Learning, Game Play, and Convergence of Behavior in Evolving Social Networks

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Abstract

I study information dissemination and opinion formation in a framework of evolving social networks. Individuals take weighted averages repeatedly to update their opinions. They also update their assessments on others' opinions, represented by an influence weight matrix. It is proven that both opinions and the influence weights are convergent. In the steady state, consensus is reached where all individuals hold the same opinion. Convergence occurs with an extended model as well, which indicates the tremendous influential power possessed by a minority group. Then I impose a dual network structure, where individuals not only collect information, but also use the information to play a coordination game with a selected group of opponents that one is connected with. All individuals update their strategies based on a naive learning process within a separate influence network in which information is disseminated. The selection of opponents also gets updated over time. I calculate the critical values of costs associated with connections for different network structures and strategies to occur in the steady state. Finally, I investigate the outcomes of social learning under various exogenous network structures. Individuals use an algorithm that takes into account both proximity of opinions and impact of neighbors. Results also show consensus, with convergence speed correlated with the network structure. In addition, an endogenous network formation in two stages that utilizes network and distance between agents' opinions is proposed. The resulting networks show power-law patterns in degree distribution.

To my parents

致我的父母

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I still remember the day when I first came to this charming little town of Blacksburg in Fall of 2005, with such mixed feelings of expectations and uncertainty. The past four years has been an amazing journey. I have learned so much in and outside of the classroom that I would never have expected. And now I am ready to graduate and start a new journey again, it feels all so familiar yet not the same. Thanks to the Department of Economics at Virginia Tech for providing an environment that enables me to grow and discover.

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Chapter 1

Social Networks in Economics

Economics is a science about decision-making. Preferences and constraints are critical determinants in the decision-marking process; we also collect information in order to make the best decision that we can. Nobody knows everything about anything. When we face a question that we cannot be sure about, we may ask someone who knows betters, discuss with family and friends, or search the Internet. On the other hand, others might consult us and get our opinions as well. That is, we exchange information on the platform of social networks to better comprehend our problems, choose a strategy, or just to satisfy our curiosity.

This dissertation intends to understand and explain the process of information dissemination and opinion formation in social networks. I model such a process with a naive social learning framework (Pan, 2008) in which individuals take weighted averages of opinions in forming their own. The basic idea is that we do not take all the information that we collect from different sources equally. We assess them based on our own criteria, then we summarize our assessments and the original information to finalize our opinion.

Also, people change. We don't always have the same attitude towards someone that we interact with; we may decide to be more or less influenced by the same person or source as time goes by. Therefore, I propose a social learning model with time-varying weights. In the basic model, weights get updated based on the difference between two individuals' opinions. With such a distance-based updating rule, one may expect diversity since individuals place little weights on those who have remote opinions. Actually, the opposite happens: consensus will be reached. All individuals converge to hold the same opinion in the steady state. Consequently, they assign the same weight on others. Additionally, in the beginning each individual is free to have any opinion within a bounded range. And they do not need to place strictly positive weight on their own opinions to reach consensus.

To better understand the influence from peers in a network, I extend the basic model by introducing a small group of individuals that are "persistent". They do not fully interact with others and insist on their initial opinions or weights, or both. It turns out, this minority group have the significant influence power over the whole community, in that everyone ends up having the same opinion that the persistent ones do.

Although a lot of times we talk to others merely for the need of communication, not in order to find the optimal way to manage our investment portfolio or get some kind of tangible payoffs, it would be interesting to integrate the opinion exchange and formation into the cost-benefit analysis of decision-making process. This is established by a dual network structure where individuals collect information from an information network, which they use to form their strategies in a coordination game that they play in an interaction network (Pan and Gilles, 2009).

In this scenario, weights that people place on others are proportional to the potential payoffs that one might get from playing the coordination game with another. The benefit from interaction also determines if costly links in the interaction network will be formed, maintained, or severed.

Analysis shows that the process leads to a steady state, where individuals' strategies and the network structure depend on cost of links and individuals' initial strategies. Especially, empty networks and complete networks are both supported, if the cost is high or low enough, respectively. In other cases, the resulting networks are indeterminant due to the randomness involved in the dynamics.

To better examine the effects of network structures, I further impose various exogenous types of network on the community of learning individuals (Pan, 2009a). I propose a "distance-network combination" updating rule that considers both difference between opinions and connections among individuals. Consensus is reached at different speeds, depending on the network structure. Generally, a community with networks with central nodes converges to consensus faster.

In addition, I present an endogenous network formation algorithm. Unlike the strategic formation model that relies on costs and benefits from links, I study a stochastic network formation mechanism. The dynamics of an evolving network also take into account both proximity of opinions and network structures. Resulting networks exhibit scale-free patterns that fit empirical evidence very well. It also shows that individuals with popular opinions are promoted to be hub nodes in the networks and possess more connections than most of the others.

This chapter consists of three parts. In the first part, I emphasize the importance of the embeddedness of social networks in economic activity. Next, I review related literatures on social learning, game play in networks, and network formation. I also discuss my research in context of these literatures. The final section shows the organization of the dissertation. Some basic notations and concepts of network theory are provided in Appendix A.

1.1 Motivation

More than twenty years ago, Kirman (1983) discussed the problem of communication in markets, in particular the problem of imperfect communication and incomplete information among individuals. Kirman suggested a modeling approach to examine the equilibrium of price that is different from the Walrasian perspective. He adopted the stochastic graph theory to capture the interaction among buyers and sellers.

Back then when Kirman's paper was published, the mainstream economic analysis did not embed the element of information and communication structure. Markets were modeled in such a way that interpersonal relations played virtually no role. One can well argue with this standpoint on economic analysis and defend the standard theoretical framework that does not emphasize networks. However, Kirman's pioneer work sheds light on a different approach that can be applied in both theory and empirical studies. Recently there is a surge of economic research that utilizes such an approach and touches some of the core questions of economics.

Although the study of embeddedness of network structure in economic activity is a relatively new development, it has quickly generated a rich literature. Generally, a social network is defined as a set of people or groups with certain forms of links among them. Business relationships and acquaintances are the two most popular forms of relationships in social network study with the focus on interaction among individuals and the consequent influences, such as knowledge sharing and reciprocal benefits.

In this section I look at a few examples to see how certain economic phenomena can be understood and examined with social network integrated into their analysis. The intention is not to present a broad survey to cover this extensive topic. Instead, my discussion will focus on three topics: labor markets, information and technology dissemination, and trade.

1.1.1 Labor Markets

Empirical research of labor markets is one of the most studied fields that focus on the importance of social networks on social-economic outcomes. Rees (1966) illustrates a clear correlation between networking and employment. He studied the labor market in Chicago area and found that about 50% of the hiring of white-collar workers was done through informal channels. The number was even higher in the case of employment of blue-collar workers, which showed 80% of jobs found through personal contacts. More recently, Topa (2001) and Munshi (2003) provide empirical evidence of importance of employment information exchange among lower-education workers and minorities.

Montgomery (1991) pointed out an important factor in this topic of personal contact and labor markets, which is referral. Especially, when referral is required for a junior worker. Montgomery modeled the process with the probability of a young worker having a reference, and the correlation between the qualities of connected individuals. Montgomery's work shows that networks affect more than one's employment: they also span a range of wage differentiation in equilibrium.

