Brjuno Numbers and Complex Dynamics

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(ABSTRACT)

The Brjuno numbers play a fundamental role in the study of the 1-dimensional Complex Dynamics Theory. In this work we examine the proof of the Brjuno theorem by using elements of Number Theory. We also examine the topological version of the proof given by J. Yoccoz and his renormalization principle.

If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, we also describe how the existence of a Siegel disk at the origin for the polynomial $P(z) = \exp(2\pi i\alpha) \cdot (z - z^2)$ implies the linearization of any germ of the form $f(z) = \exp(2\pi i\alpha) \cdot z + \mathcal{O}(z^2)$.

Dedication

To the memory of my mother: Gladys Felicita Maldonado Huapaya .

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Contents

Chapter 1

Introduction

The study of the dynamics of analytic germs in one complex variable in a neighborhood of a fixed point had been the a motivation of several works during the twentieth century. The central problem is to describe the structure of the conjugacy classes in the group G of the analytic germs that leaves the point $z = 0$ fixed. In other words, given an analytic germ $f(z) = \lambda z + \mathcal{O}(z^2)$ with $\lambda \in \mathbb{C}^*$, is there an analytic function h_f defined in some open neighborhood of $0 \in \mathbb{C}$, satisfying $h_f(0) = 0$, $h'_f(0) = 1$ and $f(h_f(z)) = h_f(\lambda z)$?

For the case $|\lambda| \neq 1$, the answer is affirmative. It was shown by G. Koenigs in 1884, who also proved that for $0 < |\lambda| < 1$ the fixed point has to be an attracting point while for $|\lambda| > 1$ the fixed point is a repelling point (see L. Carleson-T.W.Gamelin[7], A.F. Beardon[3]). But, if λ is a root of unity, i.e. $\lambda = \exp(2\pi i p/q)$ with $p/q \in \mathbb{Q}$, the answer for the question above can be negative; however Ecalle[14] and Voronin[31] have done a complete classification which guarantees whether h_f exists or not.

The most interesting case happens when $|\lambda| = 1$ and λ is not a root of unity. This means that $\lambda = \exp(2\pi i\alpha)$, where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. In order to prove that h_f can be an analytic function in some neighborhood at the origin, for the case $\lambda = \exp(2\pi i\alpha)$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, we can assume (formally) that h_f has a power series representation at the origin. By the formulas in question, it is possible to determine the coefficients of the formal power series of h_f; the denominators of these coefficients can be written as products of the form $(\lambda^n - \lambda)$ for $n \geq 2$, since α is an irrational number these products could be very small, which implies that h_f may not converge.

The first significant result, although negative, was given by H.Cremer[10] who considering the sequence of continued fractions $(p_n/q_n)_{n\geq 0}$ associated to α, proved that the arithmetical condition

$$\sup_{n\geq 0} \frac{\log(q_{n+1})}{q_n} = +\infty$$

causes the divergence of h_f.

The first positive result was given by Carl Ludwig Siegel[29] in 1942, who proved in a historical article that the formal series of h_f around the origin has positive radius of convergence (which means that h_f is analytic in some neighborhood at 0) when the irrational number α satisfies a diophantine condition, i.e. if there exist $c > 0$ and $\mu > 2$ such that

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c}{q^{\mu}}$$

then the radius of convergence of h_f is positive. In the original proof of Siegel[29], he provided direct estimates for the coefficients of h_f, but in this paper we present a different approach. Part of the organization of the proof given here has been taken from two books, A.F. Beardon[3] and L. Carleson-T.W.Gamelin[7].

In 1965, following the same ideas of the proof given by C.L. Siegel, A.D. Brjuno[4] improved the arithmetical condition using some properties of the continued fractions (see Appendix D). He proved that the arithmetical condition

$$\sum_{n\geq 0} \frac{\log(q_{n+1})}{q_n} < +\infty$$

implies the analyticity of h_f in some open neighborhood of the origin.

In this paper we present two different proofs of the Brjuno theorem. In Chapter 2, our attention is focused on the proof of the Siegel theorem and the first version of the Brjuno theorem. This first version has been taken from S. Marmi[24] who used two strong results that significantly shorten the original proof given by Brjuno. These two of results are the Davie Lemma and the famous Bieberbach Theorem which was proven by L. De Branges[5] in 1984. Basically, this version is focused on direct estimates for the coefficients of the formal power series of h_f.

In Chapter 3, we are focused on the second version of the proof of the Brjuno theorem. We follow the same techniques used by J.Yoccoz in [30] and X.Buff and A.Chéritat in [6]; however, we introduce some definitions given by S. Marmi in [24]. In Section 3.2, for any $f(z) = \lambda z + \mathcal{O}(z^2)$ univalent on the open unit disk $\mathbb{D}$, with $|\lambda| \leq 1$, we introduce the definition of stability at 0. The point $0 \in \mathbb{C}$ is called stable for the map f if and only if 0 belongs to the interior of the set $K_f = \bigcap_{n\geq 0} f^{-n}(\mathbb{D})$. In Section 3.2, we also prove that if $f(z) = \lambda z + \mathcal{O}(z^2)$ is univalent on the open unit disk $\mathbb{D}$, with $|\lambda| \leq 1$, then

$$0 \text{ is stable} \Leftrightarrow h_f \text{ is analytic at } 0.$$

If $f(z) = \lambda z + \mathcal{O}(z^2)$ is univalent on the open unit disk $\mathbb{D}$, where $\lambda = \exp(2\pi i\alpha)$ and $\alpha \in (0,1) \setminus \mathbb{Q}$, then f lifts to a map $F : \mathbb{H} \to \mathbb{C}$ by using the map $E : \mathbb{H} \to \mathbb{D}^*$ defined by $E(Z) = \exp(2\pi i Z)$; such a map F is univalent, $\mathbb{Z}$-periodic and satisfies $\lim\limits_{\Im(Z)\to+\infty}(F(Z)-Z) = \alpha$. For this reason, we define $\mathfrak{F}(\alpha)$ as the collection of the all mappings F as in the previous sentence. In Section 3.5, using the renormalization technique of Yoccoz (developed in Section

3.3 and Section 3.4) we prove that if $F \in \mathfrak{F}(\alpha)$ and α satisfies the Brjuno condition, there exists $C > 0$ such that

$$F^{\circ m}(Z) \in \mathbb{H}$$

for all $Z \in \mathbb{H}$ with $\Im(Z) > C$. As a consequence the open disk $E\{Z \in \mathbb{H} : \Im(Z) > C\}$, which contains 0, is contained in K_f. This means that 0 is an interior point for K_f and therefore 0 is stable. So, by the equivalence between stability of 0 and the analyticity of h_f, given in Section 3.2, we conclude that the radius of convergence of h_f is positive. The crucial fact in this second version is that we prove the analyticity of h_f without computing any estimates on the coefficients of the formal power series h_f.

From the definition of the set K_f given above (see also Section 3.2), we define a Siegel disk at 0 as the connected component of the interior of K_f which contains zero. In Section 3.2, we also prove that stability of f at 0 is equivalent to the existence of a Siegel disk at 0.

In Chapter 4, our attention is focused on the Quadratic Polynomial Theorem which sates that the stability at 0 for the quadratic polynomial $P_\lambda(z) = \lambda(z - z^2)$, where $\lambda = E(\alpha)$ and $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, is enough to guarantee the stability at 0 for every germ of the form $f(z) = \lambda z + \mathcal{O}(z^2)$. This result was first given by Jean Yoccoz in 1985. In this paper we provide the proof of the Quadratic Polynomial Theorem with more specific details. In Section 4.1, for any univalent map $f(z) = \lambda z + \mathcal{O}(z^2)$ defined on the open unit disk $\mathbb{D}$ with $\lambda = E(\alpha)$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, we apply the same technique used in the Straightening Theorem for polynomial-like mappings (of degree 2) given by A. Douady and J.Hubbard in [12] and prove that there exists $a_0 \in (0, 1]$ such that for all $a \in (0, a_0]$ and $b \in \mathbb{C}$ with $|b| = 10$, the map $f_{a,b}(z) = a^{-1}f(az) + bz^2$ is quasiconformally conjugate to the quadratic polynomial $\lambda z + bz^2$. This means that there exist a quasiconformal map $\phi \equiv \phi_{f,a,b}$ such that

$$f_{a,b}(\phi(z)) = \phi(\lambda z + bz^2)$$

for all $a \in (0, a_0]$ and $b \in \mathbb{C}$ with $|b| = 10$. As an immediate consequence, $f_{a,b}$ is quasiconformally conjugate to the polynomial $\lambda(z - z^2)$. Since a quasiconformal map is in particular a homeomorphism, the existence of a Siegel disk at 0 for the quadratic polynomial P_λ implies the existence of a Siegel disk at 0 for the mappings $f_{a,b}$ and therefore the analyticity of the mappings $h_{f_{a,b}}$. In Section 4.2 we describe how the analyticity of the mappings $h_{f_{a,b}}$ for $a \in (0, a_0]$, $b \in \mathbb{C}$ with $|b| = 10$ implies the analyticity of the map $h_{f_{a,0}}$ near the origin, and therefore the existence of a Siegel disk at 0 for the map $f(z) = \lambda z + \mathcal{O}(z^2)$.

Chapter 2

From Siegel to Brjuno

In this chapter we are going to prove one of the most interesting results in 1-dimensional Complex Dynamics Theory, the Brjuno theorem. A weak version of this theorem was first proven by Carl Ludwig Siegel in 1942 by using elements of Number Theory such as Diophantine numbers. The first three sections of this chapter are focused on this particular case. In the fourth section we give the definition and some properties of continued fractions which are used in the proof of Davie's Lemma[11]. Finally, in the fifth section we prove the Brjuno theorem by using Davie's Lemma and the Bieberbach theorem[9].

2.1 Small Divisors

Consider the analytic map f given by $f(z) = 3z - 4z^3$.
Is there an analytic function h defined in some open neighborhood of $0 \in \mathbb{C}$, such that $h(0) = 0$, $h'(0) = 1$ and $f(h(z)) = h(f'(0)z)$? Well , in this case the answer is affirmative because $h(z) = \sin(z)$ satisfies all of the conditions required above. So given an analytic germ f at the origin $0 \in \mathbb{C}$ with $f(0) = 0$, $f'(0) = \lambda \in \mathbb{C}^*$, the next natural question would be : Is there an analytic function h defined in some open neighborhood of $0 \in \mathbb{C}$, satisfying

$$h(0) = 0, h'(0) = 1 \text{ and } f(h(z)) = h(\lambda z) \ \ ? \tag{2.1.1}$$

For the case $|\lambda| \neq 1$ (see [7]) the answer is affirmative. But, if λ is a root of unity the answer can be negative. Details about this case may be found in Ecalle[14] and Voronin[31]. However, here we are focused on the interesting case : $|\lambda| = 1$ and λ is not a root of the unity. This is the case

$$\lambda = \exp(2\pi i\alpha), \tag{2.1.2}$$

where $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. We will consider the power series associated to f

$$f(z) = \sum_{i\geq 1} a_i z^i, \ a_1 = \lambda$$

and we assume that the formal power series of h is given by

$$h(z) = \sum_{i\geq 1} h_i z^i.$$

If h is the solution of the functional equation given by 2.1.1, the coefficients of the last series must satisfy (formally) the following recursive relation :

$$h_n = \begin{cases} 1 & \text{for } n = 1, \\ \dfrac{1}{\lambda^n - \lambda} \displaystyle\sum_{m=2}^{n} a_m \sum_{\substack{n_1+\ldots+n_m=n \\ n_i\geq 1}} h_{n_1} h_{n_2} \ldots h_{n_m} & \text{for } n \geq 2 \end{cases} \tag{2.1.3}$$

The terms of the form $\lambda^n - \lambda$ may cause the divergence of h in any open neighborhood of the origin $0 \in \mathbb{C}$ as we will see in the following result which was first given in 1917 by G.A. Pfeiffer.

Theorem 2.1.1. *There is $\lambda = \exp(2\pi i\theta)$ with $\theta \in \mathbb{R}\backslash\mathbb{Q}$ for which the equation 2.1.1 has no solution for any polynomial f of degree $d \geq 2$.*

Proof. Let $f(z) = z^d + \cdots + \lambda z$ and suppose there exists an analytic solution h of 2.1.1 defined on $\Delta(0,\delta) = \{z \in \mathbb{C} : |z| < \delta\}$. Consider the d^n fixed points of f^n; that is , the roots of the polynomial equation

$$f^n(z) - z = z^{d^n} + \ldots + (\lambda^n - 1)z = 0$$

One of these roots is $0 \in \mathbb{C}$. Label the others $\mu_1, \mu_2, \ldots, \mu_{d^n-1}$ and note that if the origin is the unique zero of $f^n(z) = h(\lambda^n h^{-1}(z))$ in $\Delta(0,\delta)$, then $\mu_j \notin \Delta(0,\delta)$. Assuming δ small ($\delta \ll 1$), we have

$$\delta^{d^n} \ll \prod_{j=1}^{d^n-1} |\mu_j| = |1 - \lambda^n| \tag{2.1.4}$$

We now construct a λ for which the last inequality is impossible, contradicting the existence of h. Suppose $q_1 < q_2 < q_3 < \ldots$ is an increasing sequence of integers such that $\theta = \sum_{k=1}^{+\infty} 2^{-q_k} \in \mathbb{R}\backslash\mathbb{Q}$, and define $\lambda = \exp(2\pi i\theta)$. It is not difficult to see that $\exp(2\pi i(\theta 2^{q_l})) = \exp(2\pi i w)$ where $w = 2^{q_l - q_{l+1}} + \sum_{k\geq l+2} 2^{q_l - q_k}$. By an elementary computation

$$|1 - \exp(2\pi i w)| = 2|\sin(\pi w)| \leq 2\pi |w|$$

and

$$|w| = w \leq 3 \cdot 2^{q_l - q_{l+1}}$$

Then

$$|1-\lambda^{2^{q_l}}| = |1-\exp(2\pi i w)| \le 6\pi \cdot 2^{q_l - q_{l+1}}$$

If h exists, replacing n by 2^{q_l} in the inequality 2.1.4 we get

$$\delta^{d^{2^{q_l}}} \le |1-\lambda^{2^{q_l}}| \le 6\pi \cdot 2^{q_l - q_{l+1}}$$

which implies that

$$q_{l+1} \le q_l + \log_2(6\pi) + [\log_2(\tfrac{1}{\delta})] d^{2^{q_l}}$$

On the other hand, since $d \in \mathbb{Z}^+$, $d \ge 2$ and $q_l \ge 2$ for all $l \ge 2$, we obtain

$$q_l < d^{2^{q_l}}$$

$$6\pi < 2^{2^{2^2}} \le 2^{d^{2^{q_l}}}$$

Hence

$$q_l + \log_2(6\pi) < 2 \cdot d^{2^{q_l}} \quad \text{for all} \quad l \ge 2$$

Then

$$q_{l+1} < c d^{2^{q_l}}$$

where $c = c(\delta) = (2 + \log_2(\dfrac{1}{\delta})) > 0$. But if we consider by induction $(q_k)_{k\ge1}$ such that $\log_2(q_{k+1}) > k2^{q_k}$ for all $k \in \mathbb{Z}^+$, we get a contradiction. □

2.2 Siegel's Theorem

Now we know that the equation 2.1.1 may not have a solution for the case $\lambda = \exp(2\pi i\alpha)$, $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. However, Siegel[29], assuming an arithmetical condition on α, proved the existence of the solution of the equation 2.1.1 .

Theorem 2.2.1. *Let $f(z) = \sum_{n\ge1} a_n z^n$ be an analytic function in a neighborhood of the origin $0 \in \mathbb{C}$ with $a_1 = \lambda = \exp(2\pi i\alpha)$. If $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and there exist constants $\widetilde{c} > 0$, $\mu > 2$ satisfying $|\alpha - \dfrac{p}{q}| \ge \dfrac{\widetilde{c}}{q^\mu}$ for all $\dfrac{p}{q} \in \mathbb{Q}$ $(q \in \mathbb{Z}^+)$, then the equation 2.1.1 has an analytic solution in some open neighborhood of the origin $0 \in \mathbb{C}$.*

In order to prove the Siegel theorem we have to do some preliminary analysis.

Proposition 2.2.2. *The arithmetical condition implies that there exist $c > 1$ and $k \in \mathbb{Z}^+$ such that*

$$\frac{1}{|\lambda^n - 1|} \le \left(\frac{c}{k!}\right) n^k \quad \textit{for all} \quad n \in \mathbb{Z}^+$$

Proof. Let n be a natural number, and let m be the closest integer to $n\alpha$. Since $n\alpha$ is irrational, $|n\alpha - m| < 1/2$. By an elementary computation

$$|\lambda^n - 1| = |e^{2\pi i(n\alpha)} - e^{2\pi i(m)}| = 2|\sin(\pi n\alpha)| = 2|\sin(\pi(n\alpha - m))|$$

By the hypothesis and since $\frac{2}{\pi}|y| \leq |\sin(y)|$ on $[-\pi/2, \pi/2]$, we have

$$\frac{\widetilde{c}}{n^\mu} \leq 4n\frac{\widetilde{c}}{n^\mu} \leq 4n|\alpha - \frac{m}{n}| = \frac{4}{\pi}|\pi(n\alpha - m)| \leq 2|\sin(\pi(n\alpha - m))| \leq |\lambda^n - 1|$$

Therefore, it is enough to consider $c > \max\{\frac{k!}{\widetilde{c}}\ ,\ 1\}$ and $k = [\![\mu]\!] + 1$, where $[\![\ ]\!]$ denotes the integer part. □

On the other hand, solving the equation 2.1.1 is equivalent to finding an analytic function φ such that $\varphi(0) = 0$, $\varphi'(0) = 1$ and $\varphi^{-1} \circ f \circ \varphi(z) = \lambda z$. Note that

$$f(z) = \lambda z + F(z) \tag{2.2.1}$$

$$\varphi(z) = z + \Phi(z) \tag{2.2.2}$$

where

$$F(z) = \sum_{i \geq 2} a_i z^i \quad \text{and} \quad \Phi(z) = \sum_{i \geq 2} b_i z^i \tag{2.2.3}$$

We have to find φ satisfying $\varphi^{-1} \circ f \circ \varphi(z) = \lambda z$, which is equivalent to getting

$$\begin{aligned} f(\varphi(z)) &= \varphi(\lambda z) \\ \lambda\varphi(z) + F(\varphi(z)) &= \lambda z + \Phi(\lambda z) \\ \lambda z + \lambda\Phi(z) + F(\varphi(z)) &= \lambda z + \Phi(\lambda z) \\ F(\varphi(z)) &= \Phi(\lambda z) - \lambda\Phi(z) \end{aligned}$$

Now the idea is to replace the last functional equation by the following simple equation:

$$F(z) = \Phi(\lambda z) - \lambda\Phi(z). \tag{2.2.4}$$

Substituting 2.2.3 into 2.2.4, we obtain

$$\sum_{j \geq 2} a_j z^j = \sum_{j \geq 2} b_j (\lambda z)^j - \lambda \sum_{j \geq 2} b_j z^j$$

So

$$|b_j| = |a_j| \frac{1}{|\lambda^{j-1} - 1|} \quad \text{for all} \quad j \geq 2$$

By Proposition 2.2.2, $|b_j| \leq |a_j| \left(\frac{c}{k!}\right) (j-1)^k$ for all $j \geq 2$. Hence

$$\limsup \sqrt[j]{|b_j|} \leq \limsup \sqrt[j]{|a_j|}.$$

From the last inequality, the radius of convergence of f is less than or equal to the radius of convergence of φ.

Consider $A = 2c(20)^{k+2}$ and $B = (30)^{k+2}$, and choose δ and θ such that

$$\begin{cases} 0 < \delta = \delta_0 < \dfrac{1}{c}\dfrac{1}{10^{k+2}}\dfrac{1}{2^{2k+4}}.\dfrac{1}{AB} \\ \theta = \theta_0 = \frac{1}{10}\frac{1}{1+(2)^0} < \frac{1}{5} \end{cases} \tag{2.2.5}$$

Note that $c\delta < \theta^{2k+4} < \theta^{k+2}$. By the continuity of F', there exists $r > 0$ such that if $|z| \le r$ then $|F'(z)| < \delta$. Hence, there exist positive real numbers δ, θ, and r satisfying :

(c1) f is analytic in $\overline{\Delta(0,r)} := \{z \in \mathbb{C} : |z| \le r\}$.
Therefore φ is analytic on $\Delta(0,r) := \{z \in \mathbb{C} : |z| < r\}$.

(c2) $\theta < \dfrac{1}{5}$

(c3) $c\delta < \theta^{k+2}$

(c4) If $|z| \le r$ then $|F'(z)| \le \delta$

Proposition 2.2.3. *Using the preceding notation, for $z \in \Delta(0,r)$ we have*

1. $|\Phi'(z)| \le c\|F'\| \left[\dfrac{r}{r-|z|}\right]^{k+1}$, *where* $\|F'\| = \sup\{|F'(z)| : |z| \le r\}$.

2. *If* $|z| < r(1-\theta)$ *then* $|\Phi'(z)| < c\dfrac{\|F'\|}{\theta^{k+1}} \le c\dfrac{\delta}{\theta^{k+1}} < \theta$.

Clearly the second part comes from the first part and the item (c3). So it is enough to prove the first part of this proposition.

Proof. From the Cauchy estimates for the power series coefficients of F' we have

$$|ja_j| \le \frac{\|F'\|}{r^{j-1}}$$

Hence

$$\begin{aligned} |\Phi'(z)| &= |\sum_{j\ge 2} jb_j z^{j-1}| \le \sum_{j\ge 2} |ja_j| \frac{c}{k!}(j-1)^k |z|^{j-1} \\ &\le c\|F'\| \sum_{j\ge 2} \frac{(j-1)^k}{k!}(\frac{|z|}{r})^{j-1} < c\|F'\| \sum_{j\ge 1} \frac{(j+k-1)!}{(j-1)!k!}(\frac{|z|}{r})^{j-1} \end{aligned}$$

Therefore

$$|\Phi'(z)| \leq c\|F'\| \left[\frac{r}{r-|z|}\right]^{k+1}$$

□

Definition 2.2.4. *For each $m \in \{0,1,2,3,4,5\}$, we define the open disks D_m by*

$$D_m := \{z \in \mathbb{C} : |z| < r(1-m\theta)\} \ .$$

It is clear that $D_5 \subseteq D_4 \subseteq \ldots \subseteq D_1 \subseteq D_0 = \Delta(0,r)$.

Proposition 2.2.5. *The mappings φ, f, φ^{-1} satisfy the following diagram :*

$$D_4 \xrightarrow{\varphi} D_3 \xrightarrow{f} D_2 \xrightarrow{\varphi^{-1}} D_1.$$

Proof. This comes from the definition above and the items (c1) and (c3). □

Since we changed the original functional equation for another simpler one, it implies that $\varphi^{-1} \circ f \circ \varphi(z) = \lambda z +$ "an analytic function" . Now we are focused on how " big " is this analytic function on D_5.

Proposition 2.2.6. *If we consider $\varphi^{-1} \circ f \circ \varphi(z) = \lambda z + G(z)$, then*

$$|G'(z)| \leq \tfrac{2c\delta^2}{\theta^{k+2}} \text{ for all } z \in D_5 \ .$$

Proof. By the equation 2.2.4, it is easy to deduce that for all $z \in D_4$,

$$G(z) = \Phi(\lambda z) - \Phi(\lambda z + G(z)) + F(z + \Phi(z)) - F(z)$$

Since $D_4 \subseteq D_1$, we can apply the second part of Proposition 2.2.3 to D_4 and obtain

$$|G(z)| \leq \theta|G(z)| + \delta \sup\{|\Phi'(\tau)| \ : \ \tau \in D_4\}|z| \leq \theta|G(z)| + \delta\frac{c\delta}{\theta^{k+1}}r$$

So

$$|G(z)| \leq \frac{c\delta^2}{\theta^{k+1}}\frac{r}{1-\theta}$$

Now, fix $z \in D_5$. Clearly $\{w \ : \ |w-z| \leq r\theta\} \subseteq D_4$. Then

$$|G'(z)| = |\frac{1}{2\pi i}\int_\gamma \frac{G(w)}{(w-z)^2}dw| \leq \frac{c\delta^2}{\theta^{k+1}}\frac{r}{1-\theta}\frac{\theta r}{\theta r^2},$$

where $\gamma(t) = z + r\theta\exp(it)$, $t \in [0, 2\pi]$. Finally, since $\theta < 1/5$ the assertion follows. □

Proposition 2.2.7. *Let $k \in \mathbb{Z}^+$ and $c > 1$, and consider $A = 2c(20)^{k+2}$ and $B = (30)^{k+2}$. Choose δ_0 with $0 < \delta_0 < \frac{1}{c}\frac{1}{10^{k+2}}\frac{1}{2^{k+2}}\frac{1}{AB}$. Define by induction $\delta_{n+1} = \frac{2c\delta_n^2}{\theta_n^{k+2}}$, where $\theta_n = \frac{1}{10}\frac{1}{1+2^n}$ for all $n \geq 0$. Then*

$$\delta_n < \frac{1}{c}(\theta_n)^{k+2} \text{ for all } n \geq 0.$$

Proof. See Appendix A. □

Proof of Theorem 2.2.1. We are going to inductively define sequences $(f_n)_{n\geq 0}$, $(F_n)_{n\geq 0}$, and $(\varphi_n)_{n\geq 0}$ satisfying $f_{n+1}(z) = \lambda z + F_n(z)$. For this, consider $f_0 = f$ and $F_0 = F$ as in the equation 2.2.1 and define φ_0 by $\varphi_0(z) = z + \Phi_0(z)$ where Φ_0 is the solution of the functional equation 2.2.4. This means

$$F_0(z) = \Phi_0(\lambda z) - \lambda\Phi_0(z)$$

By Proposition 2.2.5 $\varphi_0^{-1} \circ f_0 \circ \varphi_0$ is well defined in some open neighborhood of zero. Now define

$$\begin{aligned} f_1(z) &= \varphi_0^{-1} \circ f_0 \circ \varphi_0(z) \\ F_1(z) &= f_1(z) - \lambda z \end{aligned}$$

It is not difficult to check that f_1 and F_1 have the form given in 2.2.1. Hence we can define φ_1 from f_1 and F_1 in the same way as φ_0 was defined from f_0 and F_0. Then we can define (formally)

$$\begin{aligned} f_2(z) &= \varphi_1^{-1} \circ f_1 \circ \varphi_1(z) \\ F_2(z) &= f_2(z) - \lambda z \end{aligned}$$

Again, it is not difficult to check that f_2 and F_2 have the form given in 2.2.1. Hence we can define φ_2 from f_2 and F_2 in the same way as φ_1 was defined from f_1 and F_1. Hence, we inductively define (formally)

$$\begin{aligned} f_{n+1}(z) &= \varphi_n^{-1} \circ f_n \circ \varphi_n(z) \\ F_n(z) &= f_n(z) - \lambda z \end{aligned}$$

The well definition of these sequences depends on the right selection of the parameters δ_n, θ_n and r_n ; for this, consider δ_0 and δ_n as in proposition 2.2.7 and

$$\theta_n = \frac{1}{10}\left(\frac{1}{1+2^n}\right) \text{ and } r_n = \frac{r}{2}\left(1 + \frac{1}{2^n}\right)$$

where r is a fixed positive number such that

$$\text{if } |z| \leq r \text{ then } |F(z)| = |F_0(z)| < \delta_0$$

On the other hand, we will have "new" $D_5, D_4, \ldots, D_1, D_0$ in each step. That is why for each n we define

$$D_{n,m} = \{z \in \mathbb{C} \text{ such that } |z| < r_n(1 - m\theta_n)\}$$

where $m \in \{0,1,2,3,4,5\}$. By the definition of the sequence (r_n), we have

$$r_{n+1} = \frac{r}{2}(1+\frac{1}{2^{n+1}}) = \frac{r}{2}(1+\frac{1}{2^n})(\frac{2^n}{1+2^n})(1+\frac{1}{2^{n+1}})$$

So

$$r_{n+1} = r_n(\frac{2^n}{1+2^n} + \frac{1}{2}\frac{1}{1+2^n}) = r_n(1-\frac{1}{1+2^n} + \frac{1}{2}\frac{1}{1+2^n}) = r_n(1-5\theta_n)$$

which means that $D_{n+1,0} = D_{n,5}$. This implies that

$$V \subseteq \ldots \subseteq D_{n+1,0} = D_{n,5} \subseteq D_{n,4} \subseteq \ldots$$

where $V = \{z \in \mathbb{C} : |z| < \frac{r}{2}\}$. Therefore, by Proposition 2.2.5 $f_{n+1} = \varphi_n^{-1} \circ f_n \circ \varphi_n$ are well defined for each n. Using the condition (c4) for $z \in D_{n,0}$ we have $|F_n'(z)| \leq \delta_n$, in particular this holds on $D_{n,4}$. By Proposition 2.2.6,

$$|F_{n+1}'(z)| \leq \frac{2c\delta_n^2}{\theta_n^{k+2}} \quad \text{for all} \quad z \in D_{n,5}$$

Hence $|F_{n+1}'(z)| \leq \delta_{n+1}$ for all $z \in D_{n,5} = D_{n+1,0}$. By the selection of δ_0, θ_n and Proposition 2.2.7, it is not difficult to check that $\delta_n \to 0$ as $n \to \infty$. So $F_{n+1}'(z)$ converges uniformly to zero on V . Since each f_n is analytic on V and $F_{n+1}'(z)$ converges uniformly to zero on V, $F_n(z)$ must converge uniformly to zero on V. For each n, denote by ψ_n the function $\varphi_0 \circ \varphi_1 \circ \ldots \circ \varphi_n$. By the definition of f_{n+1} ,

$$\psi_n^{-1} \circ f \circ \psi_n(z) = f_{n+1}(z) = \lambda z + F_{n+1}(z) \tag{2.2.6}$$

for all $z \in V$. By Proposition 2.2.5, we have

$$\psi_n := \varphi_0 \circ \varphi_1 \circ \ldots \circ \varphi_n : D_{n,4} \to D_{0,3}$$

and since $V \subseteq D_{n,4}$ for all $n \geq 0$, we conclude that $\{\psi_n\}_{n\geq 0}$ is a normal family on V. So, by Montel's Theorem (see [8], [7]) the sequence $(\psi_n)_{n\geq 0}$ has a subsequence $(\psi_{n_j})_{j\in\mathbb{Z}^+}$ which converges. Let $h := \lim_{j\to\infty} \psi_{n_j}$.

