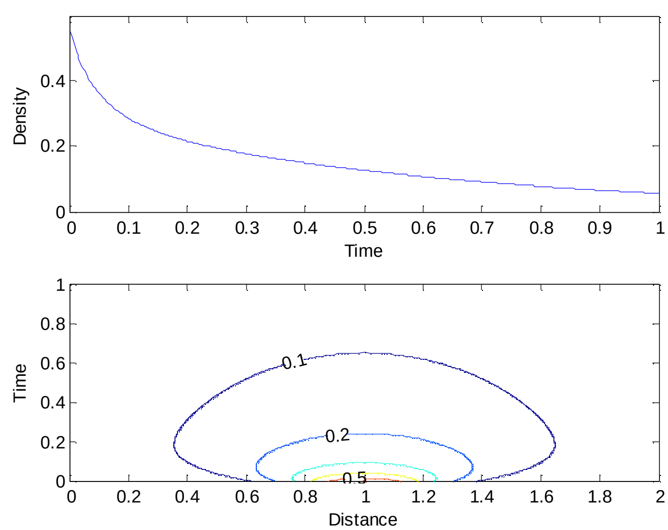


Figure 3. 5: Contours of the density response for two-equation model.

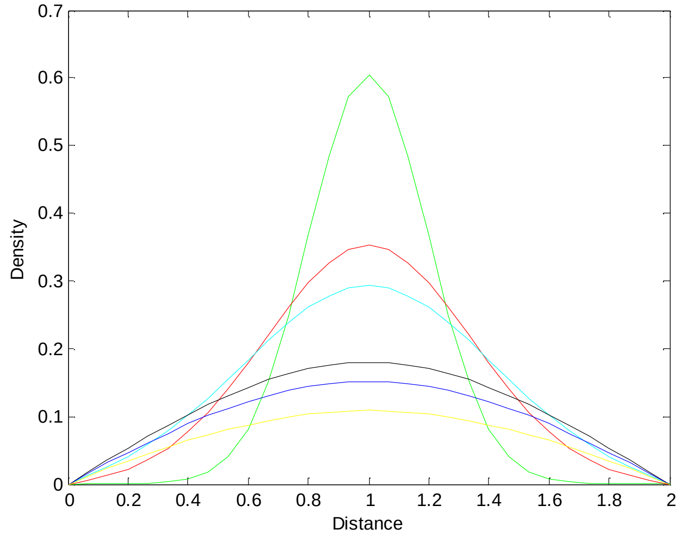


Figure 3. 6: Density response at different time instants for two-equation model.

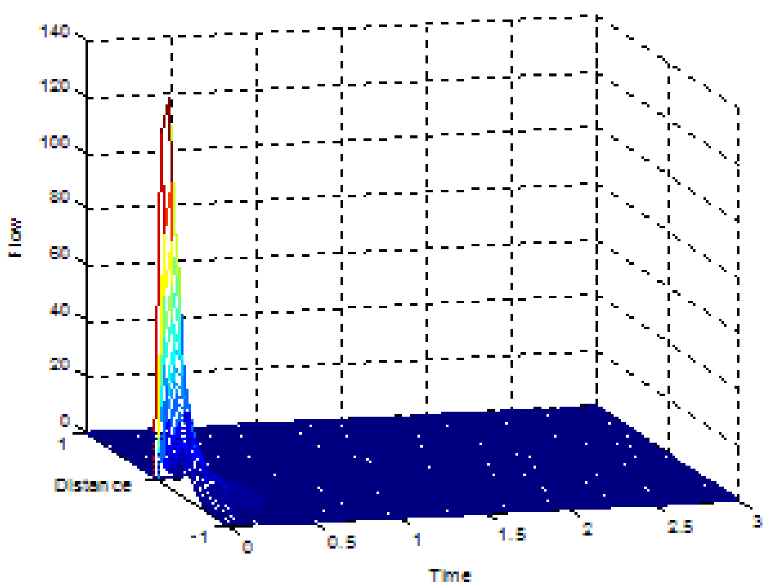


Figure 3. 7: Flow response for the two-equation model.

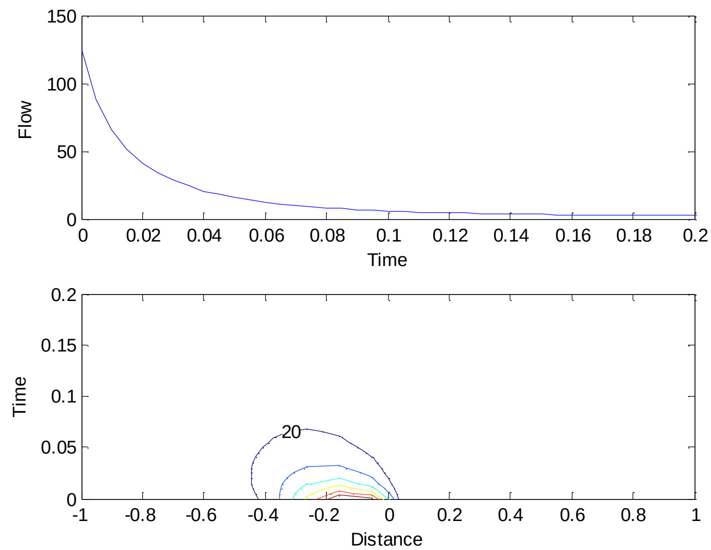


Figure 3. 8: Contours of flow response for two-equation model.

In this chapter we discussed design of feedback controllers for two infinite dimensional models representing the evacuation dynamics in one-dimensional space. We adopted the method of feedback linearization to control the movement of people. The closed loop dynamics are represented by a heat equation which dictates the change (decrease) in initial density profile with time. With the use of these controllers we have shown exponential asymptotic stability of the closed loop systems. We will modify these controllers in chapter 5 in order to add motion to the density profiles. Also the controllers discussed in this chapter are unbounded so we will modify them to take control saturation into account in chapter 5. For the two-equation model we used the modification of *backstepping* method for nonlinear ODEs.

While doing the control design we have seen that the difficulty in designing controllers for systems increases as we increase the number of partial differential equations representing the dynamics. This motivates the study of abstraction for infinite dimensional systems which will be the topic of chapter 9. The goal here is to abstract the system having higher number of partial differential equations by a system which is represented by lower number of partial differential equations and find the transformation for control design.

CHAPTER 4

FEEDBACK CONTROL DESIGN AND STABILITY ANALYSIS OF TWO DIMENSIONAL EVACUATION SYSTEM

4.1 Introduction

This chapter presents the design of nonlinear feedback controllers for two models given in chapter 2 representing evacuation dynamics in two-dimensions. The models presented here are based on the laws of conservation of mass and momentum. The first model is a one-equation model based on conservation of mass with a prescribed relationship between density and velocity given in subsection 2.3.1. The model dynamics are represented by means of a single partial differential equation. The other is a two-equation model given in subsection 2.3.2 in which the velocity is independent of density. This model is based on conservation of mass and momentum. Here the dynamics are represented by means of a set of three partial differential equations. The equations of motion in both cases are described by nonlinear partial differential equations. The system is distributed, i.e. both the state and control variables are distributed in time and space.

We address the feedback control problem for both models. The objective is to synthesize a nonlinear distributed feedback controller that guarantees stability of a closed loop system. The problem of control and stability is formulated directly in the framework of partial differential equations. Sufficient conditions for Lyapunov stability for distributed control are derived.

We are interested in designing feedback controllers to evacuate pedestrians from a two-dimensional area. We use the same methods of feedback control as in chapter 3 and extend the results to a two-dimensional case. The method of feedback linearization is used for the one-equation model which works by canceling nonlinearities in the system. For a two-equation system model the feedback control design is done by the *backstepping* approach. In both cases objective of control design is to synthesize a nonlinear distributed feedback controller that stabilizes the system and guarantees stability in closed loop system.

The organization of this chapter is as follows. In Section 4.2 we formulate the control model and present feedback control design for the one-equation model. This section also studies Lyapunov stability for this model and finally presents some simulation results. Section 4.3 presents the feedback control design and stability analysis of the two-equation system model. Simulation results for closed loop dynamics are presented.

4.2 Feedback Control for One-Equation Model

In this section we formulate the control model and present feedback control design for the one-equation model. This section also studies Lyapunov stability for this model and presents simulation results.

4.2.1 One-Equation Model

In this section we present the mathematical model of crowd dynamics for a two-dimensional single exit area of dimensions $\Omega = (0, L) \times (0, L)$. The model is given in subsection 2.3.1 by (2.12) as

$$\frac{\partial \rho(x,t)}{\partial t} + div(q(x,t)) = 0 \qquad \forall x \in \Omega \tag{4.1}$$

with initial and boundary conditions given by

$$\rho(x,t_0) = \rho_0(x) \qquad \forall \qquad x \in \Omega \tag{4.2}$$

$$\rho(x,t) = 0 \quad \forall \quad t \in [0,\infty)\,,\, x \in \partial\Omega \tag{4.3}$$

where $\partial\Omega$ denotes the boundary of the area. Here $\rho(x,t) \in H^2[\Omega,\Re]$ is the variable we want to control. For pedestrian evacuation the final density should be equal to zero; i.e. $\rho(x,t_f) = 0$. The area of the space is $\Omega = (0,L)\times(0,L)$ and the boundary condition is given by (4.3). The vector $x \in \Omega \subset \Re^2$ is expressed in terms of its coordinates as $x = [x_1, x_2]^T$. The flow rates are given as $q_1(x,t)$ and $q_2(x,t)$ in x_1 and x_2 directions. The flows q_i's are obtained as a product of density and velocity as

$$q_i = \rho v_i \;,\; i = 1,2 \tag{4.4}$$

with $v_1(x,t)$ and $v_2(x,t)$ being the velocity vector fields associated with their respective directions. The dynamics for the two dimensions are therefore given by

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho v_1)}{\partial x_1} + \frac{\partial(\rho v_2)}{\partial x_2} = 0 \tag{4.5}$$

subject to the initial conditions and boundary conditions given by (4.2) and (4.3) respectively. The velocity-density relationship as given by Greenshield's model (2.17) is

$$v_i = v_{if}(1 - \rho/\rho_{\max}) \tag{4.6}$$

where $v_{if} = v_{if}(x,t)$ is the free flow speed in x_i direction, $i = 1,2$ and $\rho_{\max}$ is the jam density. The one-equation model is therefore given by

$$\frac{\partial\rho(x,t)}{\partial t} + \sum_{i=1}^{2}\frac{\partial}{\partial x_i}\big(\rho v_{if}(1 - \rho/\rho_{\max})\big) = 0\,, \qquad i = 1,2 \tag{4.7}$$

The free flow speeds v_{1f} and v_{2f} in x_1 and x_2 directions are used as control parameters to direct crowd flow to any desired direction and are both a function of time and space.

4.2.2 One-Equation Control Model

For the control model formulation we choose the free flow velocity vector fields $v_{if} = v_{if}(x,t)$ as the distributed control variables denoted by $u_i(x,t)$. This gives us the following representation of the control system

$$\frac{\partial \rho(x,t)}{\partial t} + \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(\rho u_i (1-\rho/\rho_{\max})\right) \quad ; \quad i = 1,2 \tag{4.8}$$

where $u_i(x,t) \in H[(0,L),\Re]$ are the control variables.

4.2.3 State Feedback Control

Here we address the problem of designing distributed state feedback controllers u_1 and u_2 that stabilize the origin ($\rho(x,t) = 0$) of system (4.8) or in other words evacuate pedestrians from a two-dimensional area. We use the method of feedback linearization for PDEs as in chapter 3. More specifically we consider control law of the form

$$u_i = F_i(\rho)$$

which makes origin of the closed loop dynamics exponentially stable. Here $F_i(\rho)$ is a nonlinear operator mapping $H^2[\Omega,\Re]$ into $H^1[\Omega,\Re]$. Choose the x_1 and x_2 components of state feedback control law as

$$F_i(\rho(x,t)) = -\left[(1-\rho/\rho_{\max})\rho\right]^{-1} D \frac{\partial \rho}{\partial x_i}; \qquad i = 1,2 \tag{4.9}$$

The closed loop dynamics can be found by substituting (4.9) in (4.8) to get

$$\frac{\partial \rho}{\partial t} - D\frac{\partial^2 \rho}{\partial x_1^{\,2}} - D\frac{\partial^2 \rho}{\partial x_2^{\,2}} = 0$$

$$\frac{\partial \rho}{\partial t} - D\nabla^2 \rho = 0 \tag{4.10}$$

with boundary conditions given by (4.2) and (4.3). The closed loop dynamics represent the heat equation in two dimensions with D being the diffusion constant. (4.10) suggests that people are diffusing in both directions and the rate of diffusion is determined by the diffusion constant D. The direction of motion is important in order to have an effective evacuation. In chapter 6 we will add this component to the control law (4.9) so that there is diffusion as well as direction to the motion. The diffusion term enhances motion which becomes important due to the limitation on the control variable. The controller in (4.10) has saturation issues which will be a topic in chapter 6.

4.2.4 Lyapunov Stability Analysis

The stability problem is to establish sufficient conditions for which the origin of the closed loop dynamics (4.10) is exponentially stable. In order to check the stability of the closed loop system we use Lyapunov function analysis. For the two dimensional system, definition of stability in terms of Lyapunov can be established in a similar way to the one dimensional case as given in section 3.2.4.

The determination of conditions for which estimate (3.12) is satisfied amounts to establishing conditions for which the null state of the closed loop system (4.10) is exponentially stable with respect to the L_2 norm given by

$$\|\rho\|_2 \rightarrow \left[\int_{\Omega} |\rho|^2 \, d\Omega \right]^{1/2} \tag{4.11}$$

Norm exponential stability implies $\|\rho(t)\|_2 \rightarrow 0$ as $t \rightarrow \infty$. Let us consider the same Lyapunov functional $V(t)$ as given by (3.14) for the system (4.10). Here $V : H^2[\Omega] \rightarrow \Re_+$ is a smooth functional of the form

$$V(t) = \tfrac{1}{2}\|\rho\|_2^2 = \tfrac{1}{2}\int_{\Omega} |\rho|^2 \, d\Omega \tag{4.12}$$

Using the same procedure as in section 3.2.4 we get the time rate of change of $V(t)$ as

$$\frac{dV(t)}{dt} = -D\int_{\Omega} \sum_{i=1}^{2} \left(\frac{\partial \rho}{\partial x_i} \right)^2 d\Omega$$

Now we make use of Sobolev Inequality (3.16) which for the two-dimensional case is

$$\int_{\Omega} \sum_{i=1}^{2} \left(\frac{\partial \rho}{\partial x_i} \right)^2 d\Omega \geq C^{-1} \int_{\Omega} \rho^2 \, d\Omega$$

where C is a positive real number. The rate of change of $V(t)$ can be thus be bounded by

$$\frac{dV(t)}{dt} \leq -DC^{-1}\int_{\Omega} \rho^2 d\Omega = -2DC^{-1}V(t) = -\beta V(t)$$

It follows that $V(t) \leq V(t_0)e^{-\beta(t-t_0)}$ or

$$\|\rho(t,x)\|_2 \leq \|\rho(t_0,x)\|_2 e^{-\beta(t-t_0)}$$

with $\beta = 2DC^{-1}$. As long as $\beta > 0$, null state of (4.10) is exponentially stable and condition (3.12), $\|U(t)\|_2 \leq e^{-b(t-t_0)}$ is satisfied with $b = \beta$. Thus the equilibrium of closed loop system (4.8) using feedback control (4.9) is exponentially stable.

4.2.5 Simulation Results

This section shows simulation results for the closed loop system (4.8) using control law (4.9). The numerical method used is the Lax-Friedrichs scheme [25]. For simulation the initial distribution of density is considered to be Gaussian. The initial condition is given by

$$\rho(x,0) = G\exp(-(x_1 - a)^2 - (x_2 - b)^2)$$

with (a,b) being the centre of the Gaussian distribution and G, the highest magnitude of the distribution. The control action response for a corridor is shown in Fig.4.1. The simulation shows the density response as mesh plot snapshots. After some finite time the density has decreased to zero. The second simulation is shown as contour plot snapshots in Fig.4.2. As seen from plot, density at every point in space is decreasing exponentially with time.

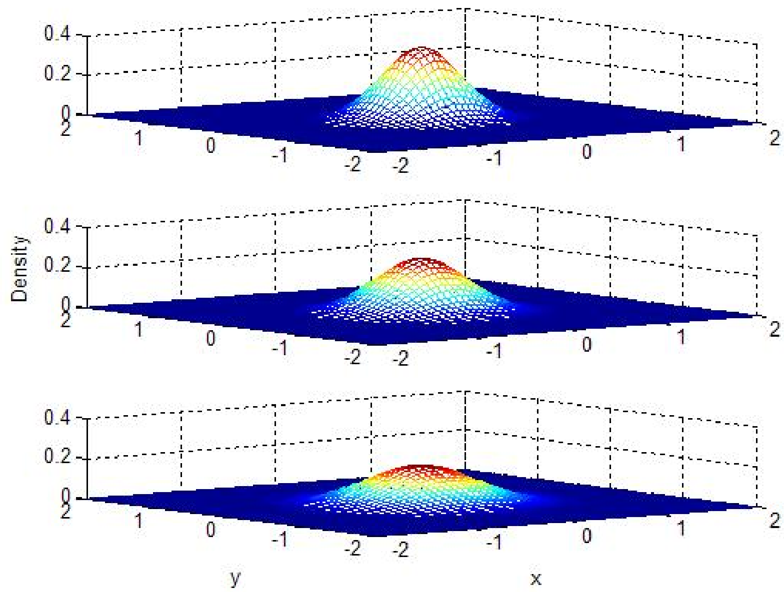


Figure 4. 1: Density response at different time instants for one-equation model.

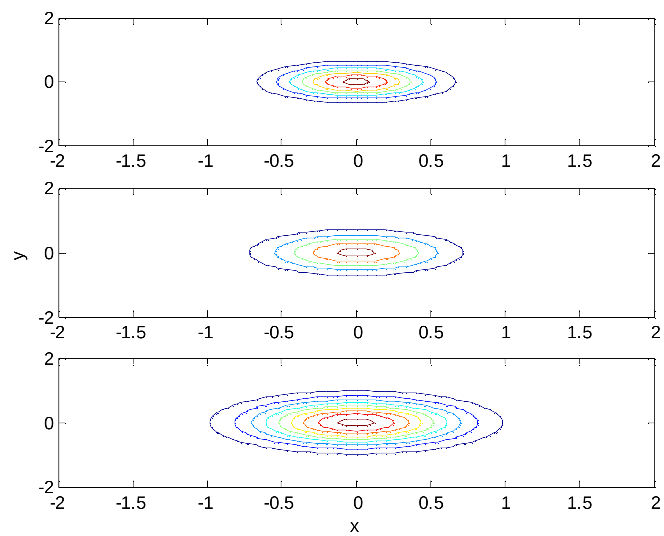


Figure 4. 2: Contours of the density response at different time instants for one-equation model.

4.3 Feedback Control for Two-Equation Model

In this section we formulate the control model and present feedback control design for the two-equation model. This section also studies Lyapunov stability for this model and presents some simulation results. In this section we design a feedback control for the two-equation model of the evacuation system given by (2.21), (2.22) and (2.23) subject to boundary conditions (2.25) using the “backstepping” approach.

4.3.1 Two Equation System Model

In this section we consider a higher order model or more precisely a system of two partial differential equations for a two dimensional space. This model consists of the conservation of mass equation coupled with a second equation based on the principle of conservation of momentum. The dynamics are given in subsection 2.3.2 by (2.21), (2.22) and (2.23) as

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{2} \frac{\partial q_i}{\partial x_i} = 0 \tag{4.13}$$

and

$$\frac{\partial q_1}{\partial t} + \frac{\partial}{\partial x_1}(q_1{}^2/\rho) + \frac{\partial}{\partial x_2}(q_1 q_2/\rho) = -\frac{\partial p}{\partial x_1} \tag{4.14}$$

$$\frac{\partial q_2}{\partial t} + \frac{\partial}{\partial x_1}(q_1 q_2/\rho) + \frac{\partial}{\partial x_2}(q_2{}^2/\rho) = -\frac{\partial p}{\partial x_2} \tag{4.15}$$

with initial conditions and subject to boundary conditions given by

$$\rho(x,t_0) = \rho_0(x), \quad q(x,t_0) = q_0(x); \qquad \forall\ x \in \Omega$$

$$\rho(x,t) = 0,\ q(x,t) = 0,\ \forall\ \ t \in [0,\infty),\, x \in \partial\Omega \tag{4.16}$$

where $p = p(x,t) \in H^1[\Omega, \Re]$ is pressure. Here ρ, q_1 and q_2 are the variables we want to control. For pedestrian evacuation the final density and flow in both directions should be equal to zero; i.e. $\rho(x,t_f) = 0$, $q_1(x,t_f) = 0$ and $q(x_2,t_f) = 0$. The area of the corridor is $\Omega = L \times L$ and the boundary condition given by (4.16) is that the space is closed from the left end and opens at the right.

4.3.2 Two-Equation Control Model

For the above two-equation model (4.13) and (4.15) we choose divergence of pressure $\partial p/\partial x_i$, $i = 1,2$ as the distributed control variables $u_i(x,t)$ which gives us the following control model

$$\frac{\partial \rho}{\partial t} = -\sum_{i=1}^{2} \frac{\partial q_i}{\partial x_i}$$

and

$$\frac{\partial q_1}{\partial t} = -\frac{\partial}{\partial x_1}(q_1^2/\rho) - \frac{\partial}{\partial x_2}(q_1 q_2/\rho) + u_1$$

$$\frac{\partial q_2}{\partial t} = -\frac{\partial}{\partial x_1}(q_1 q_2/\rho) - \frac{\partial}{\partial x_2}(q_2^2/\rho) + u_2$$

By rearranging terms, the above system can be rewritten in the following form

$$\frac{\partial \rho}{\partial t} = -\sum_{i=1}^{2} \frac{\partial q_i}{\partial x_i} \qquad (4.17)$$

$$\frac{\partial q_i}{\partial t} = \overline{u}_i \quad , \ i = 1,2 \qquad (4.18)$$

where $\overline{u}_i = -\frac{\partial}{\partial x_1}(q_1 q_2/\rho) - \frac{\partial}{\partial x_2}(q_2^2/\rho) + u_i$ are the new control variables.

4.3.3 State Feedback Control Using Backstepping

Here we address the problem of synthesizing a distributed state feedback controllers u_1 and u_2 that stabilizes origin ($\rho(x,t) = 0, q_1(x,t) = 0, q_2(x,t) = 0$) of the control system (4.17) and (4.18). More specifically we consider the components of control law

$$\overline{u}_i = \overline{F}_i(\rho, q) \qquad i = 1,2$$

such that the origin of closed loop dynamics is exponentially stable. $\overline{F}_i$ is a nonlinear operator mapping $H^2[\Omega, \Re]$ into $H^1[\Omega, \Re]$. The control strategy adopted here is similar to feedback control by backstepping PDEs discussed in section 3.3.2 which is extended to the two dimensional case here. The sequence of steps in control design is the same as before and is given as follows:

1. First we design control law for equation (4.22) where $q_1(x,t)$ and $q_2(x,t)$ can be viewed as inputs. We design the conceptual control law $q_i = G_i(\rho)$ to stabilize origin $\rho(x,t) = 0$ as

$$q_i(x,t) = G_i(\rho(x,t)) = \int_0^{x_i} \frac{\partial^2 \rho(x,t)}{\partial m^2} dm \,,\ i = 1,2 \tag{4.19}$$

G_i is a nonlinear operator mapping $H^2[\Omega,\Re]$ into $H^2[\Omega,\Re]$. The conceptual closed loop dynamics for (4.17) are

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial {x_1}^2} + D\frac{\partial^2 \rho}{\partial {x_2}^2}$$

$$\frac{\partial \rho}{\partial t} = D\nabla^2 \rho \tag{4.20}$$

which is similar to (4.10). As we have already shown the origin of this equation is asymptotically stable. In addition there exists a conceptual Lyapunov functional

$$V(t) = \tfrac{1}{2}\|\rho\|_2^2 = \tfrac{1}{2}\int_\Omega |\rho|^2 d\Omega$$

for this system which satisfies $\frac{dV(t)}{dt} \le -\beta V(t)$ and ensures its stability.

2. From the knowledge of the conceptual Lyapunov function $V(t)$ we design a smooth feedback control to stabilize the origin of the overall system. We therefore add to the conceptual Lyapunov function (4.12) a term penalizing the difference between q and $G(\rho)$. For this purpose we rewrite the dynamics (4.17) as

$$\frac{\partial \rho}{\partial t} = -\sum_{i=1}^{2} \frac{\partial G_i(\rho)}{\partial x_i} - \sum_{i=1}^{2} \frac{\partial}{\partial x_i}(q_i - G_i(\rho))$$

Defining the difference between q_i and $G_i(\rho)$ by error variables $z_i = q_i - G_i(\rho)$, we get the following modified dynamics.

$$\frac{\partial \rho}{\partial t} = -\sum_{i=1}^{2} \frac{\partial G_i(\rho)}{\partial x_i} - \sum_{i=1}^{2} \frac{\partial z_i}{\partial x_i} \tag{4.21}$$

$$\frac{\partial z_i}{\partial t} = u_{ni}, \qquad i = 1,2 \tag{4.22}$$

where $u_{ni} = \bar{u}_i - \partial G_i(\rho)/\partial t$ are the new control variables and $z_i(x,t) \in H^2[(\Omega, \Re]$.