Granovetter (1973) modeled social relations as ties and social structures as networks. He defined strength of a tie by frequency and time that the two individuals contact and spend together. He further argued that it is "the strength of weak ties" that holds together a social network and helps diffuse information among members in a social network. A more formal theoretical presentation and analysis was done by Boorman (1975). The idea is that agents decide the allocation of time and efforts among links, the difference of which lies in priority in receiving information of job openings. Two findings are quite interesting. One is that an agent will choose the optimal allocation, if all others have the same choice. The other finding is that under certain circumstances, the resulting equilibrium networks are not efficient.

Boorman's work shows its influence on more recent works by Jackson and Wolinsky (1996) and Calvó-Armengol (2004). In these models, the actual flow of information was not explicitly explored. This question is studied by Calvó-Armengol and Jackson (2004), who analyzed the dynamics of information diffusion. They found a positive relationship between the length of one's unemployment and the duration of that state. That is, the longer one has been unemployed, the longer it will take for him to find a job. The social network structure does not change this relationship, but does matter the magnitude of the affect. McEntarfer (2002) used data from the National Longitudinal Study of Youth (NLSY) to examine whether migrants who used social connections when finding their first job assimilate faster in the new region. The results were consistent with the theoretical analysis. In a way, this conclusion also conforms to the findings in Pan (2009a), which shows that consensus is always reached under different network structures, but the speed of reaching consensus depends on the underlying network structure.

1.1.2 Information and Technology Dissemination

One of the key topics in this dissertation is the diffusion of information, which is quite similar to the process of technology dissemination from a modeling aspect. Technology dissemination has long been of interest in economics. It is often studied along with fundamental economics questions such as development and growth. A pioneer work in this field was done by Griliches (1957), who studied how technological changes were generated and propagated in U.S. agriculture to account for the large cross-sectional differences in using hybrid corn seeds. Following Griliches' work, interpersonal contacts are often considered in the process of diffusing technology and innovation. Coleman, Katz, and Menzel (1966) did a remarkable survey on adopting a new drug. They asked the doctors details about the relationship and interaction that they had with other doctors, with whom they discussed the new drug. This approach more clearly incorporated a network approach on the topic. One of their main conclusions was that doctors who had more connections spread the adoption of a drug faster. With modern language of network theory, we may call those doctors hubs. This is a very interesting finding that reappears in Pan (2009a), which shows the clear influence that hubs have over the whole network with a scale-free structure.

Foster and Rosenzweig (1995) and Conley and Undry (2001) modeled dissemination as social learning and both did empirical research in the propagation of agricultural technology. The former paper has an emphasis on both productivity and social behavior; while the latter examines the availability of information and shows that communication network plays a significant role in the process, in that technology is not freely available but instead can only be obtained if one has a connection with someone who had the information.

Besides these and other empirical evidence, there is also a body of theoretical analysis of diffusion with the framework of social learning. In the next section I discuss more on the topic. For now I briefly review two models by Bala and Goyal (1998) and DeMarzo, Vayanos, and Zwiebel (2003). Both present learning as a process of belief evolution and conclude that homogenous beliefs emerge in the long run, regardless of the underlying network structure.

On the other hand, asymmetries in possession of information as well as expectations are often noticed. Consequently, diffusion of technology is found to be detained by heterogeneity, uncertainty, and inefficient communication (Foster and Rosenzweig, 1995; Conley and Undry, 2001; Munshi, 2004). An interesting note on this is the persistent agents in Pan (2008) and Pan and Gilles (2009), who have different updating rules and do not fully communicate with others. Meanwhile, it is proven that they can control the final outcomes of social learning. Especially, in Pan and Gilles (2009) where agents involve in a coordination game, persistent agent can result in an inefficient steady state, even if all normal agents start with best responses.

1.1.3 Trade

At stated, trade can be modeled in two completely different approaches: (conventional) markets and networks. No doubt we have markets where network structure and social influence have little impact, especially those that have mature infrastructure and regulation.

Markets and products with network externalities should merit the network modeling approach. For instance, one wouldn't want to buy a fax machine if none of one's clients or friends had one. Moreover, consider the case where reputation of sellers or trust affects buyers' strategy and action. Buyers observe past transactions, or consult to other buyers who have had made a purchase with certain sellers, before they decide from whom to buy a product. One example is online shopping: almost all the Internet-based merchandisers, including eBay and Amazon, have a feedback system that allows potential buyers to choose a trustworthy seller. But long before that, trust and reputation has played an important and necessary role in trading. Greif (1989) presented a study on economic institution among Mediterranean traders in the 11th century and showed the impact that social interaction had on trading activities. An empirical study on modern international trade was done by Rauch (2001).

Not only can buyers choose a seller with good reputation, in some cases they also have bargaining power (Calvó-Armengol, 2003; Corominas-Bosch, 2004; Braun and Gautschi, 2006; Polanski, 2007). Agents get other benefits through networks as well, such as insurance. With the assumption that network links are costly, trade actions and strategies within networks can be used to understand the mechanism of network formation based on benefits and costs (Kranton and Minehart, 2000, 2001; Kirman, Herreiner, and Weisbuch, 2000).

Theoretical analysis of trading normally utilizes a repeated game theoretic framework and restricted or simplified network structures. One remarkable exception is the work by Haller (1990), who presented a model of trading groups in the framework of random graphs. His work utilizes random communication structures that may be or very large scale. With the rapid development of information technology, researchers can now take advantage of the high capability of computers and various computational tools to build and analyze complex networks and economic activities. For instance, the Trade Network Game (TNG) enables users to study the formation and evolution of trade networks among strategically interacting traders operating under different specified market protocols. It extends standard matching theory and sequential game theory in that each trader must jointly determine over time both whom to seek trades with (partner selection) and how to behave in any trade interactions that take place (McFadzean, Stewart, and Tesfatsion, 2001). Computational methods are very powerful tools in studying various interactions in network settings, not only trade. In this dissertation, simulations are used as supportive evidence and basis of theoretical analysis.

1.2 Core Literature

When we think of the interaction in social networks and how that affects our actions and social-economic outcomes, it all comes down to a process of information dissemination, search and collection, and decision-making. The first two processes can be integrated into a framework of social learning; whereas game theory is most used to analyze how agents make decisions. Besides, social networks evolve. We meet new people, make new friends, or lose contact with old acquaints. The literature of network formation focuses on such dynamics of networks. Next I review some highlights in these three literatures.

1.2.1 Social Learning

Social learning is represented throughout the studies discussed above and many other fields; it is the basis for the social influence that affects individuals' opinions and consequently their behavior.

In Bala and Goyal (1998), agents are assumed to be in an arbitrary network indicated by neighbor sets. A set of neighbors is a subset of agents that can are considered connected with a specific agent. Additionally, there exists a “royal family”, whose members are connected with everyone. The essential element is belief; an agent uses her own experience as well as observations of her neighbors’ experience to update her belief, with which she chooses between two actions to maximize her utility. Bala and Goyal found that agents could be choosing suboptimal actions under the negative influence from information links.