Since the φ_n's are perturbations of the identity $z \mapsto z$, the ψ_n's are also, whence $h(z) = z + \mathcal{O}(z^2)$ and h is injective. Applying Hurwitz's Theorem [8], the sequence $\psi_{n_j}^{-1}$ converges uniformly to h^{-1}. Then, by 2.2.6

$$\psi_{n_j}^{-1} \circ f \circ \psi_{n_j}(z) = \lambda z + F_{n_j+1}(z) \quad \text{for all} \quad z \in V$$

Therefore

$$h^{-1} \circ f \circ h(z) = \lambda z \quad \text{for all} \quad z \in V.$$

□

2.3 Diophantine Numbers

We do not know yet whether there exist irrational real numbers satisfying the hypothesis of Siegel's Theorem. This means : Are there any irrational numbers $\alpha \in \mathbb{R}$ for which is possible to find constants $c = c(\alpha) > 0$ and $\mu = \mu(\alpha) > 2$ such that,

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{c}{q^{\mu}} \tag{2.3.1}$$

for all $\frac{p}{q} \in \mathbb{Q}$, $q \in \mathbb{Z}^{+}$?
The following theorem gives an affirmative answer for this question. For the details of the proof see [2], [17].

Theorem 2.3.1. *Liouville's Theorem. For every algebraic $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ of degree $n > 1$, there exists a constant $c = c(\alpha) > 0$ such that:*

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{c}{q^{n}} \tag{2.3.2}$$

for all $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^{+}$.

This result was the motivation of the following definition.

Definition 2.3.2. *We say that $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ satisfies a diophantine condition (or α is diophantine) if there exist real numbers $c > 0$, $\mu > 2$ for which the inequality 2.3.1 holds for all $\frac{p}{q} \in \mathbb{Q}$, $q \in \mathbb{Z}^{+}$.*

Well, there are two natural questions.

- Is there any diophantine real number which is not algebraic?
- Is there any other way to prove that there exist irrational numbers which are not diophantine without using Theorem 2.1.1 and Theorem 2.2.1 ?

We will answer these questions in an affirmative way after introducing the concept of continued fraction.

2.4 Continued Fractions

Definition 2.4.1. *Given $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ define the following sequences*

$$\begin{cases} a_0 = [\![\alpha]\!] \,;\ \alpha_0 = \alpha - [\![\alpha]\!] \\ a_n = \left[\!\left[\alpha_{n-1}^{-1}\right]\!\right] \,;\ \alpha_n = \alpha_{n-1}^{-1} - a_n \quad \textit{for all } n \geq 1 \end{cases} \tag{2.4.1}$$

The a_n are called partial coefficients associated to the number α. Using the preceding notation, we define $(p_n)_{n\geq 2}$, $(q_n)_{n\geq -2}$ as follows : $p_{-2} = q_{-1} = 0$, $p_{-1} = q_{-2} = 1$, and

$$\begin{cases} p_n = a_n p_{n-1} + p_{n-2} \\ q_n = a_n q_{n-1} + q_{n-2} \end{cases} \tag{2.4.2}$$

for all $n \geq 0$. The fraction $\frac{p_n}{q_n}$ is called the n-th continued fraction of α.

Proposition 2.4.2. *Properties of Continued Fractions*

1. *For all $n \geq -1$: $q_n p_{n-1} - p_n q_{n-1} = (-1)^n$.*
2. *For all $n \geq 0$: $\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}$.*
3. *$q_0 \leq q_1$ and $q_n \geq n$.*
4. *$\frac{1}{q_{n+1}+q_n} < |q_n\alpha - p_n|$.*
5. *For all $k \geq 1$: $q_k \geq \left(\frac{\sqrt{5}+1}{2}\right)^{k-1}$.*

Proof. For the proof of these facts see [2], [17] or [22]. □

Definition 2.4.3. *Given $x \in \mathbb{R}\backslash\mathbb{Q}$, define $\|x\| := \min\{|n - x| : n \in \mathbb{Z}\}$.*

Theorem 2.4.4. *The following facts hold :*

1. *$\|q_n\alpha\| = |q_n\alpha - p_n|$.*
2. *$\|q_0\alpha\| \geq \|q_1\alpha\| > \|q_2\alpha\| > \ldots > \|q_n\alpha\| > \ldots > 0$.*
3. *$\|q_n\alpha\| < |q\alpha - p|$ for all $p, q \in \mathbb{Z}$, $0 < q < q_n$ and $\frac{p}{q} \neq \frac{p_n}{q_n}$.*
4. *$\|q_n\alpha\| \leq |q\alpha - p|$ for all $p, q \in \mathbb{Z}$ and $0 < q < q_{n+1}$.*
5. *If $\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}$ then $\frac{p}{q}$ is a continued fraction of α.*

Proof. For more details see [17],[22] and [25] □

Theorem 2.4.5. *The following facts hold :*

(i) The number e is diophantine.

(ii) The number π is diophantine.

(iii) Let $a_0 = 0$, $a_n = 10^{n!}$ for $n \geq 1$, and consider $\alpha = \lim_{n\to\infty} \frac{p_n}{q_n}$ where $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$ are constructed as in 2.4.2. Then $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and α is not diophantine.

In this section, we just provide the proof of the third item of this theorem. The proof of the first item is given in Appendix B. For the proof of the second item see [23].

Proof. By the continued fractions algorithm, we can construct a bijection between the set of irrational numbers and the set

$$L = \{(a_0, a_1, \ldots, a_n, \ldots), \text{ such that } a_0 \in \mathbb{Z}, a_i \in \mathbb{Z}^+, \text{ for all } i \geq 1\}$$

The details of the construction of this bijection may be found in [17]. So we can conclude that $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. By Definition 2.4.1, one sees that :

- $q_1 = a_1 < a_1 + 1$.
- $q_2 = a_2 q_1 + q_0$, but $q_0 = 1 < q_1 = a_1 = 10$ so $q_2 < (a_2 + 1)q_1$.
- $q_3 = a_3 q_2 + q_1$, but $q_1 < q_2$ so $q_3 < (a_3 + 1)q_2$

By induction $q_n < (a_1 + 1)(a_2 + 1) \ldots (a_n + 1)$ for all $n \geq 1$, so

$$q_n < \left(1 + \frac{1}{10^{1!}}\right)\left(1 + \frac{1}{10^{2!}}\right) \ldots \left(1 + \frac{1}{10^{n!}}\right) a_1 a_2 \ldots a_n \text{ for all } n \geq 1$$

An elementary calculation shows that $\frac{1}{10^{n!}} < \frac{1}{10^{2^{n-1}}}$ for all $n > 2$, hence

$$\left(1 + \frac{1}{10^{1!}}\right)\left(1 + \frac{1}{10^{2!}}\right) \ldots \left(1 + \frac{1}{10^{n!}}\right) \leq \left(1 + \frac{1}{10^{1}}\right)\left(1 + \frac{1}{10^{2}}\right) \ldots \left(1 + \frac{1}{10^{2^{n-1}}}\right)$$

Then

$$q_n < \left(\frac{1 - \frac{1}{10^{2^n}}}{1 - \frac{1}{10}}\right) a_1 \ldots a_n < \left(\frac{1}{1 - \frac{1}{10}}\right) a_1 \ldots a_n = \left(\frac{10}{9}\right) a_1 \ldots a_n$$

So $q_n < 10 \cdot 10^{1!} \cdot 10^{2!} \ldots \cdot 10^{n!}$ and since $1 + 1! + 2! + \ldots.. + (n-1)! + n! \leq 2(n!)$ for all $n \geq 1$, we get $q_n < (10^{n!})^2 = a_n^2$. By Proposition 2.4.2(2), we have $|\alpha - \frac{p_n}{q_n}| < \frac{1}{q_n q_{n+1}}$ for all $n \geq 0$, hence

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{a_{n+1}} = \frac{1}{a_n^{n+1}} < \frac{1}{a_n^n} < \frac{1}{q_n^{n/2}}$$

If α was diophantine, there must exist $c > 0$, $\mu > 2$ such that $\left|\alpha - \frac{p}{q}\right| \geq \frac{c}{q^\mu}$ for all $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^+$. In particular, for $p = p_n$ and $q = q_n$ we get

$$\frac{c}{q_n^\mu} \leq \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^{n/2}}$$

So the inequality $\frac{c}{q_n^\mu} < \frac{1}{q_n^{n/2}}$ must hold. By Proposition 2.4.2(3), for n large we have

$$n^{n/2-\mu} < q_n^{n/2-\mu} < \frac{1}{c}$$

which is a contradiction. □

As a consequence, we can say that the last theorem has answered the two questions formulated at the end of Section 2.3 , because e is not an algebraic number and the number α from the item (iii) of this theorem is not diophantine.

2.5 Brjuno's Theorem

We start this section with a new natural question. Suppose $f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \ldots$ is analytic in some open neighborhood at the origin, and $\lambda = \exp(2\pi i\alpha)$ where α is the irrational number given in Theorem 2.4.5(iii). **Does there exist a solution for equation 2.1.1 in this case?** In 1965, following the same ideas of the proof given by Siegel, Brjuno [4] gave an answer to this question. Moreover, he gave a more general result for some subset of the set of irrational numbers.

Theorem 2.5.1. *Brjuno's Theorem. Suppose $f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \ldots$, analytic in some open neighborhood at the origin $0 \in \mathbb{C}$ with $\lambda = \exp(2\pi i\alpha)$, where $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Let $(\frac{p_n}{q_n})_{n\geq 0}$ be the sequence of continued fractions associated to α. If $\sum_{k\geq 0} \frac{\log(q_{k+1})}{q_k} < +\infty$ then the equation 2.1.1 has an analytic solution at the origin .*

The sketch of the proof can be found in [24]. The proof is based on the ideas of Brjuno[4] and Davie[11]. In order to prove this theorem, we state an important result called Davie's Lemma [11].

Lemma 2.5.2. *Davie's Lemma. If $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $\alpha = \lim_{n\to+\infty} \frac{p_n}{q_n}$ and $\lambda = \exp(2\pi i\alpha)$, then there exists a map $G : \mathbb{N} \to \mathbb{R}$ which satisfies :*

1. $G(0) = 0$

2. $G(n-1) \leq G(n)$ *for all* $n \geq 1$.

3. $G(n_1) + G(n_2) \leq G(n_1 + n_2)$ *for all* n_1, n_2.

4. $-\log|\lambda^n - 1| \leq G(n) - G(n-1)$ *for all* $n \geq 1$.

5. *There exists a universal constant* $\gamma_3 > 0$ *such that for all* $n \geq 0$ *the following inequality holds*

$$G(n) \leq n \left(\sum_{i\geq 0}^{k(n)} \frac{\log(q_{i+1})}{q_i} + \gamma_3 \right)$$

where $k(n)$ *was chosen such that* $q_{k(n)} \leq n < q_{k(n)+1}$.

Proof. For the details see Appendix C. □

Proposition 2.5.3. *For all* $m \geq 2$, *if* $n_1 + n_2 + \ldots + n_m = n$ *then*

$$G(n_1 - 1) + G(n_2 - 1) + \ldots + G(n_m - 1) \leq G(n-1) + \log|\lambda^n - \lambda|.$$

Proof. In fact, by Davie's Lemma items (2) and (3) we have

$$G(n_1 - 1) + G(n_2 - 1) + \ldots + G(n_m - 1) \leq G(n_1 + \ldots + n_m - m) \leq G(n-2)$$

By Davie's Lemma item(4) we must have

$$G(n_1 - 1) + G(n_2 - 1) + \ldots + G(n_m - 1) \leq G(n-1) + \log|\lambda^{n-1} - 1|.$$

Since $|\lambda| = 1$, the assertion follows. □

Proof of Brjuno's Theorem. Since $|f'(0)| = 1$, there exists $r > 0$ such that f is analytic and injective on $\Delta(0, r)$. Now the map $z \to \frac{1}{r} f(rz)$ is well defined in the open unit disk $\mathbb{D}$, is analytic and injective, leaves the point $z = 0$ fixed and its derivative at $z = 0$ is still λ. So without loss of generality we may assume that f is analytic and injective on the open unit disk $\mathbb{D}$. We are going to consider the Koebe function given by

$$\tilde{k}(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + 4z^4 + \ldots$$

Clearly $\tilde{k}(z) = \sum_{n\geq 1} n z^n$ and its radius of convergence is 1. Now, define $L(z,t) = z + \tilde{k}(t) - 2t$. It is not difficult to see that $L_t(z,t) \mid_{(0,0)} = -1 \neq 0$; so by the Implicit Function Theorem [18], there exists σ analytic, defined in a open neighborhood of the origin, which satisfies $L(z, \sigma(z)) = L(0,0) = 0$ and $\sigma(0) = 0$. Then

$$z + 2(\sigma(z))^2 + 3(\sigma(z))^3 + 4(\sigma(z))^4 + \ldots = \sigma(z) = \sum_{n\geq 1} \sigma_n z^n$$

It is not difficult to check that the coefficients of σ satisfy

$$\sigma_n = \begin{cases} 1 & n = 1, \\ \displaystyle\sum_{m\geq 2}^{n} m \cdot \sum_{\substack{n_1+\ldots+n_m=n \\ n_i\geq 1}} \sigma_{n_1}\sigma_{n_2}\ldots\sigma_{n_m} & \end{cases}$$

Furthermore, since σ is analytic, by the Cauchy estimate there exist positive constants γ_1 and γ_2 such that $|\sigma_n| \leq \gamma_1\gamma_2^n$ for all $n \geq 1$. On the other hand, we already know that there exists a formal power series $h = \sum_{n\geq 1} h_n z^n$ verifying

$$h_n = \begin{cases} 1 & \text{for } n = 1 \\ \dfrac{1}{\lambda^n - \lambda} \displaystyle\sum_{m=2}^{n} a_m \sum_{\substack{n_1+\ldots+n_m=n \\ n_i\geq 1}} h_{n_1}\ldots h_{n_m} & \text{for } n > 1 \end{cases}$$

By the Bieberbach–De Brange's Theorem (see [5],[9]), $|a_n| \leq n$ for all $n \geq 1$, so

$$|h_n| \leq \frac{1}{|\lambda^n - \lambda|} \sum_{m=2}^{n} m \sum_{\substack{n_1+\ldots+n_m=n \\ n_i\geq 1}} |h_{n_1}|\ldots|h_{n_m}|. \tag{2.5.1}$$

We are going to prove by induction that $|h_n| \leq \sigma_n e^{G(n-1)}$ holds for all $n \geq 1$, where h_n, $\sigma(n)$ and G are defined as above. It is clear for the case $n = 1$. Suppose the assertion $|h_{n'}| \leq \sigma_{n'} e^{G(n'-1)}$ is true for all $n' < n$ (Inductive Hypothesis). Using (2.5.1) and the inductive hypothesis we have

$$|h_n| \leq \frac{1}{|\lambda^n - \lambda|} \sum_{m,2}^{n} m \sum_{\substack{n_1+\ldots+n_m=n \\ n_i\geq 1}} \sigma_{n_1}\ldots\sigma_{n_m} e^{G(n_1-1)}\ldots e^{G(n_m-1)}$$

So, by Proposition 2.5.3 we obtain

$$|h_n| \leq \frac{1}{|\lambda^n - \lambda|} \sum_{m\geq 2}^{n} m \sum_{\substack{n_1+\ldots+n_m=n \\ n_i\geq 1}} \sigma_{n_1}\ldots\sigma_{n_m} e^{G(n-1)+\log|\lambda^n-\lambda|}$$

Then

$$|h_n| \leq e^{G(n-1)} \left(\sum_{m\geq 2}^{n} m \sum_{\substack{n_1+\ldots+n_m=n \\ n_i\geq 1}} \sigma_{n_1}\ldots\sigma_{n_m} \right)$$

Therefore, we conclude $|h_n| \leq e^{G(n-1)}\sigma_n$. Finally, since there exist positive constants γ_1 , γ_2 such that $|\sigma_n| \leq \gamma_1\gamma_2^n$ and since $G(n-1) \leq G(n)$, we have

$$\sqrt[n]{|h_n|} \leq \sqrt[n]{\gamma_1}\gamma_2 e^{\frac{1}{n}G(n)}$$

By hypothesis, $M = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{\log(q_{i+1})}{q_i}$ exists. Then, by Davie's Lemma(5)

$$\tfrac{1}{n}G(n) \leq M + \gamma_3 \ .$$

Therefore $(\sqrt[n]{|h_n|})$ is bounded, so $\lim_{n\to\infty} \sup \sqrt[n]{|h_n|} \in \mathbb{R}^+$, which proves the theorem. □

The preceding theory was the motivation for the following definition.

Definition 2.5.4. *We say that an irrational real number is a Brjuno Number if it satisfies the hypothesis of the Brjuno's theorem. We will define and denote*

$$\mathcal{B} = \{x \in \mathbb{R}\backslash\mathbb{Q} : x \textit{ satisfies the arithmetical condition of Brjuno } \}$$

Now we will state two results and we will give the answer to the question formulated at the beginning of this section. For more details about this results, the interested reader is encouraged to check the Appendix D.

Theorem 2.5.5. *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ be diophantine and $\left(\frac{p_n}{q_n}\right)$ be its sequence of continued fractions. Then*

$$\sum_{k\geq 0}^{\infty} \frac{\log(q_{k+1})}{q_k} < +\infty.$$

Theorem 2.5.6. *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and (a_n) be its sequence of partial coefficients.*

1. *If $a_{n+1} < a_n^{\rho a_n}$ for some $0 < \rho < 1$, except for finitely many n, then $\alpha \in \mathcal{B}$.*
2. *If $a_{n+1} > a_n^{\rho a_n}$ for some $\rho > 1$, except for finitely many n, then $\alpha \notin \mathcal{B}$.*

Recall that the partial coefficients of the number $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ in theorem 2.4.5(iii) are $a_0 = 0$, $a_n = 10^{n!}$ for $n \geq 1$. It is not difficult to see $(n+1)! < (1/2)\cdot(n!)\cdot(10^{n!})$, for all $n \geq 1$. Then

$$a_{n+1} = 10^{(n+1)!} < 10^{(1/2)\cdot(n!)\cdot(10^{n!})} = (a_n)^{(1/2)(a_n)}$$

for all $n \geq 1$. So, by Theorem 2.5.6 we conclude that $\alpha \in \mathcal{B}$. Finally, by Theorem 2.5.1, if $\lambda = \exp(2\pi i\alpha)$ where α is the irrational number as in Theorem 2.4.5(iii) and $f(z) = \lambda z + \sum_{n\geq 2} a_n z^n$ is analytic in some open neighborhood of the origin, then the equation 2.1.1 can be solved.

Chapter 3

Yoccoz's Theorem

The main goal in this chapter is provide a proof for the Brjuno theorem from a topological point of view by using the existence of Siegel disks. The viewpoint in the first two sections of this chapter is influenced by S.Marmi[24], and the last three sections follow the line of reasoning from X.Buff, A.Chéritat[6] and Yoccoz[30]. In the second section we give the definition of linearization and stability and we prove that for some particular conditions these concepts are equivalent.

3.1 Notations and Definitions

$\mathbb{S}$:= the space of all germs of holomorphic diffeomorphisms $f : \mathbb{D} \to \mathbb{C}$ such that:

(s1) f is univalent on $\mathbb{D}$.

(s2) $f(0) = 0$.

We will denote :

- $\mathbb{S}_\lambda$ the subspace of f such that $f'(0) = \lambda$.
- $\mathbb{S}_\mathbb{T}$ the space of f such that $|f'(0)| = 1$.

Given $\alpha \in \mathbb{R}$, we say that $F \in \mathfrak{F}(\alpha)$ if F is an analytic function on the upper half plane $\mathbb{H}$, satisfying :

(c1) F is univalent on $\mathbb{H}$.

(c2) $T \circ F = F \circ T$, where $T(Z) = Z + 1$.

(c3) $\lim_{\Im(Z)\to\infty} (F(Z) - Z) := \alpha.$

Note that this F can be written as : $F(Z) = Z + \alpha + \varphi(Z)$, where φ is a $\mathbb{Z}$-periodic function and $\lim_{\Im(Z)\to\infty} \varphi(Z) = 0$. We also define $E : \mathbb{H} \to \mathbb{D}$ by $E(Z) = \exp(2\pi i Z)$. Clearly $|E(Z)| = \exp(-2\pi\Im(Z))$.

3.2 Germs of Analytic Diffeomorphism and Linearization

Let $\mathbb{C}[[z]]$ denote the ring of formal power series and $\mathbb{C}\{z\}$ denote the ring of convergent power series.
Let $\mathbb{G}$ denote the group of germs of holomorphic diffeomorphisms of $(\mathbb{C}, 0)$ and let $\hat{\mathbb{G}}$ denote the group of formal germs of holomorphic diffeomorphisms of $(\mathbb{C}, 0)$. This means :

$$\mathbb{G} = \{f \in z\mathbb{C}\{z\}, f'(0) \neq 0\} \quad , \quad \hat{\mathbb{G}} = \{\hat{f} \in z\mathbb{C}[[z]], \hat{f}'(0) \neq 0\}$$

One has the trivial fibrations

$$\pi : \mathbb{G} = \bigcup_{\lambda\in\mathbb{C}^*} \mathbb{G}_\lambda \to \mathbb{C}^* \quad , \quad \hat{\pi} : \hat{\mathbb{G}} = \bigcup_{\lambda\in\mathbb{C}^*} \hat{\mathbb{G}}_\lambda \to \mathbb{C}^*$$

defined by

$$\pi(f) = f'(0) \quad , \quad \hat{\pi}(\hat{f}) = \hat{f}'(0)$$

where

$$\hat{\mathbb{G}}_\lambda = \{\hat{f}(z) = \sum_{n=1}^{\infty} \hat{a}_n z^n \in \mathbb{C}[[z]] \ , \ \hat{a}_1 = \lambda\} \ ,$$

$$\mathbb{G}_\lambda = \{f(z) = \sum_{n=1}^{\infty} a_n z^n \in \mathbb{C}\{z\} \ , \ a_1 = \lambda\} \ .$$

Definition 3.2.1. *Let $f \in \mathbb{G}$. We say that a germ $g \in \mathbb{G}$ is equivalent or conjugate to f if there exists $h \in \mathbb{G}_1$ such that $g = h^{-1} \circ f \circ h$.*

Definition 3.2.2. *Let $\hat{f} \in \hat{\mathbb{G}}$. We say that a germ $\hat{g} \in \hat{\mathbb{G}}$ is equivalent or conjugate to $\hat{f}$ if there exists $\hat{h} \in \hat{\mathbb{G}}_1$ such that $\hat{g} = \hat{h}^{-1} \circ \hat{f} \circ \hat{h}$.*

Let R_λ denote the germ $R_\lambda = \lambda z$. This is a simple element of $\mathbb{G}_\lambda$.

Definition 3.2.3. *A germ $f \in \mathbb{G}_\lambda$ is linearizable if there exists $h_f \in \mathbb{G}_1$ (a linearization of f) such that $h_f^{-1} \circ f \circ h_f = R_\lambda$, i.e. f is conjugate to (its linear part) R_λ.*

Let us note that there is an obvious action of $\mathbb{C}^*$ on $\mathbb{G}$ by homotheties :

$$(\mu, f) \in \mathbb{C}^* \times \mathbb{G} \mapsto Ad_{R_\mu} f = R_\mu^{-1} \circ f \circ R_\mu$$

Note that this action leaves the fibers $\mathbb{G}_\lambda$ invariant. Also, $f \in \mathbb{G}_\lambda$ is linearizable if and only if $Ad_{R_\mu} f$ is also linearizable for all $\mu \in \mathbb{C}^*$. Therefore, in order to study the problem of the existence of a linearization, it is enough to consider $\mathbb{G}/\mathbb{C}^*$, i.e. we identify two germs of holomorphic diffeomorphisms which are conjugate by a homothety.

Definition 3.2.4. *Let $f \in \mathbb{S}$. 0 is called stable if there exists a neighborhood $\mathcal{U}$ of 0 such that f^n is defined on $\mathcal{U}$ for all $n \geq 0$ and for all $z \in \mathcal{U}$ and $n \geq 0$ one has $|f^n(z)| < 1$. Note that $\mathcal{U} \subset \mathbb{D}$. To each germ $f \in \mathbb{S}$ with $|f'(0)| \leq 1$, one can associate a natural f-invariant set*

$$0 \in K_f = \bigcap_{n \geq 0} f^{-n}(\mathbb{D}) \tag{3.2.1}$$

Let $\mathcal{U}_f$ denote the connected component of the interior of K_f which contains 0. Then 0 is stable if and only if $\mathcal{U}_f \neq \emptyset$; i.e. if and only if 0 belongs to the interior of K_f.