3. Now let us modify the Lyapunov functional (4.12) by adding the error term to it thus resulting in the Lyapunov function for the overall system as

$$V_a(t) = V(t) + \tfrac{1}{2}\|z(x,t)\|_2^2 = \tfrac{1}{2}\int_\Omega |\rho|^2\, dx + \tfrac{1}{2}\int_\Omega \sum_{i=1}^{2} |z_i|^2\, d\Omega \tag{4.23}$$

The time rate of change of this functional using (4.21) and (4.22) is given as

$$\frac{dV_a(t)}{dt} = D\int_\Omega \rho \nabla^2 \rho\, d\Omega + \int_\Omega \sum_{i=1}^{2} \rho \frac{\partial z_i}{\partial x_i} d\Omega + \int_\Omega \sum_{i=1}^{2} z_i u_{ni}\, d\Omega$$

From section 4.2.4 we know that the first term is bounded by $-\beta V(t)$. Therefore

$$\frac{dV_a(t)}{dt} \le -\beta V(t) + \int_\Omega \sum_{i=1}^{2} \rho \frac{\partial z_i}{\partial x_i} d\Omega + \int_\Omega \sum_{i=1}^{2} z_i u_{ni}\, d\Omega$$

We have to choose a new control law $u_{ni}(x,t)$ such that the time derivative of the new functional or the sum of second and third terms is bounded by some negative definite function. By choosing the control law

$$u_{ni}(x,t) = -Kz_i - z_i^{-1} \rho \frac{\partial z_i}{\partial x_i} \tag{4.24}$$

we get the following result

$$\frac{dV_a(t)}{dt} \le -\beta V(t) - K\|z\|_2^2 = -2\beta V_a(t)$$

with $K = 2\beta > 0$. This shows that the origin ($\rho = 0, z_1 = 0, z_2 = 0$) of the system (4.21) and (4.22) is exponentially stable.

4. With the feedback controller (4.24) we have proved that the origin ($\rho = 0, z_1 = 0, z_2 = 0$) of (4.21) and (4.22) is exponentially stable. But we need to prove the exponential stability of the origin ($\rho = 0, q_1 = 0, q_2 = 0$) of the system

(3.20) and (3.21). The closed loop stability can be established by proposition 4.3.1 given as

Proposition 4.3.1 *The origin ($\rho = 0, q_1 = 0, q_2 = 0$) of the original system given by (4.17) and (4.18) is exponentially stable under the control law $\bar{u}_i = u_{ni} + -\partial G_i(\rho)/\partial t$ with u_{ni} given by (4.24).*

Proof: The proof is similar to that in proposition 3.3.1. The control laws (4.19) and (4.24) make the origin ($\rho = 0, z_1 = 0, z_2 = 0$) of (4.21) and (4.22) asymptotically go to zero. From the equation $z_i = q_i - G_i(\rho)$ we want to prove the asymptotic stability of ($q_1 = 0, q_2 = 0$). We will show that as $t \to \infty$ the control law $G_i(\rho) \to 0$. The control law $G_i(\rho)$ is given as

$$G_i(\rho) = -D\int_0^{x_i} \frac{\partial^2 \rho}{\partial m^2} dm \qquad i = 1,2 \tag{4.25}$$

Therefore using (4.25) we need to prove the following

$$\text{limit}_{t\to\infty} G_i(\rho(x,t)) = \text{limit}_{t\to\infty}\left(-D\int_0^{x_i} \frac{\partial^2 \rho}{\partial s^2} ds\right) = 0 \tag{4.26}$$

The proof is done in the following three steps as in one-dimensional case.

Step1: Solution of heat equation. The solution of the two-dimensional heat equation (4.20) using method of separation of variables [24] is

$$\rho(x,y,t) = \sum_{n=1}^{\infty} A_n e^{-D\lambda_n^2 t}[\sin(\lambda_n x_1) + \sin(\lambda_n x_2)] \tag{4.27}$$

where $\lambda_n = n\pi$ and the coefficients in (3.31) are given by

$$A_n = 4\int_0^{\Omega} \phi(x,y)[\sin(n\pi x_1) + \sin(n\pi x_2)]d\Omega \tag{4.28}$$

where the function $\phi(x,y) = \rho(x,y,0)$ is the initial condition. The double derivate of density with respect to x_1 and x_2 in (4.27) is given as

$$\frac{\partial^2 \rho}{\partial x_1^2} = -(\pi)^2 \sum_{n=1}^{\infty} n^2 A_n e^{-D\lambda_n^2 t} \sin(\lambda_n x_1) \tag{4.29}$$

and

$$\frac{\partial^2 \rho}{\partial x_2^2} = -(\pi)^2 \sum_{n=1}^{\infty} n^2 A_n e^{-D\lambda_n{}^2 t} \sin(\lambda_n x_2) \tag{4.30}$$

Step 2: Convergence of $\partial^2\rho/\partial x_1$ and $\partial^2\rho/\partial x_2$. The next step is to show the convergence of rate of derivatives (4.29) and (4.30). Let us construct a sequence of time denoted by t_k with $k = 1,2,......\infty$. Now for a fixed time sample t_k we have the estimate of absolute value $\partial^2\rho/\partial x_1$ given by

$$\left|\partial^2\rho/\partial x_1\right|_{t=t_k} = \pi^2 \sum_{n=1}^{\infty} \left| n^2 A_n e^{-D\lambda_n{}^2 t}{}_k \sin(\lambda_n x_1)\right| \le D\pi^2 \sum_{n=1}^{\infty} \left| n^2 A_n e^{-D\lambda_n{}^2 t}{}_k \right| \tag{4.31}$$

since $|\sin(\lambda_n x)| \le 1$ for $(x,t) \in [0,L]\times[0,\infty)$. Thus we have

$$\left|\partial^2\rho/\partial x_1\right|_{t=t_k} \le D\pi^2 \sum_{n=1}^{\infty} \left| n^2 A_n e^{-D\lambda_n{}^2 t}{}_k \right| \le D\pi^2 \sum_{n=1}^{\infty} \left| \frac{n^2 A_n}{(n^2\pi^2 t_k)^2} \right|$$

The coefficients given by (4.28) are bounded as

$$|A_n| \le 8\int_0^{\Omega} |\phi(x,y)| d\Omega = 8\int_0^{\Omega} |\rho(x,y,0)| d\Omega \le \overline{A} \tag{4.31}$$

where $\overline{A}$ is a positive constant and $\overline{A}$ exists because $\phi(x,y) = \rho(x,y0)$ is continuous on a compact (i.e; closed and bounded) interval. Therefore we have

$$\left|\partial^2\rho/\partial x_1\right|_{t=t_k} \le \frac{D\overline{A}}{\pi^2 t_k{}^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \tag{4.32}$$

Similarly, the absolute value of the double derivate of density with respect to x_1 is bounded as

$$\left|\partial^2\rho/\partial x_2\right|_{t=t_k} \le \frac{D\overline{A}}{\pi^2 t_k{}^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \tag{4.33}$$

The above equations are similar to (3.36) in proposition 3.3.1. and from there we conclude that as $t \to \infty$ the derivates $\partial^2\rho/\partial x_1$ and $\partial^2\rho/\partial x_2 \to 0$.

Step 3: Convergence of $G_i(\rho(x,t))$. So far we have shown the convergence of $\partial^2\rho/\partial x_1$ and $\partial^2\rho/\partial x_2$ to zero. From the *step 3* of proposition 3.3.1 we can show the following

$$\text{limit}_{t\to\infty} G_i(\rho(x,t)) = \text{limit}_{t\to\infty}\left(-D\int_0^{x_i}\frac{\partial^2\rho}{\partial s^2}ds\right) = 0 \quad ; \; i = 1,2 \tag{4.34}$$

This completes the proof. ◊

Thus we conclude that the origin ($\rho = 0, q_1 = 0, q_2 = 0$) for actual closed loop system (4.17) and (4.18) is exponentially stable. The feedback control laws are given by the following partial differential-integral equation

$$u_1 = u_{n1} + \frac{\partial G_1(\rho)}{\partial t} + \frac{\partial}{\partial x_1}(q_1^{\,2}/\rho) + \frac{\partial}{\partial x_1}(q_1 q_2/\rho)$$

$$u_2 = u_{n2} + \frac{\partial G_2(\rho)}{\partial t} + \frac{\partial}{\partial x_1}(q_1 q_2/\rho) + \frac{\partial}{\partial x_2}(q_2^{\,2}/\rho) \tag{4.35}$$

with $u_{ni}(x,t)$ given by (4.24) as $u_{ni}(x,t) = -Kz_i - z_i^{-1}\rho\,\partial z_i/\partial x_i$ where $z_i = q_i - G_i(\rho)$. Hence the closed loop dynamics for two equation model are exponentially stable with this control law.

4.3.4 Simulation Results

Here we show simulation results for the closed loop system (4.17) and (4.18) using controller (4.25) designed in previous section. The numerical technique used to simulate the system is same as before. For simulation the initial distribution for both density and flow is considered to be Gaussian. The density plots are shown in Fig.4.3 and Fig. 4.4 and are similar to those in Fig. 4.1 and Fig. 4.2. The flow plots are shown in Fig. 4.5 and Fig. 4.6. As seen from the plots after some finite time, the density and flows in both directions have decreased to zero.

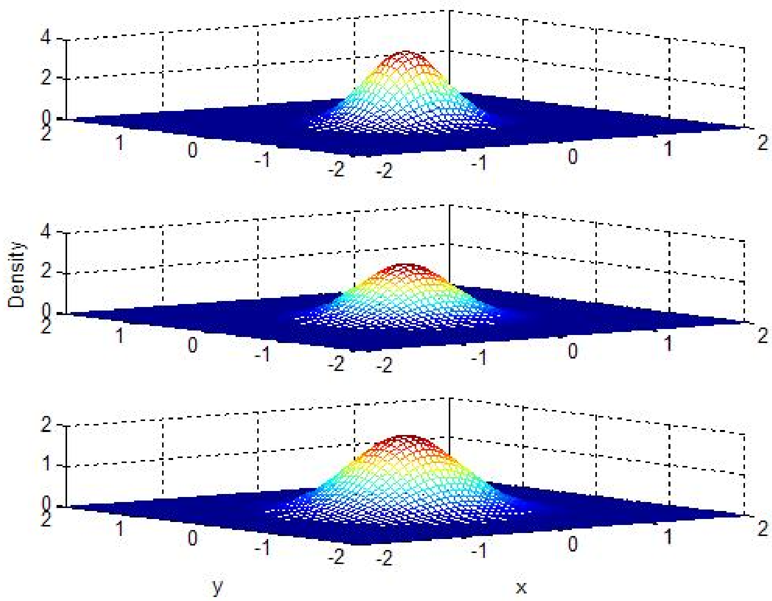


Figure 4. 3: Density response at different time instants for two-equation model.

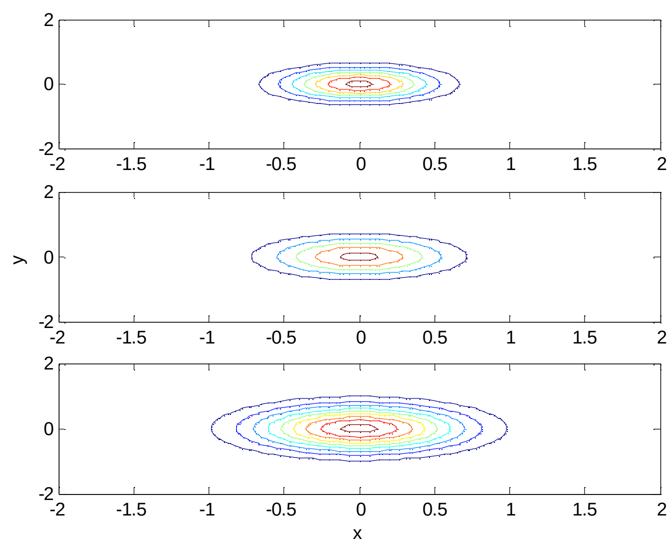


Figure 4. 4: Contours of the density response at different time instants for two-equation model.

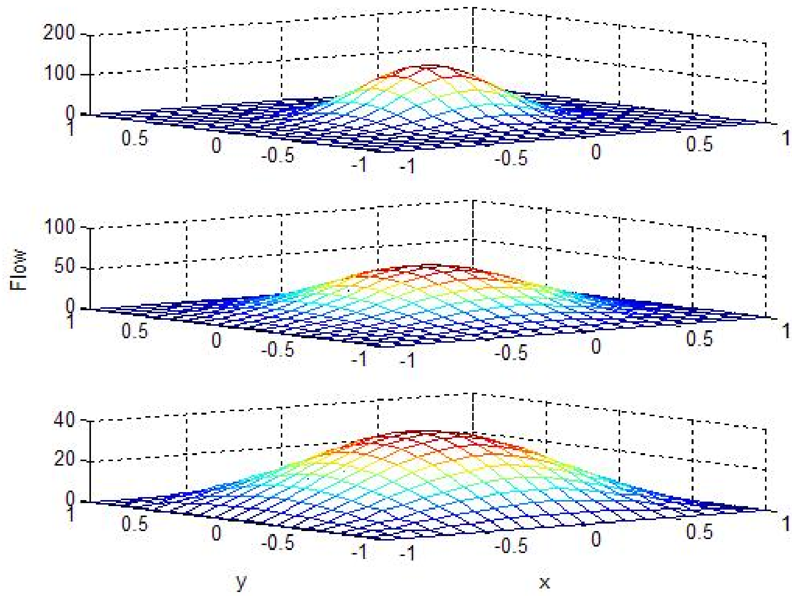


Figure 4. 5: Flow response at different time instants for two-equation model.

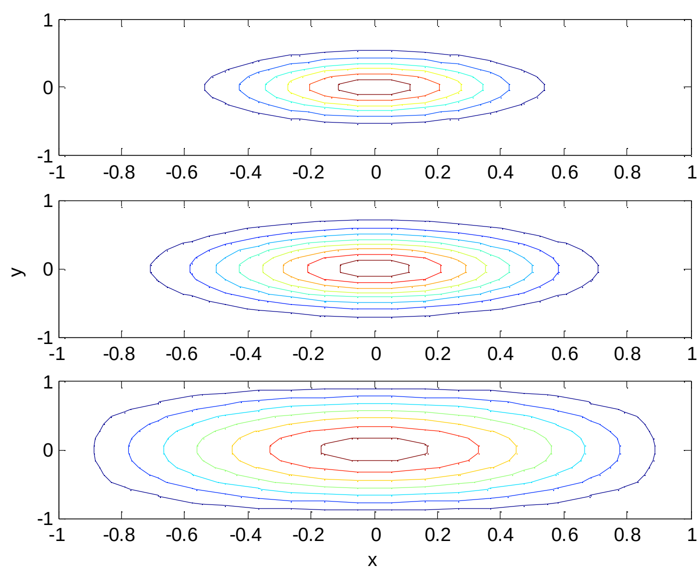


Figure 4. 6: Contours of the flow response at different time instants for two-equation model.

In this chapter we discussed the design of feedback controllers for two infinite dimensional models of evacuation in two dimensions. With the use of these controllers we have shown exponential stability of the closed loop systems. In next two chapters we will modify these controllers in order to add motion to closed-loop dynamics.

CHAPTER 5

ADVECTION FEEDBACK CONTROL DESIGN AND SATURATION OF ONE DIMENSIONAL CROWD MODELS

5.1 Introduction

This chapter presents the design of advection feedback controllers for models given in chapter 2 representing crowd dynamics in one dimension. These controllers are a modification of the controllers discussed in chapter 3. In chapter 3 we designed nonlinear feedback controllers for one dimensional evacuation models by feedback linearization. The controllers were designed such that the nonlinearity in the system got cancelled and the closed-loop dynamics were asymptotically stable. However the closed-loop dynamics represented by the heat equation suggest that the people are diffusing throughout the area of evacuation. The initial density profile is decreasing with time which means that there is diffusion but the profiles are not moving. In other words there is no direction to the flow of people. In order to have an effective evacuation, the direction of motion is important. We want to control the movement of people in such a way that the density profiles at every point are changing (decreasing) as well as moving with time. So in this chapter we modify the control laws designed in chapter 3 by adding a convective component so that there is diffusion as well as direction to the motion. Furthermore the controllers discussed in chapter 3 are assumed to be unbounded which is impossible because of control saturation. We will discuss these issues in this chapter and modify the control laws to take saturation into account.

The organization of this chapter is as follows. In Section 5.2 we modify the feedback control law for the one-equation model and present some simulation results. Section 5.3 presents the modified feedback control design and stability analysis for the two-equation system. Simulation results for closed loop dynamics are also presented.

5.2 Advection Control for One-Equation Model

The control model for the one-equation model (3.7) under the control law (3.8)

$$F(\rho(x,t)) = -[(1-\rho/\rho_{\max})\rho]^{-1} D \frac{\partial \rho}{\partial x} \tag{5.1}$$

gives the closed loop dynamics (3.9)

$$\frac{\partial \rho}{\partial t} - D\frac{\partial^2 \rho}{\partial x^2} = 0 \tag{5.2}$$

which represent the heat equation with D being the diffusion constant. (5.2) suggests that people are diffusing throughout the entire area with the rate of diffusion being determined by the diffusion constant D. Here the net movement of people is due to diffusion only and there is no direction to the motion. Direction is important for an effective evacuation. In this section we will give motion to the model by adding an advection component to the control law (5.1). First we discuss the advection control alone without a diffusion term. Next we add a diffusion term to the control law to increase the rate at which people are moving out of the evacuation area.

5.2.1 Advection Control

In this subsection we discuss the advection control law for the one-equation control model (3.7)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho u(1-\rho/\rho_{\max})\right) = 0 \tag{5.3}$$

First we discuss a convective controller given as

$$u = F(\rho(x,t)) = V\left[(1-\rho/\rho_{\max})\right]^{-1} \tag{5.4}$$

The closed loop dynamics can be found by substituting (5.4) in (5.3) to get

$$\frac{\partial \rho}{\partial t} + V\frac{\partial \rho}{\partial x} = 0 \tag{5.5}$$

The dynamics in equation (5.5) suggest that people are moving with an average speed V towards the exit. Both the direction and magnitude of this speed can be controlled. The solution of advection equation is given by

$$\tilde{\rho}(x,t) = \rho_0(x - Vt) \tag{5.6}$$

where $\rho_0(x) = \rho(x,t_0)$ is the initial density profile. (5.6) suggests that the initial density profile is moving at a velocity V. In order to study the stability of (5.5) we choose the Lyapunov function given by

$$V(t) = \int_0^L (\rho - \tilde{\rho})^2 dx = \int_0^L (\rho - \rho_0(x - Vt))^2 dx \tag{5.7}$$

The time rate of change of Lyapunov is given by $\frac{dV(t)}{dt} = -2V\int_0^L \rho \frac{\partial \rho}{\partial x} dx + 2V\int_0^L \rho_0 \frac{d\rho_0}{dx} dx$.

Since (5.6) is the solution it should satisfy (5.5) for all *x* and *t*, therefore we have $\frac{\partial \rho}{\partial x} = \frac{d\rho_0}{dx}$

and as such $\frac{dV(t)}{dt} = 0$. Thus the equilibrium of the closed loop system (5.5) using feedback control (5.4) is exponentially stable. The simulation results are shown in Fig. 5.1 in section 5. 2.3

5.2.2 Advection –Diffusion Control

The direction of motion as given by (5.4) is important in order to have an effective evacuation. However, the diffusion term aids the motion and also becomes important due to the limitation on the control variable. In this section we add the diffusion component to the control law (5.4) so that there is diffusion as well as direction to the motion. The advection-diffusion controller is a combination of (5.1) and (5.4) and is given by:

$$u = F(\rho(x,t)) = -[(1-\rho/\rho_{\max})\rho]^{-1} D\frac{\partial \rho}{\partial x} + V[(1-\rho/\rho_{\max})]^{-1} \tag{5.9}$$

$$= -[(1-\rho/\rho_{\max})\rho]^{-1}(D\frac{\partial \rho}{\partial x} - V\rho)$$

The closed loop dynamics can be found by substituting (5.9) in (5.3) to get

$$\frac{\partial \rho(x,t)}{\partial t} + V\frac{\partial \rho}{\partial x} - D\frac{\partial^2 \rho}{\partial x^2} = 0 \tag{5.10}$$

Here V is the velocity of moving currents (convection or advection) or the medium. The people move with speed V towards the exit and at the same time diffuse with diffusion constant D. The direction of motion can be controlled by changing V whereas the magnitude can be controlled by changing both V and D. The Lyapunov function as in (3.16) is given by $V(t) = \int_0^L \rho^2 dx$ and its derivative with respect to time t is

$$\frac{dV(t)}{dt} = D\int_0^L \rho\frac{\partial^2 \rho}{\partial x^2}dx - V\int_0^L \rho\frac{\partial \rho}{\partial x}dx$$

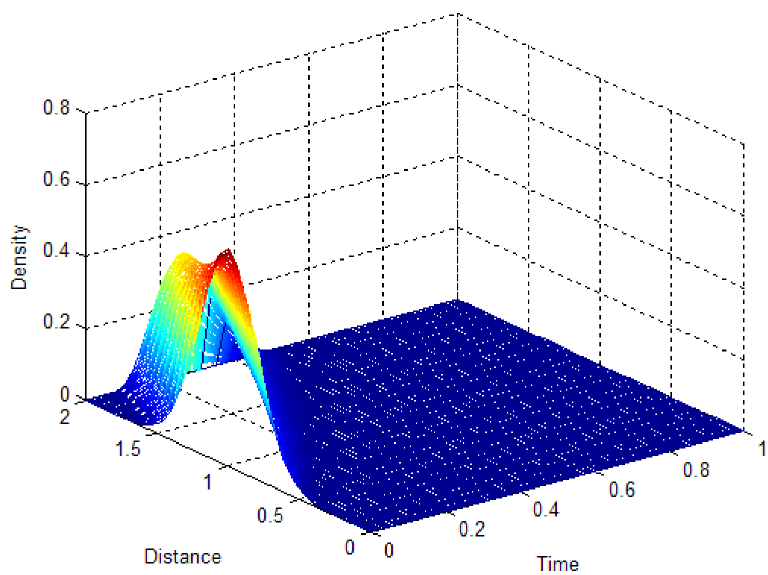
By using inequality (3.17) for first term and Holders inequality for the second we get the following result

$$\frac{dV(t)}{dt} \le -DC^{-1}\|\rho\|_2^2 - V\|\rho\|_2\|\partial\rho/\partial x\|_2 \tag{5.10}$$

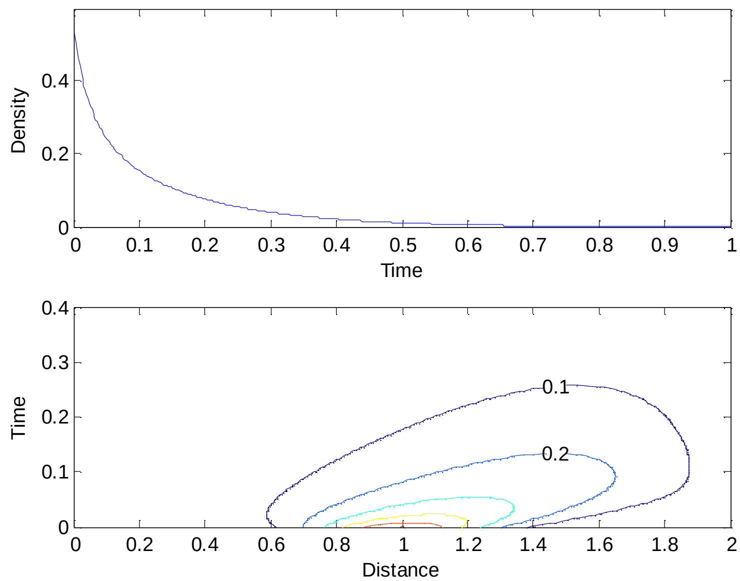
Thus for $DC^{-1} > 0$, the equilibrium of closed loop system (5.10) using feedback control (5.9) is exponentially stable. Simulation results are shown in Fig. 5.2.

5.2.3 Simulation Results

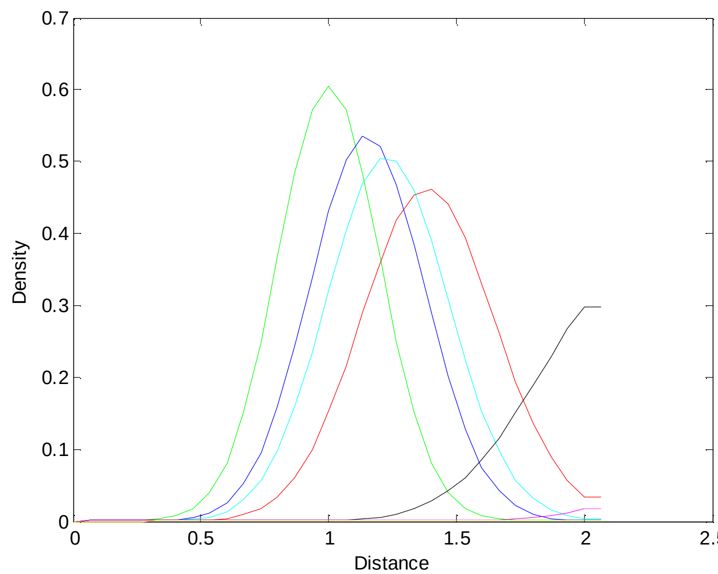
This section shows simulation results for advection and advection-diffusion control laws. The initial distribution of density is considered to be Gaussian. Simulation results are shown in Fig. 5.1, 5.2 and 5.3. Density response for advection control is shown in Fig. 5.1. In Fig 5.1(a) density response is shown as a mesh plot. We can see that the initial density profile is moving towards the right (exit) of the corridor. The second simulation is shown as a contour plot in Fig. 5.1 (b) where the contour lines vary from 0.6 to 0.1. The top plot represents the density contour at a fixed distance of $L/2$. Fig. 5.1(c) shows the density response at different time instants where the density response flattens with increase in time. The density response for advection-diffusion control is shown in Fig. 5.2. As seen from the plot in Fig. 5.2 (b) the initial density profile is decreasing as well as moving with time towards the right (exit) of the corridor at a faster rate than before. There is both advection and diffusion. The comparison between the controllers is shown in Fig. 5.3.