Bala and Goyal’s methodology is rooted in Bayesian analysis, which is widely used in learning models. Notably, there is another branch in the literature that is referred to as naive learning. Seminally proposed by French (1956), and formalized by DeGroot (1974), naive learning is a learning process that is implemented as taking weighted averages. DeMarzo, Vayanos, and Zwiebel (2003) justified the weights as “persuasive bias” and fitted the theory model with empirical findings. Golub and Jackson (2007) also adopted a naive learning framework and conform to the findings of Bala and Goyal (1998) in that influential agents can mislead the whole society into suboptimal actions.

The weights in a naive learning process is represented by a row-stochastic matrix, i.e. the sum of row elements equals to 1 for all rows and all elements are positive. Namely, for n agents we have an $n \times n$ matrix $\mathbf{T}$, where $\mathbf{T}_{ij}$ indicates the weight that agent i places on agent j. This allows us to generalize the process as:

$$\mathbf{p}^{t+1} = \mathbf{T}^t \mathbf{p}^t, \tag{1.1}$$

where $\mathbf{p}^t = (p_1^t, p_2^t, \ldots, p_n^t)^{\mathrm{T}} \in \mathbb{R}^n$ is a vector that represents opinions of all agents at time t and $\mathbf{T}^t$ is the weight matrix at time t.

Krause extended the formula with a mapping from originally collected information to opinions, as follows (Krause, 2000):

$$F(\mathbf{p}^{t+1}) = \mathbf{T}^t F(\mathbf{p}^t). \tag{1.2}$$

While (1.1) defines the learning process as taking arithmetic averages, Krause's extended formula allows other forms of means. For instance, if we can define $F_i(\mathbf{p}^t) = f(p_i^t)$, where F_i and p_i are the i-th element of the vector $F(.)$ and $\mathbf{p}^t$. Then agents are taking geometric means if we impose an algorithm with $f(r) = \ln r$ and power means with $f(r) = r^\alpha$, $(\alpha \neq 0)$.

Regardless of the algorithm of learning, studies in social learning share two key questions. Namely, consensus and efficiency (e.g. see Ellison and Fudenberg, 1993, 1995; Bala and Goyal, 1998, 2001; Gale and Kariv, 2003; Banerjee and Fudenberg, 2004; Golub and Jackson, 2007). Consensus is defined as the outcome where the learning process leads to convergence to the same opinion; it is also referred to as conformism or conformity. Efficiency is measured when there exists a "true value" of opinions (beliefs), or if agents are formalizing strategies to maximize payoffs through the learning process. These are also the questions that I explore in the following chapters.

The variation presented in this dissertation features a time-varying weight matrix that represents trust and influence (Pan, 2008), and a learning mechanism that considers both network structure and proximity of opinions (Pan and Gilles, 2009; Pan, 2009a). In particular, an influence weight matrix that changes over time is a novel setting, in that the counterparts of it is usually considered constant (French, 1956; Bala and Goyal, 1998; Friedkin and Johnsen, 1997; DeMarzo, Vayanos, and Zwiebel, 2003; Golub and Jackson, 2007). With the basic model and several extended models presented in this dissertation, consensus is reached at the steady state. This is an interesting finding for two reasons. First, there is no "true value" of opinions/beliefs in my approach. Agents start with arbitrary opinions and are not trying to have an accurate estimate of a same specific value (whereas it is the case in Bala and Goyal, 1998; DeMarzo, Vayanos, and Zwiebel, 2003; Golub and Jackson, 2007). Secondly, the principle of updating weights is that one places higher weights on opinions that are closer to one's own. Thus, since an agent does not pay much attention to remote opinions, one might guess that diversity would be the final outcome, where we have isolated island of different opinions. It is not true; we observe the opposite. And consensus in my model is proven both mathematically and by simulation.

I further show that not all the conditions of consensus proposed by Lorenz (2005) need to satisfied for conformity to occur. Lorenz argued that one should place strictly positive weight on oneself and one cannot neglect another who assigns a positive weight on him.[1] That is, the weight matrix must have strictly positive diagonal elements and zero elements should be symmetric. The positive diagonal elements assumption is also imposed in Golub and Jackson (2007). However, I show that it does not need to be the case for reaching consensus. We can have all zero diagonal elements and asymmetric zero elements, and still see agent converge to hold the same opinion through learning.

There is considerable overlap between social learning and the literature of contagion, in that information spreads with a pattern that is similar to how population get infected by disease. Steiner and Stewart (2008) stated that the two could be integrated into a unified interaction framework developed by Morris (1997). Vega-Redondo (2007) provides a survey on contagion literature at the intersection of social networks and economics. In some recent research, interpersonal influence is referred to as "social contagion" (Watts and Dodds, 2007; Gallegati, Greenwald, Richiardi, and Stiglitz, 2008; Lópes-Pintad, 2008). The basic idea is that (susceptible) agents get infected with certain probability or if a threshold of proportion of infection is reached. In a social context, this process may be interpreted as social behavior through local imitation, resulting from observation of others (Ellison and Fudenberg, 1993, 1995; Bala and Goyal, 1998). The flow which often results in a domino-like phenomenon, which is called a "cascade" (Bikhchandani, Hirshleifer, and Welch, 1998; Watts, 2002).

With regard to contagion, I introduce a minority group of "persistent" agents that have different updating rules and consistently spread some specific information across the network. The findings are that even if these agents only consist of a very small percentage of the whole community, they show great power in influencing the final outcome of the learning and interaction process. I analyzed the question of threshold in this case. The conclusion is that one persistent agent is enough to show this pattern, regardless of the size of the network. It is argued in Watts and Dodds (2007) that the key is not the power of a minority to be influential, but rather how influenceable the majority is. Consider the

[1] The third condition that needs to be satisfied is a minimum value of weights.

fact that the learning mechanism is quite naive, the argument does suggest an interesting angle to look into the results. That is, the non-persistent agents (majority) take the weighted average of opinions to update their own, regardless of the values of others' opinions or the interaction dynamics adopted by others. This indeed could be seemed as high influenceability.

As for efficiency of the final outcome, I adopt a game theory approach to examine the question. Related literature and my model will be discussed in the next subsection.

1.2.2 Games in Networks

Incorporation of network structure contributes to the study on microeconomic behavior in two ways. First, complete knowledge is one of the fundamental assumptions of game theory. However, it is problematic when one considers the complexity of social networks (Vega-Redondo, 2007). By imposing a network structure and requiring a connection between players to carry out strategic interactions, we represent the incomplete information, limited interaction, and bounded rationality. Secondly, social influence in networks enables us to examine not only the equilibrium, but also how players get to the equilibrium or disequilibrium states.

Kets (2008) presented a comprehensive study in game theory with network structures. Generally, for games in networks, each player is associated with a node (vertex) in the network, and the relationship between each pair of players is represented by links (edges). The connection may be directed or undirected to allow heterogeneity. It is reasonable to assume that a player chooses a fixed action, i.e., she takes the same action with all opponents at the same time. The work by Nowak and May (1993) shows a distinction between global and local interaction with deterministic strategy. Moreover, Berninghaus and Schwalbe (1996) proposed a theoretical model to show the impact of details of neighborhood structure on the steady states of game play in networks.