Theorem 3.2.5. *Let $f \in \mathbb{S}$ with $|f'(0)| \leq 1$. 0 is stable if and only if f is linearizable.*

Proof. The statement is non-trivial only if $\lambda = f'(0)$ has unit modulus. First, assume that 0 is stable. Then $\mathcal{U}_f \neq \emptyset$ and one can prove that it must also be simply connected (see [24], pg. 14). Now let $\zeta : \mathbb{D} \to \mathcal{U}_f$ be the unique conformal and onto map satisfying $\zeta(0) = 0$ and $\zeta'(0) > 0$. Then the map $g : \mathbb{D} \to \mathbb{D}$ defined by $g = \zeta^{-1} \circ f \circ \zeta$ leaves the point $z = 0$ fixed and verifies $g'(0) = \lambda$. By Schwarz Lemma one must have $g(z) = \lambda z$. This implies that f is analytically linearizable.

Conversely we assume h_f is analytic. Then for r small ($r > 0$), h_f maps a small disk $\Delta(0, r)$ around zero conformally into $\mathbb{D}$. Since $h_f(0) = 0$ and $|f^n(z)| < 1$ for all $z \in h_f(\Delta(0, r))$ one sees that 0 is stable. □

Definition 3.2.6. *Siegel Disk and Conformal Capacity. When $\lambda = f'(0)$ has modulus one, is not a root of unity and 0 is stable, then $\mathcal{U}_f$ is conformally equivalent to a disk and is called the Siegel disk of f (at 0).*

Thus the Siegel disk of f is the maximal connected open set containing 0 on which f is conjugate to R_λ. The conformal representation $\widetilde{h}_f : \Delta(0, c_f) \to \mathcal{U}_f$ linearizes f. Here c_f is the conformal capacity of f with respect to 0, and $\widetilde{h}_f$ satisfies $\widetilde{h}_f(0) = 0$ and $\widetilde{h}'_f(0) = 1$.

For more details of the Conformal Capacity, see Appendix E.

3.3 The Renormalization Principle

Definition 3.3.1. *Given $\delta > 0$, we denote by $\mathbb{S}_\delta(\alpha)$ the space of maps $F \in \mathfrak{F}(\alpha)$ such that*

$$\forall Z \in \mathbb{H}, \qquad |F(Z) - Z - \alpha| \leq \delta\alpha \quad \textit{and} \quad |F'(Z) - 1| \leq \delta \tag{3.3.1}$$

Such a function F is uniformly continuous on $\mathbb{H}$; this comes from the inequality $|F'(Z)| \leq 1 + \delta$. Therefore F extends continuously to $\mathbb{H} \bigcup \mathbb{R}$.

Assume $F \in \mathbb{S}_\delta(\alpha)$ and define $l = i\mathbb{R} \bigcap \overline{\mathbb{H}}$ and $l' = [0, F(0)]$. Using the inequalities in 3.3.1, for $\delta \in (0, 1/10)$, $l \bigcup l' \bigcup F(l)$ bounds an open strip $\mathcal{U}$ in $\mathbb{C}$. Gluing the curves l and $F(l)$ in the boundary of $\overline{\mathcal{U}}$ via F, we obtain a surface $\mathcal{V}$, whose remaining boundary corresponds to the segment l'. Its interior is a Riemann surface for the complex structure inherited from the set $\overline{\mathcal{U}}$ (the gluing is analytic). It is not difficult to see that this Riemann surface is biholomorphic to the punctured disk $\mathbb{D}^*$. Denote by $i_c : \overline{\mathcal{U}} \to \mathcal{V}$ the canonical map such that $i_c(l') = \partial\mathcal{V}$.

Let $\mathcal{C}$ be a Jordan curve $\in \mathbb{C}$, such that the bounded connected component V of $\mathbb{C}\backslash\mathcal{C}$ contains 0. Let $y_0 \in \mathcal{C}$; by the Caratheodory Theorem [7], there exists a unique homeomorphism $\widetilde{L} : \mathcal{V} \to \overline{V} \setminus \{0\}$, holomorphic on $\mathcal{V} \setminus \partial\mathcal{V}$ such that $\widetilde{L}(i_c(0)) = y_0$. For the particular case $\mathcal{C} = \partial\mathbb{D}$, we have that $\widetilde{L} \circ i_c : \overline{\mathcal{U}} \to \overline{\mathbb{D}} \setminus \{0\}$, so lifting via $Z \mapsto z = E(Z)$ we obtain a continuous map $L : \overline{\mathcal{U}} \to \mathbb{H} \bigcup \mathbb{R}$ whose restriction over $\mathcal{U}$ is an injective holomorphic map into $\mathbb{H}$. Furthermore, the following condition holds

$$\forall Z \in l \qquad L(F(Z)) = L(Z) + 1 \tag{3.3.2}$$

We normalize L by requiring $L(0) = 0$.

Remark 3.3.2. *Given two different points $Z_1, Z_2 \in \mathcal{U}$ satisfying $L(Z_1) - L(Z_2) \in \mathbb{Z}$, then $Z_1 \in l, Z_2 = F(Z_1)$ or $Z_2 \in l, Z_1 = F(Z_2)$. This comes immediately from the equality $1 = E(L(Z_1) - L(Z_2)) = \widetilde{L}(i_c(Z_1))/\widetilde{L}(i_c(Z_2))$.*

Theorem 3.3.3. *For all $\delta \in (0, 1/10)$, all $\alpha \in (0, 1)$, all $F \in \mathbb{S}_\delta(\alpha)$, and all $Z \in \overline{\mathcal{U}}$,*

$$\Im(Z) - 2\delta \;\leq\; \alpha\Im(L(Z)) \;\leq\; \Im(Z) + 2\delta \; . \tag{3.3.3}$$

In order to prove the theorem above, we have to prove some extra lemmas. From now on we will follow what Xavier Buff and Arnaud Chéritat did in [6]. The proofs of these lemmas and the proof of the theorem may be found in [6]; but for completeness we sketch them here with more details, especially the first lemma.

Lemma 3.3.4. *Assume $\psi : (\mathbb{D}, 0) \to (\mathbb{D}, 0)$ is a K - quasiconformal homeomorphism. Then, for all $z \in \mathbb{D}$,*

$$4^{1-K}|z|^K \;\leq\; |\psi(z)| \;\leq\; 4^{1-1/K}|z|^{1/K} \; .$$

Proof. For a given $z \in \mathbb{D}^*$, let R be the ring formed by $\mathbb{D} \setminus [0, z]$. The modulus of this ring is given by $\mu(|z|)$, see Appendix G, because R can be mapped conformally onto the ring domain $\mathbb{D} \setminus [0, |z|]$ and the modulus is invariant under conformal mappings. Since ψ fixes 0, the image of R under ψ is a ring domain which separates the points 0 and $\psi(z)$ from $\{w : |w| = 1\}$. By Grötzsh's Modulus Theorem, see Appendix G, we have that

$$M(\psi(R)) \leq \mu(|\psi(z)|)$$

Since ψ is a K-quasiconformal map, $M(B) \leq KM(\psi(B))$ for any arbitrary ring domain A with $\overline{B} \subseteq \mathbb{D}$ (see the definition of quasiconformality in [20], pg. 13) and using the continuity of the modulus, see Appendix G, we have

$$M(R) \leq KM(\psi(R))$$

where R is the ring domain $\mathbb{D} \setminus [0, z]$. We denote the modulus of R by $\mu(|z|)$ so

$$\frac{\mu(|z|)}{K} = \frac{\mu(R)}{K} \leq \mu(\psi(R)) \leq \mu(|\psi(z)|)$$

Since the map $\mu : (0, 1) \to (0, +\infty)$ is an onto and strictly decreasing map, see Appendix G, the inverse of this map is also strictly decreasing; therefore

$$|\psi(z)| \leq \mu^{-1}(\frac{\mu(|z|)}{K})$$

On the other hand, consider $s = \mu^{-1}(\frac{\mu(|z|)}{K})$. Since $K \geq 1$,

$$\mu(s) = \frac{\mu(|z|)}{K} \leq \mu(|z|)$$

So

$$s \geq |z|$$

because μ is a decreasing function on $(0, 1)$. Since the function $\mu(r)/\log(4/r))$ is strictly decreasing, see Appendix G, we have :

$$\mu(s)/\log(4/s) \leq \mu(|z|)/\log(4/|z|)$$

Then

$$K\mu(s)/\log(4/s) \leq K\mu(|z|)/\log(4/|z|)$$

So

$$\mu(|z|)/\log(4/s) \leq K\mu(|z|)/\log(4/|z|)$$

Then

$$1/\log(4/s) \leq K/\log(4/|z|)$$

So

$$\log(4/|z|) \leq K \cdot \log(4/s)$$

Hence

$$(4/|z|) \leq (4/s)^K$$

Therefore

$$|\psi(z)| \leq s \leq 4^{1-1/K}|z|^{1/K}$$

The lower bound is obtained by applying the upper bound to ψ^{-1} which is K-quasiconformal. □

Lemma 3.3.5. *If $\Psi : \mathbb{H} \to \mathbb{H}$ is a K - quasiconformal homeomorphism such that $\Psi \circ T = T \circ \Psi$, then*

$$\frac{1}{K}\Im(Z) - \frac{K-1}{2\pi K}\log(4) \;\leq\; \Im(\Psi(Z)) \;\leq\; K\Im(Z) + \frac{K-1}{2\pi}\log(4)\ .$$

Proof. Ψ is the lift, via $Z \to z = E(Z)$, of a K quasiconformal homeomorphism $\psi : (\mathbb{D}, 0) \to (\mathbb{D}, 0)$ as in the previous lemma . □

Lemma 3.3.6. *Let ε and η be any two positive real numbers. Assume $G : \mathbb{H} \to \mathbb{H}$ is a (1+ε)-quasiconformal map such that $G \circ T = T \circ G$ and*

$$K_G(X + iY) \leq 1 + \varepsilon e^{-\eta Y}$$

where $K_G = \dfrac{1 + |\overline{\partial} G/\partial G|}{1 - |\overline{\partial} G/\partial G|}$. Then

$$\Im(Z) - \frac{\varepsilon}{\eta} - \frac{\varepsilon}{2\pi(1+\varepsilon)}\log(4) \quad\leq\quad \Im(G(Z)) \quad\leq\quad \Im(Z) + \frac{\varepsilon}{\eta} + \frac{\varepsilon}{2\pi}\log(4)\ ,$$

which yields

$$|\Im(G(Z)) - \Im(Z)| \leq \frac{\varepsilon}{\eta} + \frac{\varepsilon}{2\pi}\log(4)\ .$$

Proof. Let $G_1(X + iY) = X + i\tau(Y)$, where $\tau(Y)$ is defined by

$$\tau(Y) = \frac{1}{1+\varepsilon}\left(Y - \frac{\varepsilon}{\eta}e^{-\eta Y} + \frac{\varepsilon}{\eta}\right)$$

Clearly $(1 + \varepsilon)\frac{d}{dY}\tau(Y) = 1 + \varepsilon e^{-\eta Y} > 0$, so $\tau(Y)$ is a strictly monotone increasing function. Since $\lim\limits_{Y\to+\infty} \tau(Y) = +\infty$ and $\lim\limits_{Y\to 0^+} \tau(Y) = 0$ we may conclude that $\tau : \mathbb{R}^+ \to \mathbb{R}^+$ is a diffeomorphism. Combining these facts, it is easy to see that G_1 is a diffeomorphism. On the other hand, it is not difficult to check that

$$K_{G_1}(X + iY) = \frac{1+\varepsilon}{1+\varepsilon e^{-\eta Y}} \quad\text{and}\quad \Im(G_1(Z)) \leq \frac{1}{1+\varepsilon}\left(\Im(Z) + \frac{\varepsilon}{\eta}\right)$$

Let $G_2 = G \circ G_1^{-1}$ and let μ_G, μ_{G_1} and μ_{G_2} be the respective complex dilatations of G, G_1

and G_2 ([20], pg. 23). Using the standard formula for compositions with an inverse function ([20], pg. 24), for $\zeta = G_1(Z)$ we have

$$\begin{aligned} |\mu_{G_2}(\zeta)| &\leq \left| \frac{\mu_G(Z) - \mu_{G_1}(Z)}{1 - \mu_G(Z)\overline{\mu_{G_1}(Z)}} \right| \\ &\leq \frac{|\mu_G(Z)| + |\mu_{G_1}(Z)|}{1 + |\mu_G(Z)||\mu_{G_1}(Z)|} \end{aligned}$$

Then,

$$\frac{1 + |\mu_{G_2}(\zeta)|}{1 - |\mu_{G_2}(\zeta)|} \leq \left(\frac{1 + |\mu_G(Z)|}{1 - |\mu_G(Z)|}\right)\left(\frac{1 + |\mu_{G_1}(Z)|}{1 - |\mu_{G_1}(Z)|}\right)$$

By definition of complex dilatation ([20], pg. 23) we have

$$\frac{1 + |\mu_{G_2}(\zeta)|}{1 - |\mu_{G_2}(\zeta)|} \leq K_G(Z) \cdot K_{G_1}(Z)$$

Since $K_G(X + iY) \leq 1 + \varepsilon e^{-\eta Y}$ and $K_{G_1}(X + iY) = \dfrac{1 + \varepsilon}{1 + \varepsilon e^{-\eta Y}}$, we obtain

$$\frac{1 + |\mu_{G_2}(\zeta)|}{1 - |\mu_{G_2}(\zeta)|} \leq 1 + \varepsilon$$

Applying Lemma 3.3.5 to the K-quasiconformal map G_2 with $K = 1 + \varepsilon$ and using the fact $G = G_2 \circ G_1$, we get the upper bound for $\Im(G(Z))$.

To get the lower bound, repeat the same argument writing $G = G_4 \circ G_3$ with

$$G_3(X + iY) = X + i(1 + \varepsilon)\left(Y + \frac{1}{\eta}\log\frac{1 + \varepsilon e^{-\eta Y}}{1 + \varepsilon}\right).$$

By elementary computations we have

$$K_{G_3}(X + iY) = \frac{1 + \varepsilon}{1 + \varepsilon e^{-\eta Y}} \text{ and } \Im(G_3(Z)) \geq (1 + \varepsilon)\left(\Im(Z) - \frac{\varepsilon}{\eta}\right)$$

□

As we mentioned before, the next proof may be found in [6].

Proof of Theorem 3.3.3. Since $F(Z) - Z - \alpha$ is holomorphic on the upper half plane $\mathbb{H}$ and is periodic of period 1, we may apply Schwarz lemma and obtain

$$|F(Z) - Z - \alpha| \leq \delta\alpha \exp(-2\pi\Im(Z))$$

Since $F \in \mathfrak{F}(\alpha)$ and $\delta \in (0, 1/10)$, we may apply Theorem F.3 (see Appendix F). So there exists $C_\delta > 0$ such that for all $Z \in \mathbb{H}$ with $\Im(Z) = C_\delta + t \geq C_\delta$ we have

$$|F'(Z) - 1| \leq \delta \exp(-2\pi t) = \delta \exp(2\pi C_\delta) \exp(-2\pi \Im(Z))$$

so $\lim_{\Im(Z) \to +\infty} (F'(Z) - 1) = 0$. Now, we may apply the Schwarz lemma to the periodic map $Z \to F'(Z) - 1$ (with period 1) which by hypothesis satisfies $|F'(Z) - 1| \leq \delta$. Then we have

$$|F'(Z) - 1| \leq \delta \exp(-2\pi \Im(Z))$$

Let $\mathcal{B}$ be the half strip $\{ \ Z \in \mathbb{H} \ : \ 0 < \Re(Z) < 1 \}$. Let $H : \overline{\mathcal{B}} \to \overline{\mathcal{U}}$ be the map defined by

$$H(X + iY) = i\alpha Y + X[\ F(i\alpha Y) - i\alpha Y \ ] \tag{3.3.4}$$

By straightforward computations we have

$$\frac{\partial H}{\partial X} = F(i\alpha Y) - i\alpha Y$$

and

$$\frac{\partial H}{\partial Y} = i\alpha + X[i\alpha F'(i\alpha Y) - i\alpha]$$

So

$$\partial H = \frac{1}{2}\left(\frac{\partial H}{\partial X} - i\frac{\partial H}{\partial Y}\right) = \frac{1}{2}\left(F(i\alpha Y) - i\alpha Y + \alpha + \alpha X[F'(i\alpha Y) - 1]\right)$$

and

$$\overline{\partial} H = \frac{1}{2}\left(\frac{\partial H}{\partial X} + i\frac{\partial H}{\partial Y}\right) = \frac{1}{2}\left(F(i\alpha Y) - i\alpha Y - \alpha - \alpha X[F'(i\alpha Y) - 1]\right)$$

Hence

$$|\partial H - \alpha| \leq \delta\alpha \exp(-2\pi\alpha Y) \quad \text{and} \quad |\overline{\partial} H| \leq \delta\alpha \exp(-2\pi\alpha Y)$$

Combining the last two inequalities, $Z = X + iY \in \mathbb{H}$ and $\delta \in (0, 1/10)$ we get

$$\frac{|\overline{\partial} H|}{|\partial H|} \leq \frac{\delta\alpha \exp(-2\pi\alpha Y)}{\alpha - \delta\alpha \exp(-2\pi\alpha Y)} < 1$$

Moreover, the following inequalities hold:

$$1 + \frac{|\overline{\partial} H|}{|\partial H|} \leq \frac{1}{1 - \delta \exp(-2\pi\alpha Y)} \quad \text{and} \quad 1 - \frac{|\overline{\partial} H|}{|\partial H|} \geq \frac{1 - 2\delta \exp(-2\pi\alpha Y)}{1 - \delta \exp(-2\pi\alpha Y)}$$

Now, if we set $K_H = \dfrac{1+|\overline{\partial}H/\partial H|}{1-|\overline{\partial}H/\partial H|}$, we have :

$$K_H(X+iY) \leq \frac{1}{1-2\delta\exp(-2\pi\alpha Y)} .$$

Then using the fact $0<\delta<1/10$, we have the inequality

$$K_H(X+iY) \leq 1+\frac{5}{2}\delta\exp(-2\pi\alpha Y) \leq 1+\frac{5}{2}\delta$$

Using the definition of H and since $\mathcal{U}$ is an open strip in $\mathbb{C}$ we may conclude that H is a homeomorphism. In particular, H is a $(1+\frac{5}{2}\delta)$-quasiconformal homeomorphism.

Moreover, by definition

$$H(X+iY)-\alpha(X+iY) = X[F(i\alpha Y)-i\alpha Y-\alpha]$$

So

$$|H(X+iY)-\alpha(X+iY)| = |X||\ F(i\alpha Y)-i\alpha Y\ \ -\alpha| \leq 1\cdot\delta\alpha\exp(-2\pi\alpha Y) \leq \delta\alpha$$

Then

$$\Im(H(Z))-\delta\alpha \ \leq\ \alpha\Im(Z)\ \leq\ \Im(H(Z))+\delta\alpha\ ,$$

and thus for all $Z\in\overline{\mathcal{U}}$, since $0<\alpha<1$,

$$\Im(Z)-\delta\ \leq\ \alpha\Im(H^{-1}(Z))\ \leq\ \Im(Z)+\delta \tag{3.3.5}$$

By the construction of L, given at the beginning of this section, we know that L is conformal on $\mathcal{U}$, so the map $G = L\circ H$ is quasiconformal on $\mathcal{U}$ with the same dilatation as H. By the definition of H, equation 3.3.2 and since L is defined on $\overline{\mathcal{U}}$, we have

$$G(iY+1) = L\circ H(iY+1) = L(F(i\alpha Y))$$

and

$$G(iY)+1 = L(i\alpha Y)+1 = L(F(i\alpha Y))$$

Therefore

$$G(iY+1) = G(iY)+1.$$

This implies that $i\mathbb{R}\bigcap\overline{\mathbb{H}}$ is quasiconformally removable, so G extends to a quasiconformal homeomorphism $\mathbb{H}\to\mathbb{H}$. Now, we may apply Lemma 3.3.5 with $\varepsilon = \frac{5}{2}\delta$ and $\eta = 2\pi\alpha$. Using the fact $0<\alpha<1$, we have

$$\frac{\varepsilon}{\eta}+\frac{\varepsilon}{2\pi}\log(4) = \frac{5\delta}{4\pi\alpha}(1+\alpha\log(4)) \leq \frac{\delta}{\alpha}$$

Then for all $Z \in \mathbb{H}$
$$|\Im(G(Z)) - \Im(Z)| \leq \frac{\varepsilon}{\eta} + \frac{\varepsilon}{2\pi}\log(4) \leq \frac{\delta}{\alpha}$$
So we have showed that for all $Z \in \mathbb{H}$
$$\alpha\Im(Z) - \delta \ \leq \ \alpha\Im(G(Z)) \ \leq \ \alpha\Im(Z) + \delta \tag{3.3.6}$$
Finally, by 3.3.5 and 3.3.6 it follows that
$$\Im(Z) - 2\delta \ \leq \ \alpha\Im(H^{-1}(Z)) - \delta \ \leq \ \alpha\Im(L(Z)) \ \leq \ \alpha\Im(H^{-1}(Z)) + \delta \ \leq \ \Im(Z) + 2\delta$$

□

Theorem 3.3.7. *Under the same assumptions as in Theorem 3.3.3, the map L extends to a univalent map on*
$$\mathcal{W} = \overline{\mathcal{U}} \bigcup \{Z \in \mathbb{C} : -1 \leq \Re(Z) \leq 0 \ \ and \ \ \Im(Z) \geq 4\delta\}$$
and for all $Z \in \mathcal{W}$
$$\Im(Z) - 5\delta \ \leq \ \alpha\Im(L(Z)) \ \leq \ \Im(Z) + 5\delta \tag{3.3.7}$$

The definition of $\mathcal{W}$ is such that any point $Z \in \mathcal{W}$ is eventually mapped to $\overline{\mathcal{U}}$ under iteration of F : $F^{\circ k}(Z) = Z' \in \overline{\mathcal{U}}$ for some $k \in \mathbb{N}$. In particular, L conjugates F to the translation T.

Proof. Let $\mathcal{V} = \{Z \in \mathbb{C} : -1 \leq \Re(Z) \leq 0 \text{ and } \Im(Z) \geq 4\delta\}$ and let $\mathcal{V}^* = \{Z \in \mathbb{C} : \Im(Z) \geq 4\delta|\Re(Z)|\}$. It is easy to see that $\mathcal{V} \subset \mathcal{V}^*_-$ where $\mathcal{V}^*_- = \mathcal{V}^* \bigcap \{Z \in \mathbb{C} : -1 \leq \Re(Z) \leq 0\}$. By Lemma F.4 (see Appendix F), we have that $F(\mathcal{V}^*_-) \subset \mathcal{V}^*_- \bigcup \overline{\mathcal{U}}$.

By the first part of inequality 3.3.1, $\Re(F^{\circ k}(Z)) \geq \Re(Z) + k(1-\delta)\alpha$ for all $Z \in \mathbb{H}$ whenever the iterations are well defined. If we assume that there exists $Z \in \mathcal{V}$ such that $F^{\circ k}(Z) \notin \overline{\mathcal{U}}$ for any k, then $F^{\circ k}(Z) \in \mathcal{V}^*_-$ for all k, so $\Re(F^{\circ k}(Z)) \leq 0$ for all k. This implies that $0 \geq \Re(Z) + k(1-\delta)\alpha$ for all k, which is a contradiction. Hence, for each $Z \in \mathcal{W}$ there exists $k \equiv k(Z)$ such that $F^{\circ k}(Z) \in \overline{\mathcal{U}}$. Now define
$$L(Z) = L(F^{\circ k}(Z)) - k,$$
where k is the minimum nonnegative integer such that $F^k(Z) \in \overline{\mathcal{U}}$. The continuity of this extension follow easily (see [30], pg. 28). Let $Z_1, Z_2 \in \mathcal{W}$ such that $L(Z1) = L(Z_2)$, and let $m_1, m_2 \in \mathbb{Z}$ such that $F^{\circ m_1}(Z_1), F^{\circ m_2}(Z_2) \in \overline{\mathcal{U}}$. Then
$$L(F^{\circ m_1}(Z_1)) - L(F^{\circ m_2}(Z_2)) = m_1 - m_2,$$
so by Remark 3.3.2 we have three possibilities:

(1) $m_1 = m_2$, $F^{\circ m_1}(Z_1) = F^{\circ m_2}(Z_2)$;

(2) $m_1 = m_2 + 1$, $F^{\circ m_2}(Z_2) \in l$ and $F^{\circ m_1}(Z_1) = F(F^{\circ m_2}(Z_2))$;

(3) $m_2 = m_1 + 1$, $F^{\circ m_1}(Z_1) \in l$ and $F^{\circ m_2}(Z_2) = F(F^{\circ m_1}(Z_1))$.