(a)

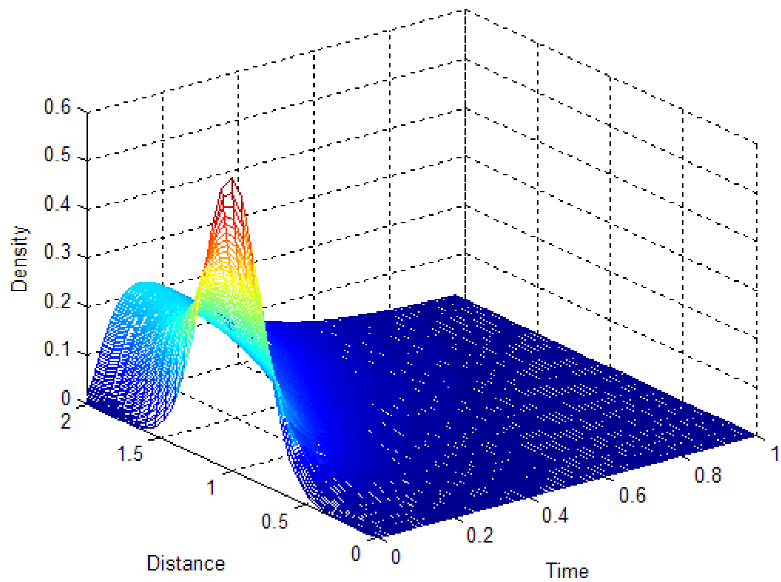


(b)



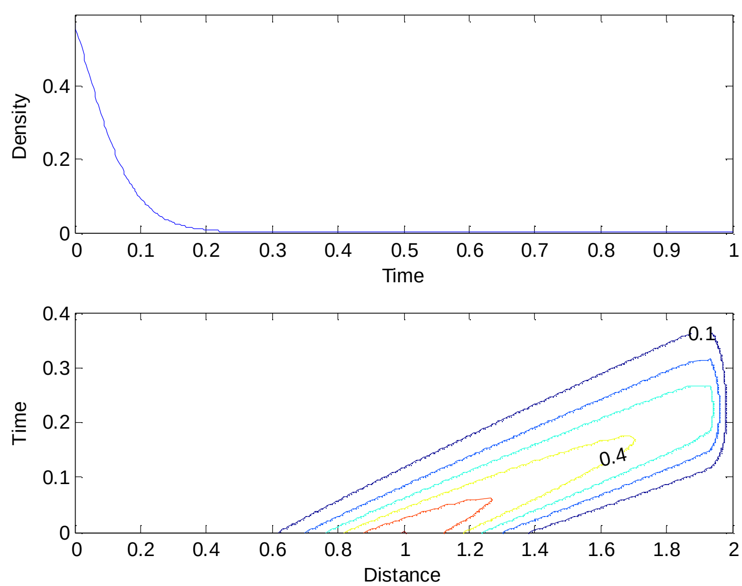
(c)

Figure 5. 1: Advection control response for one-equation model (a): Density response, (b): Contours of the response, (c): Density response at different time instants



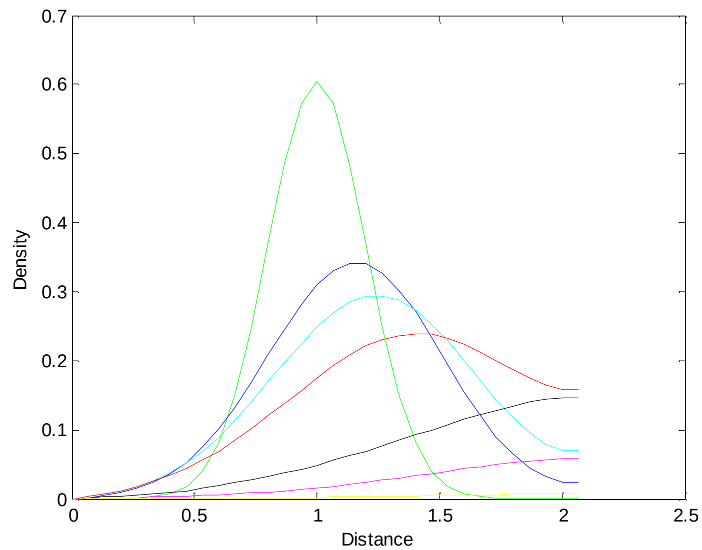
Density
0.6
0.5
0.4
0.3
0.2
0.1
0
2
1.5
1
0.5
0
Distance
0
0.2
0.4
0.6
0.8
1
Time

(a)



Density
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Time
Time
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
Distance
0.1
0.4

(b)



(c)

Figure 5. 2: Advection-diffusion control response for one-equation model (a): Density response, (b): Contours of the response, (c): Density at different time instants.

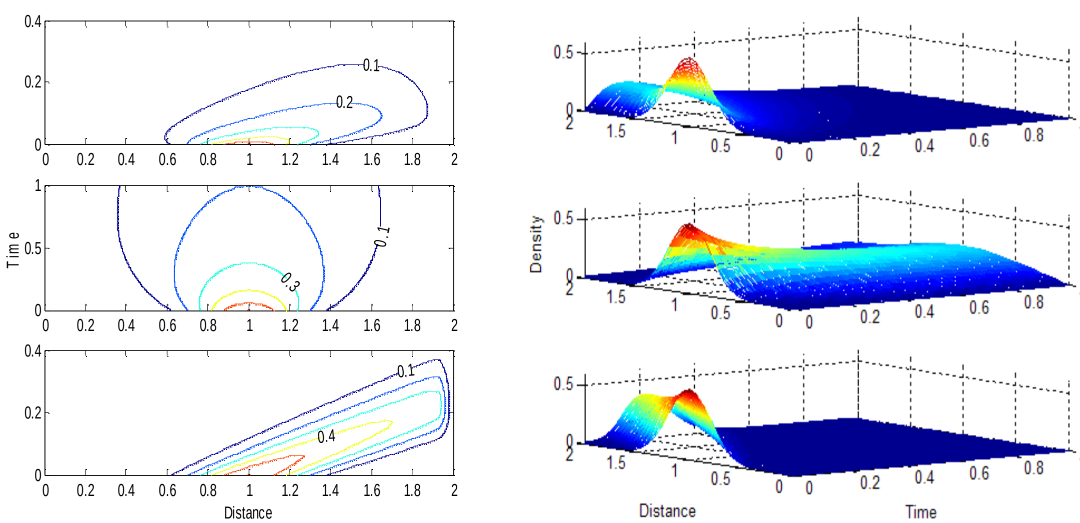


Figure 5. 3: Comparison of advection, diffusion and advection-diffusion control responses for one-equation model at different times

5.3 Advection Control for Two-Equation Model

In this section we discuss advection control for a two-equation model. A convective term added to the control law gives motion to the model. The control strategy is the same as in section 3.3.3 where we used the method of *backstepping*. First we discuss the advection control alone without a diffusion term. Next we add the diffusion term to the control law.

5.3.1 Advection Control

The two-equation control model given by (3.20) and (3.21) is

$$\frac{\partial \rho}{\partial t} = -\frac{\partial q}{\partial x} \tag{5.11}$$

$$\frac{\partial q}{\partial t} = -\frac{\partial}{\partial x}(q^2/\rho) + u = \overline{u} \tag{5.12}$$

where $u = -\frac{\partial p}{\partial x}$ is the control variable and $\overline{u} = -\frac{\partial}{\partial x}(q^2/\rho) + u$. The control law for (5.11) is given by (3.28)

$$q(x,t) = G(\rho) = -D\int_0^x \frac{\partial^2 \rho}{\partial m^2}\, dm \tag{5.13}$$

which makes the origin of the above system asymptotically stable. Now let us discuss the advection control for this system by designing the convective controller for (5.11) given as

$$q(x,t) = G(\rho) = V\int_0^x \frac{\partial \rho}{\partial m}\, dm \tag{5.14}$$

such that the closed loop dynamics for (5.11) are

$$\frac{\partial \rho(x,t)}{\partial t} + V\frac{\partial \rho}{\partial x} = 0 \tag{5.15}$$

which is same as (5.5). The above dynamics have an asymptotically stable origin and a Lyapunov function given by (5.7) which satisfies $\frac{dV(t)}{dt} = 0$. The second step is to design a feedback controller for (5.12) using *backstepping* method used in section 3.3.3. The goal is to design a feedback control law $\bar{u}$ to stabilize the overall system from the knowledge of the Lyapunov function $V(t)$ for (5.11) and modifying it. Since q is not the actual control variable, we define the difference between q and $G(\rho)$ by an error variable $z = q - G(\rho)$ and obtain the following modified dynamics.

$$\frac{\partial\rho}{\partial t} = -\frac{\partial G(\rho)}{\partial x} - \frac{\partial z}{\partial x} \tag{5.16}$$

$$\frac{\partial z}{\partial t} = u_n \tag{5.17}$$

where $u_n = \bar{u} - \partial G(\rho)/\partial t$ is the new control variable. Now let us modify the Lyapunov functional by adding the error term to (5.7) as

$$V(t) = \int_0^L (\rho - \tilde{\rho})^2 dx + \frac{1}{2}\int_0^L z^2 dx$$

Therefore the control law as given by (3.27) is

$$u_n(x,t) = -Kz - z^{-1}\rho\frac{\partial z}{\partial x} \tag{5.18}$$

with $z = q - G(\rho)$ where $G(\rho)$ is now given by (5.14) . This control ensures that the origin ($\rho(x,t) = 0, z(x,t) = 0$) of the system (5.16) and (5.17) is asymptotically stable. The final feedback control law is given by the following partial differential-integral equation

$$u = u_n + \frac{\partial G(\rho)}{\partial t} + \frac{\partial}{\partial x}(q^2/\rho) \tag{5.19}$$

with $u_n(x,t)$ given by (5.18) and $G(\rho) = V\int_0^x \frac{\partial\rho}{\partial m} dm$. To prove the asymptotic stability of the origin ($\rho(x,t) = 0, q(x,t) = 0$) of the system (5.11) and (5.12) we use the following proposition.

Proposition 5.3.1 The origin ($\rho(x,t) = 0, q(x,t) = 0$) of the original system (5.11) and (5.12) is exponentially stable under the control law $\bar{u} = u_n + \partial G(\rho)/\partial t$ with u_n given by (5.18) and $G(\rho) = V\int_0^x \frac{\partial\rho}{\partial m} dm$

Proof: The proof of this proposition is similar to the proof given for proposition 3.3.1 with the difference that here we use the solution of advection equation instead of the heat equation in step 1. The solution of an advection equation is given by (5.16) as

$$\tilde{\rho}(x,t) = \rho_0(x - Vt)$$

From the solution we see that advection for compactly supported initial data and moving in one direction will have pointwise convergence automatically. The rest of the proof follows as in proposition 3.3.1. Thus we conclude that the origin ($\rho(x,t) = 0, q(x,t) = 0$) for the actual closed- loop system is asymptotically stable.

5.3.2 Advection-Diffusion Control

In this section we add the diffusion component to the control law (5.14) so that there is diffusion as well as direction to the motion. The convective-diffusion controller given as the combination of (5.13) and (5.14) is

$$q(x,t) = G(\rho) = V\int_0^x \frac{\partial \rho}{\partial m} dm - D\int_0^x \frac{\partial^2 \rho}{\partial m^2} dm \tag{5.20}$$

The closed loop dynamics can be found by substituting (5.20) in (5.11) as

$$\frac{\partial \rho(x,t)}{\partial t} + V\frac{\partial \rho}{\partial x} - D\frac{\partial^2 \rho}{\partial x^2} = 0 \tag{5.21}$$

which is same as (5.9). The above dynamics have an asymptotically stable origin and a Lyapunov function given by $V(t) = \int_0^L \rho^2 dx$ which satisfies condition (5.10). The second step is to design a feedback controller for (5.12) using *backstepping* as before. The final feedback control law for the two-equation model is thus given by the following partial differential-integral equation

$$u = u_n + \frac{\partial G(\rho)}{\partial t} + \frac{\partial}{\partial x}(q^2/\rho)$$

with $u_n(x,t)$ given by (5.18) as

$$u_n(x,t) = -Kz - z^{-1}\rho\frac{\partial z}{\partial x}$$

with $z = q - G(\rho)$ and $G(\rho) = V\int_0^x \frac{\partial \rho}{\partial m} dm - D\int_0^x \frac{\partial^2 \rho}{\partial m^2} dm$. This control ensures that the origin ($\rho(x,t) = 0, z(x,t) = 0$) of the system (5.16) and (5.17) is asymptotically stable. To prove the asymptotic stability of the origin ($\rho(x,t) = 0, q(x,t) = 0$) of the system (5.11) and (5.12) we use the proposition 5.3.2.

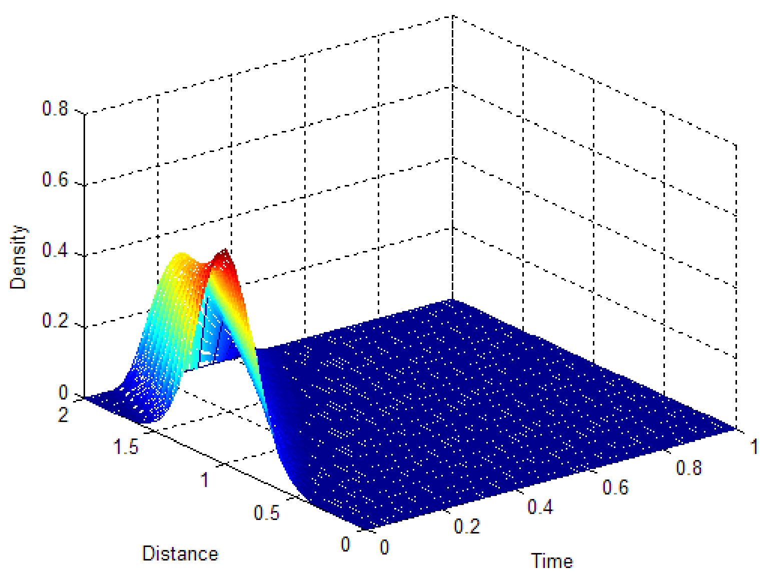
Proposition 5.3.2 *The origin (* $\rho(x,t) = 0, q(x,t) = 0$ *) of the original system (5.11) and (5.12) is exponentially stable under the control law* $\bar{u} = u_n + \partial G(\rho)/\partial t$ *with* u_n *given by (5.18) and*

$$G(\rho) = V\int_0^x \frac{\partial \rho}{\partial m} dm - D\int_0^x \frac{\partial^2 \rho}{\partial m^2} dm$$

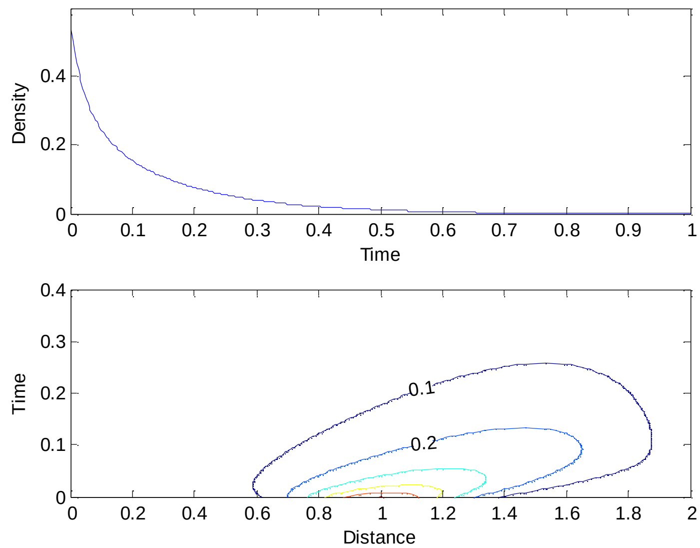
Proof: The proof follows from the proofs of proposition 3.3.1 and proposition 5.3.1.

5.3.3 Simulation Results

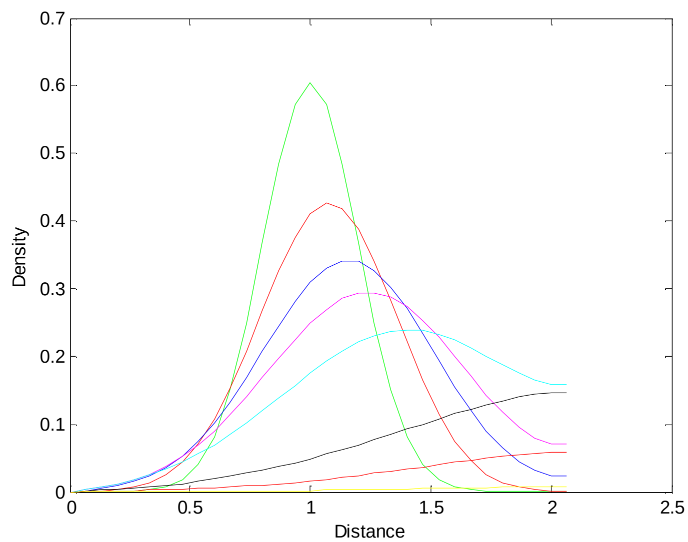
This section shows simulation results for the advection and advection-diffusion control laws for the two-equation model. The simulation results are shown in Fig. 5.4 and Fig. 5.5. As is seen from the figures, the evacuation is faster in case of advection-diffusion.



(a)

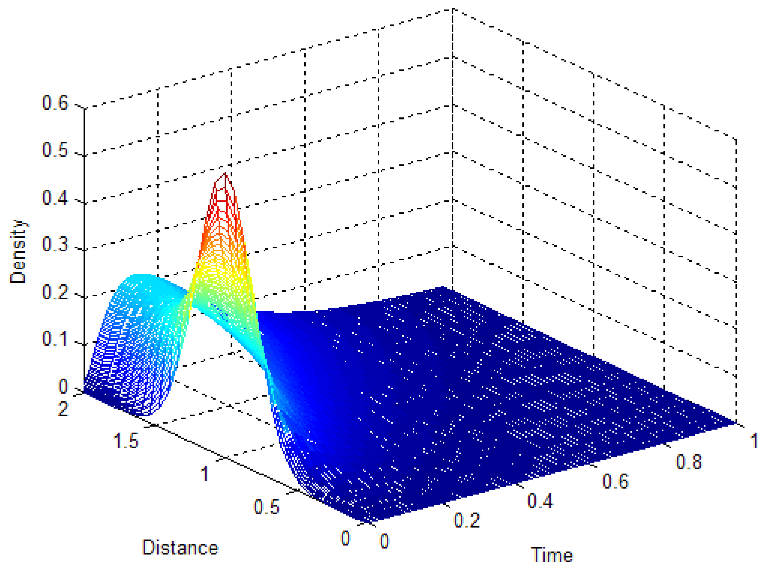


(b)



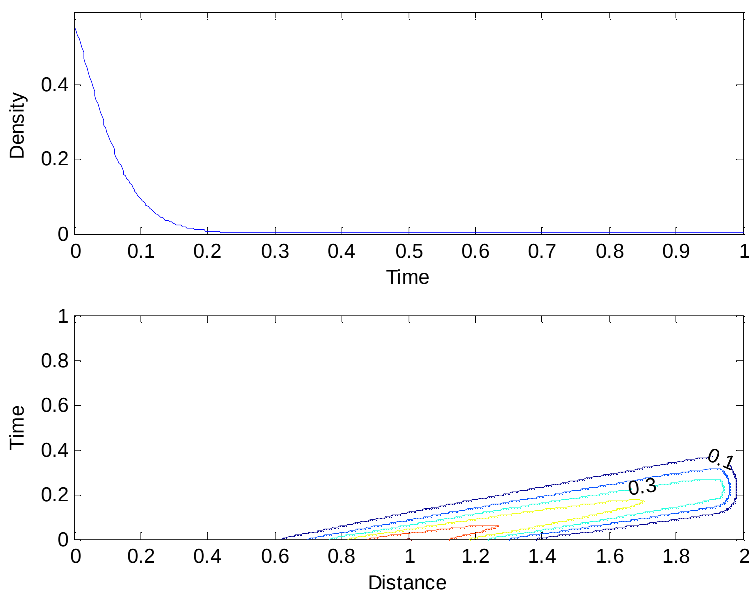
(c)

Figure 5. 4: Advection control response for two-equation model (a): Density response, (b): Contours of the response, (c): Density at different time instants.



Density
0.6
0.5
0.4
0.3
0.2
0.1
0
2
1.5
1
0.5
0
Distance
0
0.2
0.4
0.6
0.8
1
Time

(a)



Density
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Time
1
0.8
0.6
0.4
0.2
0
0.1
0.3
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
Distance

(b)

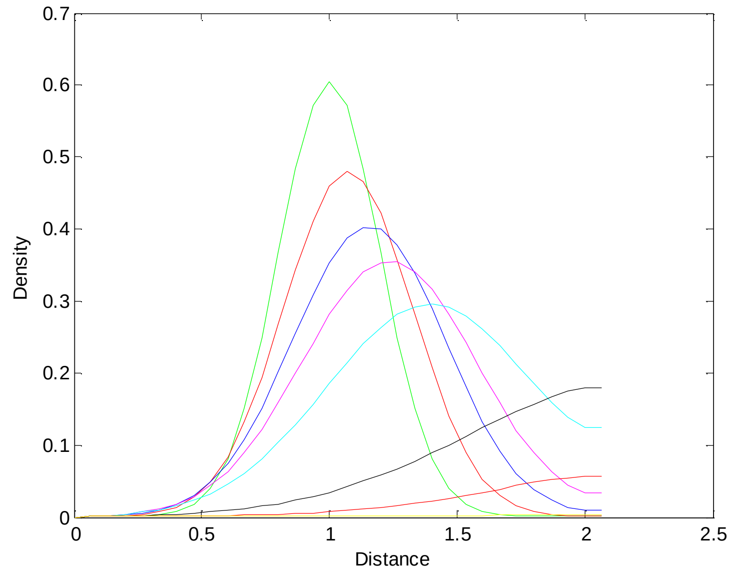


Figure 5. 5: Advection-Diffusion control response for two-equation model (a): Density response, (b): Contours of the response, (c): Density at different time instants.

5.4 Control Saturation

The feedback controllers for the one-equation model discussed in chapter 3 and in this chapter have saturation issues which will be discussed here. The control values in these controllers become unbounded. So far in the previous chapters we have assumed unbounded controls. However, the unboundedness poses a limitation on the practical implementation of the controllers. Therefore in order to take care of control saturation we redesign our controllers so that there is no unboundedness of control.

5.4.1 Saturation for One-Equation Model

In the control model (5.3) by using the control law (5.1) we have the issue of control saturation. The control law (5.1) is

$$F(\rho(x,t)) = -[(1-\rho/\rho_{\max})\rho]^{-1} D\frac{\partial \rho}{\partial x} \tag{5.21}$$

With this control we could cancel the nonlinearity in the system and convert the system into a heat equation. But the issue with this control is that it is unbounded because of an inverse term in it. At $\rho(x,t) = \rho_{\max}$ the control becomes unbounded. This situation is practically impossible as we have limitations on the implementation of the control variable. Thus in order to take care of control saturation we redesign our controllers so that there is no unboundedness of control. We choose the following control law

$$F(\rho(x,t)) = \begin{cases} \frac{1}{\rho} D\frac{\partial \rho}{\partial x} & \rho \neq 0 \\ v_{\min} & \rho = 0 \end{cases} \tag{5.22}$$

With this control law for $\rho \neq 0$ we get the following closed loop dynamics

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left((1-\rho/\rho_{\max})D\frac{\partial \rho}{\partial x}\right) = 0$$

or

$$\frac{\partial \rho}{\partial t} + (1-\rho/\rho_{\max})D\frac{\partial^2 \rho}{\partial x^2} - 1/\rho_{\max}\, D(\frac{\partial \rho}{\partial x})^2 = 0 \tag{5.23}$$

For the above closed-loop dynamics let us choose a Lyapunov functional $V(t)$ given by (5.6) as

$$V(t) = \tfrac{1}{2}\|\rho\|_2^2 = \tfrac{1}{2}\int_0^L |\rho|^2 dx$$

The time rate of change of $V(t)$ using Leibniz rule is given as $\frac{dV(t)}{dt} = \int_0^L \rho \frac{\partial \rho}{\partial t} dx$ .Using (5.23) the time derivate of Lyapunov becomes

$$\frac{dV(t)}{dt} = -D\int_0^L \rho[(1-\rho/\rho_{\max})\frac{\partial^2 \rho}{\partial x^2} - 1/\rho_{\max}\,(\frac{\partial \rho}{\partial x})^2]dx$$

$$\frac{dV(t)}{dt} = -D\int_0^L \rho(1-\rho/\rho_{\max})\frac{\partial^2 \rho}{\partial x^2}dx + D\int_0^L \rho/\rho_{\max}\,(\frac{\partial \rho}{\partial x})^2 dx \tag{5.24}$$

For the simplicity, equation (5.24) can be rewritten as

$$\frac{dV(t)}{dt} = -I_1 + I_2 \tag{5.25}$$

Integrating the first integral in (5.25) gives us

$$I_1 = D\rho(1-\rho/\rho_{\max})\frac{\partial\rho}{\partial x}\bigg|_0^L - D\int_0^L (1+2\rho/\rho_{\max})(\frac{\partial\rho}{\partial x})^2 dx \tag{5.26}$$

The first term of (5.26) vanishes by boundary condition (3.3). Thus equation (5.24) reduces to the following

$$\frac{dV(t)}{dt} = -D\int_0^L (\frac{\partial\rho}{\partial x})^2 dx - 1/\rho_{\max} D\int_0^L \rho(\frac{\partial\rho}{\partial x})^2 dx \tag{5.27}$$

For the first integral of equation (5.27) we make use of Sobolev inequality (3.16) which for our case is $\|\rho\|_2 \le C\|\nabla\rho\|_2 = C\left\|\frac{\partial\rho}{\partial x}\right\|_2$,where C is a positive real number. Using (3.16) we have

$$\int_0^L (\frac{\partial\rho}{\partial x})^2 dx \ge C^{-2}\int_0^L \rho^2 dx \tag{5.28}$$

The second integral of (5.27) can be written as $I = \int_0^L \rho(\frac{\partial\rho}{\partial x})^2 dx$. For this integral we use Young's inequality [41]

$$I = \frac{1}{2}\int_0^L \rho^2 dx + \frac{1}{2}\int_0^L (\frac{\partial\rho}{\partial x})^4 dx \tag{5.29}$$

The rate of change of V(t) can be thus be found by using (5.28) and (5.29) in (5.27) and is given by

$$\frac{dV(t)}{dt} \le -DC^{-2}\int_0^L \rho^2 dx - 1/2\rho_{\max} D\int_0^L \rho^2 dx - 1/2\rho_{\max} D\int_0^L (\frac{\partial\rho}{\partial x})^4 dx$$

Therefore we have the following bound on rate of change of Lyapunov function

$$\frac{dV(t)}{dt} \le -\alpha\int_0^L \rho^2 dx - \beta\int_0^L (\frac{\partial\rho}{\partial x})^4 dx \tag{5.30}$$

where $\alpha = DC^{-2} - D/2\rho_{\max}$ and $\beta = D/2\rho_{\max}$ with D being a positive constant. Both the first and second terms are positive definite functions, therefore as long as $\alpha > 0$ and

$\beta > 0$ we have $\frac{dV(t)}{dt} \leq 0$. It follows that null state of (5.1) is asymptotically stable using feedback control (5.22).