In the context of non-cooperative game theory, players do not take into account the structure of the network or the social position of other players. Rather, one tries to maximize one's payoffs given the behavior and strategies of those whom one is connected with.

Another scenario is that one also has some information or belief on the probability measure of the network and strategies of all, which is related to sampling and spatial games. Berninghaus, Haller, and Outkin (2007) presented a study in the setting of a threshold automata network. In their model, players play against the empirical distribution of the last strategies played by their neighbors. The best response is static with noise introduced into the system, so that players may play non-best responses by chance.

Game play in networks embeds dissemination of information based on which players formalize best responses. The information, which can be strategies, payoffs, or signals of neighbors, is collected and updated over time. Therefore, the process shares a lot in common with social learning. In previous research, the two are not really separated. Pan and Gilles (2009) proposed a dual social network framework that clearly distinguishes information collection and game play. Namely, players transmit and collect information in an information network; they play a coordination game with neighbors in an interaction network. The two networks are correlated but different in that the information network is fully connected and the interaction network is not. This dualistic system captures the fact that individuals have selected interaction while search for information from a broader source. Both networks are time-varying. The updating rule of information network combine the influence from neighbors and proximity of opinions. The updating process of interaction network represents formation and dynamics of networks, a topic that has attracted much attention.

With a game theory approach, links are formed by agents who are or of control of the nodes in networks. Benefit and cost are the main determinants in this process. Payoffs from the network such as getting information or sharing reciprocal benefits are part of the model, modeled by utility of agents. The cost of links may be born by the one who initiates the connection or both ends of a connection.

Formation of links with utilities and constraints can be traced back to modeling of personal contacts in labor markets. Boorman (1975) and Montgomery (1991) explored the tradeoff between costs and benefits from strong and weak ties. The costs of links in those works are modeled by one's limited time and other resources available; while the benefits refer to getting information or referral of job openings. With a more explicit non-cooperative game theory approach, Jackson and Wolinsky (1996) considered networks where agents derive utilities directly from the connections that they have. Agents form a new connection or maintain an existing connection if and only if payoffs from that specific link is no less than the cost of the link. Otherwise, agents either choose not to form a new link, or sever an existing link. Jackson and Wolinsky also formally defined concepts that focus on the efficiency and stability of the resulting network. A network is defined as pairwise stable if no player wants to sever a link and no two players both want to add a link. The class of pairwise stable networks can be further categorized into subsets with stronger definitions (Jackson and Watts, 2002b; Jackson, 2004; Bloch and Jackson, 2006). Gilles and Sarangi (2006) presented a model where agents have beliefs on beneficial links and anticipation of others' actions. They gave a complete characterization of the class of monadically stable networks, a strict subset of the class of pairwise stable networks and have very strong stability properties.

The second part of the definition of pairwise stable networks implies that consent need to be obtained to form a link. To the contrary, Bala and Goyal (2000) presented a noncooperative model on the network formation, which is often regarded as the Nash approach. They assumed that no consent is required for any agent to form a link with another. And while the formation of links is considerably easy, the benefit from the links can be very fruitful. However, this approach implies free access to sharing information and profits, which does not fit empirical observations well. In this dissertation, it is assumed that an agreement between two agents is needed for a link to be constructed. Network dynamics are embedded in the context of a coordination game. The conclusion is that the resulting network structure is determined by the cost of links. When the cost is sufficiently high, empty networks are stable. When the cost is sufficiently low, complete networks are stable. The model proposed in this dissertation is similar to the work by Jackson and Watts (1999), which is based on zero cost. Whereas in my model, the critical high and low values of costs can be calculated when the game presentation and initial conditions are given.

However, I show that the steady state is most likely not a Nash equilibrium. Goyal and Joshi (2006) modeled network formation from the standpoint of industrial organization.

Inefficient networks emerge from their model as well. Jackson (2006) referred to these theoretical conclusions as "a general tension between stability and efficiency". Currarini and Morelli (2000) showed more efficient networks with a formation model in the framework of sequential game to illustrate bargaining. The ability to bargain over payoffs from the network is indeed shown to be the key in determining the type of resulting networks.

Generally speaking, the modeling perspective of network dynamics based on game theory focuses on the incentives of agents, types of links, and efficiency of the resulting network. There is another literature on network dynamics that looks into the issue from a different standpoint. Namely, the literature of stochastic network formation which uses graph theory and a stochastic process. Jackson (2006) provides an insightful review on these two views on the field of network formation that have grown over the last few years and are now starting to be aware of each other. When summarizing the difference between the two approaches, Jackson states that graph theoretical modeling and analysis shows *how* observed networks at some given point in time might have resulted from some stochastic or mechanical process; whereas the economic literature with game-theoretical tools intends to answer the question of *why* certain mechanism follows that way. I review some remarkable works in the literature of stochastic network formation in the next subsection.

1.2.3 Stochastic Network Formation

Network theory has been a booming field that influences many science disciplines during the past two decades. Scholars are using complex and adaptive networks to explain many observed phenomena in such varied fields as physics, biology, and sociology. With the availability of computers and communication networks that allow data collection and analysis on a considerably large scale, there is a substantial new movement in network research. That is, the focus has been shifted away from the analysis of single small graphs and the properties of individual vertices or edges to the study of physical and statistical properties of large-scale graphs. Several representative complex networks have been studied empirically, such as the World Wide Web (Albert, Jeong, and Barabási, 1999), the network of movie stars (Barabási and Albert, 1999), and collaboration among researchers (Barabási, Jeong, Neda, Ravasza, Schubert, and Vicsek, 2002). Barabási (2003) is a good resource for those who are interested in complex networks and their properties.

One of the most famous practices on characterization of social networks is the 1967 small world experiment carried out by social psychologist Stanley Milgram which suggested that two random US citizens could be connected on average by six or less acquaintances (Travers and Milgram, 1969). This experiment also gives rise to the popular phrase "six degrees of separation". In short, in a small-world network, it is always possible for one to reach any one else by taking a relatively short path. Another characteristic of small-world networks is that the networks appear to be considerably dense. In a word, in a small-world network, your friends are very likely to be friends with each other, too.

Social networks also exhibit the scale-free structure. The term "scale-free" implies the notion "self-similarity". That is, if we take out a fraction of the network, it would represent the properties of the whole network. Scale-free networks have a degree distribution that follows power-law. In other words, the fraction $P(k)$ of nodes in the network having k connections to other nodes (i.e. with a degree of k) can be approximated with the distribution $P(k) \sim k^{-\gamma}$ where γ is a constant whose value is typically in the range between 2 and 3 in many real-world networks.

With this degree distribution in network, the "80-20 law" which we observe in many areas shows up as well: 80% of links are held by 20% of nodes. This idea is sometimes expressed more simply as the Pareto principle named after economist Vilfredo Pareto, which says that 20% of the population controls 80% of the wealth.[2] Following this observation, one of the common features of scale-free networks is the existence of hubs: some nodes have significantly high degrees while the rest have low degrees; those that dominate in edge connection are called hubs. Hubs dramatically influences the way a network operates. For example, random node failures have very little effect on a scale-free network's connectivity or effectiveness.