Since F is univalent, we conclude that $Z_1 = Z_2$. Therefore the extension L is injective. Now, for any $Z \in \mathcal{W}$ we know that $Z = F^0(Z)$, $F(Z) = F^{\circ 1}(Z)$, $F^{\circ 2}(Z)$, $F^{\circ 3}(Z)$, $\ldots$ and $F^{\circ k(F(Z))} = Z'$ are in $\mathbb{H}$. Using the first inequality in 3.3.1, we have :

$$|\Im(F^{\circ i+1}(Z) - F^{\circ i}(Z))| \leq \delta\alpha = \frac{\delta}{1-\delta}(\alpha - \delta\alpha) \leq \frac{\delta}{1-\delta}(\Re(F^{\circ i+1}(Z) - F^{\circ i}(Z)))$$

for all $i = 0, 1, 2, \ldots, k(Z)-1$. Adding all of these terms and applying the triangle inequality, we get

$$|\Im(Z' - Z)| = |\Im(F^{\circ k(Z)}(Z) - F^0(Z))| \leq \frac{\delta}{1-\delta}(\Re(Z' - Z))$$

Since $Z' \in \overline{\mathcal{U}}$ and $0 < \Re(F(it)) \leq \alpha + \delta\alpha$ for all $t \in \mathbb{R}$, $0 < \Re(Z') \leq \alpha + \delta\alpha$ must hold. Recall that $-1 \leq \Re(Z) \leq 0$. Therefore

$$|\Im(Z' - Z)| \leq \frac{\delta}{1-\delta}(\Re(Z' - Z)) \leq \frac{\delta}{1-\delta}(\alpha + \delta\alpha + 1)$$

Since $\frac{1}{1-\delta}(\alpha + \delta\alpha + 1) \leq 3$, the inequality 3.3.7 comes from a straightforward application of Theorem 3.3.3 to Z'. On the other hand, it is easy to see that $k(F(Z)) = k(Z) - 1$ when $k(Z) \geq 1$; thus

$$L(F(Z)) = L(F^{\circ k(F(Z))}(F(Z))) - k(F(Z))$$

Therefore, $L(F(Z)) = L(F^{\circ k(Z)}(Z)) - (k(Z) - 1) = L(Z) + 1$. This means that L conjugates F to the translation T, $T(Z) = Z + 1$. □

Remark 3.3.8. *From now on, L will refer to this extension.*

3.4 Construction of the n-th renormalization

Given $\delta \in (0, 1/10)$ and $F \in \mathfrak{F}(\alpha)$, we will inductively define a sequence $(F_n)_{n\geq 0}$ of univalent maps such that $F_n \in \mathfrak{F}(\alpha_n)$. Recall α_n was defined in **Chapter 2** (see 2.4.1). The construction depends on the choice at each step of some real number $t_n > 0$. We start with $F_0 = F - a_0$ (where $a_0 = [\![\alpha]\!]$) and we assume F_n has been constructed. We choose t_n such that the fundamental inequalities in 3.3.1 hold for $\Im(Z) \geq t_n$; this is always possible and the proof of this fact is given in Theorem F.3. For more details see Appendix F. It follows that $G_n : Z \mapsto F_n(Z + it_n) - it_n$ belongs to $\mathbb{S}_\delta(\alpha_n)$. For this G_n, we construct $\mathcal{U}_n$, $\mathcal{W}_n$ and L_n as we did before. Now let J_n be defined on $D_n = L_n(\{Z \in \overline{\mathcal{U}}_n : \Im(Z) > 4\delta\})$ by

$$J_n = L_n \circ T^{-1} \circ {L_n}^{-1} \tag{3.4.1}$$

Proposition 3.4.1. *If* $\Im(Z_0) > \dfrac{6\delta}{\alpha_n}$ *, there exists an integer* k *such that* $Z_0 - k$ *belongs to* D_n*, the domain of* J_n*.*

Proof. By 3.3.2 the equality $L_n(G_n(Z)) = L_n(Z) + 1$ holds for all $Z \in i\mathbb{R} \bigcap \mathbb{H}$, since the restriction of L_n to $\mathcal{U}$ is a homeomorphism onto its image and $L_n([0, G_n(0)]) = [0, 1]$we have $\mathbb{H} = \bigcup_{k \in \mathbb{Z}} (\text{Ł}_n(\overline{\mathcal{U}}_n) + k)$. Since $Z_0 \in \mathbb{H}$, there exists $k \in \mathbb{Z}$ such that $Z_0 - k$ belongs to $\text{Ł}_n(\overline{\mathcal{U}}_n)$. So there exists $y_0 \in \overline{\mathcal{U}}_n$ such that $\text{Ł}_n(y_0) = Z_0 - k$. By Theorem 3.3.3

$$\alpha_n \Im(L_n(y_0)) \leq \Im(y_0) + 2\delta$$

So

$$\alpha_n \Im(Z_0) \leq \Im(y_0) + 2\delta$$

By 2.4.1, it is easy to verify that $\alpha_n \in (0, 1)$. Since $\dfrac{6\delta}{\alpha_n} < \Im(Z_0)$, then $4\delta < \Im(y_0)$. Therefore $Z_0 - k$ belongs to D_n . □

Proposition 3.4.2. *Let* $w = L_n(it)$ *with* $t > 0$*. If* $\Im(w) > \dfrac{6\delta}{\alpha_n}$*, then*

$$t > 4\delta \text{ and } (J_n \circ T)(w) = (T \circ J_n)(w) \ .$$

Proof. Since $it \in \overline{\mathcal{U}}_n$, we can apply Theorem 3.3.3. Then

$$\Im(it) - 2\delta \leq \alpha \Im(L(it)) \leq \Im(it) + 2\delta \ .$$

Now, by hypothesis $6\delta < \alpha \Im(w) \leq \Im(it) + 2\delta$. Then $4\delta < \Im(it) = t$, so $w = L_n(it) \in D_n$ and $J_n(w)$ makes sense. By an elementary computation we have

$$J_n(w) = J_n(L_n(it)) = L_n \circ T^{-1} \circ {L_n}^{-1}(L_n(it)) = L_n(-1 + it)$$

Hence

$$T \circ J_n(w) = T \circ L_n(-1 + it) = L_n(-1 + it) + 1$$

On the other hand, since $G_n(it) \in \overline{\mathcal{U}}_n$ and $L_n(G_n(it)) = L_n(it) + 1$, we can replace it by $G_n(it)$. Repeating the same argument as above, we get $4\delta < \Im(G_n(it))$. So, $L_n(G_n(it)) \in D_n$. Then

$$\begin{aligned}
J_n \circ T(w) &= J_n(\ L_n(it) + 1\ ) \\
&= J_n(L_n(\ G_n(it)\ )) \quad \text{by Theorem 3.3.7} \\
&= L_n \circ T^{-1}(G_n(it)) \\
&= L_n(G_n(-1 + it)) \quad \text{because } G_n \circ T = T \circ G_n \quad \text{on} \quad \mathbb{H} \\
&= T \circ L_n(-1 + it\ ) \quad \text{by Theorem 3.3.7} \\
&= L_n(-1 + it) + 1
\end{aligned}$$

□

Lemma 3.4.3. *J_n extends univalently to the upper half plane :*

$$\{Z : \Im(Z) > \frac{6\delta}{\alpha_n}\}$$

Proof. By Proposition 3.4.1, $D_n + \mathbb{Z}$ contains the half plane:

$$\{Z : \Im(Z) > \frac{6\delta}{\alpha_n}\}$$

By Proposition 3.4.2, J_n commutes with the translation T on the set of the points in $L_n(i[0, +\infty))$ whose imaginary part is greater than $6\delta/\alpha_n$, so this set is analytically removable. Since $\Im(it) - 2\delta \leq \Im(L_n(it))$ for all $t > 0$, we have $\lim_{t\to+\infty} \Im(L_n(it)) = +\infty$. Combining the facts above, the lemma follows. □

Lemma 3.4.4. $J_n(w) - w \to \frac{-1}{\alpha_n} = -a_{n+1} - \alpha_{n+1}$ *, as* $\Im(w) \to +\infty$.

Proof. Let us recall that by Theorem 3.3.7, for all $Z \in \overline{\mathcal{U}}_n$ we have

$$\Im(Z) - 5\delta \leq \alpha_n \Im(L_n(Z)) \leq \Im(Z) + 5\delta$$

Then

$$\Im(Z) \to +\infty \Leftrightarrow \Im(L_n(Z)) \to +\infty \ .$$

By the last lemma, the set $\{L_n(it) : \Im(L_n(it)) > 6\delta/\alpha_n, t > 0\}$ is analytically removable, so it is enough to consider $w \in Ł_n(\overline{\mathcal{U}}_n)$ with $\Im(w) > 6\delta/\alpha_n$. By the definition of L_n, for $w = L_n(Z)$ with $Z \in \overline{\mathcal{U}}_n$ and $\Im(w) > 6\delta/\alpha_n$, we have $J_n(L_n(Z)) - L_n(Z) = L_n(Z - 1) - L_n(Z)$. So

$$\lim_{\Im(L_n(Z))\to+\infty} (\ (J_n(L_n(Z))) - L_n(Z)\ ) = \lim_{\Im(Z)\to+\infty} (\ L_n(Z-1) - L_n(Z)\ )$$

We do not know yet whether $L_n(Z - 1)$ makes sense. But, since $w = L_n(Z)$ with $Z \in \overline{\mathcal{U}}_n$ and $\Im(w) > 6\delta/\alpha_n$ we can repeat the same argument as in the proof of Proposition 3.4.2 and get $\Im(Z) > 4\delta$. Hence $-1 + Z \in \mathcal{W}_n$ and $L_n(Z - 1)$ makes sense. On the other hand, note that

$$J_n(L_n(Z))) - L_n(Z) = \frac{1}{\alpha_n}\left((\alpha_n L_n(Z-1) - (Z-1)) - (\alpha_n L_n(Z) - Z) - 1\right)$$

Following the same arguments given by Yoccoz in [30], pg. 29-32, we can find a continuous function φ such that $\lim_{\Im(Z)\to+\infty} \varphi(Z) = 0$ and

$$|\alpha_n L_n'(Z) - 1| \leq \varphi(Z)$$

Now consider the path $\gamma(t) = Z + t$ with $t \in [-1, 0]$. Then

$$\begin{aligned} |(\alpha_n L_n(Z-1) - (Z-1)) - (\alpha_n L_n(Z) - Z)| &\leq \int_\gamma |\alpha_n L_n'(\xi) - 1||d\xi| \\ &\leq \int_{-1}^{0} \varphi(\gamma(t))dt \\ &\leq \max\{\varphi(Z+t) : t \in [-1, 0]\} \end{aligned}$$

Since $\lim_{\Im(Z)\to+\infty} \varphi(Z) = 0$, we have

$$\lim_{\Im(Z)\to+\infty} (\alpha_n L_n(Z-1) - (Z-1)) - (\alpha_n L_n(Z) - Z) = 0$$

Therefore,

$$\lim_{\Im(Z)\to+\infty} (\ L_n(Z-1) - L_n(Z)\ ) = \frac{-1}{\alpha_n}$$

□

Definition 3.4.5. *For each $n \geq 0$ and $n \in \mathbb{Z}$, we set*

$$\mathcal{W}'_n = \mathcal{W}_n + it_n \tag{3.4.2}$$

and we define $\chi_n : \mathcal{W}'_n \to \mathbb{C}$

$$\chi_n(Z) = s \circ L_n(Z - it_n) - i\frac{6\delta}{\alpha_n} \tag{3.4.3}$$

where s(x+iy)=-x+iy .

Lemma 3.4.6. *On $\mathcal{W}'_n \bigcap F_n^{-1}(\mathcal{W}'_n)$, χ_n conjugates F_n to T^{-1}.*

Proof. Let $Z + it_n \in \mathcal{W}'_n \bigcap F_n^{-1}(\mathcal{W}'_n)$. Then

$$\chi_n \circ F_n(Z + it_n) = s \circ L_n(F_n(Z + it_n) - it_n) - i\frac{6\delta}{\alpha_n}$$

Recall that $G_n : Z \mapsto F_n(Z + it_n) - it_n$, so

$$\chi_n \circ F_n(Z + it_n) = s \circ L_n \circ G_n(Z) - i\frac{6\delta}{\alpha_n}$$

By the last part in the proof of Theorem 3.3.7 $L_n \circ G_n = T \circ L_n$, so

$$\chi_n \circ F_n(Z + it_n) = s(L_n(Z) + 1) - i\frac{6\delta}{\alpha_n}$$

Finally, by Definition 3.4.5, the lemma follows. □

Definition 3.4.7. *We define* F_{n+1} *by* $F_{n+1}(Z) = \chi_n \circ T^{-1} \circ \chi_n^{-1}(Z) - a_{n+1}$.

Theorem 3.4.8. *The map* F_{n+1} *is an element of* $\mathfrak{F}(\alpha_{n+1})$.

Proof. First of all, it is easy to check that

$$(L_n^{-1} \circ s)(Z + i\frac{6\delta}{\alpha_n}) = \chi_n^{-1}(Z) - it_n$$

So

$$\begin{aligned} F_{n+1}(Z) &= s \circ L_n(\chi_n^{-1}(Z) - 1 - it_n) - i\frac{6\delta}{\alpha_n} - a_{n+1} \\ &= s \circ L_n((L_n^{-1} \circ s)(Z + i\frac{6\delta}{\alpha_n}) - 1) - i\frac{6\delta}{\alpha_n} - a_{n+1} \\ &= s \circ L_n((T^{-1} \circ L_n^{-1})(s(Z) + i\frac{6\delta}{\alpha_n})) - i\frac{6\delta}{\alpha_n} - a_{n+1} \\ &= s \circ (L_n \circ T^{-1} \circ L_n^{-1}) \circ s(Z + i\frac{6\delta}{\alpha_n}) - i\frac{6\delta}{\alpha_n} - a_{n+1} \end{aligned}$$

Therefore F_{n+1} is univalent in $\mathbb{H}$. Moreover, $s(Z + i\frac{6\delta}{\alpha_n})$ is in the domain of definition of the extension J_n for all $Z \in \mathbb{H}$, so

$$F_{n+1}(Z) = s \circ J_n \circ s(Z + i\frac{6\delta}{\alpha_n}) - i\frac{6\delta}{\alpha_n} - a_{n+1} \tag{3.4.4}$$

Since $s \circ T = T^{-1} \circ s$ and the extension J_n commutes with T, we have :

$$\begin{aligned} F_{n+1}(T(Z)) &= (s \circ J_n \circ (s \circ T))(Z + i\frac{6\delta}{\alpha_n}) - i\frac{6\delta}{\alpha_n} - a_{n+1} \\ &= (s \circ J_n \circ T^{-1} \circ s)(Z + i\frac{6\delta}{\alpha_n}) - i\frac{6\delta}{\alpha_n} - a_{n+1} \\ &= (T \circ s \circ J_n \circ s)(Z + i\frac{6\delta}{\alpha_n}) - i\frac{6\delta}{\alpha_n} - a_{n+1} \\ &= T(F_{n+1}(Z)) \end{aligned}$$

Therefore $F \circ T = T \circ F$ on $\mathbb{H}$. Furthermore :

$$F_{n+1}(Z) - Z = s(J_n(s(Z + i\frac{6\delta}{\alpha_n})) - s(Z + i\frac{6\delta}{\alpha n})) - a_{n+1}$$

By Lemma 3.4.4 and Definition 2.4.1 we have

$$\lim_{\Im(Z) \to +\infty} F_{n+1}(Z) - Z = s(\frac{-1}{\alpha_n}) - a_{n+1} = \alpha_n^{-1} - a_{n+1} = \alpha_{n+1}$$

So the theorem follows. □

3.5 Proof of Brjuno's Theorem by Yoccoz

Definition 3.5.1. *For each point $Z \in \mathbb{H}$, we associate a sequence $(Z_n)_n$ as follows. We define $Z_0 = Z$. If $d_n = \Im(Z_n) \geq 4\delta + t_n$, we choose $Z'_n \in \mathbb{H}$ such that $Z_n - Z'_n \in \mathbb{Z}$ and $-1 \leq \Re(Z'_n) < 0$ and we define*

$$Z_{n+1} = \chi_n(Z'_n)$$

The sequence $(Z_n)_n$ may be finite or infinite.

Proposition 3.5.2. *If $(Z_n)_n$ is defined for some $n \geq 0$, then*

$$\Im(Z_n) - t_n - 11\delta \leq \alpha_n \Im(Z_{n+1}) \leq \Im(Z_n) - t_n - \delta$$

Proof. Applying the definition of (Z_n) and Definition 3.4.5, one sees that

$$\begin{aligned} \Im(Z_{n+1}) &= \Im(\chi_n(Z'_n)) \\ &= \Im(s \circ L_n(Z'_n - it_n) - i\frac{6\delta}{\alpha_n}) \\ &= \Im(L_n(Z'_n - it_n)) - \frac{6\delta}{\alpha_n} \end{aligned}$$

Since $\Im(Z_n) = \Im(Z'_n)$ and $Z'_n - it_n \in \mathcal{W}_n$, we can apply 3.3.7 and obtain

$$\Im(Z'_n - it_n) - 5\delta \leq \alpha_n \Im(L(Z'_n - it_n)) \leq \Im(Z'_n - it_n) + 5\delta$$

So

$$\Im(Z_n) - t_n - 5\delta \leq \alpha_n \Im(L(Z'_n - it_n)) \leq \Im(Z_n) - t_n + 5\delta$$

Now, the proposition follows by an elementary replacement. □

Let us recall that $\alpha \in (0, 1)$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and the sequence $(\alpha_n)_n$ was defined in **Chapter 1** (see definition 2.4.1).

Definition 3.5.3. *$(\beta_n)_n$ is the sequence defined by $\beta_{-1} = 1$ and $\beta_n = \alpha_0\alpha_1 \ldots \alpha_{n-1}$ for all $n \geq 0$.*

Proposition 3.5.4. *Let $(\beta_n)_n$ as above. The following facts hold:*

(i) $\beta_{n+2} \leq \frac{1}{2}\beta_n$ *for all* $n \geq 0$.

(ii) $1 + \beta_0 + \beta_1 + \beta_2 + \cdots \leq 4$.

Proof.
(i) It is enough to prove that $\alpha_{n+1}\alpha_{n+2} \le \frac{1}{2}$ for all $n \ge 0$. In fact, fix $n \ge 0$. Then there exists a positive integer p such that $\frac{1}{p+1} < \alpha_{n+1} < \frac{1}{p}$. So we have 2 subcases:

(1) If $p = 1$, then $1/2 < \alpha_{n+1} < 1$. So $1 \le [\![\alpha_{n+1}^{-1}]\!]$, which implies that

$$1/2 < \alpha_{n+1} \le \alpha_{n+1} [\![\alpha_{n+1}^{-1}]\!] = 1 - \alpha_{n+1}^{-1}\{\alpha_{n+1}^{-1}\}$$

Therefore

$$\alpha_{n+1}\alpha_{n+2} = \alpha_{n+1}\{\alpha_{n+1}^{-1}\} < 1/2$$

(2) If $p > 1$, then $p < \alpha_{n+1}^{-1} < p+1$, whence $\alpha_{n+1}p = \alpha_{n+1} [\![\alpha_{n+1}^{-1}]\!]$. Since $1-\alpha_{n+1} < p\alpha_{n+1}$ we have that $1 - \alpha_{n+1} < \alpha_{n+1} [\![\alpha_{n+1}^{-1}]\!]$. But also

$$\alpha_{n+1} < \frac{1}{p} \le \frac{1}{2} < 1$$

Therefore,

$$\frac{1}{2} < 1 - \alpha_{n+1} < \alpha_{n+1} [\![\alpha_{n+1}^{-1}]\!] = 1 - \alpha_{n+1}\alpha_{n+2}$$

(ii) By the previous fact,

$$\begin{aligned} 1 + \beta_0 + \beta_1 + \ldots &\le 1 + \beta_0 + \beta_1 + \frac{1}{2}\beta_0 + \frac{1}{2}\beta_1 \ldots \\ &\le 1 + \beta_0(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots) + \beta_1(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots) \\ &\le 1 + 2\beta_0 + 2\beta_1. \end{aligned}$$

Since $\beta_0 = \alpha_0 \in (0,1)$ and $\beta_1 = \alpha_0\alpha_1 \in (0, 1/2)$, the assertion follows. □

Proposition 3.5.5. *We use the same notation as in Definition 3.5.1. If Z_n is defined for some $n \ge 0$, then :*

$$\beta_{n-1}d_n + \sum_{i=0}^{n-1} \beta_{i-1}(t_i + \delta) \le d_0 \le \beta_{n-1}d_n + \sum_{i=0}^{n-1} \beta_{i-1}(t_i + 11\delta)$$

where the β_i's are as in Definition 3.5.3.

Proof. This follows by a direct application of Proposition 3.5.2 □

Proposition 3.5.6. *Under the same assumptions as in the last proposition, the following inequality holds :*

$$\sum_{i=0}^{n-1} \beta_{i-1}t_i \le d_0 - \beta_{n-1}d_n \le 44\delta + \sum_{i=0}^{n-1} \beta_{i-1}t_i$$

Proof. This comes from a straightforward application of the last two propositions. □

Proposition 3.5.7. *If $Z \in \mathbb{H}$ and if there exists some nonnegative integer m such that $F^{\circ m}(Z) \notin \mathbb{H}$, then the sequence $(Z_n)_n$ is finite.*

Proof. We define and denote by H_n the upper half plane $\{Z \in \mathbb{C} : \Im(Z) \geq t_n\}$. If Z_n is defined, let $1 + k_n$ be the rank of the first iterate of Z_n under $F_n : \mathbb{H} \to \mathbb{C}$ that leaves H_n; i.e. k_n is the smallest nonnegative integer such that: $\{F_n^{\circ 1}(Z_n), F_n^{\circ 2}(Z_n), \ldots, F_n^{\circ k_n}(Z_n)\} \subset H_n$ and $F_n^{\circ k_n+1}(Z_n) \notin H_n$. If $\Im(Z_n) \geq 4\delta + t_n$ then k_n cannot be zero. Otherwise $F_n(Z_n) \notin H_n$ which means that $\Im(F_n(Z_n)) < t_n$, so this implies that $4\delta < \Im(Z_n) - \Im(F_n(Z_n))$ which is impossible since $|F_n(W) - W - \alpha_n| < \delta\alpha_n < \delta$ for all $W \in H_n$. As an immediate consequence, Z_{n+1} cannot be defined when $k_n = 0$. Now assume that Z_{n+1} is defined and $k_{n+1} > 0$. Since F_n commutes with the translation $Z \to Z + 1$ it is easy to see that $\Im F_n^{\circ j}(Z_n) = \Im F_n^{\circ j}(Z'_n)$ for all $j \in \{0, 1, \ldots, k_n\}$ (here Z'_n is as in Definition 3.5.1). By using the definition of G_n at the beginning of Section 3.4, one sees that

$$G_n^{\circ j}(Z'_n - it_n) = F_n^{\circ j}(Z'_n) - it_n$$

for all $j \in \{0, 1, \ldots, k_n\}$. Therefore,

$$\Im G_n^{\circ j}(Z'_n - it_n) = \Im F_n^{\circ j}(Z'_n) - t_n$$

for all $j \in \{0, 1, \ldots, k_n\}$. Since $k_{n+1} > 0$ and $Z_{n+1} = \chi_n(Z'_n)$, by equations 3.4.4 and 3.4.3 we have

$$\Im(L_n(Z'_n - it_n)) > \frac{6\delta}{\alpha_n}$$

and

$$F_{n+1}(Z_{n+1}) = s \circ J_n(L_n(Z'_n - it_n)) - i\frac{6\delta}{\alpha_n} - a_{n+1}$$

By Proposition 3.4.1 we may assume that there is a unique integer p and a unique $y_0 \in \overline{\mathcal{U}}_n$ with $\Im(y_0) \geq 4\delta$ such that $L_n(Z'_n - it_n) = L_n(y_0) + p$. Hence

$$\begin{aligned} F_{n+1}(Z_{n+1}) &= s \circ J_n(L_n(Z'_n - it_n)) - i\frac{6\delta}{\alpha_n} - a_{n+1} \\ &= s \circ J_n(L_n(y_0)) - i\frac{6\delta}{\alpha_n} - a_{n+1} + p \\ &= s \circ L_n \circ T^{-1}(y_0) - i\frac{6\delta}{\alpha_n} - a_{n+1} + p \\ &= s \circ L_n(y_0 - 1) - i\frac{6\delta}{\alpha_n} - a_{n+1} + p \end{aligned}$$

But, by definition of the extension L_n (see the proof of Theorem 3.3.7) we have

$$L_n(Z'_n - it_n) = L_n(G_n^{\circ\sigma_1}(Z'_n - it_n)) + \sigma_1$$

where $G_n^{\circ\sigma_1}(Z_n' - it_n) \in \overline{\mathcal{U}}_n$. Hence

$$y_0 = G_n^{\circ\sigma_1}(Z_n' - it_n) \text{ and } \ \sigma_1 = p$$

So

$$F_{n+1}(Z_{n+1}) = s \circ L_n(G_n^{\circ\sigma_1}(Z_n' - it_n) - 1) - i\frac{6\delta}{\alpha_n} - a_{n+1} + \sigma_1$$

with $G_n^{\circ\sigma_1}(Z_n' - it_n) \in \overline{\mathcal{U}}_n$ and $\Im G_n^{\circ\sigma_1}(Z_n' - it_n) = \Im(y_0) \geq 4\delta$. Then $G_n^{\circ\sigma_1}(Z_n' - it_n) - 1 \in \mathcal{W}_n$. Since $\Im F_{n+1}(Z_{n+1}) \geq t_1 > 0$, it is easy to see that

$$\Im L_n(G_n^{\circ\sigma_1}(Z_n' - it_n) - 1) > \frac{6\delta}{\alpha_n}$$

Using again Proposition 3.4.1 and repeating the same idea as above, there exists a nonnegative integer σ_2 such that

$$L_n(G_n^{\circ\sigma_1}(Z_n' - it_n) - 1) = L_n(G_n^{\circ\sigma_1+\sigma_2}(Z_n' - it_n) - 1) + \sigma_2$$

where $G_n^{\circ\sigma_1+\sigma_2}(Z_n' - it_n) - 1 \in \overline{\mathcal{U}}_n$, whose imaginary part is at least 4δ. So we can write

$$F_{n+1}(Z_{n+1}) = s \circ L_n(G_n^{\circ\tau_1}(Z_n' - it_n) - 1) - i\frac{6\delta}{\alpha_n} - a_{n+1} + \rho_1$$

where $\tau_1 = \sigma_1 + \sigma_2$ and $\rho_1 = \sigma_1 - \sigma_2$. Repeating this argument k_{n+1} times, we have

$$F_{n+1}^{k_{n+1}}(Z_{n+1}) = s \circ L_n(G_n^{\circ\tau_1+\cdots+\tau_{k_{n+1}}}(Z_n' - it_n) - k_{n+1}) - i\frac{6\delta}{\alpha_n} - a_{n+1} + \rho_1 + \cdots + \rho_{k_{n+1}}$$

where $G_n^{\circ\tau_1+\cdots+\tau_{k_{n+1}}}(Z_n' - it_n) - k_{n+1} \in \overline{\mathcal{U}}_n$ and has imaginary part at least 4δ. By definition of G_n, one sees that

$$G_n^{\circ\tau_1+\cdots+\tau_{k_{n+1}}}(Z_n' - it_n) - k_{n+1} = F_n^{\circ\tau_1+\cdots+\tau_{k_{n+1}}}(Z_n') - k_{n+1} - it_n$$

So

$$\Im(F_n^{\circ\tau_1+\cdots+\tau_{k_{n+1}}}(Z_n') - k_{n+1} - it_n) > 0$$

which implies that

$$F_n^{\circ\tau_1+\cdots+\tau_{k_{n+1}}}(Z_n') - k_{n+1} \in H_n$$

Therefore

$$\tau_1 + \tau_2 + \cdots + \tau_{k_{n+1}} \leq k_n$$

Since $|F_n(Z) - Z - \alpha_n| \leq \delta\alpha_n < \frac{1}{10}\alpha_n$ on H_n, then $|F_n(Z) - Z| < \frac{11}{10}\alpha_n$. Now we have $|F_n^{\circ j+1}(Z_n' - k_{n+1}) - F_n^{\circ j}(Z_n' - k_{n+1})| < \frac{11}{10}\alpha_n$ for all $j \in \{1, 2, ..., l-1\}$ where $l = \tau_1 + \tau_2 + \cdots + \tau_{k_{n+1}}$. Adding all of these terms and applying the triangle inequality, we obtain

$$|F_n^{\circ l}(Z_n' - k_{n+1}) - (Z_n' - k_{n+1})| < \frac{11}{10}\alpha_n \cdot l \leq \frac{11}{10}\alpha_n \cdot k_n$$

Since $F_n^{\circ l}(Z_n' - k_{n+1}) - it_n = F_n^{\circ l}(Z_n') - k_{n+1} - it_n = G_n^{\circ l}(Z_n' - it_n) - k_{n+1} \in \overline{\mathcal{U}}_n$, we have $\Re(F_n^{\circ l}(Z_n' - k_{n+1})) \geq 0$. By definition of Z_n', $\Re(Z_n') < 0$. So we conclude that

$$k_{n+1} < \frac{11}{10}\alpha_n \cdot k_n$$

By Proposition 3.5.4(i) $\alpha_n\alpha_{n+1} \leq \frac{1}{2}$. This implies $k_{n+2} < \frac{121}{200} \cdot k_n$ whenever defined.