In this chapter we designed advection and advection-diffusion feedback controllers for one-dimensional evacuation system. We also modified the controllers in order to take control saturation into account. In the models used for feedback control so far we have assumed that there is no uncertainty. However, we want to modify our controllers so that we take into account the uncertainties or disturbances acting on the system. The design of robust feedback controllers is discussed in chapter 7.

CHAPTER 6

ADVECTION FEEDBACK CONTROL DESIGN OF TWO- DIMENSIONAL CROWD MODELS

6.1 Introduction

This chapter presents design of advection feedback controllers for models given in chapter 2 representing crowd dynamics in one dimension. These controllers are a modification to the controllers discussed in chapter 3. In chapter 4, we designed the nonlinear feedback controllers for one dimensional evacuation models by feedback linearization. We designed the controllers such that the nonlinearity in the system got cancelled and the closed-loop dynamics were asymptotically stable. However the closed-loop dynamics represented by the heat equation suggests that the people are diffusing throughout the area of evacuation. The motion of people is due to diffusion only and there is no direction. In order to have an effective evacuation, direction of motion is important. In this chapter we modify the control laws designed in chapter 4 by adding an advection component so that there is diffusion as well as direction to the motion.

The organization of this chapter is as follows. In Section 6.2 we modify the feedback control law for the one-equation model. This section also studies Lyapunov stability for this model and finally presents some simulation results. Section 6.3 presents the modified feedback control design and stability analysis for the two-equation system model. Simulation results for closed loop dynamics are also presented.

6.2 Advection Control for One-Equation Model

The control model for the two-dimensional one-equation model is given by (4.8)

$$\frac{\partial\rho(x,t)}{\partial t}+\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left(\rho u_i(1-\rho/\rho_{\max})\right) \quad ; \quad i=1,2 \tag{6.1}$$

where u_i is the control component in x_i. Using the feedback control law (4.9)

$$F_i(\rho(x,t))=-[(1-\rho/\rho_{\max})\rho]^{-1}D\frac{\partial\rho}{\partial x_i} \tag{6.2}$$

the closed- loop dynamics are

$$\frac{\partial\rho}{\partial t}-D\frac{\partial^2\rho}{\partial x_1{}^2}-D\frac{\partial^2\rho}{\partial x_2{}^2}=0 \tag{6.3}$$

which represents the heat equation with D being the diffusion constant. (6.2) suggests that the people are diffusing throughout the entire area with the rate of diffusion being determined by diffusion constant D. In this section we will add motion to the model by adding an advection component to the control law (6.2) so that there is diffusion as well as direction to the motion. First we shall discuss the advection control alone without a diffusion term. Next we add the diffusion term to the control law to increase the rate at which people are moving out of the evacuation area.

6.2.1 Advection Control

In this subsection we discuss the advection control law for the one-equation control model (6.1). First we discuss the convective controller given as

$$u_i=F_i(\rho(x,t))=V_i[(1-\rho/\rho_{\max})]^{-1}; \qquad i=1,2 \tag{6.4}$$

with V_i being the average velocity in x_i direction. The closed loop dynamics are found by substituting (6.4) in (6.3)

$$\frac{\partial\rho(x,t)}{\partial t}+V_1\frac{\partial\rho}{\partial x_1}+V_2\frac{\partial\rho}{\partial x_2}=0 \tag{6.5}$$

The direction and magnitude of both the speeds can be controlled for an effective evacuation. The solution of an advection equation in two-dimensions is

$$\tilde{\rho}(x,t) = \rho_0(x_1 - V_1 t, x_2 - V_2 t) \tag{6.6}$$

Now in order to study the stability of (6.5) we choose the same Lyapunov function as given by (5.7)

$$V(t) = \int_\Omega (\rho - \tilde{\rho})^2 dx = \int_\Omega (\rho - \rho_0(x_1 - V_1 t, x_2 - V_2 t))^2 d\Omega \tag{6.7}$$

The time rate of change of Lyapunov is $\frac{dV(t)}{dt} = 0$. Thus the equilibrium of closed loop system (6.5) using feedback control (6.4) is exponentially stable. The simulation results are shown in Fig. 6.1 in section 6. 2.3

6.2.2 Advection-Diffusion Control

The direction of motion as given by (6.4) is important in order to have an effective evacuation. However, the diffusion term is also important and therefore in this section we will add a diffusion component to the control law (6.4) so that there is diffusion as well as direction to the motion. The convective-diffusion controller is given as the combination of (6.2) and (6.4) as

$$u_i = -[(1 - \rho/\rho_{\max})\rho]^{-1} D \frac{\partial \rho}{\partial x_i} + V_i [(1 - \rho/\rho_{\max})]^{-1}. \tag{6.9}$$

The closed loop dynamics can be found by substituting (6.9) in (6.1)

$$\frac{\partial \rho(x,t)}{\partial t} + V_1 \frac{\partial \rho}{\partial x_1} + V_2 \frac{\partial \rho}{\partial x_2} - D \frac{\partial^2 \rho}{\partial x_1^2} - D \frac{\partial^2 \rho}{\partial x_2^2} = 0 \tag{6.10}$$

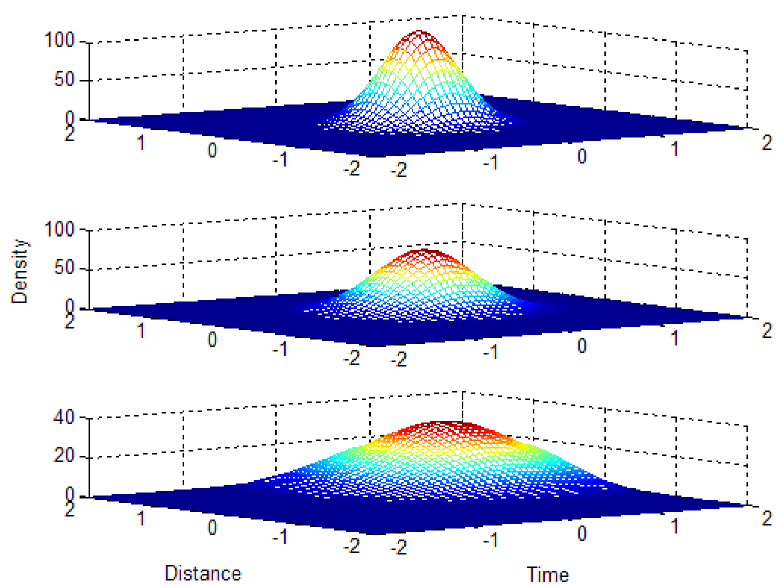
The Lyapunov function as before is given by $V(t) = \int_\Omega \rho^2 d\Omega$ and its derivative with respect to time t is $\frac{dV(t)}{dt} = D \int_\Omega \rho \sum_{i=1}^{2} \frac{\partial^2 \rho}{\partial x_i^{\;2}} d\Omega - \int_\Omega \rho \sum_{i=1}^{2} V_i \frac{\partial \rho}{\partial x_i} d\Omega$. By using (4.13) and Holders inequality we get the following result

$$\frac{dV(t)}{dt} \leq -DC^{-1}\|\rho\|_2^2 - \sum_{i=1}^{2} V_i \|\rho\|_2 \|\partial\rho/\partial x_i\|_2 \quad (6.11)$$

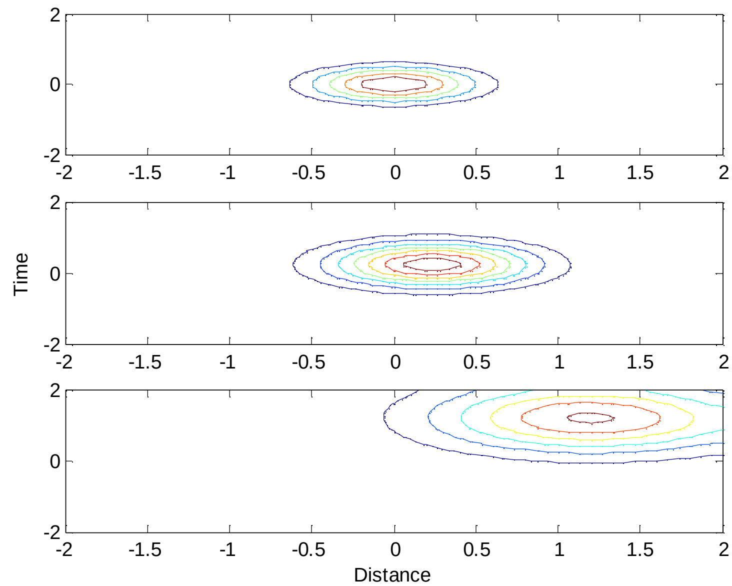
Thus for $2DC^{-1} > 0$ the equilibrium of closed loop system (6.10) using feedback control (6.9) is exponentially stable. The simulation results are shown in Fig. 6.2 in the following section.

6.2.3 Simulation Results

This section shows simulation results for the advection and advection-diffusion control laws. For simulation the initial distribution for density is considered to be Gaussian. The simulation results are shown in Fig. 6.1 and 6.2. The density response for advection control is shown in Fig. 6.1. In Fig 6.1(a) the density response is shown as a mesh plot at different time instants. The second simulation is shown as a contour plot for different time instants in Fig. 6.1 (b). The density response for advection-diffusion control is shown in Fig. 6.2. As seen from the plot in Fig. 6.2 the density is moving towards the right (exit) of the corridor at a faster rate than before.

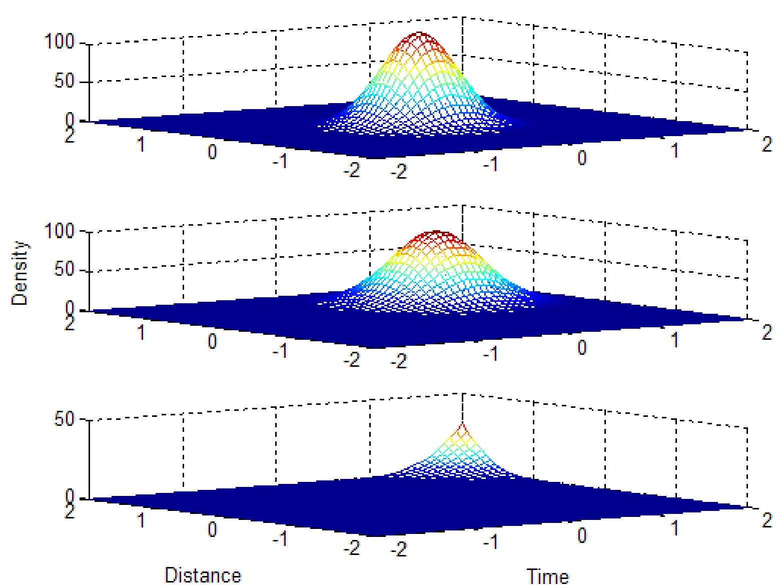


(a)

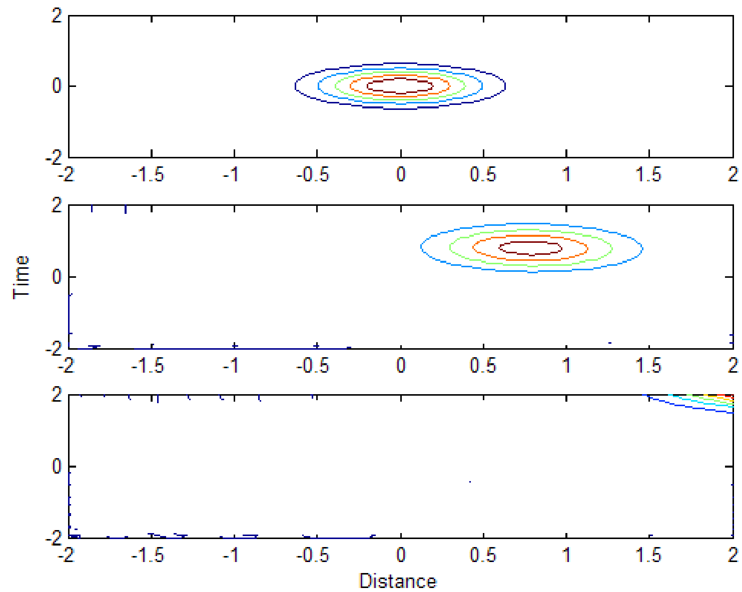


(b)

Figure 6. 1: Advection control response for one-equation model (a): Density response snapshots, (b): Contours of the response at different time instants.



(a)



(b)

Figure 6. 2: Advection-diffusion control response for one-equation model (a): Density response at different times (b): Contours of density at different time instants.

6.3 Advection Control for Two-Equation Model

In this section we discuss the advection control for a two-equation model and give motion to the model by adding the convective term to the control law. The control is done using method of *backstepping*. First we discuss the advection control alone without a diffusion term. Next we add the diffusion term to the control law.

6.3.1 Advection Control

For the two-equation control model given by (4.17) and (4.18)

$$\frac{\partial \rho}{\partial t} = -\sum_{i=1}^{2} \frac{\partial q_i}{\partial x_i} \tag{6.12}$$

$$\frac{\partial q_i}{\partial t} = \bar{u}_i \quad , \; i = 1,2 \tag{6.13}$$

where $\bar{u}_i = -\frac{\partial}{\partial x_1}(q_1 q_2 / \rho) - \frac{\partial}{\partial x_2}(q_2^{\,2} / \rho) + u_i$ are the control variables. The control law given by (4.19)

$$q_i(x,t) = G_i(\rho(x,t)) = D\int_0^{x_i} \frac{\partial^2 \rho(x,t)}{\partial m^2} dm \tag{6.14}$$

makes the origin of the equation (6.12) asymptotically stable. Now let us discuss the advection control for this system by designing the convective controller. As a first step we design the controller for (6.12) given as

$$q_i(x,t) = G_i(\rho) = -V_i \int_0^{x_i} \frac{\partial \rho(x,t)}{\partial m} dm \tag{6.15}$$

such that the closed loop dynamics for (6.12) are

$$\frac{\partial \rho(x,t)}{\partial t} + V_1 \frac{\partial \rho}{\partial x_1} + V_2 \frac{\partial \rho}{\partial x_2} = 0 \tag{6.16}$$

which is same as (6.5). The above dynamics have an asymptotically stable origin and a Lyapunov function given by (6.7) which satisfies $\frac{dV(t)}{dt} = 0$. The second step is to design a feedback controller for (6.13) using *backstepping* method. Since q_i is not an actual control variable, therefore by defining the difference between q and $G(\rho)$ as an error variable $z = q - G(\rho)$, we get the following modified dynamics.

$$\frac{\partial \rho}{\partial t} = -\sum_{i=1}^{2} \frac{\partial G_i(\rho)}{\partial x_i} - \sum_{i=1}^{2} \frac{\partial z_i}{\partial x_i} \tag{6.17}$$

$$\frac{\partial z_i}{\partial t} = u_{ni} \tag{6.18}$$

where $u_{ni} = \bar{u}_i - \partial G_i(\rho)/\partial t$ are the new control variable. Now let us modify the Lyapunov functional by adding the error term to it as

$$V(t)=\int_\Omega \left(\rho-\rho_0(x_1-V_1t,x_2-V_2t)\right)^2 d\Omega+\frac{1}{2}\int_\Omega z^2 d\Omega$$

The control law is given by (3.24) as

$$u_{ni}(x,t)=-Kz_i-z_i^{-1}\rho\frac{\partial z_i}{\partial x_i} \tag{6.19}$$

with $z_i=q_i-G_i(\rho)$ where $G_i(\rho)$ is now given by (6.14). This shows that the origin ($\rho=0,z_1=0,z_2=0$) of the system (6.17) and (6.18) is asymptotically stable. To prove the asymptotic stability of the origin ($\rho=0,q_1=0,q_2=0$) of the system (6.12) and (6.13) we use the proposition 6.3.1

Proposition 6.3.1 *The origin ($\rho(x,t)=0,q_1(x,t)=0,q_2(x,t)=0,$) of the original system (6.12) and (6.13) is exponentially stable under the control law* $\bar{u}_i=u_{ni}+\partial G_i(\rho)/\partial t$ *with* u_{ni} *given by (6.19) and* $G_i(\rho)=-V_i\int_0^{x_i}\frac{\partial\rho(x,t)}{\partial m}dm$

Proof: The proof of this proposition is similar to the proof of proposition 5.3.1 where we use the advection solution given by (6.6).

6.3.2 Advection-Diffusion Control

In this section we will add the diffusion component to the control law (6.14) so that there is diffusion as well as direction to the motion. The convective-diffusion controller is given as the combination of (6.14) and (6.15) as

$$q_i(x,t)=G_i(\rho)=-V_i\int_0^{x_i}\frac{\partial\rho(x,t)}{\partial m}dm-D\int_0^{x_i}\frac{\partial^2\rho(x,t)}{\partial m^2}dm \tag{6.20}$$

The closed loop dynamics using this control law are given as

$$\frac{\partial\rho(x,t)}{\partial t}+V_1\frac{\partial\rho}{\partial x_1}+V_2\frac{\partial\rho}{\partial x_2}-D\frac{\partial^2\rho}{\partial x_1^2}-D\frac{\partial^2\rho}{\partial x_2^2}=0 \tag{6.21}$$

which is same as (6.10). The above dynamics have an asymptotically stable origin and a Lyapunov function given by (6.6) which satisfies condition (6.11). The second step is to design a feedback controller for (6.13) using *backstepping* as before. The final feedback control law for the two-equation model is thus given by (6.19)

$$u_{ni}(x,t) = -Kz_i - z_i^{-1}\rho\frac{\partial z_i}{\partial x_i}$$

with $z_i = q_i - G_i(\rho)$ where $G_i(\rho)$ is now given by (6.20). This shows that the origin ($\rho = 0, z_1 = 0, z_2 = 0$) of the system (6.17) and (6.18) is asymptotically stable. To prove the asymptotic stability of the origin ($\rho = 0, q_1 = 0, q_2 = 0$) of the system (6.12) and (6.13) we use the proposition 6.3.2.

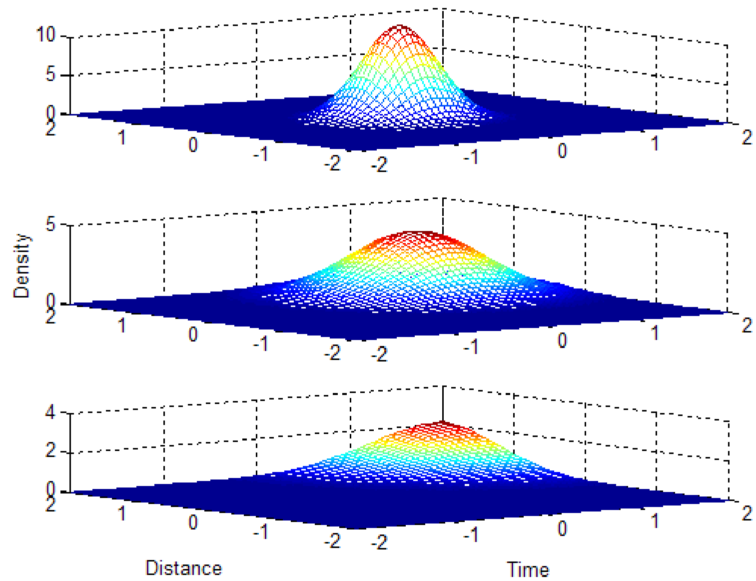
Proposition 6.3.2 *The origin* ($\rho(x,t) = 0, q_1(x,t) = 0, q_2(x,t) = 0,$) *of the original system (6.12) and (6.13) is exponentially stable under the control law* $\bar{u}_i = u_{ni} + \partial G_i(\rho)/\partial t$ *with* u_{ni} *given by (6.19) and* $G_i(\rho) = -V_i\int_0^{x_i}\frac{\partial\rho(x,t)}{\partial m}dm - D\int_0^{x_i}\frac{\partial^2\rho(x,t)}{\partial m^2}dm$

Proof: The proof follows from the proofs for proposition 4.3.1 and 6.3.1.

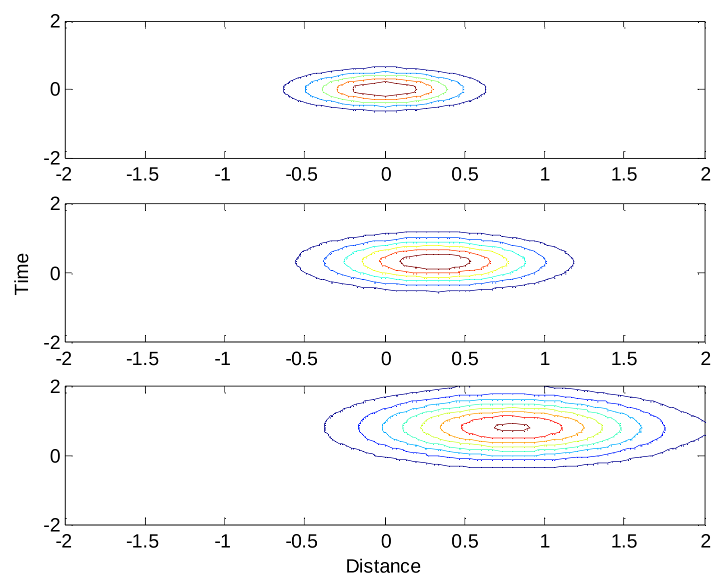
Thus we conclude that the origin ($\rho(x,t) = 0, q_1(x,t) = 0, q_2(x,t) = 0$) for actual closed- loop system is asymptotically stable.

6.3.3 Simulation Results

This section shows simulation results for the advection and advection-diffusion control laws for two-equation model. The simulation results are shown in Fig. 6.4 for advection law and in Fig.6.5 for advection-diffusion law. As is seen from the figures the evacuation is faster in case of advection-diffusion.

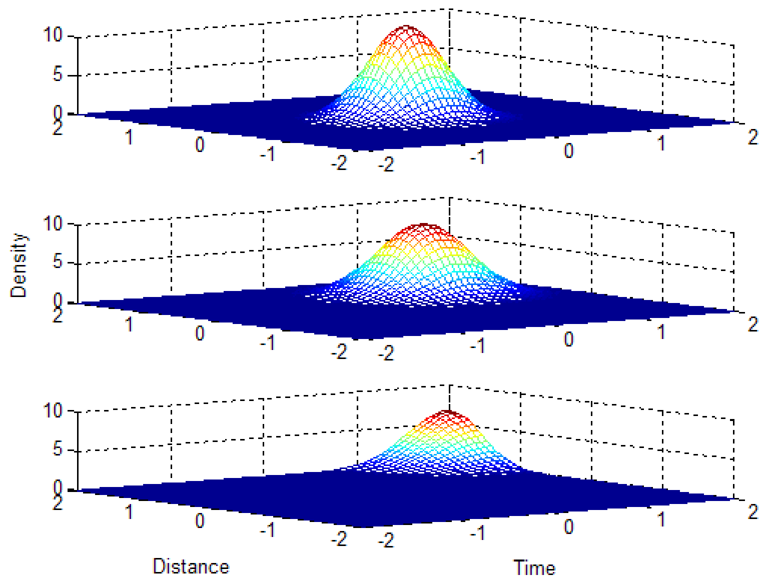


(a)

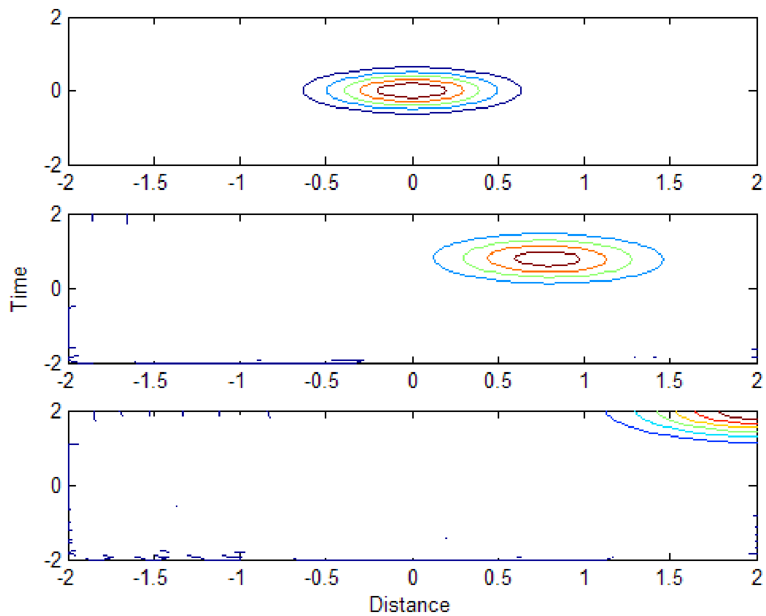


(b)

Figure 6. 3: Advection control response for two-equation model (a): Density response snaphots, (b): Contours of the response at different time instants.