[2]Outside the field of economics, the power-law distribution may also be referred to as the Bradford distribution.

Consequently, one would wonder what is behind the fascinating features of those networks. The origin and dynamics of networks appeals to many. The models of network formation in the literature on stochastic graph theory build networks through a stochastic process where generation of links is associated with some probability or distribution, some additional rules or algorithms may be imposed with the random emergence of links. Armed with statistical physics, this approach examines the characteristics of complex networks of large scale.

A stochastic graph is a graph that is generated by some random process. The earliest model of stochastic network formation is the Erdős-Rényi random graph model (Erdős and Rényi, 1959). Namely, agents meet each other, and links will be formed with probability θ. The resulted random graph $\mathbf{g} = (V, E)$[3] will have m edges with probability $\theta^m(1-\theta)^{M-m}$, where $M = \frac{1}{2}n(n-1)$ is the maximum possible number of edges. As for the degree distribution, the probability that any given node i has k links equals to:

$$\begin{pmatrix} n-1 \\ k \end{pmatrix} \theta^k (1-\theta)^{n-1-k}. \tag{1.3}$$

Although the calculation of fraction of nodes with a certain degree involves correlation across nodes in that each link is associated with one node, with a large size n this kind of correlation vanishes. In this case, the possible link between nodes is only one out the $n-1$ that each might have. Namely, for large n and small θ, the binomial expression in (1.3) can be approximated by a Poisson distribution. Therefore, the fraction of nodes that have k links is approximately

$$\frac{e^{-(n-1)\theta}((n-1)\theta)^k}{k!}. \tag{1.4}$$

With the mechanism based on probability θ of creating links, a natural topic to investigate into is the relationship between θ and the properties of resulting networks. A useful indicator is the average number of links that a node in the network has, which is denoted by $\langle k \rangle \sim \theta n$. If $\langle k \rangle < 1$, a typical graph is composed of isolated trees. If

[3] Here V is the vertex (node) set and E is the collection of edges (links) in $\mathbf{g}$.

$\langle k \rangle > 1$, a giant cluster appears. The diameter of the graph equals the diameter of the giant cluster if $\langle k \rangle \geq 3.5$, and is proportional to $\ln(n)/\ln(\langle k \rangle)$. If $\langle k \rangle \geq \ln(n)$, almost every graph is totally connected. The diameters of the graphs having the same n and $\langle k \rangle$ are concentrated on a few values around $\ln(n)/\ln(\langle k \rangle)$ (Albert and Barabási, 2002).

Overall, Erdős-Rényi random networks exhibit a small average shortest path length along with a low clustering coefficient.[4] Recall the small-world pattern that we have in social networks. From a graph theory perspective, small-world networks have a small average shortest path length and a high clustering coefficient. Apparently, the Erdős-Rényi algorithm can not explain those characteristics.

Watts and Strogatz (1998) proposed a network formation mechanism with a rewiring process that generates high clustering. The Watts-Strogatz model takes two steps to construct a network that has small-world features. The first step is to construct a regular ring lattice, a graph with n nodes each connected to k neighbors ($\frac{k}{2}$ on each side). Then the next step is to rewire the existing edges with a probability θ. That is, we replace a node in an edge with another node that is chosen with a uniform probability.

The idea of Watts-Strogatz model is to start with a network that has high clustering, then the random rewiring process reduces the diameter of the network. The rewiring can be interpreted as linking individuals from long geographic distance away or two social groups, which provides individuals that are connected with those two access to each other. Of course, if we remove too many links during rewiring will harm the high clustering that we want to keep. How many links to rewire and whether it is appropriate for the two small-world features are based on the probability θ and the size n.

Watts (1999) pointed out a threshold defined by θ and n that determines the scale of the diameter of the network l. His finding implies that there exists a θ-dependent crossover size n^* such that if $n < n^*$, $l \sim n$; but if $n > n^*$, $l \sim \ln(n)$. The clustering coefficient of Watts-Strogatz networks takes the form with respect to mean instead of actual number of links and is proportional to $C(0)(1-\theta)^3$, where $C(0)$ is the clustering coefficient with no rewiring. Albert and Barabási (2002) did a more in-depth review on the issue.

[4] Please refer to Appendix A for definitions and explanations of path length and clustering coefficient.

The Watts-Strogatz model and several other variations (Newman, 2000) fit well with the statistical properties of small-world networks. But we are not able to generate networks with a power-law degree distribution by adjusting parameters of this family of models. In order to construct scale-free networks, two key elements are required: a growing size, and preferential attachment (Barabási and Albert, 1999).

Preferential attachment reflects the "rich get richer" property of scale-free networks, i.e., the nodes with high degree are more likely to get connected and consequently increase their degree. Barabási-Albert algorithm models preferential attachment as the probability of getting connected to newcoming nodes that join the network one at a time. Such probability of a node that has a degree of k_i is calculated as:

$$\theta(k_i) = \frac{k_i}{\sum_j k_j}. \tag{1.5}$$

Jackson and Rogers (2007) modeled preferential attachment with a two-phase network-based algorithm. During phase one, a newcomer meets a group of existing agents at random. Then during phase two, he meets a group of existing agents that are connected with those that he meets in the previous stage. While meeting both groups, the newcomer forms links with existing agents with certain probability. In this model, agents with more connections are more likely to meet the newcomer and thus more likely to get connected. In other words, the network-based mechanism realizes preferential attachment. The Jackson-Rogers model unifies networks of various types by adjusting the probabilities of meeting and connecting agents.

In this dissertation, I present a variation of Jackson-Rogers model. Instead of forming links with a given probability, agents choose to connect with those whose opinions are close enough to their own. The model attempts to mimic the thinking process of individuals in a social situation, in that people tend to get along better with others who share much in common than those who don't. When the distribution of opinion is known, this new rule of link formation can be transferred into probabilities. The resulting networks exhibit scale-free property. Agents with opinions closer to the mean value are promoted to become hubs with this algorithm.

1.3 Outline of the Dissertation

In Chapter 2, a naive social learning model is presented. Agents share opinions with each other and update their own by taking weighted average of all opinions. Over time, agents also update the influence weights that they place on others, primarily according to the distance between opinions. It is proven that the learning process leads to consensus. A group of persistent agents are introduced into the model. They do not perform a full update that others do. Although the proportion of persistent agents is small, the group shows significant influence over the final opinion of the whole society in that they are able to drag others to have the same opinion that they have.

A coordination game is incorporated into the learning process in Chapter 3. Players of the game interact only with those that they are connected in an interaction network. On the other hand, they collect information from all players in order to update their strategy. The updating process of strategy is similar to the social learning mechanism in Chapter 2, which includes updating process of the information network. The interaction network is also time-varying. Players form or sever links based on cost and benefit associated with the link. When cost is low enough, players end up building a complete network and converge to choose the same mixed strategy. Persistent players exist in this model as well. And they may be diverse in the sense that they have different initial strategies; whereas the persistent agents in Chapter 2 are uniform and have the same initial opinions. Diverse persistent players critically determine other players' final strategies.