Finally, by definition $F_0(W) = F(W) - a_0$, so if $F^{\circ m}(Z) \notin \mathbb{H}$ for some $Z \in \mathbb{H}$ and some positive integer m, then k_0 is finite, whence by the analysis above the proposition follows. □

Proposition 3.5.8. *Assume we can choose a sequence of nonnegative real numbers $(t_n)_n$ so that the $n-th$ renormalization F_n verifies the fundamental inequalities given in* 3.3.1 *when $\Im(Z) \geq t_n$ and so that $\tau = \sum_{n=0}^{\infty} \beta_{n-1}t_n < +\infty$. Then*

$$F^{\circ m}(Z) \in \mathbb{H}$$

for all $m \geq 0$ and for all $Z \in \mathbb{H}_\tau$, where

$$\mathbb{H}_\tau = \{Z \in \mathbb{C} : \Im(Z) > \tau + 44\delta\}$$

Proof. Let $Z \in \mathbb{H}_\tau$ and assume Z_n is defined. We will prove that Z_{n+1} is also defined. By Proposition 3.5.5, we have :

$$d_0 \leq \beta_{n-1}d_n + \sum_{i=0}^{n-1} \beta_{i-1}(t_i + 11\delta)$$

So

$$\begin{aligned}
\beta_{n-1}d_n &\geq d_0 - \sum_{i=0}^{n-1} \beta_{i-1}(t_i + 11\delta) \\
&= d_0 - \sum_{i=0}^{n-1} \beta_{i-1}t_i - (1 + \beta_0 + \cdots + \beta_{n-2})(11\delta) \\
&\geq d_0 - (\tau - \sum_{i=n}^{\infty} \beta_{i-1}t_i) + (4 - (1 + \cdots + \beta_{n-1}) - 4 + \beta_{n-1})11\delta \\
&\geq (d_0 - 44\delta - \tau) + \sum_{i=n}^{\infty} \beta_{i-1}t_i + (4 - (1 + \cdots + \beta_{n-1}) + \beta_{n-1})11\delta
\end{aligned}$$

By Proposition 3.5.4, $4 - (1 + \cdots + \beta_{n-1}) > 0$ and since $d_0 = \Im(Z) > \tau + 44\delta$, we have

$$\beta_{n-1}d_n \geq \sum_{i=n}^{\infty} \beta_{i-1}t_i + \beta_{n-1}(11\delta)$$

So

$$\beta_{n-1}d_n \geq \beta_{n-1}t_n + \beta_{n-1}(11\delta)$$

Then

$$\Im(Z_n) = d_n \geq t_n + 11\delta > t_n + 4\delta$$

Thus, we can define Z_{n+1}. This implies that the sequence $(Z_n)_n$ is infinite, therefore the proposition follows from Proposition 3.5.7. $\square$

Theorem 3.5.9. *Brjuno Theorem by Yoccoz. Let* $f \in \mathbb{S}_\lambda$ *where* $\lambda = \exp(2\pi i\alpha)$ *and* $\alpha \in (0,1) \setminus \mathbb{Q}$. *If* $\displaystyle\sum_{n=0}^{\infty} \beta_{n-1} \log(\frac{1}{\alpha_n}) < +\infty$, *then f is linearizable.*

Proof. Fix $\delta \in (0, 1/10)$ and let $F \in \mathfrak{F}(\alpha)$ such that $f(E(Z)) = E(F(Z))$ for all $Z \in \mathbb{H}$. By Theorem F.3 (see Appendix F), there exists a universal constant $C > 0$, just depending on δ, such that

$$|F_n(Z) - Z - \alpha_n| \leq \delta\alpha_n \quad \text{and} \quad |F_n'(Z) - 1| \leq \delta$$

for all $Z \in \mathbb{H}$ with $\Im(Z) \geq \dfrac{1}{2\pi}\log\dfrac{1}{\alpha_n} + C$, where F_n is the n-th renormalization of F. Putting $t_n = \dfrac{1}{2\pi}\log\dfrac{1}{\alpha_n} + C$, it is easy to see that $\tau = \displaystyle\sum_{n=0}^{\infty} \beta_{n-1}t_n < +\infty$. So by the last proposition $F^{\circ m}(Z) \in \mathbb{H}$ for all $m \geq 0$ and for all $Z \in \mathbb{H}_\tau$. Then

$$f^{\circ m}(E(Z)) = E(F^{\circ m}(Z)) \in \mathbb{H}$$

for all $Z \in \mathbb{H}_\tau$ and all $m \geq 0$. Hence

$$f^{\circ m}(z) \in \mathbb{D}$$

for all $z \in E(\mathbb{H}_\tau) = \Delta(0, \exp(-2\pi(\tau + 44\delta)))$. This means that 0 is stable. Therefore, by Theorem 3.2.5 we conclude that f is linearizable. $\square$

Remark 3.5.10. *It is not difficult to see that* $\displaystyle\sum_{n=0}^{\infty} \beta_{n-1}\log(\frac{1}{\alpha_n})$ *converges if only of the series* $\displaystyle\sum_{n=0}^{\infty} \frac{q_{n+1}}{q_n}$ *converges. The idea of the proof of this fact is given in [30] pg. 13.*

Chapter 4

Linearization of Quadratic Polynomials

The main goal in this chapter is to prove that the stability of the origin for the polynomial $P_\lambda(z) = \lambda(z - z^2)$ is enough to guarantee the stability of the origin for any germ $f \in \mathbb{S}_\lambda$ when $\lambda = E(\alpha)$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. The point of view in this chapter has been mainly influenced by Yoccoz[30] and Lehto[20].

4.1 Douady - Hubbard Theorem

In order to get our main result, in this first section we are focused on prove that there exist $a_0 \in (0, 1]$ such that for all $a \in (0, a_0]$ and $b \in \mathbb{C}$ with $|b| = 10$ the map $a^{-1}f(az) + bz^2$ is quasiconformally conjugate to the quadratic polynomial $P(z) = \lambda z + bz^2$, for any $f \in \mathbb{S}_\lambda$ where $\lambda = E(\alpha)$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Definition 4.1.1. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Let $\lambda = E(\alpha)$ and let f be an element of $\mathbb{S}_\lambda$. This means f is a holomorphic and injective map which is defined on the open unit disk $\mathbb{D}$ such that $f(0) = 0$ and $f'(0) = \lambda$.*

Definition 4.1.2. *Let f be an element of $\mathbb{S}_\lambda$ as above. For each $b \in \mathbb{C}$, we denote by f_b the holomorphic map on $\mathbb{D}$ defined by :*

$$f_b(z) = f(z) + bz^2.$$

Proposition 4.1.3. *Let H_b be the formal linearization of f_b. Then H_b has the form*

$$H_b(z) = z(1 + \sum_{n \geq 1} P_n(b) z^n)$$

where, for each n, the function $P_n(b)$ is a polynomial of degree n in b .

Proof. This comes from a straightforward inspection of the coefficients of H_b following the recursive relation given in 2.1.1. □

Definition 4.1.4. *A triple (U, U', F) is called polynomial-like of degree 2 if U, U' are open simply connected domains of $\mathbb{C}$, neither equal to $\mathbb{C}$, U' is relatively compact in U and $F : U' \to U$ is holomorphic and proper of degree 2 .*

Definition 4.1.5. $\mathcal{W} = \{z : |z| < \frac{13}{36}\}$, $\mathcal{W}_b = \{z : |z| \le 1/3\ ,\ f_b(z) \in \mathcal{W}\}$

Lemma 4.1.6. *For $|b| \ge 10$, $(\mathcal{W}, \mathcal{W}_b, f_b)$ is a polynomial-like of degree 2 .*

Proof. Since f is univalent on $\mathbb{D}$, we have :

$$|f(z)| \le \frac{|z|}{(1-|z|)^2} \quad \text{for all} \quad z \in \mathbb{D}$$

In particular, $|f(z)| \le 3/4$ for $|z| = 1/3$. On the other hand, for $|b| \ge 10$:

$$|bz^2| \ge \frac{10}{9} = \frac{13}{36} + \frac{3}{4} \quad \text{for all} \quad |z| = \frac{1}{3}$$

Given any $y \in \mathcal{W}$, the following inequality holds for all $|z| = 1/3$:

$$|f_b(z) - (y + bz^2)| = |f(z) - y| \le |f(z)| + |y| < \frac{3}{4} + \frac{13}{36} = \frac{10}{9} \le |bz^2|$$

Thus, for all $|z| = 1/3$ we have $|(f_b(z) - y) - bz^2| < |bz^2|$. Then by Rouché's theorem there are exactly two solutions for the equation $f_b(z) = y$ inside the set $\{\ z : |z| < 1/3\}$. This proves that $f_b : \Delta(0, 1/3) \bigcap f_b^{-1}(\mathcal{W}) \to \mathcal{W}$ is surjective. Furthermore, since f is defined in $\mathbb{D}$, the last inequality shows that $\{z : |z| = 1/3\} \bigcap f_b^{-1}(\mathcal{W})$ is empty. Hence

$$\mathcal{W}_b = \Delta[0, 1/3] \bigcap f_b^{-1}(\mathcal{W}) = \Delta(0, 1/3) \bigcap f_b^{-1}(\mathcal{W})$$

Now let K be a compact subset of $\mathcal{W}$ and let $\{z_n\}$ be a sequence of elements in $f_b^{-1}(\mathrm{K})$. Without loss of generality, since K is compact we may assume that $\{f_b(z_n)\}$ converges to a point $y \in \mathrm{K}$. But since $f_b^{-1}(\mathrm{K}) \subset \mathcal{W}_b$, $\{z_n\}$ is a bounded sequence, so there is a subsequence $\{z_{n_j}\}$ converging to a point z with $|z| \le 1/3$; since f_b is defined in the whole unit disk by continuity $\{f_b(z_{n_j})\}$ converges to $f_b(z)$ which implies that $f_b(z) = y$. But by the last inequality above $|z|$ must be less than 1/3, thus $z \in\ f_b^{-1}(\mathrm{K})$. Therefore we conclude that $f_b : \mathcal{W}_b \to \mathcal{W}$ is a proper map of degree 2. Since f_b is proper, it maps the boundary of $\mathcal{W}_b$ into the boundary of $\mathcal{W}$. This implies that the image under f_b of each connected component of $\mathcal{W}_b$ is the set $\mathcal{W}$. In consequence, since the degree of f_b is 2, if we assume that $\mathcal{W}_b$ is not connected then the connected component containing zero will be holomorphically equivalent to the set $\mathcal{W}$. Since the connected component $\mathcal{C}_0$ containing zero is contained in $\mathcal{W}$ and $|\lambda| = 1$, by Schwarz lemma $f_b(z) = \lambda z$ for all $z \in \mathcal{C}_0$, which implies that $\mathcal{W} = \mathcal{C}_0 \subset \mathcal{W}_b \subset \mathcal{W}$. But this is a contradiction because we assumed that $\mathcal{W}_b$ was not connected. Therefore, $\mathcal{W}_b$ must be connected. □

We denote by Γ the circle $\{\ b \in \mathbb{C} : |b| = 10\}$. For $a \in (0,1]$, $b \in \Gamma$, and $f \in \mathbb{S}_\lambda$, we set

$$f_{a,b} = a^{-1} f(az) + bz^2 \tag{4.1.1}$$

The Straightening Theorem for polynomial-like maps (see [12]) guaranties that, for any $b \in \Gamma$, the map f_b is conjugated on $\mathcal{W}_b$ to a polynomial of degree 2 by a quasiconformal homeomorphism which is defined on the whole complex plane. However, adapting the proof of the Straightening Theorem to the family $\{\widetilde{f}_{a,b} : f \in \mathbb{S}_\lambda,\ a \in (0,1],\ b \in \Gamma\}$ we can show that there exist an interval $(0, a_0]$ such that $\widetilde{f}_{a,b}$ is linearizable for all $a \in (0, a_0]$. Our goal in this section is to provide the complete proof of this version of the Straightening Theorem following the sketch given by Yoccoz[30], pg. 61.

We start by choosing a $\mathcal{C}^\infty$ function η defined on the whole real line $\mathbb{R}$ such that $\eta(\mathbb{R}) = [0,1]$, η is 1 on $(-\infty, 1/3]$ and η is 0 on $[13/36, +\infty)$. Now we define, for $a \in (0,1]$, $b \in \Gamma$, and $f \in \mathbb{S}_\lambda$, a mapping $\widetilde{f}$ by the formula

$$\widetilde{f}_{a,b}(z) = \eta(|z|) f_{a,b}(z) + (1 - \eta(|z|))(\lambda z + bz^2).$$

Clearly these mappings are well defined over the whole complex plane $\mathbb{C}$. Furthermore, each is exactly $f_{a,b}$ on the disk $\{z : |z| \le 1/3\}$, is $\lambda z + bz^2$ outside the disk $\{z : |z| < 13/36\}$ and it is also a $\mathcal{C}^\infty$ function. This means

$$\widetilde{f}_{a,b}(z) = \begin{cases} f_{a,b}(z) = a^{-1} f(az) + bz^2 & \text{on } \ \Delta[0, 1/3], \\ \eta(|z|) f_{a,b}(z) + (1 - \eta(|z|))(\lambda z + bz^2) & \text{on } \ \{z, 1/3 \le |z| \le 13/36\}, \\ \lambda z + bz^2 & \text{on } \ \mathbb{C} \setminus \Delta(0, 13/36) \end{cases} \tag{4.1.2}$$

Lemma 4.1.7. *The net of mappings $f_{a,b}$ converges to the polynomial $P(z) = \lambda z + bz^2$ as a tends to zero, uniformly in $b \in \Gamma$ and $f \in \mathbb{S}_\lambda$ on every compact subset of the open unit disk $\mathbb{D}$. In particular, it holds over the set $\{z : 1/3 \le |z| \le 13/36\}$.*

Proof. Let K be a compact subset of the open unit disk $\mathbb{D}$. Then there exists $r \equiv r(K)$ such that $r \in (0,1)$ and $|z| \le r$ for all $z \in K$.

Let $f \in \mathbb{S}_\lambda$ with power series $\sum_{n=1}^{\infty} a_n z^n$. Then

$$|f_{a,b}(z) - (\lambda z + bz^2)| = |a^{-1} f(az) + bz^2 - \lambda z - bz^2|$$

So

$$|f_{a,b}(z) - (\lambda z + bz^2)| = a^{-1} |a_2 (az)^2 + a_3 (az)^3 + a_4 (az)^4 + \dots|$$

By D'Branges theorem ([5], [9]),

$$|f_{a,b}(z) - (\lambda z + bz^2)| \le a^{-1} (2|az|^2 + 3|az|^3 + 4|az|^4 + \dots)$$

So
$$|f_{a,b}(z) - (\lambda z + bz^2)| \leq a^{-1}(\frac{|az|}{(1-|az|)^2} - |az|)$$
Then
$$|f_{a,b}(z) - (\lambda z + bz^2)| \leq (\frac{|z|}{(1-|az|)^2} - |z|)$$
Hence
$$|f_{a,b}(z) - (\lambda z + bz^2)| \leq r(\frac{1}{(1-ar)^2} - 1) \text{ for all } z \in K$$
Finally, taking the limit as a tends to zero, the lemma follows. □

Corollary 4.1.8. *The net of mappings* $\widetilde{f}_{a,b}$ *converges to the polynomial* $P(z) = \lambda z + bz^2$ *as* a *tends to zero, uniformly in* $b \in \Gamma$ *and* $f \in \mathbb{S}_\lambda$ *on every compact subset of the complex plane* $\mathbb{C}$.

Proof. This follows from the equation 4.1.2 and the lemma above. □

Corollary 4.1.9. *The net of mappings* $\frac{\partial}{\partial \bar{z}}\widetilde{f}_{a,b}$ *converges to the map zero as a tends to zero, uniformly in* $b \in \Gamma$ *and* $f \in \mathbb{S}_\lambda$ *on every compact subset of the complex plane* $\mathbb{C}$.

Proof. By the equality 4.1.2 and the corollary above, the assertion holds on every compact subset of the complement of the set $\{z : 1/3 \leq |z| \leq 13/36\}$. Therefore, we just have to analyze over $\{z : 1/3 \leq |z| \leq 13/36\}$. In fact, by an elementary computation
$$\frac{\partial}{\partial \bar{z}}\widetilde{f}_{a,b}(z) = \eta'(|z|) \cdot \frac{z}{2|z|} \cdot f_{a,b}(z) - \eta'(|z|) \cdot \frac{z}{2|z|} \cdot (\lambda z + bz^2)$$
Since η' is continuous on the compact interval $[1/3, 13/36]$ there must be a constant $C > 0$ such that
$$|\frac{\partial}{\partial \bar{z}}\widetilde{f}_{a,b}(z)| \leq C \cdot |f_{a,b}(z) - (\lambda z + bz^2)|$$
Therefore, the corollary follows from Lemma 4.1.7. □

Corollary 4.1.10. *The net of mappings* $\frac{\partial}{\partial z}\widetilde{f}_{a,b}$ *converges to the polynomial* $Q(z) = \lambda + 2bz$ *as* a *tends to zero, uniformly in* $b \in \Gamma$ *and* $f \in \mathbb{S}_\lambda$ *on every compact subset of the complex plane.*

Proof. As in Corollary 4.1.9, it is enough to analyze over the closed annulus $\{z : 1/3 \leq |z| \leq 13/36\}$. In fact, by an elementary computation we have
$$\frac{\partial}{\partial z}\widetilde{f}_{a,b}(z) - Q(z) = \eta'(|z|) \cdot \frac{z}{2|z|} \cdot (f_{a,b}(z) - (\lambda z + bz^2)) \ + \ \eta(|z|) \cdot (\frac{\partial}{\partial z} f_{a,b}(z) - Q(z))$$

Consider $C_1 > \sup\{\eta(r) : 1/3 \leq r \leq 13/36\} \geq 0$. Since η' is continuous on the compact interval $[1/3, 13/36]$ we can guarantee the existence of C_1; and also consider the positive constant C given in the proof of Corollary 4.1.9. Then

$$|\frac{\partial}{\partial z}\widetilde{f}_{a,b}(z) - Q(z)| \leq C \cdot |f_{a,b}(z) - (\lambda z + bz^2)| + C_1 \cdot |\frac{\partial}{\partial z}f_{a,b}(z) - Q(z)|$$

Hence, we only have to prove that $\frac{\partial}{\partial z}f_{a,b}(z)$ converges to $Q(z)$ uniformly on b and f. By definition of $f_{a,b}$, it is not difficult to see that

$$|\frac{\partial}{\partial z}f_{a,b}(z) - Q(z)| = |f'(az) - \lambda|$$

Now, if we set $f(z) = \sum_{n=1}^{\infty} a_n z^n$ with $a_1 = \lambda$, by D'Branges theorem

$$|f'(az) - \lambda| = |\sum_{n=2}^{\infty} na_n(az)^{n-1}| \leq \sum_{n=2}^{\infty} n^2|az|^{n-1}$$

Since $|z| \leq 13/36$, we have

$$|f'(az) - \lambda| \leq \frac{1 + \frac{13a}{36}}{(1 - \frac{13a}{36})^3} - 1$$

for any $f \in \mathbb{S}_\lambda$. Therefore, the corollary follows. □

Lemma 4.1.11. *There exist a_0 and a continuous map $k : [0, a_0] \to [0, 1)$ with $k(0) = 0$, such that for all $f \in \mathbb{S}_\lambda$, $b \in \Gamma$ and $a \in (0, a_0]$, the map $\widetilde{f}_{a,b}$ is a branched covering of degree 2 of $\mathbb{C}$ satisfying*

$$|\frac{\partial}{\partial \bar{z}}\widetilde{f}_{a,b}(z)| \leq k(a)|\frac{\partial}{\partial z}\widetilde{f}_{a,b}(z)|$$

for all $1/3 \leq |z| \leq 13/36$.

Proof. The fact that the map $\widetilde{f}_{a,b}$ is a branched covering of degree 2 comes from the equation 4.1.1, because $a^{-1}f(az)$ is also an element of $\mathbb{S}_\lambda$ and we already proved that for any $|b| \geq 10$ and any $F \in \mathbb{S}_\lambda$, $(\mathcal{W}, \mathcal{W}_b, F_b)$ is polynomial-like of degree 2 (see Lemma 4.1.6). To get the inequality in the assertion, we set $L = \{z : 1/3 \leq |z| \leq 13/36\}$, $P(z) = \lambda z + bz^2$ and $\|f_{a,b} - P\| = \max\{|f_{a,b}(z) - P(z)| : z \in L\}$. The existence of the map k comes from Corollaries 4.1.9 and 4.1.10 as follows:
Given $\epsilon > 0$ there exists $a_0 \in (0, 1]$ such that, if $a \in (0, a_0]$ then

$$|\frac{\partial}{\partial z}\widetilde{f}_{a,b}(z) - (\lambda + 2bz)| < \epsilon$$

and

$$|\frac{\partial}{\partial \bar{z}}\widetilde{f}_{a,b}(z)| \leq C\|f_{a,b} - P\| < \epsilon$$

for all $f \in \mathbb{S}_\lambda$ and all $b \in \Gamma$ and for all $z \in L$. Note that the constant C comes from the proof in Corollary 4.1.9. By the minimum modulus principle

$$-\epsilon + \frac{17}{3} \leq -\epsilon + |\lambda + 2bz| < |\frac{\partial}{\partial z}\widetilde{f}_{a,b}(z)|$$

So, for very small ϵ we have

$$\frac{1}{|\frac{\partial}{\partial z}\widetilde{f}_{a,b}(z)|} < \frac{1}{\frac{17}{3} - \epsilon}$$

Then

$$\frac{|\frac{\partial}{\partial \bar{z}}\widetilde{f}_{a,b}(z)|}{|\frac{\partial}{\partial z}\widetilde{f}_{a,b}(z)|} \leq \frac{C}{\frac{17}{3} - \epsilon} \cdot \|f_{a,b} - P\| < \frac{\epsilon}{\frac{17}{3} - \epsilon}$$

for all $f \in \mathbb{S}_\lambda$ and for all $b \in \Gamma$. Therefore if we consider $0 < \epsilon < 17/6$ (fixed) we have

$$\frac{C}{\frac{17}{3} - \epsilon} \cdot \|f_{a,b} - P\| < 1$$

Then, we can define

$$k(a) = \begin{cases} 0 & \text{for } \ a = 0 \\ \frac{C}{\frac{17}{3}-\epsilon} \cdot \sup\{\|f_{a,b} - P\| : f \in \mathbb{S}_\lambda, b \in \Gamma\} & \text{for } \ a \in (0, a_0] \end{cases} \tag{4.1.3}$$

By Lemma 4.1.7, this map is continuous at zero. The proof of the continuity of $k(a)$ on $(0, a_0]$ is as follows. First of all, note that

$$|\ \|f_{a,b} - P\| - \|f_{s,b} - P\|\ | \leq \|f_{a,b} - f_{s,b}\|$$

where $\|f_{a,b} - f_{s,b}\| = \sup\{|f_{a,b}(z) - f_{s,b}(z)| : z \in L, b \in \Gamma\}$. Now we prove that $\|f_{a,b} - f_{s,b}\|$ goes to zero when s is approaching a.

In fact, assume $f(z) = \sum_{i=1}^{\infty} a_i z^i$. Then

$$|f_{a,b}(z) - f_{s,b}(z)| = |\frac{1}{a}\sum_{i=1}^{\infty} a_i(az)^i - \frac{1}{s}\sum_{i=1}^{\infty} a_i(sz)^i| = |\sum_{i=2}^{\infty} a_i((a)^{i-1} - (s)^{i-1})z^i|$$

Hence

$$|f_{a,b}(z) - f_{s,b}(z)| = |a - s||a_2 z^2 + a_3(a + s)z^3 + a_4(a^2 + as + s^2)z^4 + \ldots|$$

By D'Branges theorem and since $a, s \in (0, a_0] \subset (0, 1]$, we get

$$|f_{a,b}(z) - f_{s,b}(z)| \leq |a - s|(2 \cdot 1|z|^2 + 3 \cdot 2|z|^3 + 4 \cdot 3|z|^4 + 5 \cdot 4|z|^5 + \ldots\ldots)$$

Then
$$|f_{a,b}(z) - f_{s,b}(z)| \leq |a-s|(2^2|z|^2 + 3^2|z|^3 + 4^2|z|^4 + 5^2|z|^5 + ...)$$
On the other hand, it is not difficult to check that
$$\sum_{i=1}^{\infty} i^2 t^i = \frac{(1+t)t}{(1-t)^3}$$
So
$$\sum_{i=2}^{\infty} i^2 t^i = (\frac{(1+t)}{(1-t)^3} - 1)t$$
Therefore
$$|f_{a,b}(z) - f_{s,b}(z)| \leq |a-s| \cdot (\frac{1+|z|}{(1-|z|)^3} - 1)|z|$$
Since $1/3 \leq |z| \leq 13/36$, there must be a constant $M > 0$ such that
$$|f_{a,b}(z) - f_{s,b}(z)| \leq |a-s| \cdot M$$
for all $f \in \mathbb{S}_\lambda$, $b \in \Gamma$ and $z \in L$. So
$$|\ \|f_{a,b} - P\| - \|f_{s,b} - P\|\ | \leq \|f_{a,b} - f_{s,b}\| \leq M \cdot |a-s|$$
for all $f \in \mathbb{S}_\lambda$, and $b \in \Gamma$. Then
$$\frac{C}{\frac{17}{3} - \epsilon} \cdot |k(a) - k(s)| \leq M \cdot |a-s|$$
for any $a, s \in (0, a_0]$. Therefore k is continuous on $[0, a_0]$. □

Remark 4.1.12. *For each open set* U $\subset \mathbb{C}$*, we identify the Beltrami forms on* U *as the functions* $\mu \in L^\infty$ *with norm* $\|\mu\|_{L^\infty} < 1$.

Definition 4.1.13. *If* $f : \mathrm{U} \to \mathrm{V}$ *is a* $\mathcal{C}^1$ *mapping, and* μ *is a Beltrami form on* V*, the form* $f^*\mu$ *is defined by :*
$$(f^*\mu)(z) = \frac{\overline{\frac{\partial}{\partial z} f(z)} \mu(f(z)) + \frac{\partial}{\partial \bar{z}} f(z)}{\overline{\frac{\partial}{\partial \bar{z}} f(z)} \mu(f(z)) + \frac{\partial}{\partial z} f(z)}$$

Proposition 4.1.14. *Let* $a \in (0, a_0]$*. There exists a unique Beltrami form* $\mu = \mu_{f,a,b}$ *on* $\mathbb{C}$ *invariant under* $\widetilde{f}_{a,b}$ *, null on* $\mathbb{C} \setminus \mathcal{W}$ *and null on the Julia set* $\bigcap_{n \geq 0} f_{a,b}^{-n}(\mathcal{W}_b)$ *of* $f_{a,b}$*. Furthermore,* $\|\mu\|_{L^\infty} \leq k(a)$ *holds.*

Proof. Let ν the Beltrami form defined by

$$\nu(z) = \begin{cases} \dfrac{\frac{\partial}{\partial \bar{z}} \widetilde{f}_{a,b}(z)}{\frac{\partial}{\partial z} \widetilde{f}_{a,b}(z)} & , \ \{z \in \mathbb{C} : 1/3 \leq |z| \leq 13/36\} \\ & \\ 0 & , \qquad \text{otherwise} \end{cases}$$

Now define the recursive sequence $\xi_n = \varphi^* \xi_{n-1}$ for all $n \geq 1$, where $\xi_0 = \nu$ and $\varphi(z) = \widetilde{f}_{a,b}(z)$. By elementary computations, $\mu = \lim_{n \to \infty} \xi_n$ verifies the conditions required, so the support of μ is contained in $\mathcal{W}$. By Lemma 4.1.6 $f_{a,b}$ is analytic on $\Delta(0, 1/3) \bigcap f_{a,b}^{-1}(\mathcal{W})$. Then applying Lemma 4.1.11 to the corresponding $\widetilde{f}_{a,b}$, the inequality $\|\mu\|_{L^\infty} \leq k(a)$ follows. □

By the Ahlfors-Bers Theorem, there exists a unique quasiconformal homeomorphism $\phi \equiv \phi_{f,a,b}$ satisfying the following properties :

(i) For almost all $z \in \mathbb{C}$: $\frac{\partial}{\partial \bar{z}} \phi(z) = \mu(z) \frac{\partial}{\partial z} \phi(z)$.