(a)



(b)

Figure 6. 4: Advection-diffusion control response for two-equation model (a): Density response snapshots, (b): Contours of the response at different time instants.

In this chapter we designed advection and advection-diffusion control laws for two-dimensional evacuation system. In next chapter we will discuss the design of robust feedback controllers where the control design objective is to minimize the effect of uncertainty on closed-loop system response.

CHAPTER 7

ROBUST FEEDBACK CONTROL DESIGN AND STABILITY ANALYSIS OF ONE DIMENSIONAL CROWD MODELS

7.1 Introduction

In previous chapters we addressed the control of crowd dynamic systems without accounting for the presence of uncertainty in the design of the controller. The uncertainty is a mismatch between the model used for controller design and the actual process model. The uncertain function may include uncertain model parameters or external disturbances. Here we consider the case where we have an uncertainty in the input u to the system. The uncertainty is due to the mismatch between the control command and the actual control command followed by people and is distributed in space. The objective is to develop a framework for the synthesis of distributed robust controllers that handle the effect of this uncertain variable. A distributed robust controller is derived that guarantees boundedness of state and achieves asymptotic stabilization with arbitrary degree of asymptotic attenuation of the effect of uncertain variables on the output of the closed-loop system. The controller is designed constructively using Lyapunov's direct method [31] and requires the existence of known bounding functions that capture the magnitude of the uncertain term. This chapter presents the design of robust nonlinear feedback controllers for two models given in chapter 2 representing crowd dynamics in one-dimension with uncertainty in the control input. The models are based on the laws of conservation of mass and momentum and are given in subsection 2.2.1. In both cases the objective of control design is to synthesize a nonlinear distributed feedback controller that stabilizes the system and guarantees stability of the closed loop system in presence of uncertainty.

The organization of this chapter is as follows. In Section 7.2 we formulate the uncertain control model and present robust control design for the one-equation crowd model. This section also studies Lyapunov stability for this model. Simulation results for closed loop dynamics are presented where the developed control method is tested with a disturbance in the system. Section 7.3 presents the robust control and stability analysis for the two-equation system model.

7.2 Robust Control for One-Equation Model

In this section we formulate the uncertain control model and present robust control design for the one-equation model. The problem is to design a state feedback control that guarantees the desired performance of the closed-loop system irrespective of uncertain elements. The one-equation model is given in subsection 2.2.1 by (2.7) as

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial (q(x,t))}{\partial x} = 0 \tag{7.1}$$

Here $\rho(x,t)$ is the variable we want to control. The flow $q(x,t)$ is obtained as a product of density and velocity as $q(x,t) = \rho(x,t)v(x,t)$ and the velocity-density relationship as given by Greenshield's model is

$$v = v_f (1 - \rho/\rho_{\max}) \tag{7.2}$$

where $v_f = v_f(x,t)$ is the free flow speed and $\rho_{\max}$ is the maximum or jam density. The one-equation model is therefore given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho v_f (1 - \rho/\rho_{\max})\right) = 0 \tag{7.3}$$

By choosing free flow velocity vector field $v_f = v_f(x,t)$ as the distributed control variable denoted by u we get the following control model

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho u(1-\rho/\rho_{\max})\right) = 0 \tag{7.4}$$

7.2.1 Input Uncertain Control Model

Let us consider that the uncertain variable for this system is the control input or the free flow velocity. This means that there is an error in the control command and the actual input command followed by people. The uncertain model is therefore given as

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}[(1-\rho/\rho_{\max})\rho(u+\theta)] = 0 \tag{7.5}$$

In above equation $\theta = \theta(t,u,\rho)$ denotes the unknown function which takes care of uncertainty in the input u to the system. The uncertain term here satisfies an important structural property namely it enters the state equation exactly at the point where the control variable enters. This property will be referred to as the matching condition. The nominal model (the system without uncertainty) for this system is described by

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}[(1-\rho/\rho_{\max})\rho u] = 0$$

The first step is to design a stabilizing feedback controller for the nominal model. With the feedback controller (3.8) $u = F(\rho)$ given in chapter 3 as

$$F(\rho(x,t)) = -[(1-\rho/\rho_{\max})\rho]^{-1} D \frac{\partial \rho}{\partial x} \tag{7.6}$$

we get the nominal closed-loop system as

$$\frac{\partial \rho}{\partial t} - D\frac{\partial^2 \rho}{\partial x^2} = 0 \tag{7.7}$$

Thus with $|\theta(t,u,\rho)| = 0$ and $u = F(\rho)$ the nominal closed loop model (7.7) has exponentially stable origin and there exists a Lyapunov function

$$V(t) = \tfrac{1}{2}\|\rho\|_2^2 = \tfrac{1}{2}\int_0^L |\rho|^2 dx$$

which satisfies $\frac{dV(t)}{dt} \le -\beta V(t) = -2DC^{-1}\|\rho\|_2^2$ with $\beta = 2D$.

7.2.2 Robust Control by Lyapunov Redesign Method

In this section we consider the system of (7.5) and address the problem of synthesizing a distributed state feedback controller that stabilizes the closed-loop system irrespective of the uncertainty. The controller is designed constructively using Lyapunov's direct method where we use the Lyapunov function for the system to design feedback control. A standard method for finding a Lyapunov function for an uncertain system is developed in [42] and is known as *Lyapunov redesign*. This technique has been incorporated in various books like [31], [43]. The key idea of this method is to employ a Lyapunov function for the nominal system as Lyapunov function for the uncertain system. This re-use of the same Lyapunov function is referred to by the term "redesign". The Lyapunov redesign technique uses a Lyapunov functional of a nominal system to design an additional control component to robustify the design to a class of large uncertainties that satisfy the matching condition; i.e., the uncertain terms enter the state equation at the same point as the input. Lyapunov redesign can be used to achieve robust stabilization. The goal is to design feedback control law for (7.5) as

$$u = F(\rho) + G(\rho) \tag{7.8}$$

such that we achieve closed loop stability and asymptotic attenuation of $\theta(t,u,\rho)$ where $G(\rho)$ is a nonlinear operator mapping $H^2[(0,L),\Re]$ into $H^1[(0,L),\Re]$. In equation (7.8) $F(\rho)$ achieves closed loop stability and $G(\rho)$ asymptotically attenuates the effect of $\theta(t,u,\rho)$. The function $F(\rho)$ will be designed by the previous approach and $G(\rho)$ will be designed using *Lyapunov redesign* method. The design of function $G(\rho)$ is known as Lyapunov redesign [31] and is done on the basis of the assumption that we have a bounding function that captures the size of the disturbance. Let us assume with controller (7.8) there exists a known smooth function which bounds the magnitude of uncertain variables as:

$$\|\theta(t,u,\rho)\| = \|\theta(t,\rho,(F(\rho)+G(\rho)\| \le \gamma(t,\rho) + \kappa\|G(\rho)\| \tag{7.9}$$

where γ is a nonnegative H^1 function and is a measure of size of uncertainty $\theta(t,u,\rho)$. From estimate (7.9), the only requirement is the knowledge of γ which doesn't necessarily have to be small. From the knowledge of Lyapunov function $V(t)$ and functions γ and κ the goal is to design $G(\rho)$ and apply $u = F(\rho) + G(\rho)$ to the actual system (7.5) such that the overall closed loop system is stabilized in presence of uncertainty. By using the control law (7.6) the feedback control law (7.8) is given by

$$u = -[(1-\rho/\rho_{\max})\rho]^{-1} D\frac{\partial \rho}{\partial x} + G(\rho) \tag{7.10}$$

Under the feedback control law (7.10) the closed-loop dynamics for (7.5) take the form

$$\frac{\partial \rho}{\partial t} - D\frac{\partial^2 \rho}{\partial x^2} - \frac{\partial}{\partial x}[\rho(1-\rho/\rho_{\max})\, G(\rho)] = 0 \tag{7.11}$$

Thus the error in the closed loop dynamics (7.10) and (7.13) due to input uncertainty is given by

$$\theta = \frac{\partial}{\partial x}[\rho(1-\rho/\rho_{\max})\, G(\rho)] \tag{7.12}$$

Let us denote the component $G(\rho)$ of control input (7.8), by v and $w(\rho) = \rho(1-\rho/\rho_{\max})$ we can rewrite (7.12) as

$$\theta = -\frac{\partial(wv)}{\partial x} = v\,(\partial w/\partial x) - w\,(\partial v/\partial x) \tag{7.13}$$

where $\partial w/\partial x = (1-2\,\rho/\rho_{\max})\,\partial\rho/\partial x$. The magnitude of this uncertain term with respect to L_2 norm is given by

$$\|\theta(t,\rho,u)\|_2^{\,2} == \int_0^L |v\,(\partial w/\partial x)|^2\,dx + \int_0^L |w\,(\partial v/\partial x)|^2\,dx + 2\int_0^L |v\,(\partial w/\partial x)||w\,(\partial v/\partial x)|\,dx \tag{7.14}$$

The bound on the magnitude of uncertainty (7.14) can be found by applying Holders inequality [41] to each integral term in the above equation, which results in

$$\|\theta(t,\rho,u)\|_2^2 \le \|v\|_2^2\,\|(\partial w/\partial x)\|_2^2 + \|w\|_2^2\,\|(\partial v/\partial x)\|_2^2 + 2\|v\,(\partial w/\partial x)\|_2\,\|v\,(\partial w/\partial x)\|_2$$

Therefore from the above inequality we have the following bound on uncertainty

$$\|\theta(t,\rho,u)\|_2 \le \|v\|_2\,\|(\partial w/\partial x)\|_2 + \|w\|_2\,\|(\partial v/\partial x)\|_2$$

$$= \gamma(\rho,t) + \kappa(\rho,t)\bar{v} \tag{7.15}$$

where $\kappa(\rho,t)=\|(\partial w/\partial x)\|_2$, $\gamma(\rho,t)=\|w\|_2$ and $\bar{v}=\|v\|_2/\|(\partial v/\partial x)\|_2$. This bound will be utilized to design the control law $G(\rho)$. Let us now apply control law (7.8) to the input-uncertain system (7.5) so that the closed-loop dynamics take the form

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}[\rho(1-\rho/\rho_{\max})(F(\rho)]+\frac{\partial}{\partial x}[\rho(1-\rho/\rho_{\max})(G(\rho)+\theta)]=0 \tag{7.16}$$

As can be seen the system (7.16) is a perturbation of the nominal closed-loop system (7.7), the third term being the perturbation. Let us choose the Lyapunov function $V(t)$ for (7.7) same as before. To design $G(\rho)$ we find the rate of change of Lyapunov function as

$$\frac{dV(t)}{dt}=\int_0^L\rho\frac{\partial\rho}{\partial t}dx\le-k\|\rho\|_2^2+\int_0^L\rho\frac{\partial}{\partial x}[\rho(1-\rho/\rho_{\max})(G(\rho)+\theta)]dx \tag{7.17}$$

The first inequality is because of the asymptotic stability of nominal closed-loop system (7.7). We need to choose a control law $G(\rho)$ so as to cancel the destabilizing effort of $\theta(t,u,\rho)$ on $dV(t)/dt$. The law $G(\rho)$ should be such that the second term in (7.17) is negative semi-definite in order to have asymptotic stability. By using $w(\rho)=\rho(1-\rho/\rho_{\max})$ and denoting $G(\rho)$ by v we can rewrite (7.17) as

$$\frac{dV(t)}{dt}\le-k\|\rho\|_2^2+\int_0^L\rho\frac{\partial(wv)}{\partial x}dx+\int_0^L\rho\frac{\partial(w\theta)}{\partial x}dx \tag{7.18}$$

Since the disturbance in our case is a function of space and time that is $\theta=\theta(t,u,\rho)$, let us define $\hat{\theta}=\partial(w\theta)/\partial x$ and a new control variable $\hat{v}=\partial(wv)/\partial x$. Therefore we have

$$\frac{dV(t)}{dt}\le-k\|\rho\|_2^2-(\int_0^L\rho\hat{v}dx+\int_0^L\rho\hat{\theta}dx) \tag{7.19}$$

Now let us apply the Holders inequality to the second integral in the second term of (7.19). This results in the following

$$\frac{dV(t)}{dt}\le-k\|\rho\|_2^2-(\int_0^L\rho\hat{v}dx+\|\rho\|_2\|\hat{\theta}\|_2) \tag{7.20}$$

By using Sobolev inequality (3.16) we can show that $\|\hat{\theta}\|_2=\|\partial(w\theta)/\partial x\|_2\ge C^{-1}\|w\theta\|_2$ and by using (7.15) we have the rate of change of Lyapunov as

$$\frac{dV(t)}{dt} \le -k\|\rho\|_2^2 - \left(\int_0^L \rho \hat{v} dx + \|\rho\|_2 \|w\|_2 (\gamma + \kappa \bar{v}) \right) \quad (7.21)$$

Now let us choose the state feedback control as following

$$\hat{v} = \frac{-\eta(\rho,t)}{\kappa(\rho,t)} w(x,t)$$

or

$$v = \frac{1}{w} \int_0^L \left(\frac{-\eta(\rho,t)}{\kappa(\rho,t)} w(x,t) \right) dx \quad (7.23)$$

where $\eta(\rho,t) \ge \gamma(\rho,t)$ is a nonnegative function. By using this control law we can show that $\bar{v} = \frac{\|v\|_2}{\|\partial v / \partial x\|_2} \ge \frac{\|v\|_2}{C\|v\|_2} = C^{-1}$. Using this relation we can write (7.21) as

$$\frac{dV(t)}{dt} \le -k\|\rho\|_2^2 - \left(\int_0^L \rho \hat{v} dx + C^{-1} \|\rho\|_2 \|w\|_2 (\gamma + \kappa C^{-1}) \right) \quad (7.24)$$

By using the control law (7.23) and then applying Holders inequality to the integral term in above equation it becomes $\int_0^L \rho \hat{v} dx \le C\|\rho\|_2 \|w\|_2 \|\eta/k\|_2$. Hence (7.24) can be written as

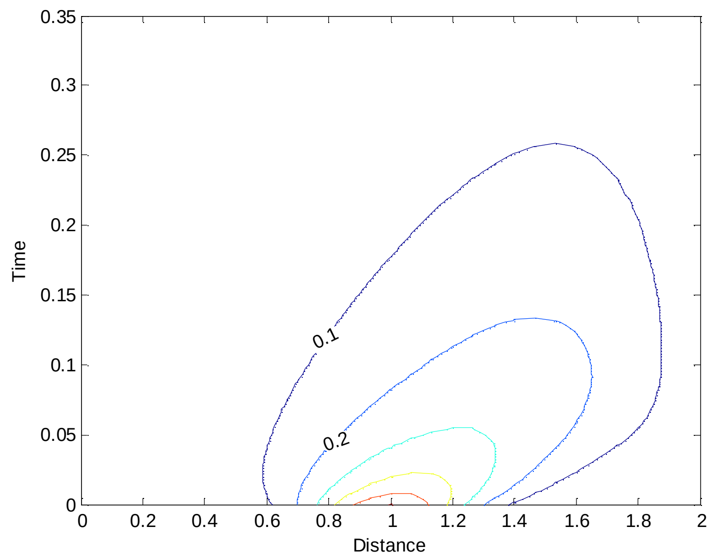
$$\frac{dV(t)}{dt} \le -k\|\rho\|_2^2 - \left(C^{-1} \|\rho\|_2 \|w\|_2 \gamma(\rho,t) - C^{-1} \|\rho\|_2 \|w\|_2 (C\|\eta/k\|_2 - \kappa C^{-1}) \right)$$

As long as $(C\|\eta/k\|_2 - \kappa C^{-1}) \le \gamma$, we have asymptotic stability in presence of the disturbance. The robust control law for (7.5) is therefore given as

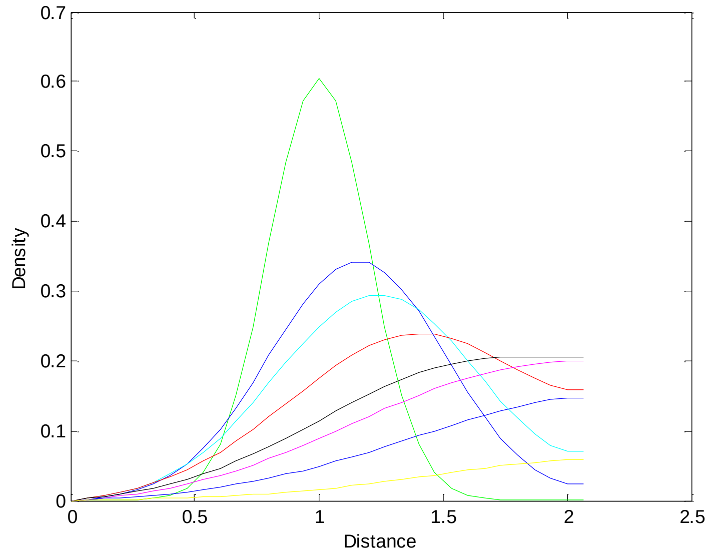
$$u = -[(1 - \rho/\rho_{\max})\rho]^{-1} D \frac{\partial \rho}{\partial x} + \frac{1}{w} \int_0^L \left(\frac{-\eta(\rho,t)}{\kappa(\rho,t)} w(x,t) \right) dx \quad (7.25)$$

7.2.3 Simulation Results

This section shows simulation results for the system (7.5) using the robust control law (7.25). For simulation the initial distribution of density is considered to be Guassian. The simulation results are shown in Fig. 7.1, 7.2 and 7.3. The density response for nominal control model (model without uncertainty) is shown in Fig. 7.1. In Fig 7.1(a) the density response is shown as a contour plot which shows people moving towards the right (exit) of the corridor. The simulation in Fig. 7.1(b) shows the density response at different time instants where the response flattens with time. The density response for uncertain model is shown in Fig. 7.2. We have added a step function (with respect to time over all distance) disturbance to the system. As seen from the plot in Fig. 7.2 (b) the density is moving back because of the disturbance. In Fig. 7.3 we show the response with the robust controller (7.25) added to the system. As is seen from the figure the response has improved and the effect of disturbance is cancelled.

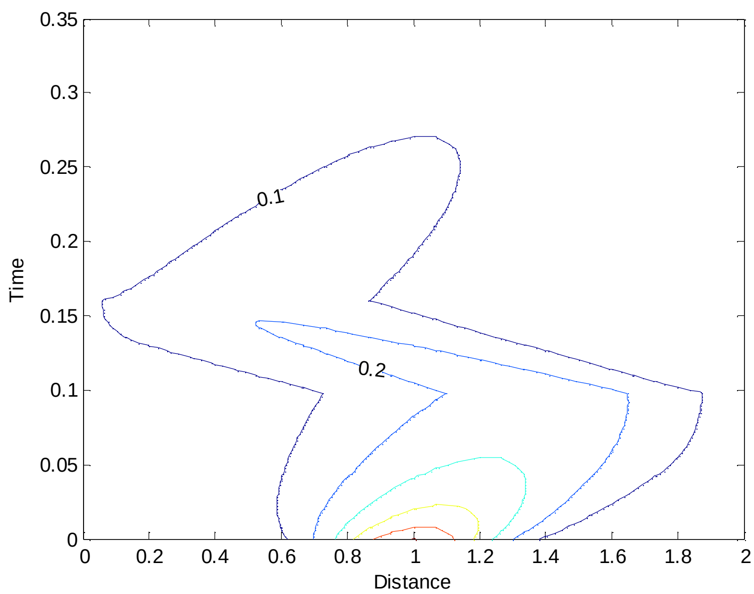


(a)

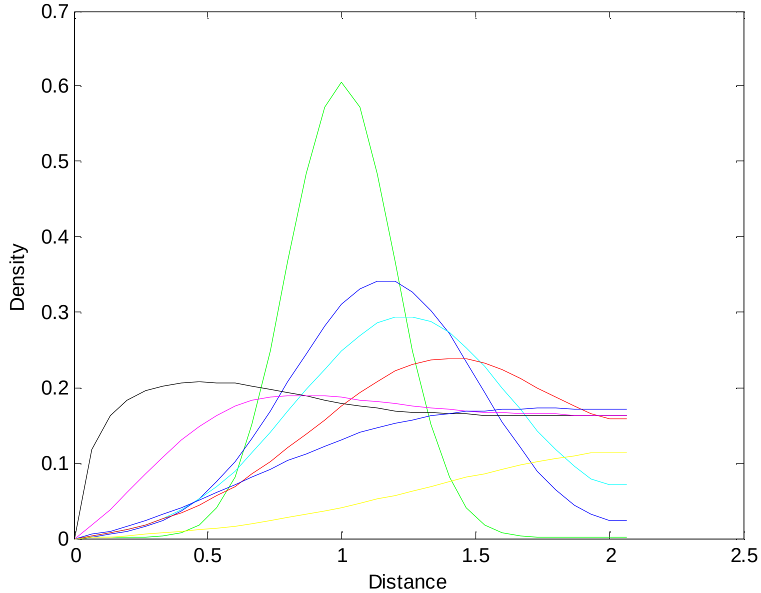


(b)

Figure 7. 1: Density response for one-equation nominal-model, (a): Contours of the response, (b): Density at different time instants.

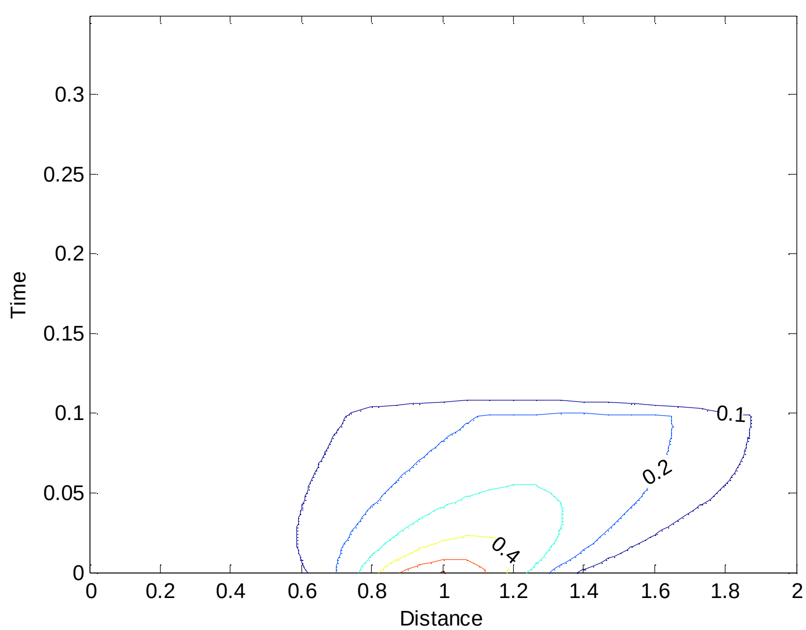


(a)

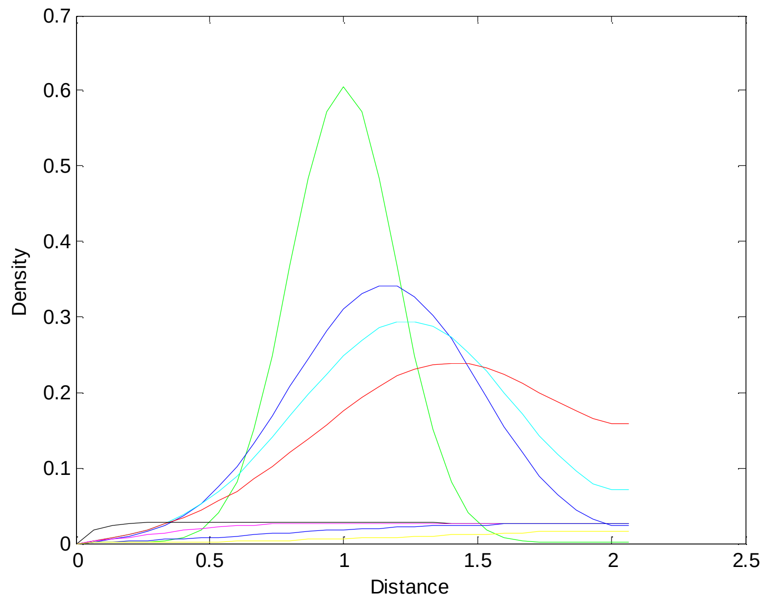


(b)

Figure 7. 2: Density response for one-equation uncertain-model, (a): Contours of the response, (b): Density at different time instants



(a)



(b)

Figure 7. 3: Robust control response for one-equation uncertain-model, (a): Contours of the response, (b): Density at different time instants

7.3 Robust Control for Two-Equation Model

In this section we formulate the control model and present robust control design for the two-equation model. This section also studies Lyapunov stability for this model and presents some simulation results. We also consider the uncertainty in the input to the system. The control design is done using a combination of both *Lyapunov redesign* and *backstepping* and the technique is referred to as *robust backstepping.* Let us consider the two-equation model for a one dimensional corridor. The model is a higher order model or more precisely a system of two partial differential equations and consists of conservation of mass equation coupled with a second equation based on the principle of conservation of momentum. The model is given in subsection 2.2.3 by (2.9) and (2.10) as

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{7.25}$$

$$\frac{\partial q}{\partial t}+\frac{\partial}{\partial x}(q^2/\rho)=-\frac{\partial p}{\partial x} \tag{7.26}$$

Here ρ and q are the variables we want to control. For pedestrian evacuation the final density and flow should be equal to zero; that is $\rho(x,t_f)=0$ and $q(x,t_f)=0$. For this model we choose divergence of pressure $\partial p/\partial x$ as distributed control variable u thus giving us the following representation

$$\frac{\partial \rho}{\partial t}=-\frac{\partial q}{\partial x} \tag{7.27}$$

$$\frac{\partial q}{\partial t}=\bar{u} \tag{7.28}$$

where $\bar{u}=-\frac{\partial}{\partial x}(q^2/\rho)+u$ is the new control variable. Now let us consider the uncertainty in the control input for this system. We can have both matched as well as unmatched uncertainties depending upon which equation contains the uncertainty. We can have uncertainty in (7.27) where it enters the equation through the flow variable q which means that there is an error between the flow control command and the actual people flow. For this case the uncertainty is unmatched since q is the "conceptual" control input to this equation and is not the actual control input (7.28). We can also have uncertainty in the control variable $\partial p/\partial x$ in which case the uncertainty is matched and we can have both. The control design scheme in all cases is that of robust *backstepping* [43] where we do a combination of backstepping and Lyapunov redesign. Now let us discuss these cases one by one.