Chapter 4 focuses on a naive learning process with various exogenous network structures. Agents consult their neighbors when deciding on influence weights to place on others. They also consider the distance between opinions of their neighbors and their own. This is called "network-distance combination" updating rule. The learning process leads to a steady state of conformity with convergence speed dependent on the network structure. In Chapter 4 we also study an endogenous network formation model that considers both the network effect and proximity between opinions. The algorithm results in a power-law degree distribution.

Chapter 2

Trust, Influence, and Convergence of Behavior in Social Networks [1]

We do not live like Robinson Crusoe, and interaction with others has an important role in our day to day lives (Barabási, 2003). The big successes of Facebook, MySpace and other similar online community utilities are clear evidence of how people want to be connected. Moreover, a social network acts as a platform and backbone of social activities which assist and enable dissemination of information that influences individuals' decision making process as well as the development of economies. Examples of how social networks affect individuals and communities arise in the diffusion of information and knowledge (Granovetter, 1973), stock market (Shiller, 1995), marketing (Chan and Misra, 1990; Vernette, 2004), and politics (Roch, 2005), just to name a few.

I propose a social learning model where agents continuously update their opinions by taking weighted averages of those of others'. Social influence during the learning process is modeled by a row-stochastic matrix, i.e. the sum of the elements in each row equals to 1. This kind of naive learning mechanism was seminally proposed by French (1956) and subsequently formalized by DeGroot (1974), which has been employed in multiple models since its introduction and recently adopted by Golub and Jackson (2007). DeMarzo, Vayanos, and Zwiebel (2003) justified this simplified framework as a

[1] This chapter is based on Pan (2008).

reflection of persuasion bias, which conforms to empirical findings. My variation of the model can be distinguished from others with three basic assumptions.

First, one novel feature of my model is a time-varying influence weight matrix. The influence indicator, or variables that play similar roles in social learning, is in most cases assumed to be constant over time (French, 1956; Bala and Goyal, 1998; DeMarzo, Vayanos, and Zwiebel, 2003; Friedkin and Johnsen, 1997; Golub and Jackson, 2007). My main hypothesis is that agents redistribute the influence weights they place on others and the influence matrix continuously gets updated over time. This assumption reflects the changes in attitude that people tend to make during social interaction. There has been studies on time-varying weight matrices, such as Hegselmann and Krause (2002) model and Weisbuch, Deffuant, Amblard, and Nadal (2002) model. The difference between their models and mine is that agents in their models only place positive weights on those with close opinions; whereas agents in my framework place positive weights even on the most remote opinion holders.

Second, in many social learning models, evolution of opinions is examined according to accuracy of estimation of a predesignated value or state (Bala and Goyal, 1998; DeMarzo, Vayanos, and Zwiebel, 2003; Golub and Jackson, 2007). Thus, in those models agents' opinions are referred to as *beliefs* in the sense that they are essentially estimates of the true value. In this chapter, there is no "true value" and agents are not aiming at getting close to such a "true" state. The reason is that a true value does not always exist in real life. For instance, when asked about opinions on food and music, there is no "right" answer. Even for some seemingly less subjective topics such as justifications of a war or creditability of a presidential candidate, the "truth" may never be known. Also, this assumption has made the convergence outcome more interesting, in the sense that individuals come to an agreement with a wider diversity of initial opinions.

Third, a lot of previous studies focused on observational learning. That is, agents observe others' payoff-maximizing decisions and update their own opinions accordingly (Bala and Goyal, 2001; Banerjee and Fudenberg, 2004; Gale and Kariv, 2003). The examination of convergence is thus based on optimality with regard to agents' utility functions. The basic learning model presented in this chapter differs from them in that there is no preference or utility function involved.[2] The assumption is made in views that individuals do not always share information for a specific payoff-related purpose. Rather, communication with others stems from our psychological needs for social interaction. Consequently, the main finding is that the convergence of my model crucially relies on the details of the interaction dynamics.

Social learning studies have focused on the final outcome of the learning process. Especially, whether the opinions and behavioral decisions of members in a society conform (Bala and Goyal, 1998; Banerjee and Fudenberg, 2004; Ellison and Fudenberg, 1993, 1995; Golub and Jackson, 2007). The core of this chapter is the convergence of both opinions and the influence distribution. There are two main questions: 1) what are the conditions for convergence; and 2) what are the characterizations of the final outcomes. For instance, if we observe consensus in the steady state.

With a time-invariant matrix, the matrix is defined to be convergent if and only if the opinions are (Golub and Jackson, 2007). In contrast to that, I prove that convergence of either factor does not guarantee that of the other when both change over time. On the other hand, convergence of both occurs when the details of updating rules are specified. The patterns of final opinions show strong conformism. That is, not only do opinions converge, all agents' opinions converge to the same value. With the given interaction dynamics, the influence weights converges to be equally distributed among agents.

With a time-varying matrix, Lorenz (2005) generalized three conditions for such conformity to emerge. He argued that the sequence of weight matrices need to satisfy all the three conditions to reach consensus, including positive diagonal elements of the matrix and symmetric zero elements. However, I show that in the steady state of my model, diagonal elements converge to zero. Also, even if the diagonal elements are always zero during each period and we have asymmetric zero elements, consensus can still be reached under certain circumstances.

An extended model was examined to better understand the interaction effects. The idea is to introduce a group of so-called "persistent" agents that only perform limited interactions with others. It is shown that persistence on opinion affects more on the

[2] Learning and game play will be studied in the next chapter.

convergence path than persistence on influence weights does. Moreover, persistent agents can significantly alter the final opinion of the whole society; while it is assumed that they only constitute a minority group of the whole population (less than 10%). For that reason, my model shows cascade-like phenomena and support the *opinion leaders* argument of Katz and Lazarsfeld (1955).

I have simulated the complete updating processes for both the basic and extended models. The outcomes greatly support the theoretical prediction. The convergence behavior is very robust, consistent, and independent of initial conditions.

The chapter is organized as follows. The basic model and updating rules are developed in section 2. Section 3 formalizes the patterns and conditions of equilibrium and convergence. In section 4, the persistent agent experiment is introduced. Section 5 illustrates simulation results. Section 6 concludes. Proofs and discussions are presented in Appendix B.

2.1 The Model

2.1.1 Agents, Opinions, and Influence

A finite set N={1, 2, ..., n} of agents interact and share *opinions*, which are represented by a n×1 vector $\mathbf{p}^t = (p_1^t, p_2^t, \ldots, p_n^t)^{\mathrm{T}} \in \mathbb{R}^n$, where p_i^t is agent i's opinion at time t.[3]

A n×n nonnegative matrix $\mathbf{T}$ is referred to as the *influence* matrix. $\mathbf{T}^t$ captures the interaction patterns at time t, i.e., for all $i, j \in N$, $\mathbf{T}_{ij}^t \in [0,1]$ indicates the influence weight that agent i places on agent j's opinion at time t. Also, the influence matrix is row-stochastic, i.e.

$$\sum_{j=1}^{n} \mathbf{T}_{ij}^t = 1 \text{ and } \mathbf{T}_{ij}^t \geq 0, \text{ for all } i, j \in N, \text{ for all } t. \tag{2.1}$$

Moreover, $\mathbf{T}^t$ may be asymmetric, so that $\mathbf{T}_{ij}^t \neq \mathbf{T}_{ji}^t$ for some i, j.