(ii) $\phi(0) = 0$.

(iii) The function $\phi(z) - z$ is bounded .

(iv) $\text{supp}(\mu) \subset \mathcal{W}$.

Lemma 4.1.15. *Douady-Hubbard Theorem (Yoccoz's Version). Assuming the same notation as above :*

$$\phi \circ \widetilde{f}_{a,b} \circ \phi^{-1}(z) = \lambda z + b z^2 \ .$$

Proof. The map $\phi \circ \widetilde{f}_{a,b} \circ \phi^{-1}$ is quasi-regular and preserves the null Beltrami form of $\mathbb{C}$. Hence, this is a holomorphic map. It is also a branched covering of degree 2 with 0 as fixed point, so

$$\phi \circ \widetilde{f}_{a,b} \circ \phi^{-1}(z) = \lambda' z + b' z^2$$

for some λ' and $b' \neq 0$. By the equation 4.1.2, for $|z| \geq 13/36$ we have

$$\phi(\lambda z + b z^2) = \lambda' \phi(z) + b' (\phi(z))^2$$

By item (iii) above, $\lim_{|z| \to \infty} \dfrac{\phi(z)}{z} = 1$, so

$$\lim_{|z| \to \infty} \frac{\phi(\lambda z + b z^2)}{\lambda z + b z^2} = \lambda' \lim_{|z| \to \infty} \frac{\phi(z)}{z} \frac{1}{\lambda + b z} + b' \lim_{|z| \to \infty} (\frac{\phi(z)}{z})^2 \frac{1}{\frac{\lambda}{z} + b}$$

Then

$$1 = \lambda' \cdot 1 \cdot 0 + b' \cdot 1 \cdot \frac{1}{b}$$

So

$$b = b'$$

On the other hand, for each $t \in (0, 1]$ define :

$$T = \phi(t) \text{ and } \phi_t(z) = \frac{1}{T}\phi(tz)$$

Clearly $\phi_t(0) = 0, \phi_t(1) = 1$ and $\phi_t(\infty) = \infty$. Hence, $\{\phi_t\}_{t\in(0,1]}$ is a normal family of quasiconformal homeomorphisms (see [21], pg. 70). We set

$$\widetilde{f}_t(z) = \frac{1}{t}\widetilde{f}_{a,b}(tz)$$

It is not difficult to check that $\phi_t^{-1}(w) = \frac{1}{t}\phi^{-1}(Tw)$, so

$$\phi_t \circ \widetilde{f}_t \circ \phi_t^{-1}(z) = \frac{1}{T}\phi(t(\widetilde{f}_t \circ \phi_t^{-1})(z))$$

and

$$\phi_t \circ \widetilde{f}_t \circ \phi_t^{-1}(z) = \frac{1}{T}\phi(\widetilde{f}_{a,b}(\phi^{-1}(Tz)))$$

But we already know that $\phi \circ \widetilde{f}_{a,b} \circ \phi^{-1}(z) = \lambda' z + bz^2$. Then

$$\phi_t \circ \widetilde{f}_t \circ \phi_t^{-1}(z) = \frac{1}{T}(\lambda' Tz + b(Tz)^2) = \lambda' z + bTz^2$$

Since $\{\phi_t\}_{t\in(0,1]}$ is a normal family, let ψ be the limit of some sequence $\{\phi_{t_n}\}_{n\in\mathbb{Z}^+}$ with $\lim_{n\to\infty} t_n = 0$. So ψ is also a quasiconformal homeomorphism ([21], pg. 73-76), $\lim_{n\to\infty} \phi(t_n) = 0$ and $\lim_{n\to\infty} \widetilde{f}_{t_n}(z) = \lambda z$. Thus

$$\lim_{n\to\infty} \phi_{t_n} \circ \widetilde{f}_{t_n} \circ \phi_{t_n}^{-1}(z) = \lim_{n\to\infty} \lambda' z + b\phi(t_n)z^2 = \lambda' z$$

Then

$$\psi(\lambda\psi^{-1}(z)) = \lambda' z$$

so

$$\psi(\lambda z) = \lambda' \psi(z)$$

From this equality one sees that $\psi((\lambda)^n) = (\lambda')^n$ for all $n \in \mathbb{Z}^+$. Since ψ is a homeomorphism, λ' has unit modulus and is not a root of unity. Furthermore $\psi : \mathrm{S}^1 \to \mathrm{S}^1$ is an orientation-preserving homeomorphism, because $\{\lambda^n : n \in \mathbb{Z}^+\}$ is dense in S^1. Thus, $F, \widetilde{F} : \mathrm{S}^1 \to \mathrm{S}^1$ given by $F(z) = \lambda z$ and $\widetilde{F}(z) = \lambda' z$ are conjugate by the orientation-preserving homeomorphism ψ. So, by the theorem on the topological invariance of the rotation number ([7], pg. 47-48) we get $\lambda = \lambda'$. □

Remark 4.1.16.
Recall that each $\phi \equiv \phi_{f,a,b}$ is $K(a)$-quasiconformal homeomorphism and satisfies:

- *(i) For almost all $z \in \mathbb{C}$: $\frac{\partial}{\partial \bar{z}}\phi(z) = \mu(z)\frac{\partial}{\partial z}\phi(z)$.*
- *(ii) $\phi(0) = 0$.*
- *(iii) The function $\phi(z) - z$ is bounded .*
- *(iv) $supp(\mu) \subset \mathcal{W}$.*

From the last item each ϕ is univalent on $\mathbb{C} \setminus \mathcal{W} = \{z : |z| \geq 13/36\}$. Now, for each ϕ let g_ϕ be the map defined by

$$g_\phi(z) = \frac{1}{\phi(\frac{1}{z})}.$$

It is easy to see that each g_ϕ is well-defined and is univalent on the disk $\Delta(0, 36/13)$. Then each g_ϕ is univalent on $\mathbb{D}$. By the items above, one sees that $g_\phi(0) = 0$ and $g'_\phi(0) = 1$. So we may apply Theorem E.6 (see Appendix E) to each g_ϕ and obtain

$$\frac{|z|}{(1+|z|)^2} \leq |g_\phi(z)| \leq \frac{|z|}{(1-|z|)^2}.$$

In particular, for any z in the closed ring $\{z : 1/2 \leq |z| \leq 3/4\}$ we have

$$\frac{8}{49} \leq |g_\phi(z)| \leq 12.$$

Hence, for any z_3 in the closed ring $\{4/3 \leq |z_3| \leq 2\}$, the following inequality holds:

$$\frac{1}{12} \leq |\phi(z_3)| \leq \frac{49}{8}.$$

Now, for each $a \in (a, a_0]$, we set

$$K(a) = \frac{1+k(a)}{1+k(a)}$$

Here a_0 and $k(a)$ are the same as in Lemma 4.1.11.

Lemma 4.1.17. *Fix $a \in (0, a_0]$. There exists a constant $C(a)$, depending only on a, such that:*

$$|\phi_{f,a,b}(z_1) - \phi_{f,a,b}(z_2)| \leq C(a)|z_1 - z_2|^{1/K(a)}$$

for all $z_1, z_2 \in \mathbb{D}$.

Proof. First of all, fix $z_3 \in \mathbb{R}$ with $4/3 \leq z_3 \leq 2$. Now, for each $\phi_{f,a,b}$ define

$$\Lambda_{f,a,b}(w) = \frac{\phi_{f,a,b}(z_3 z)}{\phi_{f,a,b}(z_3)}.$$

By an elementary computation (see [7], pg. 17), the dilatations $\mu_\Lambda \equiv \mu_{\Lambda_{f,a,b}}$ $\mu_\phi \equiv \mu_{\phi_{f,a,b}}$ satisfy

$$\mu_\Lambda(w) = \mu_\phi(3w)$$

for almost all w. By Proposition 4.1.14 we conclude that $\|\mu_\Lambda\|_{L^\infty} \leq k(a)$. Therefore the maps $\Lambda_{f,a,b}(w)$ are $K(a)$-quasiconformal homeomorphisms for all $a \in (0, a_0]$, $b \in \mathbb{C}$ with $|b| = 10$. Moreover, the maps $\Lambda_{f,a,b}$ fix 0,1 and ∞. So $\{\Lambda_{f,a,b} : f \in \mathbb{S}_\lambda, b \in \Gamma\}$ is a normal family of $K(a)$-quasiconformal homeomorphisms (see[21], pg. 73). Since the equicontinuity, Holder continuity and normality of a family of K-quasiconformal mappings are equivalent concepts ([21], pg. 72) and the closed disk $\{w : |w| \leq \frac{3}{4}\}$ is compact, there must be a constant $\widetilde{c}(a)$ just depending on a such that :

$$|\Lambda_{f,a,b}(w_1) - \Lambda_{f,a,b}(w_2)| \leq \widetilde{c}(a)|w_1 - w_2|^{1/K(a)}$$

for all $w_1, w_2 \in \{w : |w| \leq \frac{3}{4}\}$. Then

$$|\phi_{f,a,b}(z_3 w_1) - \phi_{f,a,b}(z_3 w_2)| \leq \widetilde{c}(a) \cdot |\phi_{f,a,b}(z_3)| \cdot |w_1 - w_2|^{1/K(a)}$$

for all $w_1, w_2 \in \{w : |w| \leq \frac{3}{4}\}$. Since $4/3 \leq z_3 \leq 2$ we may apply the last inequality in Remark 4.1.16. In particular we have

$$|\phi_{f,a,b}(z_3 w_1) - \phi_{f,a,b}(z_3 w_2)| \leq \widetilde{c}(a) \cdot \frac{49}{8} \cdot |w_1 - w_2|^{1/K(a)}$$

for all $w_1, w_2 \in \{w : |w| \leq \dfrac{1}{z_3}\}$. By the elementary change of variable $z = z_3 w$ and using the inequality $4/3 \leq z_3$, the lemma follows. □

4.2 The Quadratic Polynomial Theorem

Definition 4.2.1. *For $\lambda \in \mathbb{C}$ with modulus 1 and not a root of unity, we denote by P_λ the quadratic polynomial*

$$P_\lambda(z) = \lambda(z - z^2)$$

Definition 4.2.2. *For $|\lambda| = 1$ not a root of unity we set H_λ to be the unique formal series that satisfies $H_\lambda(0) = 0, H'_\lambda(0) = 1$ and*

$$H_\lambda(\lambda z) = P_\lambda(H_\lambda(z))$$

Remark 4.2.3. *Assuming that the radius of convergence of H_λ is strictly positive, then the polynomial P_λ is linearizable. Thus it has a Siegel disk $\mathcal{U}_\lambda$ at 0 . Let $c = c(\mathcal{U}_\lambda, 0)$ be the conformal capacity of $\mathcal{U}_\lambda$ at 0. By Theorem E.4 (see Appendix E) $\Delta(0, c/4) \subset \mathcal{U}_\lambda$. Considering the map $g_b(z) = -\frac{b}{\lambda}z$, it is easy to check that $g_b(\lambda z + bz^2) = P_\lambda(g_b(z))$ i.e., P_λ is conjugated to the polynomial $\lambda z + bz^2$ for all $b \in \mathbb{C}^*$, so the polynomial $\lambda z + bz^2$ has as Siegel disk $\mathcal{D}_\lambda = g_b^{-1}(\mathcal{U}_\lambda)$. Therefore, we have $\Delta(0, \frac{c}{4|b|}) = g_b^{-1}(\Delta(0, c/4)) \subset \mathcal{D}_\lambda$. In particular $\Delta(0, \frac{c}{40}) \subset \mathcal{D}_\lambda$ for all b with $|b| = 10$. By Lemma 4.1.15, the map $\phi_{f,a,b}$ conjugates the polynomial $\lambda z + bz^2$ to the map $\widetilde{f}_{a,b}$ for all b with $|b| = 10$. So, since a quasiconformal conjugacy is in particular a topological conjugacy, the map $\widetilde{f}_{a,b}$ has as Siegel disk $\mathcal{D'}_\lambda = \phi_{f,a,b}^{-1}(\mathcal{D}_\lambda)$. Therefore the mappings $\widetilde{f}_{a,b}$ are linearizable for all b with $|b| = 10$.*

Lemma 4.2.4. *(i) The Siegel disk $\mathcal{D'}_\lambda$ is contained in $\mathcal{W}_b$.*

(ii) For all b with $|b| = 10$, the maps $f_{a,b}$ are linearizable and their Siegel disks are the respective $\mathcal{D'}_\lambda$.

(iii) The disk $\{z, |z| \leq [\frac{c}{40C(a)}]^{K(a)}\}$ is contained in $\mathcal{D'}_\lambda$.

Proof. Using the equation 4.1.2, it is not difficult to see that if $z \in \mathbb{C} \setminus \mathcal{W}_b$ then $\lim_{n\to\infty} \widetilde{f}^n_{a,b}(z) = \infty$. So item (i) follows. Here $\widetilde{f}^n_{a,b}$ is the n-th iteration of $\widetilde{f}_{a,b}$.

By the equation 4.1.2, $\widetilde{f}_{a,b} = f_{a,b}$ on $\Delta(0, 1/3)$; then item (ii) comes from item (i). Finally, let z such that $|z| \leq [\frac{c}{40C(a)}]^{K(a)}$. Then

$$C(a)|z - 0|^{1/K(a)} < \frac{c}{40}$$

By Lemma 4.1.17

$$|\phi_{f,a,b}(z) - 0| \leq C(a)|z - 0|^{1/K(a)} < \frac{c}{40}$$

Then $\phi_{f,a,b}(z) \in \Delta(0, \frac{c}{40})$. So, by the remark above $\phi_{f,a,b}(z) \in \mathcal{D}_\lambda$. Then $z \in \mathcal{D'}_\lambda = \phi_{f,a,b}^{-1}(\mathcal{D}_\lambda)$ and item (iii) follows. □

Theorem 4.2.5. *The Quadratic Polynomial Theorem* . *Let $f \in \mathbb{S}_\lambda$, $\lambda \in \mathbb{C}$ with modulus 1 and not a root of unity. Assume the radius of convergence of H_λ is a positive number. Then f is linearizable.*

Proof. By the previous remark and lemma, $f_{a,b}(z) = a^{-1}f(az) + bz^2$ is linearizable . Note that the conformal representation $h_{a,b} : \Delta(0, c(\mathcal{D'}_\lambda, 0)) \to \mathcal{D'}_\lambda$ linearizes $f_{a,b}$. Since the map $f_{a,b} \in \mathbb{S}_\lambda$, its formal linearization $H_{a,b}$ (see 4.1.3) is analytic on $\Delta(0, c(\mathcal{D'}_\lambda, 0))$. Thus the power series of $h_{a,b}$ and $H_{a,b}$ coincide.

Let $R_a = [\frac{c}{40C(a)}]^{K(a)}$. By the last lemma $\Delta(0, R_a)$ is strictly contained in $\mathcal{D}'_\lambda$. Then, by Theorem E.3 (see Appendix E), we have $R_a < c(\mathcal{D}'_\lambda, 0)$. Therefore the power series $H_{a,b}$ is well defined on the closed disk $\Delta[0, R_a]$. Recall that for each $b \in \mathbb{C}$, $H_{a,b}$ has the form

$$H_{a,b}(z) = z(1 + \sum_{n\geq 1} P_n(a,b) z^n)$$

where the function $P_n(a, b)$ is a polynomial of degree n in b. Since $H_{a,b}$ is analytic on $\Delta(0, R_a)$ and continuous on $\Delta[0, R_a]$ for all $|b| = 10$, its coefficients satisfy the following inequality

$$|P_n(a,b)| \leq \frac{\max\{|H_{a,b}(z)| : |z| \leq R_a\}}{R_a^{n+1}}$$

Thus by the maximum principle one has $|P_n(a, 0)| \leq M(a)(1/R_a)^n$ for all $n \in \mathbb{Z}^+$, for all $a \in (0, a_0]$, where $M(a) = \max\{|H_{a,b}(z)| : |z| \leq R_a\}$. Then $H_{a,0}$ is analytic which means that $f_{a,0}$ is linearizable for all $a \in (0, a_0]$ (recall that a_0 comes from Lemma 4.1.11) . Therefore f is linearizable. $\square$

Appendix A

Arithmetical Lemmas

Lemma A.1. *Let $(\delta_n)_{n\geq 0}$ be a sequence of positive real numbers such that $\delta_{n+1} \leq AB^n\delta_n^2$ for all n where $A > 1$ and $B > 1$. Then $AB^{n+1}\delta_n \leq (AB\delta_0)^{2^n}$ for all nonnegative integer n.*

Proof. The case $n = 0$ is obvious. Suppose is true for $n = l$ (inductive hypothesis). Since $\delta_{l+1} \leq AB^l\delta_l^2$, we have

$$AB^{l+2}\delta_{l+1} \leq A^2B^{2(l+1)}\delta_l^2 = (AB^{l+1}\delta_l)^2 \leq ((AB\delta_0)^{2^l})^2$$

Therefore $AB^{(l+1)+1}\delta_{l+1} \leq (AB\delta_0)^{2^{l+1}}$ □

Corollary A.2. *Using the same notation as above, if $AB\delta_0 < 1$ then $\delta_n < (AB\delta_0)^{n+1}$ for all $n \geq 0$.*

Proof. In fact, the case $n = 0$ follows from the hypothesis $A > 1$ and $B > 1$. Moreover, we have $1 < AB^{n+1}$ when $n \geq 0$. In particular

$$(AB\delta_0)^{n+1} < AB^{n+1}(AB\delta_0)^{n+1}$$

for all $n \geq 1$. By the last lemma, $AB^{n+1}\delta_n \leq (AB\delta_0)^{2^n}$ for all $n \geq 1$. Then $AB^{n+1}\delta_n \leq (AB\delta_0)^{2^n} \leq (AB\delta_0)^{n+1} < (AB^{n+1})(AB\delta_0)^{n+1}$. Therefore $\delta_n < (AB\delta_0)^{n+1}$ for all $n \geq 0$. □

Lemma A.3. *Let $k \in \mathbb{Z}^+$ and $c > 1$, and consider $A = 2c(20)^{k+2}$, $B = (30)^{k+2}$. Choose δ_0 with $0 < \delta_0 < \frac{1}{c} \cdot \frac{1}{10^{k+2}} \cdot \frac{1}{2^{k+2}} \cdot \frac{1}{AB}$. Define by induction the sequence $\delta_{n+1} = \frac{2c\delta_n^2}{\theta_n^{k+2}}$, where $\theta_n = \frac{1}{10} * \frac{1}{1+2^n}$ for all $n \geq 0$. Then*

(i) $\delta_{n+1} \leq AB^n\delta_n^2$

(ii) $\delta_n < (AB\delta_0)^{n+1}$

(iii) $\delta_n < \frac{1}{c}(\theta_n)^{k+2}$.

Proof of (i). By the choice of A, the case $n = 0$ follows. For the case $n \geq 1$, since $[10(1 + 2^n)]^{k+2} \leq (30^{k+2})^n$, one sees that $(\frac{1}{\theta_n})^{k+2} \leq B^n$. Then

$$\frac{2c\ {\delta_n}^2}{\theta_n^{k+2}} \leq 2cB^n \delta_n^2$$

But $2c < A$, so $\frac{2c\delta_n^2}{\theta_n^{k+2}} < AB^n\delta_n^2$. Therefore, $\delta_{n+1} \leq AB^n\delta_n^2$ for all $n \geq 0$. □

Proof of (ii). By the choice of δ_0, $AB\delta_0 < 1$. Furthermore, it is easy to see that $\delta_n > 0$ for all nonnegative integers n. So the claim follows from a direct application of the last corollary. □

Proof of (iii). By the choice of δ_0, $2^{k+2}AB\delta_0 < \dfrac{1}{c}\dfrac{1}{10^{k+2}} < 1$. Then

$$(AB\delta_0)^{n+1} < \frac{1}{c}\frac{1}{10^{k+2}}\left(\frac{1}{2^{k+2}}\right)^{n+1}$$

for all $n \geq 1$. Moreover, since $1 + 2^n \leq 2^{n+1}$ for all $n \geq 1$, one sees that

$$(AB\delta_0)^{n+1} < \frac{1}{c}\left(\frac{1}{10}\frac{1}{1+2^n}\right)^{k+2} = \frac{1}{c}(\theta_n)^{k+2}$$

for all $n \geq 1$. So, by (ii) we have $\delta_n < \dfrac{1}{c}(\theta_n)^{k+2}$ for all $n \geq 1$. Finally, for the case $n = 0$ it is enough to see that $\delta_0 < \dfrac{1}{c}(20)^{-(k+2)} = \dfrac{1}{c}(\theta_0)^{k+2}$. □

Appendix B

e is diophantine

Proposition B.1. *Let* $\theta \in \mathbb{R} \backslash \mathbb{Q}$ *and let* $\frac{p_n}{q_n}$ *be its sequence of continued fractions. Then* $\frac{1}{q_{n+1}+q_n} < |\theta q_n - p_n| < \frac{1}{q_{n+1}}$ *for all* $n \geq 1$*.*

Proposition B.2. *Let* $\theta \in \mathbb{R} \backslash \mathbb{Q}$ *and let* $\frac{p_n}{q_n}$ *be its sequence of continued fractions. Then* $q_n \geq (\frac{3}{2})^n$ *for all* $n \geq 5$*.*

Proposition B.3. *If* $|\theta - \dfrac{a}{b}| < \dfrac{1}{2b^2}$ *then* $\dfrac{a}{b}$ *is a continued fraction of* θ*.*

Proposition B.4. *The partial coefficients of the number e verify :*

- $a_0 = 2$.
- $a_n = 1$ *for* $n = 3k$ *and* $n = 3k - 2$ *with* $k \geq 1$.
- $a_n = 2k$ *for* $n = 3k - 1$ *, with* $k \geq 1$.

This means $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, \ldots]$

For details of the proofs of the facts above see [2], [17], [22]. For the rest of this section $\frac{p_n}{q_n}$ means the n-th continued fraction of e.

Lemma B.5. *Given* $\varepsilon > 0$ *and* $d > 0$*, the set* $\tilde{S} = \left\{ n \, : \, \left| e - \dfrac{p_n}{q_n} \right| < \dfrac{1}{d q_n^{2+\varepsilon}} \right\}$ *is finite.*

Proof. By definition $q_{n+1} = a_{n+1} q_n + q_{n-1}$, and since $q_{n-1} \leq q_n$ we have

$$q_{n+1} + q_n = q_n(a_{n+1} + 1 + \tfrac{q_{n-1}}{q_n}) \leq q_n(a_{n+1} + 2)$$

By Proposition B.1 and the last inequality, we have

$$\frac{1}{(a_{n+1}+2)} \leq \frac{q_n}{q_{n+1}+q_n} \leq \left| e - \frac{p_n}{q_n} \right| * {q_n}^2$$

By Proposition B.4, one sees that $a_{n+1} \leq n-2$ holds for all $n \geq 8$. Therefore

$$\frac{1}{nq_n^2} < \left| e - \frac{p_n}{q_n} \right|$$

for all $n \geq 8$. On the other hand, it is clear that the inequality $d^{1/n}(\frac{3}{2})^{\varepsilon} > n^{1/n}$ holds for n large. So there exists N_0 such that $d^{1/n}(\frac{3}{2})^{\varepsilon} > n^{1/n}$ for all $n > N_0$. Now, by Proposition B.2 we have $dq_n^{\varepsilon} > n$ for all $n > N_0$. Finally, if $n \in \tilde{S}$ and $n \geq 8$

$$\frac{1}{nq_n^2} < \left| e - \frac{p_n}{q_n} \right| < \frac{1}{dq_n^{2+\varepsilon}} \; .$$

Then $dq_n^{\varepsilon} < n$, and thus $n \in \{1, 2, \ldots, N_0\}$. Therefore $\tilde{S} \subset \{1, 2, \ldots, N_0\}$. □

Corollary B.6. *For any given $\varepsilon > 0$ and $d > 0$, the set*

$$S = \left\{ \frac{a}{b} \in \mathbb{Q} : \left| e - \frac{a}{b} \right| < \frac{1}{db^{2+\varepsilon}} \right\}$$

is finite.

Proof. Assume S is an infinite set. In this case S contains elements which have large denominators. Now consider the partition $\{S_1, S_2\}$ of S given by $S_1 = \{\frac{a}{b} \in S : db^{\varepsilon} < 2\}$ and $S_2 = \{\frac{a}{b} \in S : db^{\varepsilon} > 2\}$. It is clear that S_1 is finite and $S_2 \neq \emptyset$. For any $\frac{a}{b} \in S_2$, we have

$$\left| e - \frac{a}{b} \right| < \frac{1}{db^{2+\varepsilon}} < \frac{1}{2b^2}$$

So, by Proposition B.3 the fraction $\frac{a}{b}$ is a continued fraction of e. This means that $\frac{a}{b} = \frac{p_n}{q_n}$ for some n, hence $\left| e - \frac{a}{b} \right| = \left| e - \frac{p_n}{q_n} \right| < \frac{1}{dq_n^{2+\varepsilon}}$ and thus $n \in \tilde{S}$. Now, by the last lemma there must be a finite number of fractions $\dfrac{a}{b}$ for which $db^{\varepsilon} > 2$. This means that S_2 is finite. Therefore S is finite, which is a contradiction. □

Even though e is not an algebraic number, we have proved that (taking d=1) $S_e = \left\{ \frac{a}{b} \in \mathbb{Q} : \left| e - \frac{a}{b} \right| < \frac{1}{b^{2+\varepsilon}} \right\}$ is finite .

Recall that $\theta \in \mathbb{R} \backslash \mathbb{Q}$ satisfies a diophantine condition if there exist $c > 0$, $\mu > 0$ such that $\left| \theta - \frac{m}{n} \right| \geq \frac{c}{n^{2+\mu}}$ for all $\frac{m}{n} \in \mathbb{Q}$ and $n \in \mathbb{Z}^+$.

Theorem B.7. *e satisfies a diophantine condition.*

Proof. Fix $\varepsilon > 0$. Since S_e is finite, $S_e = \left\{\frac{a_1}{b_1}, \ldots, \frac{a_s}{b_s}\right\}$, so we can choose a large ν such that $\frac{1}{b_i^{2+\nu}} < |e - \frac{a_i}{b_i}|$ for $i = 1, 2, \ldots, s$. On the other hand, if $\frac{a}{b} \notin S_e$ we have $\frac{1}{b^{2+\varepsilon}} \leq \left|e - \frac{a}{b}\right|$. Therefore, taking $\mu = \max\{\varepsilon, \nu\}$

$$\left|e - \frac{a}{b}\right| \geq \frac{1}{b^{2+\mu}}$$

for all $\frac{a}{b} \in \mathbb{Q}$, $b \in \mathbb{Z}^+$. □

Appendix C

Davie's Lemma

Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Through this Appendix we denote by $\frac{p_n}{q_n}$ the n-th continued fraction of α and we consider $\|n\alpha\| = \min\{|n\alpha - m| \ : \ m \in \mathbb{Z}\}$.