7.3.1 Robust *Backstepping*: Unmatched Uncertainty

Let us first consider the uncertainty in the input q to the nominal system (7.27). This means that there is an error in the actual flow control command q and the actual people flow of people. The control design strategy here is robust backstepping which is the combination of *backstepping* and *Lyapunov redesign.* The uncertain model is given as

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(q + \theta) \tag{7.29}$$

$$\frac{\partial q}{\partial t} = \bar{u} \tag{7.30}$$

where $\theta = \theta(t,u,\rho)$ is the unknown function which takes care of the disturbance in the input q to (7.27). The uncertain term for the overall system (7.30) is unmatched as it does not enter the system at the same point as the input $\bar{u}$ even though it enters equation (7.27) through the conceptual input q. As a first step we design a robust control for equation (7.29) using *Lyapunov redesign technique* and in the next step find the control law for the overall system (7.30) using *backstepping.*

1. ***Lyapunov redesign**:* The first step is to design a robust controller for (7.29) using Lyapunov redesign method. The goal is to design robust control law for (7.29) as

$$q = G(\rho) + v = \overline{G}(\rho) \tag{7.31}$$

such that we achieve closed loop stability and asymptotic attenuation of $\theta(t,u,\rho)$. $G(\rho)$ in (7.31) achieves closed loop stability and v asymptotically attenuates the effect of $\theta(t,u,\rho)$. The function $G(\rho)$ is designed from the feedback linearization of nominal model (7.27) for (7.29) and v will be designed by using Lyapunov redesign method. The nominal model (7.27) is given as

$$\frac{\partial \rho}{\partial t} = -\frac{\partial q}{\partial x}$$

The stabilizing feedback controller $q = G(\rho)$ for this nominal model from (3.22) given in chapter 3 is

$$G(\rho) = -D\int_0^x \frac{\partial^2 \rho}{\partial m^2} dm \tag{7.32}$$

which gives the conceptual nominal closed-loop system as

$$\frac{\partial \rho}{\partial t} - D\frac{\partial^2 \rho}{\partial x^2} = 0 \tag{7.33}$$

The nominal closed loop model (7.33) has an exponentially stable origin and there exists a Lyapunov function $V(t)$ given by (3.16) which satisfies $\frac{dV(t)}{dt} \le -\beta V(t)$ with $\beta = 2DC^{-1}$.

Now by using (7.32) the control law (7.31) becomes

$$q = -D\int_0^x \frac{\partial^2 \rho}{\partial m^2} dm + v \tag{7.34}$$

which gives the following closed-loop dynamics for (7.27)

$$\frac{\partial \rho}{\partial t} - D\frac{\partial^2 \rho}{\partial x^2} - \frac{\partial v}{\partial x} = 0 \tag{7.35}$$

Thus the error in the closed loop dynamics (7.33) and (7.35) due to input uncertainty is given by

$$\theta(t, \rho, u) = -\frac{\partial v}{\partial x} \tag{7.36}$$

The magnitude of this uncertain term with respect to L_2 norm is given by $\|\theta(t, \rho, u)\|_2 = -\|\partial v/\partial x\|_2$ which due to Sobolev inequality (3.16) is

$$\|\theta(t, \rho, u)\|_2 \le \gamma \|v\|_2 \tag{7.37}$$

where $\gamma = C^{-1}$. Let us now apply control law (7.33) to the input-uncertain system (7.29) so that the closed-loop dynamics take the form

$$\frac{\partial \rho}{\partial t} - D\frac{\partial^2 \rho}{\partial x^2} - \frac{\partial v}{\partial x} - \frac{\partial \theta}{\partial x} = 0 \tag{7.38}$$

Let us choose the Lyapunov function $V(t)$ for (7.29) same as before. To design v we find the rate of change of Lyapunov function as

$$\frac{\partial V(t)}{\partial t} \le -D\|\rho\|_2^2 - \int_0^L \rho \frac{\partial v}{\partial x} dx - \int_0^L \rho \frac{\partial \theta}{\partial x} dx$$

Now let us define $\hat{\theta} = \partial\theta/\partial x$ and a new control variable $\hat{v} = \partial v/\partial x$. Therefore we have

$$\frac{\partial V(t)}{\partial t} \le -D\|\rho\|_2^2 - \int_0^L \rho\hat{v} dx - \int_0^L \rho\hat{\theta} dx \tag{7.39}$$

Now let us apply the Holders inequality to the second integral of (7.39). This results in the following

$$\frac{\partial V(t)}{\partial t} \leq -D\|\rho\|_2^2 - \int_0^L \rho\hat{v}dx - \|\rho\|_2 \left\|\hat{\theta}\right\|_2$$

From Sobolev inequality (3.16) we have $\left\|\hat{\theta}\right\|_2 = \left\|\frac{\partial \theta}{\partial x}\right\|_2 \geq C^{-1}\|\theta\|_2$, thus by using this relation in conjunction with (7.37) we get the rate of change of Lyapunov function as

$$\frac{\partial V(t)}{\partial t} \leq -D\|\rho\|_2^2 - \int_0^L \rho\hat{v}dx - C^{-1}\|\rho\|_2 \|\hat{v}\|_2 \tag{7.40}$$

Now let us choose the control law as

$$\hat{v} = -\frac{\eta}{1-\kappa}\rho(x,t)$$

or

$$v = -\int_0^L \frac{\eta}{1-\kappa}\rho dx \tag{7.41}$$

where $\eta(\rho,t) \geq \gamma(\rho,t)$. Using this control law in (7.40) we get

$$\frac{dV(t)}{dt} \leq -D\|\rho\|_2^2 - \|\rho\|_2^2 \left\|\frac{\eta}{1-\kappa}\right\|_2 \left(C^{-1} - 1\right)$$

which is negative semi-definite as long as $C^{-1} - 1 \geq 0$ thus giving us asymptotic stability. Therefore robust control that stabilizes the uncertain model (7.29) is given as

$$q = -D\int_0^x \frac{\partial^2 \rho}{\partial m^2} dm - \int_0^L \frac{\eta}{1-\kappa}\rho dx \tag{7.42}$$

2. ***Control by backstepping method:*** The next step is to design a feedback controller for overall system (7.30) using *backstepping* method used in section 3.3.2. The goal is to design feedback control law $\bar{u}$ to stabilize the overall system from the knowledge of Lyapunov function $V(t)$ for (7.29) and modifying it. We proceed with the control design in a similar way as in section 3.3.2 as follows

1. First design a conceptual control law $q = \overline{G}(\rho)$ for (7.29) from the previous step to stabilize origin $\rho(x,t) = 0$. With this control law the conceptual closed loop dynamics for (7.29) are asymptotically stable and there exists a Lyapunov function to (3.14) which satisfies $\frac{dV(t)}{dt} \le -\beta V(t)$.

2. Since q is not the actual control variable, defining the difference between q and $\overline{G}(\rho)$ by an error variable $z = q - \overline{G}(\rho)$, we get the following modified dynamics.

$$\frac{\partial \rho}{\partial t} = -\frac{\partial \overline{G}(\rho)}{\partial x} - \frac{\partial z}{\partial x} \tag{7.43}$$

$$\frac{\partial z}{\partial t} = u_n \tag{7.44}$$

where $u_n = \overline{u} - \partial \overline{G}(\rho)/\partial t$ is the new control variable.

3. Now let us modify the Lyapunov functional by adding the error term to it as in (3.26) which is given by

$$V_a(t) = V(t) + \frac{1}{2}\|z\|_2^2 = \frac{1}{2}\int_0^L |\rho|^2\, dx + \frac{1}{2}\int_0^L |z|^2\, dx$$

with $z = q - \overline{G}(\rho)$. The control law u_n is given by (3.27) as

$$u_n(x,t) = -Kz - z^{-1}\rho \frac{\partial z}{\partial x} \tag{7.45}$$

This control ensures that the origin of the system (7.43) and (7.44) is asymptotically stable.

Thus the final robust feedback control law for (7.29) and (7.30) is given by the following partial differential-integral equation

$$u = u_n + \frac{\partial \overline{G}(\rho)}{\partial t} + \frac{\partial}{\partial x}(q^2/\rho) \tag{7.46}$$

with $u_n(x,t)$ given by (7.45) and $\overline{G}(\rho) = -D\int_0^x \frac{\partial^2 \rho}{\partial m^2} dm - \int_0^L \frac{\eta}{1-\kappa}\rho dx$.

7.3.2 Robust Control: Matched Uncertainty

Let us consider the uncertainty in the input to the overall system (7.26) which means that there is an error between the actual value and control command for gradient of pressure $\partial p/\partial x$. By assuming the uncertainty in input u in second equation we have our input uncertain model as

$$\frac{\partial \rho}{\partial t} = -\frac{\partial q}{\partial x} \tag{7.47}$$

$$\frac{\partial q}{\partial t} = -\frac{\partial}{\partial x}(q^2/\rho) - \frac{\partial}{\partial x}(p+\theta) \tag{7.48}$$

Here the uncertainty is matched as it enters the system at the same point as actual control input u. With $|\theta(t,u,\rho)| = 0$ the nominal model for this system is given by (7.27) and (7.28). As a first step towards the control design we design feedback control for the nominal model (the system without disturbance) by applying backstepping technique. After that the robust control for the uncertain system (7.48) is designed using *Lyapunov redesign technique*. We proceed with the control design as follows

1. ***Control by backstepping method:*** The first step is to design a feedback controller $u = -\partial p/\partial x$ for the nominal two-equation system (7.27) and (7.28) using *backstepping* technique. The controller as designed in section 3.3.2 is given by (3.28) as

$$u = -\frac{\partial p}{\partial x} = u_n + \frac{\partial G(\rho)}{\partial t} + \frac{\partial}{\partial x}(q^2/\rho) \tag{7.49}$$

where u_n is given by (3.27) as $u_n = -Kz - z^{-1}\rho\frac{\partial z}{\partial x}$, with $z = q - G(\rho)$ and $G(\rho) = -D\int_0^x \frac{\partial^2 \rho}{\partial m^2}dm$.

The nominal system (7.27) and (7.28) is asymptotically stable and there exists a Lyapunov function $V_a(t)$ given by (3.26) as

$$V_a(t) = V(t) + \tfrac{1}{2}\|z\|_2^2 = \tfrac{1}{2}\int_0^L |\rho|^2 dx + \tfrac{1}{2}\int_0^L |z|^2 dx \tag{7.50}$$

which satisfies $\frac{dV_a(t)}{dt} \le -2\beta V_a(t)$ with $\beta = 2DC^{-1}$

2. ***Lyapunov redesign:*** The next step is to design a robust control for the uncertain model (7.47) and (7.48) using Lyapunov redesign method. The goal is to design feedback control law

$$\hat{u} = u + v \tag{7.51}$$

such that we achieve closed loop stability and asymptotic attenuation of $\theta(t,u,\rho)$. u is given by (7.49) and v will be designed by using Lyapunov redesign method. Now with the feedback controller $\hat{u} = u + v$ or $p = \int_0^L \hat{u}dx + v$ the error in the closed-loop dynamics for the nominal and actual models due to the input uncertain term is

$$\theta(t,\rho,u) = -\frac{\partial v}{\partial x}$$

The magnitude of this uncertain term with respect to L_2 norm is given by

$$\|\theta(t,\rho,u)\|_2 = -\|\partial v/\partial x\|_2$$

which by using Sobolev inequality can be shown as

$$\|\theta(t,\rho,u)\|_2 \le -\gamma\|v\|_2 \tag{7.52}$$

where $\gamma = C^{-1}$. After applying control law (7.51) the rate of change of Lyapunov function $V_a(t)$ is

$$\frac{dV_a(t)}{dt} \le -D\|\rho\|_2^2 - K\|z\|_2^2 - \int_0^L \rho\frac{\partial v}{\partial x}dx - \int_0^L \rho\frac{\partial \theta}{\partial x}dx$$

or

$$\frac{dV_a(t)}{dt} = -2\beta V_a(t) - \int_0^L \rho\frac{\partial v}{\partial x}dx - \int_0^L \rho\frac{\partial \theta}{\partial x}dx \tag{7.53}$$

Let us define $\hat{v} = \partial(v)/\partial x$ and $\hat{\theta} = \partial(\theta)/\partial x$, therefore (7.53) becomes

$$\frac{dV_a(t)}{dt} \le -2\beta V_a(t) - \int_0^L \rho\hat{v}dx - \int_0^L \rho\hat{\theta}dx \tag{7.54}$$

By applying Holders inequality and Sobolev inequality (3.16) to the third term in the above equation and then by using (7.52) we have

$$\frac{dV_a(t)}{dt} \le -2\beta V_a(t) - \int_0^L \rho\hat{v}dx - C^{-1}\|\rho\|_2\|\hat{v}\|_2 \tag{7.55}$$

Let us choose the control law as before

$$\hat{v} = -\frac{\eta}{1-\kappa}\rho(x,t)$$

or

$$v = -\int_0^L \frac{\eta}{1-\kappa}\rho dx \tag{7.56}$$

where $\eta(\rho,t) \geq \gamma(\rho,t)$. Using this control law (7.54) becomes

$$\frac{dV_a(t)}{dt} \leq -2\beta V_a(t) - \|\rho\|_2^2 \left\|\frac{\eta}{1-\kappa}\right\|_2 \left(C^{-1}-1\right)$$

Hence we have asymptotic stability as long as ($C^{-1}-1 \geq 0$). Thus the robust feedback control is given as $\hat{u} = u + v$ with u given by (7.49) and v is given as

$$v = -\int_0^L \frac{\eta}{1-\kappa}\rho dx \tag{7.57}$$

7.3.3 Robust Control: Both Matched and Unmatched Uncertainties

Now let us consider both matched and unmatched uncertainties for the two-equation model. This means that there is uncertainty in the input $\partial p/\partial x$ to the overall system (7.25) and (7.26) and also in the input q to (7.25) in the flow. Therefore we have our uncertain model as

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(q + \theta_1) \tag{7.58}$$

$$\frac{\partial q}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{q^2}{\rho}\right) + \frac{\partial}{\partial x}(p + \theta_2) \tag{7.59}$$

Here the uncertainties are both matched as well as unmatched. The errors θ_1 and θ_2 are given by (7.36) and (7.52) as $\theta_1(t,\rho,u) = -\partial v_1/\partial x$ and $\theta_2(t,\rho,u) = -\partial v_2/\partial x$ respectively and are bounded as $\|\theta_1(t,\rho,u)\|_2 \leq \gamma\|v_1\|_2$ and $\|\theta_2(t,\rho,u)\|_2 \leq \gamma\|v_2\|_2$. The control design scheme will be a combination of the designs for matched and unmatched uncertainties obtained in sections 7.3.1 and 7.3.2. We proceed with the control design as follows

1. ***Control by Lyapunov redesign:*** The first step is to design a robust feedback controller $q = G(\rho) + v_1$ for (7.58) using *Lyapunov redesign* technique. The control is given by (7.42) as

$$q = -D\int_0^x \frac{\partial^2 \rho}{\partial m^2} dm - \int_0^L \frac{\eta}{1-\kappa} \rho dx \tag{7.60}$$

with $v_1 = -\int_0^L \frac{\eta}{1-\kappa} \rho dx$. The controller stabilizes equation (7.58) in presence of uncertainty θ_1. The Lyapunov function is given by (7.33) as $V(t) = -1/2\|\rho\|_2^2$.

2. ***Control by backstepping:***: The next step is to design a controller for the nominal model (model without uncertainty θ_2) using the *backstepping* approach.. The model given by (7.29) and (7.30) is the nominal model

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(q + \theta_1)$$

$$\frac{\partial q}{\partial t} = -\frac{\partial}{\partial x}(\frac{q^2}{\rho}) + \frac{\partial p}{\partial x} = -\frac{\partial}{\partial x}(\frac{q^2}{\rho}) + u \tag{7.61}$$

The controller $u = -\frac{\partial p}{\partial x}$ is given by(7.46) as the following partial differential-integral equation

$$u = u_n + \frac{\partial \overline{G}(\rho)}{\partial t} + \frac{\partial}{\partial x}(q^2/\rho) \tag{7.62}$$

The control law $u_n(x,t)$ is given by (7.45) as $u_n(x,t) = -Kz - z^{-1}\rho \frac{\partial z}{\partial x}$ with $z = q - \overline{G}(\rho)$ and $\overline{G}(\rho) = -D\int_0^x \frac{\partial^2 \rho}{\partial m^2} dm - \int_0^L \frac{\eta}{1-\kappa} \rho dx$. The modified Lyapunov function for this system is

$$V_a(t) = V(t) + \tfrac{1}{2}\|z\|_2^2 = \tfrac{1}{2}\int_0^L |\rho|^2 dx + \tfrac{1}{2}\int_0^L |z|^2 dx$$

3**. *Control by Lyapunov redesign:*** In the final step we design the robust feedback control $\hat{u} = u + v_2$ for overall system (7.60) using Lyapunov redesign obtained in section 7.3.2. The feedback control is thus given as

$$\hat{u} = u + v_2 = u - \int_0^L \frac{\eta}{1-\kappa} \rho dx \tag{7.63}$$

The control law u is given by (7.62). The Lyapunov function for this system is same as in the previous step. Therefore we achieve asymptotic stability and disturbance attenuation with this controller.

In this chapter we designed robust feedback controllers for one-dimensional case. We designed controllers for uncertainty in the input. For one-equation model the uncertainty is matched and we used the method of Lyapunov redesign which achieved the attenuation of disturbance. For the two-equation model we discussed both matched as well as unmatched uncertainties and a combination of both. We used the method of *robust backstepping* which is a combination of Lyapunov redesign and *backstepping*. In all the controllers we achieved both asymptotic stability and disturbance attenuation.

In all the feedback controllers designed so far we have observed that the difficulty in control design for systems increases as we increase the number of partial differential equations representing the dynamics. However, as we increase the number of PDEs representing the dynamics the accuracy of the model increases. This motivates the study of abstraction for infinite dimensional systems which will be the topic of chapter 9. The goal here is to abstract the evacuation system having higher number of partial differential equations by a system which is represented by lower number of partial differential equations and find the transformation for control design. In working towards the study of abstraction and the issues related with control design transfer for an infinite dimensional system we will first study the abstraction for finite dimensional case and extend the results to infinite dimensional case from there. In finite dimensional abstraction we discuss an example of a robotic car for abstraction. In chapter 8 we will first present some mathematical preliminaries to build a framework for finite dimensional abstraction and in chapter 9 we will consider the abstraction example for a robotic car. The goal is to extend the same framework for abstraction to get similar result for the evacuation case.

CHAPTER 8

ABSTRACTION PRELIMINARIES

8.1 Introduction

In this chapter the mathematical background needed for finite dimensional abstractions is presented. Some differential geometric concepts that are used in later chapters are introduced. A full treatment of these concepts can be found in [44], [45], and [46]. We also introduce the notions of Φ-related vector fields and Φ-related control systems and present some important results on abstracted linear control systems from [17].

8.2 Definitions

The *tangent space* to smooth manifold M at p is the space of real linear mappings X_p, whose domain is all smooth functions defined on a neighborhood of p, satisfying

$$X_p(fg) = X_p(f)g(p) + f(p)X_p(g) \tag{8.1}$$

for all smooth f, g defined on a neighborhood of p. The tangent space associated with $p \in M$ is denoted by T_pM. The union of all tangent spaces denoted by

$$TM = \bigcup_{p \in M} T_pM \tag{8.2}$$

is called the *tangent bundle* of M .

The *canonical projection* map $\pi : TM \to M$ takes a tangent vector $X_p \in T_pM \subset TM$ to the point $p \in M$. If $M = \Re^n$ locally, then $T_p\Re^n = \Re^n$ and $T\Re^n = \Re^n \times \Re^n$.

Let $\Phi : M \to N$ be a smooth map between two smooth manifolds M and N. The *tangent map* $T_p\Phi : T_pM \to T_qN$ takes forward tangent vectors from T_pM to T_qN where $p \in M$ and $q = \Phi(p) \in N$. The union of all *tangent maps* is denoted by

$$T\Phi = \bigcup_{p \in M} T_p\Phi \, . \tag{8.3}$$

If M and N are both Euclidean spaces then $T_p\Phi = \frac{\partial \Phi}{\partial p}\big|_{p \in M}$ and $T\Phi = d\Phi$ is the total derivative of Φ.

A *vector field* X is a smooth map $X : M \to TM$ that assigns to every point $p \in M$ a tangent vector $X(p) \in T_pM \subset TM$. A smooth curve $c: I \to M$ that satisfies

$$c'(t) = X(c(t)) \tag{8.4}$$

for all $t \in I \subseteq \Re$, is called an *integral curve*.

Given two vector fields X and Y on M, the product denoted by $[X,Y]$ and given by

$$[X,Y]_p(f) = X_p(Y(f)) - Y_p(X(f)) \tag{8.5}$$

is called a *Lie bracket.*

A *distribution* Δ on M assigns to each $p \in M$ a subspace of T_pM. The distribution generated by vector fields $X_1,X_2,\ldots,X_k$ is given by $\Delta = \text{span}[X_1,X_2,\ldots,X_k]$. A vector field X belongs to a distribution Δ if $X(p) \in \Delta(p)$ for every $p \in M$.

Given a distribution Δ, Lie(Δ) is the *Lie algebra* generated by Δ. It is obtained by taking the span of iterated Lie brackets of vector fields in Δ.

The following definition ensures that for any vector field X, the tangent map $T\Phi(X(p))$ is well-defined for all $p \in M$.

Definition 8.1 (Φ-Related Vector Fields): Let $\Phi : M \rightarrow N$ be a smooth map between manifolds M and N. Let X and Y be vector fields on M and N, respectively. *Then X and Y are Φ-related if for all* $p \in M$

$$T\Phi(X(p)) = Y(\Phi(p)). \tag{8.6}$$

Equivalently, if $\Phi : M \rightarrow N$ is a smooth surjection, *then for any vector field X, TΦ(X) is well-defined on N only if*

$$T_{p_1}\Phi(X(p_1)) = T_{p_2}\Phi(X(p_2)) \tag{8.7}$$

whenever $\Phi(p_1) = \Phi(p_2)$ *for any* $p_1, p_2 \in M$.

Fig. 8.1 shows the relationship between vector fields in manifolds M and N. The vector fields X and Y are Φ-related if $T\Phi(X(p)) = Y(\Phi(p))$ for all $p \in M$.

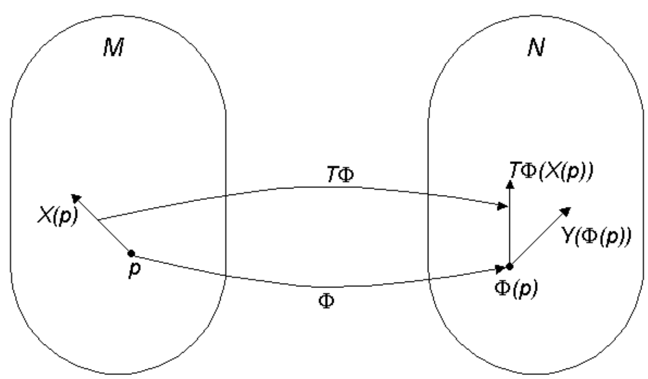


Figure 8. 1: The relationship between vector fields in M and N. The vector fields X and Y are Φ-related if $T\Phi(X(p)) = Y(\Phi(p))$.

It is useful to view Φ-relatedness of vector fields in terms of their integral curves. The condition on the integral curves of two Φ-related vector fields is given by the following theorem from [44].

Theorem 8.2*:* Suppose X and Y are vector fields on M and N respectively and let $\Phi : M \rightarrow N$ be a smooth map, *then X and Y are Φ-related if and only if for every integral curve c of X,* $\Phi \circ c$ *is an integral curve of Y.*

Next, the concept of Φ-relatedness of vector fields is extended to control systems. In order to have a coordinate free and general description of control systems, the concept of fiber bundles is used.

Let us consider nonlinear control systems described by the ordinary differential equations

$$\dot{x}(t) = f(x(t), u(t))$$

$$y(t) = h(x(t), u(t)) \tag{8.8}$$

where $x(t) \in \Re^n$ denotes the state of the system, $u(t) \in \Re^m$ denotes the control input, and $y(t) \in \Re^p$ denotes the output of the system. The control system in (8.8) can be described using the commutative diagram in Fig.8.2 adapted from [46].

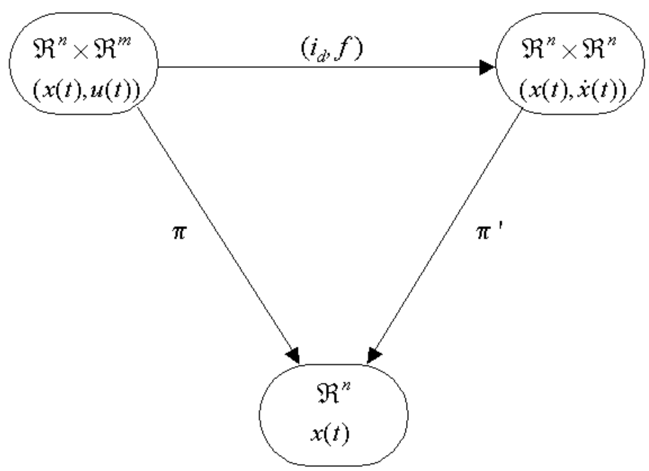


Figure 8. 2: Local description of a control system.

In the figure,

$$(i_d, f)(x(t), u(t)) = (x(t), \dot{x}(t))$$

$$\pi(x, u) = x$$

$$\pi'(x, \dot{x}) = x \tag{8.9}$$

This control system representation can be generalized from the above local description to a global one on a manifold M as shown in Fig.8.3 adapted from [46].