[3]Here p_i^t is scalar; whereas in DeMarzo, Vayanos, and Zwiebel (2003), each agent has a vector of opinions and does multiple estimates, which are interpreted as multidimensional opinions. Their paper also shows that the multidimensional opinions are convergent and thus can be represented on a unidimensional scale.

At this stage, the basic framework settings of agents, opinions and influence are on the same page as Golub and Jackson (2007), except that $\overline{\mathbf{T}}$ in their paper, referred to as the *interaction* matrix, is constant over time.[4]

2.1.2 Updating Processes

At $t = 0$, both opinions and influence weights are arbitrary. That is, each agent i is endowed with an arbitrary initial opinion p_i^0 drawn from a given range. Unlike other social learning models, such as Golub and Jackson (2007), opinions in this model are not correlated with a predesignated "true value" or "real state". Also, each agent receives an arbitrary influence assignment, which is a row vector with all elements adding up to be 1. The initialization can be formalized as below:

$$\mathbf{p}^0 = (p_1^0, \ldots, p_n^t)^{\mathrm{T}}, \text{ where } p_i^0 \in [\underline{M}, \overline{M}] \text{ for all } i \in N; \tag{2.2}$$

$$\sum_{j=1}^{n} \mathbf{T}_{ij}^0 = 1, \quad \mathbf{T}_{ij}^0 \in [0, 1] \text{ for all } i, j \in N. \tag{2.3}$$

Updating Influence Weights

For $t > 0$, the key assumption in redistribution of influence is that an agent places more weights on others who share similar opinions with him. This idea is represented by the concept of distance, which is a simple way to measure the closeness of two agents' opinions. At time t, the distance between agents i and j is defined as:

$$d_{ij}^t = |p_i^t - p_j^t|.$$

And the basic idea of redistributing influence is:

$$\mathbf{T}_{ij}^{t+1} \propto \frac{1}{d_{ij}^t}, \text{ for } j \neq i. \tag{2.4}$$

Note that d_{ii}^t is always zero. Also, for agents i, j with the same opinion at time t,

[4] I use $\overline{\mathbf{T}}$ to denote a time-invariant influence matrix and $\mathbf{T}$ as a influence matrix in general.

$d_{ij}^t = 0$. Applying equation (2.4) is problematic in these cases. To solve this problem of zero d_{ii}^t, it is further assumed that an agent i does not directly change the influence he places on himself. Instead, he only updates the weights on other agents according to the distances. Then agent i normalizes all the weights (including that he puts on his own) to satisfy equation (2.1), which gets $\mathbf{T}_{ii}^t$ modified. As for the cases where agents have very close opinions, distances less than a small positive number $\underline{d}$ are taken as the same as $\underline{d}$. Thus, the redistribution rule can be written as:

$$\mathbf{T}_{ii}^{t+1} = \frac{\mathbf{T}_{ii}^t}{\mathbf{T}_{ii}^t + \sum_{j \in N_{-i}} w_{ij}^t}, \qquad \text{for all } i \tag{2.5}$$

$$\mathbf{T}_{ij}^{t+1} = \frac{w_{ij}^t}{\mathbf{T}_{ii}^t + \sum_{j \in N_{-i}} w_{ij}^t}, \qquad \text{for all } i \text{ and } j \neq i, \tag{2.6}$$

where $w_{ij}^t = \dfrac{1}{d_{ij}^t} = \dfrac{1}{\max(\underline{d}, |p_i^t - p_j^t|)}$, $\underline{d}$ is a very small positive number that prevents taking inversion of 0 when $p_i^t = p_j^t$.

Updating Opinions

Then, for $t > 0$, each agent takes a weighted average of others' current opinions in forming his own for the next period. That is, we have the opinions updating rule as:

$$\mathbf{p}^t = \mathbf{T}^t \mathbf{p}^{t-1} \qquad \text{for } t > 0. \tag{2.7}$$

The updating rule in Golub and Jackson (2007) takes a similar form, which is:

$$\mathbf{p}^t = \overline{\mathbf{T}}\mathbf{p}^{t-1} = \overline{\mathbf{T}}^t \mathbf{p}^0 \qquad \text{for } t > 0.$$

However, note that in their model, $\overline{\mathbf{T}}^t$ refers to the t-th power of $\overline{\mathbf{T}}$, i.e. $\overline{\mathbf{T}}^t = \underbrace{\overline{\mathbf{T}} \times \overline{\mathbf{T}} \times \ldots \times \overline{\mathbf{T}}}_{t}$. Here in this model, $\mathbf{T}^t$ represents the influence matrix at time t, which changes over time and is not computed by raising some initial matrix to powers. The Golub-Jackson rule can be as one of the equivalent forms of my updating rule. That is, by assuming that $\mathbf{T}$ does not change over time, $\mathbf{T}^t = \overline{\mathbf{T}}$, for all t, then $\mathbf{p}^t = \overline{\mathbf{T}}\mathbf{p}^{t-1} = \mathbf{T}^t\mathbf{p}^0$

coincides with (2.7). More details on redistribution process of influence will be discussed later.

In DeMarzo, Vayanos, and Zwiebel (2003), the learning process is summarized as

$$\mathbf{p}^t = [(1-\lambda_t)\mathbf{I} + \lambda_t\overline{\mathbf{T}}]\mathbf{p}^{t-1},$$

which can also be transformed to (2.7) by defining $\mathbf{T}^t = (1-\lambda_t)\mathbf{I} + \lambda_t\overline{\mathbf{T}}$.

The updating rule in Friedkin and Johnsen (1997) is:

$$\mathbf{p}^t = \mathbf{D}\overline{\mathbf{T}}\mathbf{p}^{t-1} + (\mathbf{I}-\mathbf{D})\mathbf{p}^0,$$

where $\mathbf{D}$ is an $n \times n$ matrix with positive diagonal and zero elements elsewhere. The formula can be rewritten as:

$$\mathbf{p}^t = [\mathbf{D}\overline{\mathbf{T}}]^t\mathbf{p}^0 + \sum_{i=0}^{t-1}[\mathbf{D}\overline{\mathbf{T}}]^i(\mathbf{I}-\mathbf{D})\mathbf{p}^0.$$

Note that (2.7) can be written as:

$$\mathbf{p}^t = \mathbf{T}^t \cdot \mathbf{T}^{t-1} \cdots \mathbf{T}^1\mathbf{p}^0.$$

The Friedkin-Johnsen setup can be interpreted as a special case of equation (2.7) with $\mathbf{T}^t = f(t) \times [f(t-1)]^{-1}$, where $f(t) = [\mathbf{D}\overline{\mathbf{T}}]^t + \sum_{i=0}^{t-1}[\mathbf{D}\overline{\mathbf{T}}]^i(\mathbf{I}-\mathbf{D})$.[5]

A period ends after both the opinion vector and influence matrix are properly updated.[6] The updating process repeats during each period, as shown in Figure 2.1. So then the question is: does the learning process go anywhere? Or in other words, do opinions and/or the influence matrix converge?