Lemma C.1. *If $r \in \mathbb{Z}$ and $0 < r < q_k$ then $\|r\alpha\| \geq \frac{1}{2q_k}$.*

Proof. In fact, if $\|r\alpha\| < \frac{1}{2q_k}$, by definition there exists $p \in \mathbb{Z}$ such that $|r\alpha - p| = \|r\alpha\| < \frac{1}{2q_k} < \frac{1}{2r}$. This implies that

$$\left|\alpha - \frac{p}{r}\right| < \frac{1}{2r^2}$$

So, by Proposition B.3, $\frac{p}{r}$ is a continued fraction of α. This means that $p = p_j$ and $r = q_j$ for some j. Since $(q_n)_n$ is an increasing sequence and $q_j < q_k$, $j < k$ must hold. By Proposition B.1 we have

$$\frac{1}{2q_{j+1}} \leq \frac{1}{q_{j+1} + q_j} < |q_j\alpha - p_j| < \frac{1}{q_{j+1}}$$

So $\frac{1}{2q_{j+1}} < \|q_j\alpha\| = \|r\alpha\| < \frac{1}{2q_k}$. Then $q_j < q_k < q_{j+1}$ which is a contradiction. □

Corollary C.2. *Let k be a non-negative integer . If $n \in \mathbb{Z}^+$ and $\|n\alpha\| \leq \frac{1}{4q_k}$, then $n \geq q_k$ and either q_k divides n or $n \geq \frac{q_{k+1}}{4}$.*

Proof. Clearly if n satisfies the hypothesis of the lemma above we get an obvious contradiction, so $n \geq q_k$. Now, if q_k does not divide n and $n < \frac{q_{k+1}}{4}$ then $n = mq_k + r$ where $0 < r < q_k$. It is easy to check that $m < \frac{q_{k+1}}{4q_k}$ holds. By Proposition B.1 we know that $\|q_k\alpha\| < \frac{1}{q_{k+1}}$, so $\|mq_k\alpha\| < \frac{m}{q_{k+1}} < \frac{1}{4q_k}$. Now let $\tau_1, \tau_2 \in \mathbb{Z}$ satisfying $\|n\alpha\| = |(mq + r)\alpha - \tau_1|$ and $\|mq_k\alpha\| = |mq_k\alpha - \tau_2|$. Then $\|n\alpha\| = |(mq + r)\alpha - \tau_1| \geq |r\alpha - (\tau_1 - \tau_2)| - |mq\alpha - \tau_2|$, so we get $\|n\alpha\| \geq \|r\alpha\| - \|mq\alpha\| > \frac{1}{2q_k} - \frac{1}{4q_k} = \frac{1}{4q_k}$, which is a contradiction. □

Definition C.3. *Let q be a positive integer and E a real number with $E \geq q$. Suppose A is a set of non-negative integers such that $0 \in A$ and such that whenever j_1 , $j_2 \in A$ and q does not divide $j_1 - j_2$ we have $|j_1 - j_2| \geq E$.*

Definition C.4. *Let* A^* *be a set of non-negative integers, namely the set of* j *such that either* $j \in A$ *or* $j_1 < j < j_2$ *and* q *divides* $j - j_1$*, for some* j_1 , $j_2 \in A$ *with* $j_2 - j_1 < E$*. Note that if the second condition holds, then* q *also divides* $j_2 - j_1$ *which implies that* q *divides* $j_2 - j$*.*

Claim C.4.1. *If* $j, j' \in A^*$ *with* $q < j' - j < E$ *there must exist* $j'' \in A^*$ *with* $j < j'' < j'$*.*

Proof. We have 4 cases :

1. If $j, j' \in A$, since $q < j' - j < E$, it is enough to consider $j'' = j + q$.
2. If $j \in A$ and $j' \in A^* \backslash A$, there exist $j_1, j_2 \in A$ such that $j_1 < j' < j_2$ and q divides $j' - j_1$ with $j_2 - j_1 < E$. If $j_1 \leq j$, take $j'' = j' - q$; this j'' verifies the second part of the definition of A^*. Otherwise, if $j_1 > j$ consider $j'' = j_1$.
3. If $j \in A^* \backslash A$ and $j' \in A$, there exist $j_1, j_2 \in A$ such that $j_1 < j < j_2$ and q divides $j - j_1$ with $j_2 - j_1 < E$. If $j_2 \geq j'$, take $j'' = j + q$; this j'' verifies the second part of the definition of A^*. Otherwise, if $j_2 < j'$ consider $j'' = j_2$.
4. If $j \in A^* \backslash A$ and $j' \in A^* \backslash A$ there exist $j_1, j_2, \tau_1, \tau_2 \in A$ such that $j_1 < j < j_2$, $\tau_1 < j' < \tau_2$, q divides $j - j_1$ with $j_2 - j_1 < E$ and q divides $j' - \tau_1$ with $\tau_2 - \tau_1 < E$. If $j < j_2 < j'$ or $j < \tau_1 < j'$, it is enough to consider $j'' = j_2$ or $j'' = \tau_1$. Otherwise, since $\tau_1 < j' < \tau_2$, q divides $j' - \tau_1$, $\tau_2 - \tau_1 < E$ and $q < j' - j < E$, it is easy to see that $j'' = j' - q \in A^*$.

□

Claim C.4.2. *Let* $m_n = \max\{ j \ : \ 0 \leq j \leq n \ , \ j \in A^* \}$.

1. *If* $n \in A^*$ *and* $m_{n-1} + q \in A^*$*, then* $m_{n-1} = n - q$*.*
2. *If* $n \in A^*$ *and* $m_{n-1} + q \notin A^*$*, then* $n - m_{n-1} \geq E$ *.*

Proof of 1. The assumption $m_{n-1} + q \leq n - 1$ contradicts the definition of m_{n-1}, whence $m_{n-1} + q \geq n$. Assuming that $m_{n-1} + q > n = m_n$, we will have 4 subcases :

1. Case I : $m_{n-1} \in A$ and $n \in A$.
   Since $0 < n - m_{n-1} < q$, q does not divide $n - m_{n-1}$, so $n - m_{n-1} \geq E$. Then $q > E$, which is a contradiction .
2. Case II : $m_{n-1} \in A^* \backslash A$ and $n \in A$.
   By definition of A^*, there exist $j_1, j_2 \in A$ such that $j_1 < m_{n-1} < j_2$ and q divides $m_{n-1} - j_1$ with $j_2 - j_1 < E$. Since there is not any element of A^* between m_{n-1} and n, $j_2 \geq n$ must hold.

If q does not divide $n - j_1$, by definition of A we have $n - j_1 \geq E$, which implies $E > j_2 - j_1 \geq n - j_1 \geq E$. Hence q divides $n - j_1$ and since q divides $m_{n-1} - j_1$, q divides also $n - m_{n-1}$. But this is a contradiction because $0 < n - m_{n-1} < q$.

3. Case III : $m_{n-1} \in A$ and $n \in A^* \backslash A$.
By definition of A^*, there exist $j_1, j_2 \in A$ such that $j_1 < n < j_2$ and q divides $n - j_1$ with $j_2 - j_1 < E$. Since there is not any element of A^* between m_{n-1} and n, $j_1 \leq m_{n-1}$ must hold.
If q does not divide $j_2 - m_{n-1}$, by definition of A we have $j_2 - m_{n-1} \geq E$, which implies $E > j_2 - j_1 \geq j_2 - m_{n-1} \geq E$. Hence q divides $j_2 - m_{n-1}$ and since q divides $j_2 - n$, q also divides $n - m_{n-1}$. But this is a contradiction again.

4. Case IV : $m_{n-1} \in A^* \backslash A$ and $n \in A^* \backslash A$.
By definition, there exist $j_1, j_2, \tau_1, \tau_2 \in$ A such that $j_1 < n < j_2$, $\tau_1 < m_{n-1} < \tau_2$, q divides $n - j_1$ with $j_2 - j_1 < E$ and q divides $m_{n-1} - \tau_1$ with $\tau_2 - \tau_1 < E$. In particular $j_1 < \tau_2$. Since q divides $\tau_2 - m_{n-1}$ then $m_{n-1} + q \leq \tau_2$. By the same argument $j_1 \leq n - q$.
Suppose q does not divide $\tau_2 - j_1$. By definition of A, $\tau_2 - j_1 \geq E$ holds. This implies that $j_1 < \tau_1$ and $j_2 < \tau_2$. If q does not divide $\tau_1 - j_1$, then $E \geq j_2 - j_1 > \tau_1 - j_1 > E$, which is a contradiction . Hence, q must divide $\tau_1 - j_1$. Therefore q divides $n - m_{n-1}$ which is a contradiction again .
Suppose q divides $\tau_2 - j_1$. Since q divides $\tau_2 - m_{n-1}$ and $n - j_1$, we conclude that q divides $n - m_{n-1}$ which is impossible. That is why this case never happens.

Since all of these cases imply contradictions then our initial assumption $m_{n-1} + q > n$ fails to be true. Therefore $m_{n-1} + q = n$ holds. □

Proof of 2. By hypothesis $n \in A^*$ and $m_{n-1} + q \notin A^*$, hence $m_{n-1} \neq n - q$. We have 4 subcases :

1. Case I : $m_{n-1} \in A$ and $n \in A$.
If $0 < n - m_{n-1} < q$, q does not divide $n - m_{n-1}$, so $n - m_{n-1} \geq E$.
If $q < n - m_{n-1}$, then $n - m_{n-1} \geq E$. Otherwise by Claim C.4.1 there must exist $j'' \in A^*$ such that $m_{n-1} < j'' < n$. But this contradicts the definition of m_{n-1}.

2. Case II : $m_{n-1} \in A^* \backslash A$ and $n \in A$.
By definition of A^*, there exist $j_1, j_2 \in A$ such that $j_1 < m_{n-1} < j_2$ and q divides $m_{n-1} - j_1$ with $j_2 - j_1 < E$. The inequality $n - m_{n-1} < q$ does not hold. Otherwise it implies that q does not divides $n - m_{n-1}$, which implies that q does not divide $n - j_1$; but as in item *1*, Case II, this is a contradiction . Hence $q < n - m_{n-1}$. Following the same argument as in Case I above, $n - m_{n-1} \geq E$ must hold.

3. Case III : $m_{n-1} \in A$ and $n \in A^* \backslash A$.
By definition of A^*, there exist $j_1, j_2 \in A$ such that $j_1 < n < j_2$ and q divides

$n - j_1$ with $j_2 - j_1 < E$. If $n - m_{n-1} < q$, then q does not divide $n - m_{n-1}$, and hence q does not divide $m_{n-1} - j_1$. By definition of A, we have $m_{n-1} - j_1 \geq E$. Thus $E > j_2 - j_1 \geq j_2 - m_{n-1} \geq E$, hence $n - m_{n-1} \geq q$. Following the same argument as in Case I above, $n - m_{n-1} \geq E$ must hold.

4. Case IV : $m_{n-1} \in A^* \backslash A$ and $n \in A^* \backslash A$.
If $q < n - m_{n-1} < E$, then by Claim C.4.1 there must exist $j'' \in A^*$ such that $m_{n-1} < j'' < n$ But this contradicts the definition of m_{n-1}. So either $n - m_{n-1} \geq E$ or $n - m_{n-1} < q$. If $n < m_{n-1} + q$, repeating exactly the same steps as in item *1*, Case IV, we can prove that this case never happens. Therefore $n - m_{n-1} \geq E$ holds.

□

Definition C.5. *For each $n \geq 0$, $n \in \mathbb{Z}$ define*

$$l(n) = \max\left\{(1+\tau)\frac{n}{q} - 2, \frac{\tau m_n + n}{q} - 1\right\}$$

where $m_n = \max\{j : 0 \leq j \leq n, j \in A^*\}$ *and* $\tau = 2\dfrac{q}{E}$.
Now we define

$$h(n) = \begin{cases} q^{-1}(m_n + \tau n) - 1, & m_n + q \in A^* \\ l(n), & m_n + q \notin A^* \end{cases}$$

Claim C.5.1. *By Claim C.4.2 and Definition C.5, one sees that:*

1. $(1+\tau)\dfrac{n}{q} - 2 \leq h(n) \leq (1+\tau)\dfrac{n}{q} - 1$ *for all n.*
2. *If $n \in A^*$ and $n > 0$, then $h(n) \geq h(n-1) + 1$.*
3. *$h(n) \geq h(n-1)$ for all $n > 0$.*
4. *$h(n+q) \geq h(n) + 1$ for all n.*

Definition C.6. *For each k, let A_k be the set defined by*

$$A_k = \left\{n \geq 0 : \|n\alpha\| \leq \frac{1}{8q_k}\right\}$$

By Corollary C.2 these sets satisfy the definition C.3 where $q = q_k$, $E = \max\{q_k, \frac{q_{k+1}}{4}\}$, $\tau_k = 2\frac{q_k}{E} \leq \frac{8q_k}{q_{k+1}}$. So, for each k we can define h_k as in Definition C.5. Now, for each k we define $g_k(n) := \max\{h_k(n) \ , \left[\left[\frac{n}{q_k}\right]\right]\}$.

Remark C.7.

1. *By C.5.1,* $h_k(0) \in [-2,-1]$. *So we have* $g_k(0) = 0$.

2. $g_k(1) \geq \left[\left[\frac{1}{q}\right]\right] \geq 0$

Proposition C.8. *The function* g_k *is non-negative, and using Claim C.5.1 it is easy to check that* g_k *satisfies :*

1. $g_k(n) \leq (1+\tau)q^{-1}n$.

2. $g_k(n_1) + g_k(n_2) \leq g_k(n_1 + n_2)$.

3. *If* $n \in A$ *and* $n > 0$, *then* $g_k(n) \geq g_k(n-1) + 1$

Definition C.9. *For each* $n \geq 0$, $n \in \mathbb{Z}$, *define*

$$G(n) = 2n\log(2) + \sum_{k=0}^{k(n)} g_k(n)\log(2q_{k+1}) \ ,$$

where $k(n)$ *is defined by the condition*

$$q_{k(n)} \leq n < q_{k(n)+1}.$$

Since $q_n \in \mathbb{Z}^+$ and $1 = q_0 \leq q_1 < q_2 < q_3 \ldots$, $k(n)$ is well defined.

Proposition C.10. *The function* $G(n)$ *satisfies :*

1. $G(0) = 0$

2. $G(n-1) \leq G(n)$, *for all* $n \geq 1$.

3. $G(n_1) + G(n_2) \leq G(n_1 + n_2)$, *for all* $n_1, n_2 \in \mathbb{N}$.

4. *There exists a universal constant* $\gamma_3 > 0$ *such that for all* $n \in \mathbb{Z}_0^+$ *the following inequality holds*

$$G(n) \leq n\left(\sum_{i=0}^{k(n)} \frac{\log q_{i+1}}{q_i} + \gamma_3\right)$$

5. $-\log|\lambda^n - 1| \leq G(n) - G(n-1)$, *for all* $n \geq 1$, *where* $\lambda = \exp(2\pi i\alpha)$ *and* $\alpha = \lim_{n\to\infty} \frac{p_n}{q_n}$.

Proof of 1. This follows by definition. □

Proof of 2. Use the fact $k(n-1) \leq k(n)$ and the definition of $G(n)$. □

Proof of 3. $G(n_i) = 2n_i \log(2) + \sum_{k=0}^{k(n_i)} g_k(n) \log(2q_{k+1})$ where $i \in \{1,2\}$. Consider $r = k(n_1+n_2)$. By definition of $k(n)$ it is not difficult to see that $k(n_1) < r$ and $k(n_2) < r$. Then

$$G(n_1) + G(n_2) \leq 2(n_1+n_2)\log(2) + \sum_{k=0}^{r} (g_k(n_1) + g_k(n_2)) \log(2q_{k+1})$$

So, by Proposition C.8 the assertion follows. □

Proof of 4. By the first item of proposition C.8, $g_k(n) \leq (1+\tau_k)q_k^{-1}n$ where $\tau_k = 2\frac{q_k}{E_k} \leq 8\frac{q_k}{q_{k+1}}$. So

$$G(n) \leq 2n\log 2 + \sum_{k=0}^{k(n)} \left[\frac{n}{q_k} + \frac{8n}{q_{k+1}}\right] \log(2q_{k+1}).$$

Then

$$G(n) \leq n \left\{ 2\log 2 + \sum_{k=0}^{k(n)} \frac{\log(2) + \log(q_{k+1})}{q_k} + 8\sum_{k=0}^{k(n)} \frac{\log 2q_{k+1}}{q_{k+1}} \right\}$$

By induction, it is not difficult to check that $q_{k+2} \geq 2q_k$ for all $k \geq 0$. So

$$\sum_{k=0}^{\infty} \frac{1}{q_k} \leq 4$$

On the other hand, by induction is also easy to check that $q_k \geq \left(\frac{\sqrt{5}+1}{2}\right)^{k-1}$ for all $k \geq 1$.

Since $\log(t)/t$ is decreasing on $(e, +\infty)$ and $\left(\frac{\sqrt{5}+1}{2}\right)^3 > e$, we have

$$\frac{\log q_{k+1}}{q_{k+1}} \leq k \cdot \frac{\log(\frac{\sqrt{5}+1}{2})}{\left(\frac{\sqrt{5}+1}{2}\right)^k}$$

for all $k \geq 3$. Since q_1, q_2 and q_3 are positive integers we have $\log(q_i)/q_i < 1$ for $i \in \{1,2,3\}$. Now by elementary computations is easy to see that

$$\sum_{k=0}^{\infty} \frac{\log q_{k+1}}{q_{k+1}} < 3 + \log(\frac{\sqrt{5}+1}{2}) \cdot \sum_{k=3}^{\infty} k \left(\frac{2}{\sqrt{5}+1}\right)^k$$

Therefore, there exists a universal constant $\gamma_3 > 0$ such that

$$G(n) \leq n \left\{ \sum_{k=0}^{k(n)} \frac{\log q_{k+1}}{q_k} + \gamma_3 \right\}$$

□

Proof of 5. By an elementary computation $|\sin(t)| \geq \frac{2}{\pi}|t|$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\|n\alpha\| \leq \frac{1}{2}$ for all $n \geq 0$, where $\|n\alpha\| = \min\{|n\alpha - m| : m \in \mathbb{Z}\}$. Let $m \in \mathbb{Z}$ such that $\|n\alpha\| = |n\alpha - m|$. So $2\|n\alpha\| = \frac{2}{\pi}|\pi(n\alpha - m)| \leq |\sin(\pi(n\alpha - m))|$. Hence

$$2\|n\alpha\| \leq |\sin(n\pi\alpha)|$$

for all $n \geq 0$. Then $|\lambda^n - 1|^{-2} = (2\sin(n\pi\alpha))^{-2} \leq (4\|n\alpha\|)^{-2}$.

Fix $n \in \mathbb{Z}^+$. We have two possibilities :

(i) If $\|n\alpha\| \geq \dfrac{1}{8} = \dfrac{1}{8q_0}$ then $|\lambda^n - 1|^{-2} \leq 4$, and hence $-\log|\lambda^n - 1| \leq \log 2$. By Definition C.9, one sees that $2\log 2 \leq G(n) - G(n-1)$. Then

$$-\log|\lambda^n - 1| \leq G(n) - G(n-1)\ .$$

(ii) If $\dfrac{1}{8q_{k_0+1}} \leq \|n\alpha\| < \dfrac{1}{8q_{k_0}}$, then $n \in A_{k_0}$ and $\|n\alpha\| < \dfrac{1}{4q_{k_0}}$. By Corollary C.2 $n \geq q_{k_0}$. Hence $k(n) \geq k_0$.

By Proposition C.8(3), we have that $g_{k_0}(n-1)+1 \leq g_{k_0}(n)$. Thus $g_{k_0}(n-1)\log(2q_{k_0+1}) + \log(2q_{k_0+1}) \leq q_{k_0}(n)\log(2q_{k_0+1})$. Since $k_0 \in \{1, 2, \ldots, k(n)\}$, we obtain

$$\log(2q_{k_0+1}) \leq \sum_{k=0}^{k(n)} [g_k(n) - g_k(n-1)]\log(2q_{k+1})$$

Therefore $-\log|\lambda^n - 1| \leq \log(2q_{k_0+1}) \leq G(n) - G(n-1)$

□

Appendix D

Proof of Theorem 2.5.5 and Theorem 2.5.6

Proof of Theorem 2.5.5. By hypothesis, there exist constants $c > 0$ and $\mu > 2$ such that

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c}{q^\mu}$$

for all $p \in \mathbb{Z}$, $q \in \mathbb{Z}^+$. By Proposition B.1, $\left|\alpha - \frac{p_k}{q_k}\right| < \frac{1}{q_k q_{k+1}}$ for all k. Then

$$\frac{\log q_{k+1}}{q_k} < \frac{\log c^{-1}}{q_k} + (\mu - 1)\frac{\log q_k}{q_k}$$

By Proposition B.2, $q_k \geq \left(\frac{3}{2}\right)^k$ for all $k \geq 5$, so we have $\sum_{k=0}^{\infty} \frac{\log c^{-1}}{q_k} < +\infty$. On the other hand, since $\frac{\log t}{t}$ is decreasing on $(e, +\infty)$, we obtain

$$\frac{\log q_k}{q_k} < \frac{k \log\left(\frac{3}{2}\right)}{\left(\frac{3}{2}\right)^k}$$

So $\sum_{k=0}^{\infty} \frac{\log q_k}{q_k} < +\infty$. Therefore $\sum_{k=0}^{\infty} \frac{\log q_{k+1}}{q_k} < +\infty$. □

Lemma D.1. *Let $(a_k)_{k \geq 1}$ be a sequence of positive integers. Then*

$$\log\left(a_k + \frac{1}{a_{k-1}}\right) < \log(a_k) + 1.$$

Proof. See also [19]. An elementary computation shows that $x > \frac{1}{e-1}$ implies $\log(x+1) < \log x + 1$. Since $a_k \geq 1 > \frac{1}{e-1}$, we have $\log(a_k+1) < \log(a_k) + 1$. □

Lemma D.2. *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $(a_k)_{k\geq 1}$ its sequence of partial coefficients and $(\frac{p_k}{q_k})$ its sequence of continued fractions. Then*

$$\sum_{k\geq 0}^{\infty} \frac{\log q_{k+1}}{q_k} < +\infty \quad \textit{if and only if} \quad \sum_{k\geq 0}^{\infty} \frac{\log a_{k+1}}{q_k} < +\infty.$$

Proof. See also [19].

[$\Rightarrow$]This follows from the fact $a_{k+1} \leq q_{k+1}$.

[$\Leftarrow$]By equation (2.4.2), $q_{k+1} = a_{k+1}q_k + q_{k-1}$ and $q_k = a_k q_{k-1} + q_{k-2} > a_k q_{k-1}$ hold for all $k \geq 2$, so

$$q_{k+1} = \left(a_{k+1} + \frac{q_{k-1}}{q_k}\right) q_k < \left(a_{k+1} + \frac{1}{a_k}\right) q_k.$$

By Lemma D.1, we have $\dfrac{\log q_{k+1}}{q_k} < \dfrac{\log a_{k+1}}{q_k} + \dfrac{1}{q_k} + \dfrac{\log q_k}{q_k}$ for all $k \geq 2$. By hypothesis $\sum \dfrac{\log a_{k+1}}{q_k} < +\infty$, and since the series $\sum \dfrac{1}{q_k}$, $\sum \dfrac{\log q_k}{q_k}$ converge (see the proof of Theorem 2.5.5), we obtain

$$\sum_{k\geq 0}^{\infty} \frac{\log q_{k+1}}{q_k} < +\infty \ .$$

□

Proof of Theorem 2.5.6. See also [19].
(i) We will apply the ratio test to the series $\sum_{n\geq 0} \dfrac{\log a_{n+1}}{q_n}$. It is clear that $\dfrac{\log a_{n+1}}{q_n} \dfrac{q_{n-1}}{\log a_n} < \rho \dfrac{a_n \log a_n}{q_n} \dfrac{q_{n-1}}{\log a_n}$, so $\dfrac{\log a_{n+1}}{q_n} \dfrac{q_{n-1}}{\log a_n} < \rho a_n \dfrac{q_{n-1}}{q_n}$. Since $q_n = a_n q_{n-1} + q_{n-2} > a_n q_{n-1}$, we have

$$\frac{\log a_{n+1}}{q_n} \frac{q_{n-1}}{\log a_n} < \rho < 1.$$

So $\sum_{n\geq 0}^{\infty} \dfrac{\log a_{n+1}}{q_n} < +\infty$. Applying Lemma D.2, we have $\sum_{n\geq 0} \dfrac{\log q_{n+1}}{q_n} < +\infty$. Therefore $\alpha \in \mathcal{B}$.
(ii) Note that $q_n = a_n q_{n-1} + q_{n-2} = a_n(q_{n-2}a_{n-1} + q_{n-3}) + q_{n-2}$ for all $n \geq 2$, so

$$\frac{1}{a_n a_{n-1}} > \frac{q_{n-2}}{q_n}$$

By hypothesis $a_{n+1} > a_n^{\rho a_n}$ for some $\rho > 1$, except for finitely many n. Therefore the coefficients a_n are growing quickly, so

$$\frac{q_{n-2}}{q_n} \to 0 \ .$$

On the other hand one sees that $\dfrac{\log a_{n+1}}{q_n}\dfrac{q_{n-1}}{\log a_n} > \rho \dfrac{a_n \log a_n}{q_n}\dfrac{q_{n-1}}{\log a_n}$, so

$$\frac{\log a_{n+1}}{q_n}\frac{q_{n-1}}{\log a_n} > \rho a_n \frac{q_{n-1}}{q_n} = \rho\left(1 - \frac{q_{n-2}}{q_n}\right) .$$

Now consider $\varepsilon \in (0, 1 - \frac{1}{\rho})$. So there is n_0 such that if $n \geq n_0$ then $\dfrac{q_{n-2}}{q_n} < \varepsilon$. Then,

$$\frac{\log a_{n+1}}{q_n}\frac{q_{n+1}}{\log a_n} > \rho(1-\varepsilon) > 1 \quad \text{for} \quad n \geq n_0$$

thus the series $\sum_{n\geq 0} \dfrac{\log a_{n+1}}{q_n}$ diverges. Therefore, by Lemma D.2 we conclude that $\alpha \notin \mathcal{B}$. $\square$

Appendix E

Conformal Capacity and Koebe Inequalities

Theorem E.1. ***Riemann Mapping Theorem.*** *Given any simply connected region Ω which is not the whole plane, and a point $z_0 \in \Omega$, there exists a unique conformal map (the Riemann map) $\zeta : \mathbb{D} \to \Omega$ such that ζ is onto, $\zeta(0) = z_0$ and $\zeta'(0) > 0$.*

The Riemann Mapping Theorem is a classical result in the one-dimensional complex theory. For more details see [8].

Definition E.2. *The number $\zeta'(0)$ is called the conformal capacity of Ω with respect to z_0 and will be denoted by $c(\Omega, z_0)$.*

Considering the map $h : \Delta(0, c(\Omega, z_0)) \to \Omega$ defined by $h(z) = \zeta(\frac{z}{\zeta'(0)})$, we have $h(0) = z_0$ and $h'(0) = 1$. This map h is called the conformal representation of Ω at z_0.