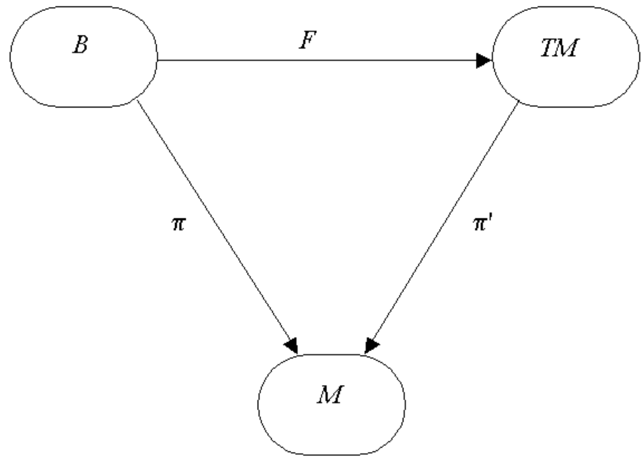


Figure 8. 3: Global description of a control system.

The components of the control system can be described as follows:

- M - the state space manifold
- $\pi : B \to M$ - a fiber bundle
- $\pi^{-1}(x)$ - the state dependent input spaces where $x \in M$
- TM - the tangent space of M
- $\pi' : TM \to M$ - the canonical projection of TM on M
- $F : B \to TM$ - the system dynamics

In other words, for any point (x,u) in B, $F(x,u) = (x,f(x,u))$ where $f(x,u)$ is the velocity vector at the point $x \in M$. Using the local coordinates of the manifolds, this representation is exactly the same as given above.

The *fiber bundle* $\pi : B \to M$ is often simply $M \times U$ for some input space U. However, this is generally not the case because the input space may be dependent on the state. The use of product $M \times U$ is then insufficient to describe the space. The product space can be generalized into the form of a fiber bundle, defined as follows.

Definition 8.3 (Fiber Bundles) *$(B,M,\pi,U,\{O_i\}_{i \in I})$ is called a fiber bundle if the following holds.*

- *B is a smooth manifold called the total space.*
- *M is a smooth manifold called the base space.*
- *U is a smooth manifold called the standard fiber.*
- $\pi : B \rightarrow M$ *is a surjective submersion.*
- $\{O_i\}_{i \in I}$ *is an open cover for M such that for every* $i \in I$*, there exists a diffeomorphism* Ψ_i: $\pi^{-1}(O_i) \rightarrow O_i \times U$ *such that* $\pi_o \circ \Psi_i = \pi$*, where* π_o *is the projection from* $O_i \times U$ *to* O_i*. The submanifold* $\pi^{-1}(p)$ *is called the fiber at* $p \in M$*.*

Fig. 8.4 illustrates the relationships of a fiber bundle. The definition of fiber bundle is given in [46].

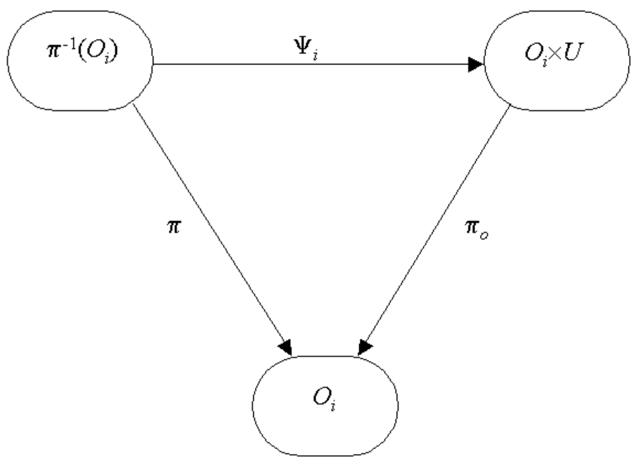


Figure 8. 4: Fiber bundle.

Definition 8.4 (Control Systems) *A control system S = (B,F) consists of the following*

- *A fiber bundle* $\pi : B \rightarrow M$ *called the control bundle.*
- *A smooth map* $F : B \rightarrow TM$ *which is fiber preserving.*

A map is fiber preserving if $\pi'(F) = \pi$, where $\pi' : TM \rightarrow M$ is the tangent bundle projection.Fig.8.3 illustrates, globally, the relationships of a control system and Fig.8.2 shows the local relationships for an *n*-dimensional system with *m* inputs.

Definition 8.5 (Trajectories of control systems) Let $I \subseteq \Re$. *A smooth curve $c : I \to M$ is called the trajectory of the control system (B, F) if there exists a curve $c^B : I \to B$ such that $\pi(c^B) = c$.*

This definition translates into local coordinates as the well-known concept of trajectories in a control system given by $\dot{x} = f(x, u, t)$. Locally, a trajectory is a curve $x(t)$ for which there exists an input $u(t)$ such that $\dot{x}(t) = f(x(t), u(t), t)$.

Definition 8.6 (Φ-Related Control Systems) Let $\Phi : M \to N$ be a smooth map and let $S_M = (B_M, F_M)$ and $S_N = (B_N, F_N)$ be two control systems with $\pi_M : B_M \to M$ and $\pi_N : B_N \to N$. *Then control systems S_M and S_N are Φ-related if*

$$T\Phi \circ F_M(\pi_M^{-1}(p)) \subseteq F_N(\pi_N^{-1}(\Phi(p))) \tag{8.10}$$

for all $p \in M$.

The following example, motivated by the double integrator example in [17], illustrates concept of Φ-related control systems in higher dimensions.

Example 8.7 Consider the control system, S_M, given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ u \end{bmatrix} \tag{8.11}$$

where $M = \Re^3$ and $u \in \Re$. Let Φ-relation be the projection map defined by $\Phi(x_1,x_2,x_3) = (x_1,x_2)$. Define the control system, S_N, to be

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ v \end{bmatrix} \tag{8.12}$$

where $N = \Re^2$ and $v \in \Re$. Let $p = (x_1,x_2,x_3)$ be a point in M. Then vector fields in S_N mapped from vector fields in S_M are found by the following process.

$$p = (x_1, x_2, x_3) \in M$$

$$\Downarrow \pi_M^{-1}$$

$$(x_1, x_2, x_3, u) \in B_M$$

$$\Downarrow F_M$$

$$\left(x_1, x_2, x_3, [x_2, x_3, u]^T\right) \in TM$$

$$\Downarrow T\Phi$$

$$\left(x_1, x_2, [x_2, x_3]^T\right) \in TN$$

Similarly, vector fields in S_N originating from $\Phi(p)$ are found as follows.

$$p = (x_1, x_2, x_3) \in M$$

$$\Downarrow \Phi$$

$$(x_1, x_2) \in N$$

$$\Downarrow \pi_N^{-1}$$

$$(x_1, x_2, v) \in B_M$$

$$\Downarrow F_N$$

$$\left(x_1, x_2, [x_2, v]^T\right) \in TN$$

Setting $v = x_3$ generates the corresponding vector field in N. Because of the freedom in the control input v, S_N can generate any vector field that is mapped from S_M as illustrated in Fig.8.5. So it follows that $T\Phi \circ F_M(\pi_M^{-1}(p)) \subseteq F_N(\pi_N^{-1}(\Phi(p)))$ and control systems S_M and S_N are Φ-related because $v \in \Re$.

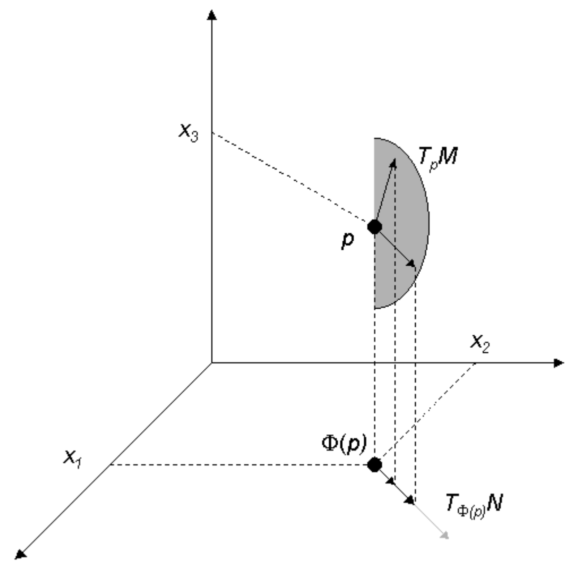


Figure 8. 5: The mapping of vector fields in S_M to S_N

In addition to constructing abstractions it is also desirable to propagate properties between systems. In particular the property of controllability is reviewed here. The following definitions [47] of reachability are utilized towards that end.

Definition 8.8 (Reachability) Let $S = (B,F)$ be a control system on a manifold M. *Given a point $p \in M$, the reachable set in time $t \in I \subseteq \Re$ is defined as*

$$\text{Reach}(p,S) = \{p' \in M \mid \mid c : I \to M \text{ such that}$$

$$c(t_1) = p \text{ and } c(t_2) = p', \text{ for } t_1, t_2 \in I\}. \qquad (8.13)$$

Furthermore, reachability of a set can be defined for $A \subseteq M$ as

$$\text{Reach}(A,S) = \bigcup_{p \in A} \text{Reach}(p,S). \qquad (8.14)$$

Using the concept of reachability, controllability can be restated as follows.

Definition 8.9 (Controllability) *A control system $S = (B,F)$ is controllable if for every $p \in M$, Reach$(p,S) = M$*

One of the important results from [17] is the following theorem that provides a condition for an abstracted system S_N to be controllable.

Theorem 8.10 (Reachability for Φ-Related Systems) Consider two control systems $S_M = (B_M, F_M)$ and $S_N = (B_N, F_N)$ which are Φ-related with respect to some surjective submersion *$\Phi : M \to N$. Then for all $p \in M$,*

$$\Phi(\text{Reach}(p, S_M)) \subseteq \text{Reach}(\Phi(p), S_N) \tag{8.15}$$

Thus it follows that *if S_M is controllable then S_N is controllable.*

We have introduced some fundamental topics in differential geometry that will be used in the next chapter. In the next chapter, nonlinear abstractions are considered for finite dimensional case by extending the results given in [17]. In that chapter, the abstraction of a robotic car to a rolling disk that preserves controllability properties, in particular local accessibility is reviewed. The results and framework are then extended to an evacuation problem.

CHAPTER 9

FINITE AND INFINITE DIMENSIONAL ABSTRACTION

9.1 Introduction

This chapter discusses the abstraction for finite and infinite dimensional cases. For finite dimensional case the chapter iscusses the rolling disk as an abstraction of a robotic car as an example. This chapter introduces the concepts of *traceability*, *ε-traceability*, *ε-consistency*. To reduce complexity of analysis, simplified models that capture the behavior of interest in the original system can be obtained. These simplified models, called abstractions, can be analyzed more easily than the original complex model. Hierarchies of consistent abstractions can significantly reduce the complexity in determining the reachability properties of nonlinear systems. Such consistent hierarchies of reachability, preserving nonlinear abstractions are considered for the robotic car by extending the results given in [17].

Not only can these abstractions be analyzed with respect to some behavior of interest, these can also be used to transfer control design for complex models to simplified models. This chapter is an initial step towards that study. In this chapter, the abstraction of a robotic car to a rolling disk that preserves controllability properties, in particular local accessibility, are reviewed. In this framework, showing the local accessibility of the abstracted rolling disk is equivalent to showing the local accessibility of the robotic car. Working towards the study of control design, it is seen that there are certain classes of trajectories that exist in the rolling disk system that cannot be achieved by the robotic car.

However, the robotic car can achieve trajectories that are arbitrarily close to those rolling disk trajectories. Such classes of trajectories are a concern when designing a controller for the abstracted system that is to be transformed to a controller in the original system. In order to account for these cases, new concepts of traceability, ε-traceability, and ε-consistency are defined and controllability equivalence between the systems is expressed in terms of these new concepts.

In the later part of the chapter we present the abstraction of an evacuation system by extending the results and framework for finite case. The organization of the chapter is as follows. In Section 9.2 differential geometric concepts discussed in chapter 8 are generalized for control systems and some results on linear control systems from [17] are reviewed. Sections 9.3 discusses rolling disk as an abstraction to a car-like robot. Section 9.4 revises the concepts in Section 9.3 for our application and introduces concept of traceable control systems. Section 9.5 discusses the abstraction of an evacuation system.

9.2 Abstracted Control Systems

Theorem 8.10 allows us to propagate controllability from the original system to abstracted system. However it is of more interest to study the propagation of controllability in the reverse direction. If the relationship is known, then the controllability of the original system can be determined from that of the abstracted one. The definitions and theorems presented in this section are from [17] and are needed to study the propagation of controllability in the reverse direction. The complete proofs to these theorems are given in [17].

Definition 9.2.1 (Implementability) Let $\Phi : M \rightarrow N$ be a smooth surjection. *A control system $S_M = (B_M, F_M)$ is implementable by control system $S_N = (B_N, F_N)$ if whenever there exists a trajectory in S_N connecting $q_1, q_2 \in N$, then there exist points $p_1 \in \Phi^{-1}(q_1)$ and $p_2 \in \Phi^{-1}(q_2)$ and a trajectory in S_M connecting them.*

Implementability ensures the existence of trajectories in the original system. It is dependent upon a particular element chosen from the equivalence class in original system corresponding to elements of abstracted one. A trajectory of abstracted system that is implementable by one element of the equivalence class may not be implementable by rest of its elements

Theorem 9.2.2 (Implementability Condition) Let $\Phi : M \rightarrow N$ be a smooth surjection. *A control system* $S_M = (B_M,F_M)$ *is implementable by control system* $S_N = (B_N,F_N)$ *if and only if for all* $q \in N$, $Reach(q,S_N) \subseteq \Phi(Reach(\Phi^{-1}(q),S_M))$.

For implementability of Φ-related systems, the inclusion becomes equality.

We have seen that implementability depends upon a particular element chosen from an equivalence class. In order to propagate controllability from the abstracted system to the original system, this dependence is to be removed. To remove the dependence concept of consistency, the definition below is used.

Definition 9.2.3 (Consistency) Let $\Phi : M \rightarrow N$ be a smooth surjection. *A control system* $S_M = (B_M,F_M)$ *is consistent with respect to* Φ *whenever the following holds. If there exists a trajectory in* S_M *connecting* $p_1,p_2, \in M$*, then for all* $p_1' \in \Phi^{-1}(\Phi(p_1))$ *and* $p_2' \in \Phi^{-1}(\Phi(p_2))$ *there exists a trajectory in* S_M *connecting* p_1' *to* p_2'.

Consistency removes the dependence on a particular element chosen from an equivalence class. Whenever some element in original system is implementable by abstracted system, then all the elements of its equivalence class are implementable.

Theorem 9.2.4 (Consistency Condition) Let $\Phi : M \rightarrow N$ be a smooth surjection. *A control system* $S_M = (B_M,F_M)$ *is consistent with respect to* Φ *if and only if for all* $p \in M$, $Reach(p,S_M) = \Phi^{-1}(\Phi(Reach(\Phi^{-1}(\Phi(p)),S_M)))$.

When implementability and consistency are combined, it provides a powerful result for controllability in reverse direction. This is given in the following theorem.

Theorem 9.2.5 (Controllability Equivalence) *Let* $\Phi : M \rightarrow N$ *be a smooth surjection and suppose control systems* $S_M = (B_M,F_M)$ *and* $S_N = (B_N,F_N)$ *be* Φ*-related. Furthermore,* *suppose that S_N is implementable by S_M and S_M is consistent with respect to Φ. Then S_N is controllable if and only if S_M is controllable.*

9.3 Abstraction of Finite Dimensional System

In this section we want to apply the notion of Φ-relatedness to the car-like robot with respect to some aggregation map Φ. Consider the robotic car to be the complex control system. We want to find Φ such that the abstracted system is a rolling disk. The car kinematic model is given as (see Fig.9.1)

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ \frac{\tan\phi}{l} \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2 \tag{9.1}$$

where u_1 is the rear wheel's linear velocity, u_2 is the steering angular velocity, $x,y \in \Re$, $\theta \in [-\pi,\pi]$, and $\phi \in (-\pi/2,\pi/2)$. Note that there is a singularity at $|\phi| = \pi/2$. Denote the car control system as S_M defined on manifold M.

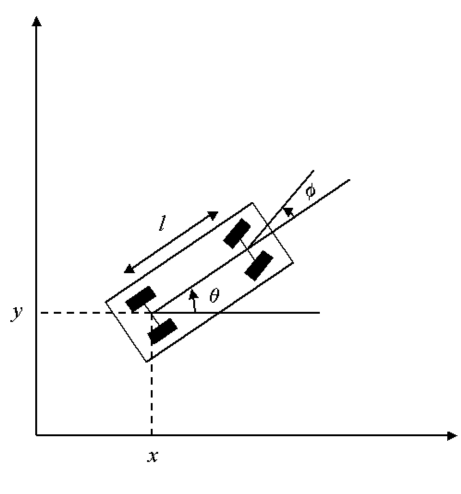


Figure 9. 1: The variables used for the car.

Controllability of the Car The system represented by equation (9.1) can be written as

$$\dot{X} = g_1 u_1 + g_2 u_2 \tag{9.2}$$

where $X = [x\ y\ \theta\ \phi]^T$, $g_1 = \begin{bmatrix} \cos\theta & \sin\theta & \frac{\tan\phi}{l} & 0 \end{bmatrix}^T$, and $g_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$.

The system in (9.2) is a driftless control system. To check the controllability of the system, the Lie algebra rank condition (LARC) is used, which for our case is:

$$\text{rank}\ [g_1\ \ g_2\ \ [g_1,g_2]\ \ [g_1,[g_1,g_2]]\ \ [g_2,[g_1,g_2]]\ \dots\ ] = 4 \tag{9.3}$$

where $[g_1,g_2]$ is the Lie bracket of vector fields g_1 and g_2.

Upon computation we see that the rank of the matrix in (9.3) is 4 except when $\phi = \pm\pi/2$. Thus the car is controllable away from those steering angles.

Consider the abstracting map Φ that is the projection from the car's space to the rolling disk's space. Define $\Phi : S_M \to S_N$ by $\Phi(x,y,\theta,\phi) = (x,y,\theta)$. Choosing this Φ gives the following abstracted system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2 . \tag{9.4}$$

This is same as the kinematic model of the rolling disk (see Fig.9.2) that is given by

$$\begin{bmatrix} \dot{x}_u \\ \dot{y}_u \\ \dot{\theta}_u \end{bmatrix} = \begin{bmatrix} \cos\theta_u \\ \sin\theta_u \\ 0 \end{bmatrix} v_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v_2 \tag{9.5}$$

where v_1 is the linear velocity, v_2 is the angular velocity, $x_u,y_u \in \Re$, and $\theta_u \in [-\pi,\pi]$. Denote the rolling disk control system as S_N defined on manifold N.

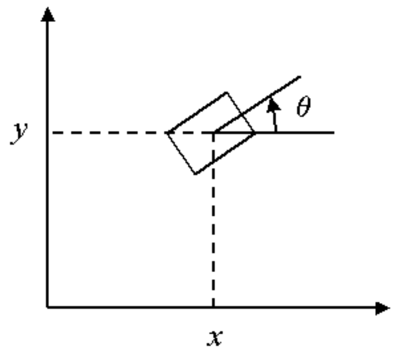


Figure 9. 2: The variables used for the rolling disk.

Controllability of the Rolling Disk The model of rolling disk as represented by equation (9.5) can be rewritten as

$$\dot{X} = g_1 u_1 + g_2 u_2 \tag{9.6}$$

with $X = [x\ y\ \theta]^T$, $g_1 = [\cos\ \theta\ \ \sin\ \theta\ \ 0]^T$, and, $g_2 = [0\ 0\ 1]^T$.

Again to check the controllability of the system (9.6) the Lie algebra rank condition is used. For (9.5), the LARC is

$$\text{rank}\ [g_1\ \ g_2\ \ [g_1,g_2]\ \ [g_1,[g_1,g_2]]\ \ [g_2,[g_1,g_2]]\ \ldots\ ] = 3 \tag{9.7}$$

Performing the computations gives that the rank of the above matrix is 9. Thus the rolling disk is controllable everywhere.

Proposition 9.3.1 *The car and rolling disk are Φ-related control systems with Φ defined for each $(x,y,\theta,\phi) \in M$ by*

$$\Phi(x,y,\theta,\phi) = (x,y,\theta). \tag{9.8}$$

Proof: Consider any point $(x_0,y_0,\theta_0,\phi_0) \in M$. Then

$$\pi_M^{-1}(x_0, y_0, \theta_0, \phi_0) = ((x_0, y_0, \theta_0, \phi_0), (u_1, u_2)) \quad . \tag{9.9}$$

Thus,

$$T\Phi \circ F_M(\pi^{-1}((x_0, y_0, \theta_0, \phi_0))) = ((x_0, y_0, \theta_0), (\dot{x}_0, \dot{y}_0, \dot{\theta}_0)). \tag{9.10}$$

For any u_1 and u_2, letting $v_1 = u_1$ and $v_2 = \dot{\theta}_0$ gives

$$(x_0, y_0, \theta_0) = (x_u, y_u, \theta_u) \in F_N(\pi_N^{-1}(\Phi(x_0, y_0, \theta_0, \phi_0))). \tag{9.11}$$

So $$T\Phi(F_M(\pi_M^{-1}((x_0, y_0, \theta_0, \phi_0))) \subseteq F_N(\pi_N^{-1}(\Phi((x_0, y_0, \theta_0, \phi_0)).$$ ◊

Proposition 9.3.2 *The car is consistent with respect to the Φ given by (9.8)*

Proof: Let $p_1 = (x_1, y_1, \theta_1, \phi_1)$ and $p_2 = (x_2, y_2, \theta_2, \phi_2)$ be two points in M with a trajectory in S_M connecting them. Let $\hat{u}_1(t)$ and $\hat{u}_2(t)$ be the control inputs defined for $0 \le t \le T$ that generate that trajectory. Then for any $\phi_0, \phi_3 \in (-\pi/2, \pi/2)$, the points $p_1' = (x_1, y_1, \theta_1, \phi_0)$ and $p_2' = (x_2, y_2, \theta_2, \phi_3)$ are in $\Phi^{-1}(\Phi(p_1))$ and $\Phi^{-1}(\Phi(p_2))$ respectively. For any p_1' and p_2', define the control inputs by:

- For $t \in [0,1)$, $u_1(t) = 0$, $u_2(t) = \phi_1 - \phi_0$
- For $t \in [1, T+1)$, $u_1(t) = \hat{u}_1(t-1)$, $u_2(t) = \hat{u}_2(t-1)$
- For $t \in [T+1, T+2]$, $u_1(t) = 0$, $u_2(t) = \phi_3 - \phi_2$

These control inputs generate trajectories between p_1' and p_2'. ◊

Proposition 9.3.3 *The rolling disk is implementable by the car.*

Proof: Consider any two points $(x_1, y_1, \theta_1), (x_2, y_2, \theta_2) \in N$ with a trajectory between them. The points $(x_1, y_1, \theta_1, 0), (x_2, y_2, \theta_2, 0) \in M$ are in $\Phi^{-1}(x_1, y_1, \theta_1)$ and $\Phi^{-1}(x_2, y_2, \theta_2)$ respectively. Since the car is controllable, there is a trajectory in S_M connecting the two points. ◊

The previous three propositions show that the conditions for theorem 9.2.5 are satisfied. But there is a class of trajectories in case of a rolling disk for which there exists no corresponding trajectory in car.

Consider the rolling disk trajectory given by $\dot{x}_u = \dot{y}_u = 0$, $v_1 = 0$, and v_2 nonzero, i.e., the rolling disk is rotating about its axis. The car trajectory that corresponds to this case must have $\phi = \pm\pi/2$, making $\dot{\theta}$ indeterminate. Thus the car cannot realize this particular class of trajectories.

However, there exists a car trajectory which, given a distance function defined on N, maps to a rolling disk trajectory that is arbitrarily close to the rotating rolling disk. So the rotating rolling disk corresponds to a limiting case of car trajectories, namely, as $\phi \rightarrow \pm\pi/2$, the car trajectory is mapped to trajectories that approach the rotating rolling disk. This concept leads to the definitions in the next section.

9.4 Traceable Control Systems

In order to account for the trajectories of the rolling disk which do not have any corresponding trajectories in a car, we define the following notion of *traceability*.

Definition 9.4.1 (Traceability) Consider a control system $S_M = (B_M, F_M)$ and a smooth surjection $\Phi : M \rightarrow N$. Let $c_N : I \rightarrow N$ be a trajectory of S_N where $I \subseteq \Re$. *Then S_N is traceable by S_M if for all trajectories c_N there exists a trajectory in S_M, $c_M : I \rightarrow M$, such that $\Phi(c_M(t)) = c_N(t)$ for all $t \in I$.*

Traceability is a stronger condition than implementability since traceability requires that there exists a trajectory in S_M for which the entire trajectory maps to the trajectory in S_N. Implementability only requires that there exists a trajectory in S_M for which its endpoints map to the endpoints of the trajectory in S_N. Any control system, S_N, that is traceable by S_M is implementable by S_M. However, the converse does not hold. The rolling disk is an example of a system that is implementable by the car, but not traceable. However, as the above development shows, the rolling disk is *almost* traceable by the car. This gives rise to the idea of ε-traceability. But first we define the norm of a trajectory and give an example.

Proposition 9.4.2 (Norm in N) Let $c_N : I \rightarrow N$ be a trajectory. *Then*

$$\| c_N \| = \sup_{t \in I} \sup_{i} | x_i(t) | \tag{9.12}$$

is a norm where x_i are the states in N.

Proof: The definition in (9.12) satisfies the properties of a norm.