Theorem E.3. *Let $\mathcal{U}$ be an open simply connected set and let $x_0 \in \mathcal{U}$. If $\mathcal{V}$ is an open simply connected subset of $\mathcal{U}$ with $x_0 \in \mathcal{V}$ then $c(\mathcal{V}, x_0) \leq c(\mathcal{U}, x_0)$ with equality if and only if $\mathcal{U} = \mathcal{V}$.*

Proof. Let $\zeta : \mathbb{D} \to \mathcal{U}$ and $\eta : \mathbb{D} \to \mathcal{V}$ be as in the theorem above . Then $\zeta^{-1} \circ \eta : \mathbb{D} \to \mathbb{D}$ is analytic and has 0 as a fixed point. So, the theorem follows from a simple application of the Schwarz lemma. □

Theorem E.4. ***Koebe One-Quarter Theorem.*** *If $f \in \mathbb{S}_1$, then the image of f covers the open disk centered at 0 of radius one-quarter, that is $f(\mathbb{D}) \supset \Delta(0, 1/4)$.*

Proof. If we consider $f(z) = \sum_{n=1}^{\infty} a_n z^n$, by Bieberbach's Theorem (see [13], pg.30), we have

$|a_2| \leq 2$. Now fix a point τ and suppose $\tau \in \mathbb{C} \setminus f(\mathbb{D})$. Then

$$\frac{\tau f(z)}{\tau - f(z)} = z + (a_2 + \frac{1}{\tau})z^2 + \dots$$

belongs to $\mathbb{S}_1$. Applying the Bieberbach's Theorem twice, we obtain

$$\frac{1}{|\tau|} \leq |a_2| + |a_2 + \frac{1}{\tau}| \leq 2 + 2 = 4$$

□

Theorem E.5. *Let* $\zeta : \mathbb{D} \to \Omega$ *and* $z_0 \in \Omega$ *as in the Riemann Mapping Theorem. Then* $\Omega \supset \Delta(z_0, \frac{c(\Omega,z_0)}{4})$.

Proof. In fact, define $f(z) = \frac{1}{\zeta'(0)}(\zeta(z) - z_0)$. Clearly f belongs to $\mathbb{S}_1$ and by the theorem above

$$\frac{1}{\zeta'(0)}(\Omega - z_0) \supset \Delta(0, 1/4)$$

Then,

$$\Omega - z_0 \supset \Delta(z_0, \zeta'(0)/4)$$

Therefore

$$\Omega \supset \Delta(z_0, \frac{c(\Omega, z_0)}{4})$$

□

The next result is the well-known **Koebe Distortion Theorem**.

Theorem E.6. *If* $f \in \mathbb{S}_1$ *and* $|z| < 1$, *then*

$$\frac{1 - |z|}{(1 + |z|)^3} \leq |f'(z)| \leq \frac{1 + |z|}{(1 - |z|)^3}$$

and

$$\frac{|z|}{(1 + |z|)^2} \leq |f(z)| \leq \frac{|z|}{(1 - |z|)^2}.$$

Proof. For the details of the proof see [9], pg. 65 . □

Appendix F

Controlling the height of renormalization

Lemma F.1. *For all $f \in \mathbb{S}_1$,*

$$|z\frac{f'(z)}{f(z)} - 1| \leq \frac{2|z|}{1-|z|}$$

Proof. For all $f \in \mathbb{S}_1$ and for each fixed z in the open unit disk, the following inequality holds

$$|\log(\frac{zf'(z)}{f(z)})| \leq \log(\frac{1+|z|}{1-|z|})$$

The proof of this fact comes from the Goluzin inequalities. This was brought to my attention by Dr. Peter Duren and Dr. Leslie Kay. The interested reader may find the Goluzin inequalities in [13], pg. 126. Now let $w = \log(\frac{zf'(z)}{f(z)})$. Then

$$e^{|w|} \leq \frac{1+|z|}{1-|z|}$$

On the other hand it is not difficult to check that

$$|\frac{zf'(z)}{f(z)} - 1| = |e^w - 1| = |\sum_{n\geq 0} \frac{w^n}{n!} - 1| \leq \sum_{n\geq 1} \frac{|w|^n}{n!} = e^{|w|} - 1$$

Therefore

$$|\frac{zf'(z)}{f(z)} - 1| \leq e^{|w|} - 1 \leq \frac{2|z|}{1-|z|}$$

□

Theorem F.2. *Let $F(Z) = Z + \alpha + \varphi(Z)$ be a element of $\mathfrak{F}(\alpha)$. For all $Z \in \mathbb{H}$:*

$$|\varphi'(Z)| \leq \frac{2\exp(-2\pi\Im(Z))}{1-\exp(-2\pi\Im(Z))}$$

$$|\varphi(Z)| \leq \frac{-1}{\pi}\log(1-\exp(-2\pi\Im(Z)))$$

Proof. Let $f \in \mathbb{S}_1$ such that $f(E(Z)) = E(Z+\varphi(Z))$ for all $Z \in \mathbb{H}$, where $E(Z) = \exp(2\pi i Z)$. Then

$$\varphi'(Z) = E(Z)\frac{f'(E(Z))}{f(E(Z))} - 1$$

Therefore, the first inequality follows from the last lemma. Now, in order to prove the second inequality we can apply the first inequality and obtain

$$|\varphi(Z) - \varphi(Z+ni)| \leq \int_0^n |\varphi'(Z+si)|ds \leq \int_0^n \frac{2\exp(-2\pi(\Im Z + s))}{1-\exp(-2\pi(\Im Z+s))}ds$$

Then

$$|\varphi(Z) - \varphi(Z+ni)| \leq \frac{1}{\pi}\log(1-\exp(-2\pi\Im(Z))\exp(-2\pi s))]_0^n$$

So

$$|\varphi(Z) - \varphi(Z+ni)| \leq \frac{1}{\pi}\log\left(\frac{1-\exp(-2\pi\Im(Z))\exp(-2\pi n)}{1-\exp(-2\pi\Im(Z))}\right)$$

But, since $\exp(-2\pi n) \to 0$ as n tends to infinity and $\lim_{\Im(Z)\to\infty}\varphi(Z) = 0$, we obtain the second inequality . □

Theorem F.3. *For all $\delta \in (0, 1/10)$, there exists a constant C_δ such that for all $F \in \mathfrak{F}(\alpha)$,*

$$\Im(Z) \geq C_\delta \;\Rightarrow\; |F'(Z) - 1| \leq \delta$$

and

$$\Im(Z) \geq \frac{1}{2\pi}\log(\frac{1}{\alpha}) + C_\delta \;\Rightarrow\; |F(z) - z - \alpha| \leq \delta\alpha$$

Proof. Since $F(Z) = Z + \alpha + \varphi(Z)$, we can use the last theorem and obtain

$$|F'(Z) - 1| \leq \frac{2\exp(-2\pi\Im Z)}{1-\exp(-2\pi\Im(Z))}$$

$$|F(Z) - Z - \alpha| \leq \frac{-1}{\pi}\log(1-\exp(-2\pi\Im(Z)))$$

Let C_δ be the positive constant given by $C_\delta = \frac{-1}{2\pi}\log(\frac{\delta}{2+\delta})$. Then for all $Z \in \mathbb{H}$ such that $\Im(Z) = C_\delta + t \geq C_\delta$ we have

$$|F'(Z) - 1| \leq \delta\exp(-2\pi t)$$

Since $C_\delta = \frac{-1}{2\pi}\log(\frac{\delta}{2+\delta}) > \frac{-1}{2\pi}\log(1-\frac{1}{e^{\pi\delta}})$, then $\frac{-1}{\pi}\log(1-k_\delta) \leq \delta$ where $k_\delta = \frac{\delta}{2+\delta}$.
On the other hand, note that for any $t > 0$

$$\log(1-\exp(-2\pi t)\cdot k_\delta) \geq \log(1-k_\delta) \geq \log(1-k_\delta)\exp(-2\pi t)$$

Hence, for any $t > 0$

$$\frac{-1}{\pi}\log(1-\exp(-2\pi t)\cdot k_\delta) \leq \frac{-1}{\pi}\log(1-k_\delta)\cdot\exp(-2\pi t) \leq \delta\exp(-2\pi t)$$

If $\Im(Z) = t + C_\delta \geq C_\delta$, by the second inequality of the last theorem and the previous inequality, we have

$$|F(Z) - Z - \alpha| \leq \delta\exp(-2\pi t)$$

Therefore, if $\Im(Z) \geq C_\delta$ then $|F'(Z) - 1| \leq \delta$.
Furthermore, if $\Im(Z) \geq \frac{1}{2\pi}\log(\frac{1}{\alpha}) + C_\delta$ then $|F(Z) - Z - \alpha| \leq \delta\alpha$. □

Remark. In particular, F can not have fixed points above $\frac{1}{2\pi}\log(\frac{1}{\alpha})$ plus some universal constant.

Lemma F.4. *Suppose $F \in \mathbb{S}_\delta(\alpha)$ with $\delta \in (0, 1/10)$ and consider the corresponding sets $\mathcal{V}$, $\mathcal{V}^*$, $\mathcal{V}^*_-$ as in the proof of Theorem 3.3.6. Then*

$$F(\mathcal{V}^*_-) \subset \mathcal{V}^*_- \bigcup \overline{\mathcal{U}}\ .$$

Proof. By the definition of $\mathbb{S}_\delta(\alpha)$, $|F(Z) - Z - \alpha| \leq \delta\alpha$ holds. Then $\Im(F(Z)) \geq \Im(Z) - \delta\alpha$ for all $Z \in \mathbb{H}$. In particular $\Im(F(Z)) \geq \Im(Z) - \delta\alpha \geq 4\delta(-\Re(Z)) - \delta\alpha$ for all $Z \in \mathcal{V}^*_-$.
Since $F \in \mathbb{S}_\delta(\alpha)$ we also have $\Re(F(Z) - Z) - \alpha \geq -\delta\alpha$. Hence, if $\Re(F(Z)) < 0$ we have,

$$\Im(F(Z)) \geq 4\delta(-\Re(Z)) - \delta\alpha \geq 4\delta(-\Re(F(Z))) + 4\delta(\alpha - \delta\alpha) - \delta\alpha \geq 4\delta(-\Re(F(Z)))$$

This means that for all $Z \in \mathcal{V}^*_-$ with $\Re(F(Z)) < 0$, $F(Z)$ has to be in $\mathcal{V}^*_-$.
Since F is univalent on $\mathbb{H}$ and periodic with period 1, the sets $\overline{\mathcal{V}}$, $F(\overline{\mathcal{V}})$ are homeomorphic, and the following union

$$F(\{-1+it : t \geq 4\delta\}) \bigcup F(\{t + i4\delta : -1 \leq t \leq 0\}) \bigcup F(\{it : t \geq 4\delta\})$$

bounds the open strip $F(\mathcal{V})$. By definition of $\mathbb{S}_\delta(\alpha)$, it is easy to see that $\lim_{t\to+\infty}(F(-1+it) - it) = -1 + \alpha < 0$, so $\Re(F(-1+it)) < 0$ for t large. Therefore, we have two possibilities. If $\Re(F(-1+it)) < 0$ for some $t \geq 4\delta$, by the first conclusion $F(-1+it) \in \mathcal{V}^*_-$ because $-1 + it \in \mathcal{V}^*_-$ when $t \geq 4\delta$.

If $\Re(F(-1+it)) \geq 0$ then $0 \leq \Re(F(-1+it)) = -1 + \Re(F(it)) < \Re(F(it))$ which implies that the line segment $[F(-1+it), F(it)]$ is contained in $\overline{\mathcal{U}}$ because $F(-1+it) = F(it) - 1$ for all $t \geq 4\delta$; therefore the inclusion $F(\{-1+it : t \geq 4\delta\}) \subset \mathcal{V}_-^* \bigcup \overline{\mathcal{U}}$ holds. On the other hand, it is obvious that $F(\{it : t \geq 4\delta\}) \subset \mathcal{V}_-^* \bigcup \overline{\mathcal{U}}$ and using the definition of $\mathbb{S}_\delta(\alpha)$ one sees that $\Im(F(s+i4\delta)) \geq \Im(s+i4\delta) - \delta\alpha \geq 4\delta - \delta\alpha$. Then we conclude that $F(\mathcal{V}) \subset \mathcal{V}_-^* \bigcup \overline{\mathcal{U}}$.

Since $-1 + i4\delta \in \mathcal{V}_-^*$, from one of the inequalities above $\Im F(-1+i4\delta) \geq 4\delta - \delta\alpha$. On the other hand it is easy to see $\Im(F(0)) \leq \delta\alpha$. Now consider the map $\varphi(t) = \Im(F(t - i4\delta t))$ where $t \in [-1, 0]$. By the previous statements $\varphi(-1) > \varphi(0)$. Clearly φ is continuous on $[-1, 0]$, and differentiable on $(-1, 0)$. Moreover $\varphi'(t)$ does not change sign on $(-1, 0)$; otherwise there exists some $t^* \in (-1, 0)$ such that $0 = \varphi'(t^*) = \Im(F'(t^* - i4\delta t^*) \cdot (1 - i4\delta))$. Since $F \in \mathbb{S}_\delta(\alpha)$, we have $|F'(t^* - i4\delta t^*) - 1| \leq \delta$. Then

$$|F'(t^* - 4\delta t^* i)(1 - i4\delta) - (1 - i4\delta)| \leq |1 + i4\delta|\delta$$

So

$$4\delta = \Im(F'(t^* - i4\delta t^*)(1 - i4\delta) - (1 - i4\delta)) \leq (1 + 4\delta)\delta$$

which is a contradiction because $\delta \in (0, 1/10)$. Since $\varphi(-1) > \varphi(0)$ and $\varphi'(t) \neq 0$ we can say that φ is strictly decreasing with minimum at 0, so $\Im(F(t - i4\delta t)) \geq \Im(F(0))$ for all $t \in [-1, 0]$. Thus the curve $F(t - i4\delta t)$ cannot intersect the segment line $[0, F(0)]$.
Now, if $\Re(F(t - i4\delta t)) < 0$ then $F(t - i4\delta t) \in \mathcal{V}_-^*$, because $t - i4\delta t \in \mathcal{V}_-^*$. If $\Re(F(t - i4\delta t)) > 0$, the point $F(t - i4\delta t)$ cannot be at the right side of the image of the imaginary axis under F; otherwise the curve φ intersects the image of the imaginary axis under F in at least one point. But that is impossible because F is injective, so the point $F(t - i4\delta t) \in \overline{\mathcal{U}}$. Therefore, the image of the curve φ is also in $\mathcal{V}_-^* \bigcup \overline{\mathcal{U}}$.
Finally we will analyze the remainder of the set $\mathcal{V}_-^*$. This means the closure of the set $\mathcal{V}_-^* \setminus \mathcal{V}$ which is $\{Z : 4\delta \geq \Im(Z) \geq -\Re(Z) \; ; \; -1 \leq \Re(Z) \leq 0\}$. Since the boundary of this set is mapped into $\mathcal{V}_-^* \bigcup \overline{\mathcal{U}}$ and F is univalent, in particular a homeomorphism, we may conclude that $F(\mathcal{V}_-^* \setminus \mathcal{V})$ is also a subset of $\mathcal{V}_-^* \bigcup \overline{\mathcal{U}}$. $\square$

Appendix G

Modulus of a Ring Domain and Quasiconformal Mappings

Definition G.1. *A plane domain G is called simply connected if its complement is connected. If the complement of G consists of n components ($1 < n < \infty$), then G is called n-tuply connected. A doubly connected domain is called a ring domain or ring.*

The definition of the modulus of a ring domain is based on the following mapping theorem: Every ring domain B can be mapped conformally onto an annulus $\{z : 0 \leq r_1 < |z| < r_2 \leq \infty\}$. Such a mapping is called a *canonical mapping* and the corresponding annulus a canonical image of B. If $r_1 > 0$ and $r_2 < \infty$ for one canonical image of B, then the ratio r_2/r_1 is the same for all canonical images of B, see ([27], Pg. 292). The number

$$M(B) = log(\frac{r_2}{r_1}),$$

which then determines the conformal equivalence class of B, is called the modulus of B. On the other hand, if $r_1 = 0$ or $r_2 = \infty$ for any canonical image, then we define $M(B) = \infty$. This happens if and only if at least one boundary component of B consists of a single point.

Definition G.2. ***Continuity of the Modulus***. *A sequence of sets $\{E_n\}$ is said to converge uniformly to a set E if, for each $\epsilon > 0$, there exists an N such that $n > N$ implies each point of E_n lies within distance ϵ of E and each point of E lies within distance ϵ of E_n.*

The next result is know as the continuity property for the modulus of a ring. For more details see [16], pg. 367 .

Lemma G.3. *Let R_n be a sequence of rings and let R be a bounded ring. If each of the components of ∂R_n converges uniformly to the corresponding component of ∂R, then*

$$M(R) = \lim_{n\to\infty} M(R_n)$$

Definition G.4. ***Quasiconformal Mappings***. *A sense preserving homeomorphism f of a domain A is a K-quasiconformal ($K \geq 1$) if and only if $M(B) \leq KM(f(B))$ for any arbitrary ring domain B, $\overline{B} \subseteq A$.*

For more details about this definition see [20], pg. 13 . For these kinds of mappings we have the following generalized extension theorem. For the details of the proof see [21], pg. 42.

Theorem G.5. ***Extension Theorem for Quasiconformal Mappings***. *A quasiconformal mapping of a Jordan domain onto another Jordan domain can be extended to a homeomorphism between the closures of the domains.*

Lemma G.6. *Assume $\psi : (\mathbb{D}, 0) \to (\mathbb{D}, 0)$ is a K-quasiconformal homeomorphism. For each $r \in (0, 1)$ set $R = \mathbb{D} \setminus [0, r]$. Then*

$$M(R) \leq KM(\psi(R))$$

Proof. Fix $r \in (0, 1)$. There exists n_0 such that $n \geq n_0$ implies $\frac{1}{n} < 1 - r$. For $n \geq n_0$ let R_n be the bounded ring domain whose boundary is the disjoint union of the line segment $[0, r]$ and the circle $\{z : |z| = 1 - \frac{1}{n}\}$. Clearly R_n converges uniformly to the bounded ring domain R. Moreover by the definition of quasiconformal map, we have

$$M(R_n) \leq KM(\psi(R_n))$$

From this inequality and the continuity of the modulus it is enough to prove that the boundary of $\psi(R_n)$ converges uniformly to the boundary of $\psi(R)$. But from the definition of the sequence $\{R_n\}_{n \geq n_0}$ it is enough to show that $\psi(\{z : |z| = 1 - \frac{1}{n}\})$ converges uniformly to $\partial\mathbb{D}$. Since $\mathbb{D}$ is a Jordan domain and ψ is quasiconformal we can extend ψ homeomorphically to the boundary; we also call this extension ψ. On the other hand, let E_n be the set defined by $E_n = \psi(\{z : |z| = 1 - \frac{1}{n}\})$.
Given $\epsilon > 0$ there exists N such that $n > N$ implies $dist(x, \partial\mathbb{D}) < \epsilon$ for all $x \in E_n$. Otherwise there exist $\epsilon_0 > 0$ and a sequence (z_{n_k}) satisfying

$$|z_{n_k}| = 1 - \frac{1}{n_k} \quad \text{and} \quad dist(\psi(z_{n_k}), \partial\mathbb{D}) > \epsilon_0;$$

we may assume (z_{n_k}) converges to a point $a \in \partial\mathbb{D}$. So using the extension of ψ we get $dist(\psi(a), \partial\mathbb{D}) = \lim_{k\to\infty} dist(\psi(z_{n_k}), \partial\mathbb{D}) \geq \epsilon_0$, which is a contradiction since $\psi(a) \in \partial\mathbb{D}$. Now, given $y \in \partial\mathbb{D}$ by the extension there exists a unique $x \in \partial\mathbb{D}$ such that $\psi(x) = y$. Since the sequence of circles $\{z : |z| = 1 - \frac{1}{n}\}$ converges uniformly to $\partial\mathbb{D}$ there exists a sequence (z_n) which converges to x and satisfies $|z_n| = 1 - \frac{1}{n}$. Then $\lim_{n\to\infty} dist(y, E_n) \leq \lim_{n\to\infty} |\psi(x) - \psi(z_n)| = 0$. Define $g_n(y) = dist(y, E_n)$; clearly this is a sequence of continuous functions on $\partial\mathbb{D}$ and for each $y \in \partial\mathbb{D}$ the sequence $(g_n(y))$ converges monotonically (decreasing) to zero. Since $\partial\mathbb{D}$ is compact we can apply Dini's Theorem. Thus $dist(y, E_n)$ converges uniformly in y to zero. Therefore E_n converges uniformly to $\partial\mathbb{D}$ and the lemma follows. □

The following result is known as Grötzsch's Modulus Theorem. For the proof see [21], pg. 54.

Theorem G.7. ***Grötzsch's Modulus Theorem*** *Suppose $r \in \mathbb{R}$ with $0 < r < 1$. If the ring domain B separates the points 0 and r from the circle $|z| = 1$ then*

$$M(B) \leq M(B(r))$$

where $B(r)$ is the ring domain whose boundary consists of the circle $|z| = 1$ and the segment $0 \leq x \leq r$ of the real axis.

This ring domain is called the Grötzsch extremal domain. From now on, we denote $M(B(r))$ by $\mu(r)$. For any $r \in (0,1)$, the following equality holds :

$$\mu(r) = \frac{\pi}{2} \frac{K(\sqrt{1-r^2})}{K(r)} ,$$

where $K(r)$ is the Legendre normal integral given by

$$K(r) = \int_0^1 \frac{dt}{\sqrt{1-t^2}\sqrt{1-r^2t^2}}$$

This means that $\mu(r)$ can be expressed in terms of elliptic integrals. For the proof of the last equality above see [21], pg. 59-60. Now, it is easy to check that μ is a continuous function on $(0,1)$ satisfying $\mu(0+) = +\infty$ and $\mu(1-) = 0$. By an elementary computation (see [1], pg. 50)

$$\frac{d}{dr}\left(\frac{K(\sqrt{1-r^2})}{K(r)}\right) = \frac{-\pi}{2r(1-r^2)(K(r))^2}$$

Therefore, we conclude that μ is a strictly decreasing homeomorphism from $(0,1)$ onto $(0,+\infty)$.

Lemma G.8. *The function $h(r) = \mu(r) + \log(\frac{r}{1+r'})$, where $r' = \sqrt{1-r^2}$, is strictly decreasing.*

Proof. . By a computation

$$\frac{d}{dr}(h(r)) = \frac{\pi^2}{4rr'K^2}\left(\frac{4r'K^2}{\pi^2} - 1\right) < 0$$

where $K(r)$ is the Legendre normal integral . □

Lemma G.9. *The function $f(r) = \mu(r) + \log(r)$ is strictly decreasing .*

Proof. Clearly $f(r) = h(r) + \log(1+r')$ where $r' = \sqrt{1+r^2}$. Since $\log(1+r')$ is strictly decreasing, the monotonicity of f follows . □

Lemma G.10. *The function $\mu(r)/\log(4/r)$ is strictly decreasing on (0,1).*

Proof. This follows from the lemma above, since

$$\frac{\log(4/r)}{\mu(r)} = 1 + \frac{\log(4) - (\mu(r) + \log(r))}{\mu(r)}$$

□

List of symbols
$\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.
$\mathbb{D} = \Delta(0,1) := \{z \in \mathbb{C} : \, |z| < 1\}$.
$\mathbb{H} = UpperHalfPlane := \{Z \in \mathbb{C} : \Im(Z) > 0\}$.
$\Delta(z_0, r) := \{z \in \mathbb{C} : \, |z - \; z_0| < r\}$.
$\Delta[z_0, r] := \{z \in \mathbb{C} : \, |z - \; z_0| \leq r\}$.
(a, b):= the open interval with extremes a and b
$[0, z]$:= the closed line segment with extremes 0 and z .
$l = i\mathbb{R} \bigcap \overline{\mathbb{H}} := \{it \in \mathbb{C} : t \geq 0\}$.
e=exp(1)
$f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, means that f leaves the point z=0 fixed.
$[\![\alpha]\!]$:=integer part of α.
$\{\alpha\}$:=fractional part of α
$||\alpha||$:=min$\{|n - \alpha| : n \in \mathbb{Z}\}$.
$dist(y, V)$:=inf$\{|y - v| : v \in V\}$

Bibliography

[1] G.D.Anderson, M.K.Vamanamurthy and M.K.Vuorinen . *Conformal invariants , inequalities , and quasiconformal maps* . Canadian Math. Soc. Series of Monographs and Advanced Texts , John Wiley & Jones , New York (1997)

[2] A. Baker, *A concise introduction to the theory of numbers*. Cambridge University Press (1984).

[3] A.F. Beardon, *Iteration of Rational Functions*. Springer – Verlag (1991).

[4] A.D. Bjruno, *Analytical form of differential equations*. Trans. Moscow Mathematic Society 25(1971), 131-288 .

[5] L. de Branges, *A Proof of the Bieberbach Conjecture*. Acta Math. 154(1985), 137-152.

[6] X. Buff, A. Chéritat, *The Brjuno Function continuously estimates the size of quadratic Siegel Disks* . Annals of Mathematics 164(2006), 265-312.

[7] L. Carleson – T. W. Gamelin, *Complex Dynamics*. Springer–Verlag (1993).

[8] J.B. Conway. *Functions of one complex variable I*. Springer–Verlag (1978).

[9] J.B. Conway. *Functions of one complex variable II*. Springer–Verlag (1995).

[10] H. Cremer, *Über die Häufigkeit der Nichtzentren*. Math. Ann. 115(1938), 573-580.

[11] A.M. Davie, *The critical function for the semistandard map*. Nonlinearity 7(1990), 21-37.

[12] A. Douady, J.H. Hubbard, *On the Dynamics of Polynomial-Like Mappings*, Ann.scient.Éc.Norm.Sup. , 4 série, t.18 (1985) p.287 à 343.

[13] P.L. Duren, *Univalent Functions*. Springer–Verlag (1983).

[14] J. Ecalle, *Les fonctions résurgentes et leurs applications*. t. I, II, III, Publications mathématiques d'Orsay n° 81-05, 81-06, 85-05.

[15] A. Fletcher, V. Markovic, *Quasiconformal Maps and Teichmuller Theory*. Oxford (2007)

[16] F.W. Gehring, *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc. 103(1962), 353–393.

[17] G.H. Hardy, E.M. Wright, *An introduction to the theory of numbers.* Fifth edition Oxford Univ. Press (1954).

[18] S.G. Krantz, *Function theory of several complex variables.* Wadsworth Books (1992).

[19] R.E. Lee de Ville, *Brjuno Numbers and the symbolic dynamics of the complex exponential.* Qualitative theory of dynamical systems 1(1999), 71-82.

[20] O. Lehto, *Univalent Functions and Teichmüller Spaces.* Springer Verlag (1987).

[21] O. Lehto, K.I. Virtanen, *Quasiconformal Mappings in the Plane.* Springer Verlag (1973).

[22] Y. Lequain, *Aproximaciones diofantinas.* Décimo Noveno Coloquio Sociedad Brasileña de Matemática.

[23] K. Mahler, *On the approximation of* π . Nederl. Akad. Wetensch. Proc. Ser. A. 56-Indagationes Math. 15(1953), 30–42.

[24] S. Marmi, *Linearization of analytic and non analytic germs of diffeomorphisms of* $(\mathbb{C}, 0)$. Bolletin Societé Mathématique de France, march (2000).

[25] C.G. Moreira, *Introducao a teoria dos numeros.* Monografias del IMCA (2002).

[26] R. Perez Marco, *Sur la dynamique des germes de difféomorphismes holomorphes de (C,0) et des difféomorphismes analytiques du cercle, Thèse Université de Paris-Sud*, décembre 1990.

[27] W. Rudin, *Real and Complex Analysis.* Third Edition, McGraw-Hill Publishing Company (1987).

[28] C.L. Siegel, J.K. Moser, *Lectures on Celestial Mechanics.* Springer – Verlag (1971).

[29] C.L. Siegel, *Iteration of analytic functions.* Annals of mathematics 43(1942), 807-812.

[30] J.C. Yoccoz, *Théoreme de Siegel, polynomes quadratiques et nombres de Brjuno.* Asterisque 231(1995), 3-88.

[31] S.M. Voronin, *Classification analytique des germes d'applications conformes* $(C, 0) \to (C, 0)$ *tangentes à l'identité.* Functional Analysis 15/1 (1981),p. 1-17.