Since $|x_i(t)| \geq 0$ for all i and all $t \in I$, then $||c_N|| \geq 0$. If $||c_N|| = 0$, then $|x_i(t)| = 0$ for all i and all $t \in I$. So $c_N = 0$. Similarly if $c_N = 0$, then $x_i(t) = 0$ for all i and all $t \in I$ and it follows that $||c_N|| = 0$. For any constant α,

$$
\begin{aligned}
\|\alpha c_N\| &= \sup_{t\in I} \sup_i |\alpha x_i(t)| \\
&= \sup_{t\in I} \sup_i |\alpha||x_i(t)| \\
&= |\alpha| \sup_{t\in I} \sup_i |x_i(t)| \\
&= |\alpha| \|c_N\|
\end{aligned}
$$

Given any other trajectory $c'_N : I \to N$ corresponding to states $x'_i(t)$, then

$$
\begin{aligned}
\|c_N + c'_N\| &= \sup_{t\in I} \sup_i |x_i(t) + x'_i(t)| \\
&\leq \sup_{t\in I} \sup_i |x_i(t)| + \sup_{t\in I} \sup_i |x'_i(t)| \\
&= \|c_N\| + \|c'_N\|
\end{aligned}
\qquad \lozenge
$$

For the following development, car trajectories will be denoted by $X(t) = (x(t),y(t),\theta(t),\phi(t))$ and rolling disk trajectories by $Y(t) = (x_u(t),y_u(t),\theta_u(t))$. The following proposition shows that for a rolling disk trajectory, c_N, that has no corresponding car trajectory, c_M, with $c_N(t) = \Phi(c_M(t))$, there exists a car trajectory with $\Phi(c_M(t))$ arbitrarily close to $c_N(t)$.

Proposition 9.4.3: *Given a rolling disk trajectory, $Y(t) = (\alpha,\beta,\hat{\theta}(t))$, where α and β are constants (i.e., $\dot{x}_u = \dot{y}_u = 0$), and given any $\varepsilon > 0$, there exists a car trajectory, $X(t)$, such that $||Y(t)-\Phi(X(t))|| < \varepsilon$.*

Proof: Choose the car's initial conditions as $x(0) = \alpha$, $y(0) = \beta$, $\theta(0) = \theta_u(0)$, and $\phi(0) \in (-\pi/2,\pi/2)$ so that $\tan\phi(0) > \frac{2l}{\varepsilon}$. Let $u_1(t) = \frac{l\dot{\theta}_u(t)}{\tan\phi(0)}$ and $u_2(t) = 0$. The car's resulting trajectory, $X(t)$, is

$$
\begin{aligned}
x(t) &= \alpha + \tfrac{l}{\tan\phi(0)} \sin\theta_u(t) \\
y(t) &= \beta + \tfrac{l}{\tan\phi(0)} (1 - \cos\theta_u(t)) \\
\theta(t) &= \theta_u(t) \\
\phi(t) &= \phi(0).
\end{aligned}
\tag{9.13}
$$

By choice of ϕ, $|x(t)\text{-}x_u(t)| < \varepsilon$ and $|y(t)\text{-}y_u(t)| < \varepsilon$ for all t. Hence $||Y(t)\text{-}\ \Phi(X(t))|| < \varepsilon$. ◊

The concept of trajectories in S_M that map arbitrarily close to those in S_N is formalized in the following definition of ε-traceability.

Definition 9.4.4 (ε-Traceability) Let $S_M = (B_M,F_M)$ and $S_N = (B_N,F_N)$ be two control systems and let $\Phi : M \rightarrow N$ be a smooth surjection. *Then S_N is ε-traceable by S_M if given $\varepsilon > 0$ and a trajectory $c_N : I \rightarrow N$ then there exists a trajectory $c_M : I \rightarrow M$ such that $||c_N\text{-}\ \Phi(c_M)|| < \varepsilon$.*

An immediate consequence of this definition is the relationship between traceability and ε-traceability.

Theorem 9.4.5 *S_N is ε-traceable by S_M for $\varepsilon = 0$ if and only if S_N is traceable by S_M.*

Proof: Let c_N be a trajectory in N. When $\varepsilon = 0$, the definition for traceable is satisfied since there exists a trajectory c_M in M with $||\Phi(c_M) - c_N|| = 0$. By definition of the norm in N, it is necessary that $\Phi(c_M) = c_N$.

If S_N is traceable by S_M, then there exists a trajectory c_M with $\Phi(c_M) = c_N$. Thus $||\Phi(c_M) - c_N|| = 0$. ◊

As with consistency and implementability, it is useful to describe the condition for ε-traceability in terms of reachability.

Theorem 9.4.6 (ε-Traceability Condition) *S_N is ε-traceable by S_M if and only if, given $\varepsilon > 0$, for any $q \in N$ for which there exists $p \in M$ with*

$$\text{Reach}(q,S_N) \subseteq \Phi(\text{Reach}(p,S_M)) \tag{9.14}$$

then there is a trajectory $c_N(t)$ in $Reach(q,S_N)$ and a trajectory $c_M(t)$ in $Reach(p,S_M)$ such that

$$||c_N(t) - \Phi(c_M(t))|| < \varepsilon. \tag{9.15}$$

In the case of the ε-traceability of Φ-related systems, the inclusion in (9.14) becomes equality.

Proof: In case the case that S_M and S_N are Φ-related, we have from (8.15), $\Phi(\text{Reach}(p,S_M)) \subseteq \text{Reach}(\Phi(p),S_N)$. Let $q = \Phi(p) \in N$ and $p \in \Phi^{-1}(q)$, therefore (9.14) can be written as

$$\Phi(\text{Reach}(p,S_M)) \subseteq \text{Reach}(q,S_N). \tag{9.16}$$

Then the equality follows from (9.14) and (9.16). ◊

With ε-traceability, we have a weaker condition on the relationship between the Φ-related systems S_M and S_N than what implementability gives. For controllability to propagate between the two systems, a condition stronger than consistency is needed. A restriction is placed on the system to require that if there are converging sequences of trajectories in S_N then a corresponding trajectory exists in S_M. This leads to the following definition of ε-consistency stated precisely below.

Definition 9.4.7 (ε-Consistency) Let $S_M = (B_M, F_M)$ be a control system on M and let $\Phi : M \to N$ be a smooth surjection. *Then S_M is ε-consistent with respect to Φ whenever the following holds. If for any $p_1, p_2 \in M$ there exist sequences p_{1_n} and p_{2_n} in M such that*

1. *There exist trajectories connecting $\Phi(p_{1_n})$ and $\Phi(p_{2_n})$ for all n and*

2. *$\Phi(p_{1_n}) \to \Phi(p_1)$ and $\Phi(p_{2_n}) \to \Phi(p_2)$,*

then there is a trajectory connecting p_1 and p_2.

Unlike consistency, which requires that trajectories exist between all members of $\Phi^{-1}(\Phi(p_1))$ and $\Phi^{-1}(\Phi(p_2))$ based on the existence of only one trajectory, ε-consistency gives the condition for a trajectory to exist between p_1 and p_2. Consistency does not involve trajectories in S_N, but for ε-consistency, the existence of a single trajectory relies upon a sequence of trajectories in S_N whose endpoints map arbitrarily close to $\Phi(p_1)$ and $\Phi(p_2)$. The following example shows that S_M can be ε-consistent with respect to Φ even if it is not controllable.

Example 9.4.8 Consider the system in S_M given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} u \\ 0 \end{bmatrix} \tag{9.17}$$

and the system in S_N given by

$$\dot{x}_2 = 0 \tag{9.18}$$

with the Φ-map given by $\Phi(x_1,x_2) = x_2$. For $p_1 = (x_1,\alpha)$ and $p_2 = (x_1',\beta)$, let $p_{1_n} = \alpha$ and $p_{2_n} = \beta$ for all n. If $\alpha = \beta$, then there exist trajectories in S_N connecting $\Phi(p_{1_n})$ and $\Phi(p_{2_n})$ for all n and a trajectory in S_M connecting p_1 and p_2. However, if $\alpha \neq \beta$, there is no trajectory in S_M connecting p_1 and p_2 and no trajectories in S_N connecting $\Phi(p_{1_n})$ and $\Phi(p_{2_n})$.

Controllability of S_M is thus not a necessary condition for ε-consistency. However, the following theorem shows that Φ-relatedness between S_M and S_N and controllability of S_M are sufficient conditions for S_M to be ε-consistent with respect to Φ.

Theorem 9.4.9 Consider control systems $S_M = (B_M,F_M)$ and $S_N = (B_N,F_N)$ which are Φ-related with respect to smooth surjection $\Phi : M \rightarrow N$. *If S_M is controllable, then S_M is ε-consistent with respect to Φ.*

Proof: Consider any $p_1, p_2 \in M$. Let $p_{1_n} = p_1$ and $p_{2_n} = p_2$ for all n. Since S_M is controllable, there exist trajectories connecting p_{1_n} and p_{2_n} for all n. Denote each such trajectory by c_M. Then $T\Phi(c_M)$ is also a trajectory in S_N. The sequences p_{1_n} and p_{2_n} satisfy the conditions of Definition 9.9.7. Thus it follows that the car is ε-consistent with respect to the Φ mapping defined by (9.8). ◊

As with implementability and consistency, the relationship between ε-traceability and ε-consistency provide a means to propagate controllability between control systems S_M and S_N.

Theorem 9.4.10 (Controllability Equivalence) *Consider control systems $S_M = (B_M,F_M)$ and $S_N = (B_N,F_N)$ which are Φ-related with respect to smooth surjection $\Phi : M \to N$. Assume that S_N is ε-traceable by S_M and S_M is ε-consistent with respect to Φ. Then S_N is controllable if and only if S_M is controllable.*

Proof: If S_M is controllable then S_N is controllable since the control systems are Φ-related.

Now assume S_N is controllable and consider any points p_1, $p_2 \in M$. Because S_N is controllable, there is a trajectory, c_N, connecting $\Phi(p_1)$ and $\Phi(p_2)$. Then for each n, ε-traceability with $\varepsilon = 1/n$ gives a trajectory, c_{M_n}, such that $\|c_N - \Phi(c_{M_n})\| < 1/n$. Denote the endpoints of each c_{M_n} by p_{1_n} and p_{2_n}. These endpoints provide sequences in M with $\Phi(p_{1_n}) \to \Phi(p_1)$ and $\Phi(p_{2_n}) \to \Phi(p_2)$. Because S_N is controllable, there exist trajectories connecting $\Phi(p_{1_n})$ and $\Phi(p_{2_n})$ for all n. Since S_M is ε-consistent, there is a trajectory connecting p_1 and p_2. Thus S_M is controllable. ◊

9.5 Abstraction of an Evacuation System

In this section we would like to address the abstraction of an evacuation system With the above presented notions, a framework has been provided that gives conditions for the existence of trajectories in the original system and allows controllability propagation between the two systems. We want to extend these results to obtain the similar result for an evacuation system. In infinite dimensional case abstraction would mean to obtain the model with a lesser number of PDEs than the original one.

The preceding chapters discussed the design of feedback controllers for two infinite dimensional models of evacuation. With the use of these controllers we have shown exponential asymptotic stability of the closed loop system. However we observed that the difficulty in designing controllers for system increases as we increase the number of partial differential equations representing its dynamics. This motivates the study of abstractions for infinite dimensional systems. Evidently, control design in the abstracted system is a simpler task than in the original system. This work is a first step in the process of finding a method to transform control designs in the abstracted system back to the original system for an evacuation system. The future goal of this work is to find the transformation that will transfer the control design for an abstracted one equation model to the two equation model. The goal here is to abstract the system having higher number of partial differential equations by a system which is represented by lower number of partial differential equations. The abstraction is constructed in such a way that it enables to preserve certain properties of the system. As a part of that research we want to study abstraction which preserves controllability for evacuation dynamics. For that purpose we need to modify the notions of consistent abstractions introduced in [6] for infinite dimensional systems.

For a one-dimensional case we can state the abstraction problem more specifically as: Find an abstracting map Φ from a two equation control model

$$\frac{\partial \rho}{\partial t} = -\frac{\partial q}{\partial x}$$

$$\frac{\partial q}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{q^2}{\rho}\right) + \hat{u}(x,t) \tag{9.19}$$

to a one-equation control model

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}\left(\rho(1-\rho/\rho_{\max})u\right) \tag{9.20}$$

such that (9.19) and (9.20) are Φ-related according to definition 8.6. Not only do we want to construct this mapping we also want to preserve the controllability property of two equation model. In other words we want to find map Φ such that (9.19) is a consistent abstraction of (9.20). In this framework, showing the controllability of the abstracted one equation model is equivalent to showing controllability of the two equation model. Let us first construct a mapping to abstract equation (9.19)

Consider the abstracting map Φ that is the mapping from two-equation model manifold M to the one-equation model manifold N. Here M is a Banach manifold [48] of mappings (or functions) from one finite dimensional manifold to another for a two-equation system. We can express it locally as $(L^1 \cap L^\infty \cap BV)\times(L^1 \cap L^\infty \cap BV)$ where BV

represents a space of bounded variations. Similarly N is a Banach manifold of mappings for one-equation system and can be expressed locally as $(L^1 \cap L^\infty \cap BV)$. Here $L^1[(0,L),\Re]$ is the infinite dimensional Hilbert space of one dimensional vector function defined for an interval $[0,L]$.

Define a mapping $\Phi : S_M \to S_N$ by $\Phi(\rho,q)=\rho$. Choosing this Φ gives the following abstracted system

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(q)$$

This in conjunction with the relation $q=\rho v$ and Greenshield's model gives the following dynamics

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}\left(\rho(1-\rho/\rho_{\max})u\right)$$

which is the one-equation model.

Proposition 9.5.1 *The one- equation and two-equation models given by (9.19) and (9.20) are* Φ *-related control systems with* Φ *defined for each* $f=(\rho,q\in M)$ *by*

$$\Phi(\rho,q)=\rho \tag{9.21}$$

Proof: Consider any function $f=(\rho_0,q_0)\in M$. Then

$$\pi_M^{-1}(\rho_0,q_0)=((\rho_0,q_0),(u_1))$$

with u_1 being the input to the system (9.19). Thus we have

$$T\Phi \circ F_M(\pi_M^{-1}(\rho_0,q_0)) = F_1(\rho_o,v_o) = \frac{\partial q_o(x,t)}{\partial x}$$

where v_0 is the input to system (9.20) corresponding to this function f. Here $T\Phi$ is the tangent mapping from tangent space $T_f M$ to tangent space $T_g N$, where $T_f M$ is the tangent space of Banach manifold M at function $f\in M$ and $T_g N$ is the tangent space of M at function $g\in N$. Now we have

$$\pi_N^{-1}(\Phi(\rho_0,q_0)) = ((\rho_0),(u_2))$$

where u_2 is the input to one-equation model (9.20) and is the free flow velocity v_f. Thus we have

$$F_N(\pi_N^{-1}(\Phi(\rho_0,q_0)))=\overline{F}_2(\rho,u_2)=-\frac{\partial}{\partial x}\left(F_2(\rho)u_2\right)$$

with $F_2(\rho)=\rho(1-\rho/\rho_{\max})$. Now for any u_1, letting $u_2=F_2^{-1}(\rho_o)F_1(\rho_o,q_o)$ we have

$$T\Phi\circ F_M(\pi_M^{-1}(f))\subseteq F_N(\pi_N^{-1}(\Phi(f))$$

The two equation model and one equation model are Φ-related control systems with respect to map $\Phi(\rho,q)=\rho$ according to (8.10).

For a two-dimensional case the problem of abstraction can be stated as follows: Find an abstracting map Φ from a two equation control model represented by

$$\frac{\partial\rho}{\partial t}=-\sum_{i=1}^{2}\frac{\partial q_i}{\partial x_i}$$

and

$$\frac{\partial q_1}{\partial t}=-\frac{\partial}{\partial x_1}\left(\frac{q_1^{\ 2}}{\rho}\right)-\frac{\partial}{\partial x_2}\left(\frac{q_1q_2}{\rho}\right)+\hat{u}_1$$

$$\frac{\partial q_2}{\partial t}=-\frac{\partial}{\partial x_2}\left(\frac{q_2^{\ 2}}{\rho}\right)-\frac{\partial}{\partial x_1}\left(\frac{q_1q_2}{\rho}\right)+\hat{u}_2 \qquad (9.22)$$

to a one equation control model

$$\frac{\partial\rho}{\partial t}=-\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left(\rho(1-\rho/\rho_{\max})u_i\right) \qquad (9.23)$$

such that (9.22) and (9.23) are Φ-related according to definition 8.6.

For a two dimensional case consider the abstracting map Φ given by $\Phi: S_M\to S_N$ defined as $\Phi(\rho,q_1,q_2)=\rho$. Choosing this Φ gives the following abstracted system

$$\frac{\partial\rho}{\partial t}=-\sum_{i=1}^{2}\frac{\partial q_i}{\partial x_i}$$

This in conjunction with the relation $q=\rho v$ and Greenshield's model gives the following dynamics

$$\frac{\partial \rho}{\partial t} = -\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(\rho(1-\rho/\rho_{\max})u_i\right)$$

which is the one-equation model. In this case M is a Banach manifold of mappings for a two-equation system and we can express it locally as $(L^1 \cap L^\infty \cap BV)\times(L^1 \cap L^\infty \cap BV)$ .Similarly N is a Banach manifold of mappings for one-equation system and can be expressed locally as $(L^1 \cap L^\infty \cap BV))$ .Here $L^1[\Omega,\Re]$ is the infinite dimensional Hilbert space of two-dimensional vector function defined for an interval Ω .

Proposition 9.5.2 *The one- equation and two-equation models given by (9.22) and (9.23) are* Φ *-related control systems with* Φ *defined for each* $f = (\rho, q \in M)$ *by*

$$\Phi(\rho, q_1, q_2) = \rho \tag{9.24}$$

Proof: Consider any function $f = (\rho_0, q_{10}, q_{20}) \in M$. Then

$$\pi_M^{-1}(\rho_0, q_{10}, q_{20}) = ((\rho_0, q_{10}, q_{20}), (u_1, u_2))$$

with u_1 and u_2 being the inputs to the system (9.22). Thus we have

$$T\Phi \circ F_M(\pi_M^{-1}(\rho_0, q_{10}, q_{20})) = F_1(\rho_o, v_{1o}) + F_2(\rho_0, v_{2o}) = \frac{\partial q_{1o}}{\partial x_1} + \frac{\partial q_{2o}}{\partial x_2}$$

where v_{1o} and v_{2o} are the inputs to system (9.22) corresponding to this function f . Here $T\Phi$ is the tangent mapping from tangent space $T_f M$ to tangent space $T_g N$. Now we have

$$\pi_N^{-1}(\Phi(\rho_0, q_{10}, q_{20})) = ((\rho_0), (u_3, u_4))$$

where u_3 and u_4 are the inputs to one-equation model (9.20) and are the components of free flow velocity in two directions. Thus we have

$$F_N(\pi_N^{-1}(\Phi(\rho_0, q_0))) = \overline{F}_3(\rho, u_3) + \overline{F}_4(\rho, u_4) = -\frac{\partial}{\partial x_1}(F_3(\rho)u_3) - \frac{\partial}{\partial x_2}(F_4(\rho)u_4)$$

with $F_3(\rho) = \rho(1-\rho/\rho_{\max})$. Now for any u_1 and u_2 letting $u_3 = F_3^{-1}(\rho_o)F_1(\rho_o, q_{1o})$ and $u_4 = F_4^{-1}(\rho_o)F_2(\rho_o, q_{2o})$ we have

$$T\Phi \circ F_M(\pi_M^{-1}(f)) \subseteq F_N(\pi_N^{-1}(\Phi(f))$$

The two equation model and one equation model are Φ-related control systems with respect to map $\Phi(\rho, q_1, q_2) = \rho$ according to (8.10).

We have shown how to obtain abstraction maps for both one and two-dimensions. However, not only do we want to construct these mapping we also want to preserve the controllability property of two equation model. In other words we want to find map Φ such that (9.22) is a consistent abstraction of (9.23). In this framework, showing the controllability of the abstracted one equation model is equivalent to showing controllability of the two equation model. The future goal is to find the transformation that will transfer the control design for an abstracted one equation model to the two equation model.

This chapter discussed the application of abstracted control systems to the robotic car and rolling disk. In the analysis of these systems, it was found that they exhibited a behavior not completely captured by the existing notions of consistency and implementability. This led to the definition of traceability, ε-traceability, and ε-consistency. These new notions provide conditions for trajectories to exist in the original system when a trajectory in the abstracted system is present. When the abstracted control system is ε-traceable by the original control system and the original control system is ε-consistent with respect to the mapping between them, then the controllability is propagated between the two systems. With these notions, a framework has been provided that gives conditions for the existence of trajectories in the original system and allows controllability propagation between the two systems. Then we discussed abstraction for an evacuation system and found a mapping to construct an abstracted system. The future goal is to extend this work and obtain the similar results for an evacuation system. Presumably, control design in the abstracted system is a simpler task than in the original system. As indicated, this work is a first step in the process of finding a method to transform control designs in the abstracted system back to the original system for the evacuation system.

CHAPTER 10

CONCLUSIONS AND FUTURE WORK

10.1 Conclusions

In this work we discussed the feedback control design for an evacuation problem. We designed the feedback controllers to control the movement of people during evacuation in order to avoid jams. In this dissertation we designed diffusion and advection feedback controllers to control the movement. We also designed the robust feedback controllers to cancel the effect of uncertainties or disturbance in the system. While doing the control design we observed two points

- The accuracy of models representing evacuation dynamics increases with the increase in partial differential equations in the system.
- The difficulty of designing controllers increases with increase in partial differential equations in the system.

This motivates the abstraction of evacuation systems where we can abstract certain number of PDEs to obtain a simpler system and then transform the control design back to the original system. In order to address this issue of abstraction for evacuation systems we first developed a framework for abstraction for finite dimensional case with robotic car as an application. Then we extended this framework to infinite dimensional case.

In chapter 2 we presented two macroscopic models for representing the motion during evacuation in both one and two dimensions. The models obtained were represented by nonlinear hyperbolic PDEs. We presented two models for each case; one was called as one-equation model and the other one as two-equation model.

In chapter 3 we presented design of nonlinear feedback controllers for two different models representing evacuation dynamics in one-dimension. The model dynamics for this case were represented by one partial differential equation. We addressed the feedback control problem for both models in distributed setting and were able to synthesize nonlinear distributed feedback controllers that achieved exponential stability. The closed-loop dynamics were represented by a heat equation where the density profiles are decreasing with time. The problem of control and stability was formulated directly in the framework of partial differential equations. Sufficient conditions for Lyapunov stability for distributed control were also derived. We used the method of feedback linearization for the control design and for two-equation model we used *backstepping* approach. We also motivated the problem of abstraction for infinite dimensional systems.

In chapter 4 we presented design of nonlinear feedback controllers for two different models representing evacuation dynamics in two-dimensions. The model dynamics were represented by means of a set of three partial differential equations. We addressed the feedback control problem for both models and were able to synthesize nonlinear distributed feedback controllers that achieved exponential stability. Here again we used the method of feedback linearization.

In chapter 5 we presented design of nonlinear advection feedback controllers for two different models representing evacuation dynamics in one-dimension. We added motion to the controllers such that the density profiles are not only decreasing with time but are moving also. We also addressed the control saturation issues for these controllers.

In chapter 6 we presented design of nonlinear advection feedback controllers for two different models representing evacuation dynamics in two-dimensions. We added motion to the controllers and achieved asymptotic stability.

In chapter 7 we presented design of nonlinear robust feedback controllers for two different models representing evacuation dynamics in one-dimension. We presented the Lyapunov redesign and *robust backstepping* methods for the feedback control of PDEs and designed controllers for both matched and unmatched uncertainties. We achieved attenuation of uncertainty or disturbance in the system.

In chapter 8 we presented the framework for abstractions in finite dimensional case. This chapter serves as the preliminary for infinite dimensional abstraction framework.

In chapter 9 we discussed the application of abstracted control systems to a robotic car and rolling disk. In the analysis of these systems, it was found that they exhibited a behavior not completely captured by the existing notions of consistency and implementability. This led to the definition of traceability, ε-traceability, and ε-consistency. These new notions provide conditions for trajectories to exist in the original system when a trajectory in the abstracted system is present. When the abstracted control system is ε-traceable by the original control system and the original control system is ε-consistent with respect to the mapping between them, then the controllability is propagated between the two systems. With these notions, a framework was provided that gives conditions for the existence of trajectories in the original system and allows controllability propagation between the two systems. In this chapter we also discussed the abstraction for an evacuation system in both one and two dimensions. The goal here was to abstract the system having higher number of partial differential equations by a system which is represented by lower number of partial differential equations. The abstraction was constructed in such a way that the one-equation model is an abstraction of a two-equation model. The chapter also introduced some differential geometric preliminaries for the infinite dimensional case.

10.2 Future Work

In the analysis of abstraction of a robotic car, it was found that the system exhibited a behavior not completely captured by the existing notions of consistency and implementability. This led to the definition of traceability, ε-traceability, and ε-consistency. With these notions, a framework has been provided that gives conditions for the existence of trajectories in the original system and allows controllability propagation between the two systems. As a part of future research we want to study the similar problem for evacuation. We intend to study abstraction which preserves controllability property for evacuation dynamics. For that purpose we need to modify the notions of consistent abstractions introduced in [17] for infinite dimensional systems.

The long term goal of this work is the process of finding a method to transform control designs in the abstracted system back to the original system. As a part of future work we plan to do the following

- Obtain more detailed and accurate models for evacuation dynamics.
- Design and address other types of feedback control problems like adaptive and sliding mode control.
- Develop the framework similar to one in chapter 9 for infinite dimensional case and obtain the abstraction which preserves controllability property for evacuation dynamics.
- Find a method to transform control design for higher order evacuation models to lower order ones.
- Apply the results so far done for finite dimensional case to other well motivated problems like autonomous underwater vehicles.

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VITA

Sabiha Wadoo was born in Kashmir, India. She received her Bachelor's degree in Electrical Engineering from Regional Engineering College, Kashmir, India in June 2001. She received a Master's degree in Electrical Engineering in December 2002 and a Master's degree in Mathematics in May 2005, both from Virginia Tech. She started pursuing PhD in Electrical Engineering with Dr. Pushkin Kachroo as her advisor from January 2003.