[5]The $\mathbf{T}$ matrices in Golub and Jackson (2007) and DeMarzo, Vayanos, and Zwiebel (2003) are both stochastic as well. However, it is not necessarily the case in Friedkin and Johnsen (1997).

[6]Initially, following the convention of evolution literature, it was also assumed that mutation actions would be taken. That is, with probability θ, an agent will violate the redistribution rules when updating weights. For example, he might double the influence he places on certain agents, regardless of the actual distances. However, simulation results have shown sound evidence that this mutation process has no significant effects on the final outcomes. Thus, in this model $\theta = 0$.

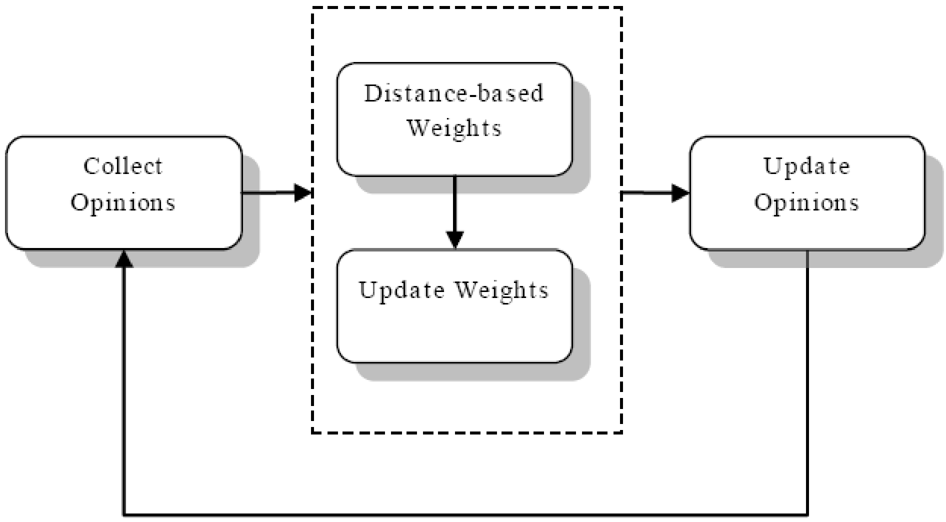


Figure 2.1: Updating process of the basic learning model

2.2 Convergence

2.2.1 Preliminaries

With the basic ideas of influence redistribution, agents tend to place higher weights on similar opinions and those that are relatively different from one's current opinions will have lower weights. So one might guess that diversity would emerge and that it would be possible to sustain islands of different opinions, since an agent does not pay much attention to faraway opinion. However, the opposite is true. That is, although there is no "true value" of opinions in this framework, eventually agents will approach to hold approximately the same opinion. First, convergence of opinion and the characteristics of the final outcome are defined as follows.

Definition 1. An opinion vector sequence $\{\mathbf{p}^t\}_{t=1}^{\infty}$ is *convergent* if $\lim_{t\to\infty} \mathbf{p}^t$ exists. i.e. there exists $\mathbf{p}^* \in \mathbb{R}^n$ s.t. for all $\epsilon > 0$, there exists $t^* > 0$ s.t. $\parallel \mathbf{p}^t - \mathbf{p}^* \parallel \leq \epsilon$ for $t > t^*$, where for a $1 \times n$ vector x, $\parallel \mathrm{x} \parallel = \sum_{i=1}^{n} x_i^2$ is defined as its norm.

Definition 2. An opinion vector sequence $\{\mathbf{p}^t\}_{t=1}^{\infty}$ is *conforming* if there exists $p^* \in \mathbb{R}$ s.t. for all $\epsilon > 0$, there exists $t^* > 0$ s.t. $|p_i^t - p^*| \leq \epsilon$ for $t > t^*$, for all $i \in N$.

Note that $\mathbf{p}^*$ is a vector, whereas p^* is a number. That is, conformism means that in the long run, all agents' opinions converges to the same p^*, which is a stronger definition than convergence. Especially, if at certain time t we have $p_i^t = p^*$ for all i, then the influence distribution will no longer have any effect on opinions, in that the weighted average will always remain to be p^*. That is, the society reaches a steady stage in a common opinion.

On the other hand, since each element in the influence matrix represents the level of influence one has on others, it is interesting to examine the convergence of influence for two reasons. First, whether the influence matrix shows convergence or not is an appealing question. Recall that we have an arbitrary initial $\mathbf{T}^0$, the question becomes whether order emerges out of chaos. Second, if $\mathbf{T}^t$ converges, the patterns of the final influence distribution can be interpreted as different forms of institutions, including culture and social contexts. For instance, if some agents receive higher weights than others, or if we have exclusive groups of agents that only place positive weights on other members in the same group.

The definition of convergence of influence mainly focuses on the first question, i.e. whether $\mathbf{T}^t$ stabilizes alone the updating process.

Definition 3. A influence matrix sequence $\{\mathbf{T}^t\}_{t=1}^{\infty}$ is *convergent* if $\lim_{t\to\infty} \mathbf{T}^t$ exists. i.e. there exists $\mathbf{T}^*$ such that for all $\epsilon > 0$, there exists $t^* > 0$ s.t. $\parallel \mathbf{T}^t - \mathbf{T}^* \parallel \leq \epsilon$ for $t > t^*$, where for a $l \times m$ matrix M, $\parallel \mathrm{M} \parallel = \sum_{i=1}^{l} \sum_{j=1}^{m} (\mathrm{M}_{ij})^2$ is defined as its norm.

Based on these definitions, conditions for convergence and characterizations of the final patterns are discussed in the next subsection.

2.2.2 Characterizations and Conditions

In the standard Markov theory literature, it is addressed that a transition matrix is convergent if and only if $\lim_{t\to\infty} \overline{\mathbf{T}}^{(t)}\mathbf{p}$ exists for all vectors $\mathbf{p}$ [7]. As it has been pointed out

[7] Here $\overline{\mathbf{T}}^{(t)}$ is the power function of $\mathbf{T}$ (Golub and Jackson, 2007), whereas the notation $\mathbf{T}^t$ in this model indicates $\mathbf{T}$ at time t. The two do not necessarily equal.

in Section 2, $\overline{\mathbf{T}}^{(t)}\mathbf{p}$ can be interpreted as the updated $\mathbf{p}^t$ with a time-invariant $\overline{\mathbf{T}}$ matrix and initial value of $\mathbf{p}$. On the other hand, in my model where $\mathbf{T}^t$ is time-varying, $\mathbf{p}^t = \mathbf{T}^t\mathbf{p}^{t-1} = \mathbf{T}^t \cdot \mathbf{T}^{t-1} \cdots \mathbf{T}^1\mathbf{p}^0$, which makes the setting different and we can not directly apply the convergent condition in this case.

Theorem 2.1. *For* $\mathbf{p}^t = \prod_{\tau=t}^{1} \mathbf{T}^\tau \mathbf{p}^0$*, the existence of* $\lim_{t\to\infty} \mathbf{T}^t$ *does not imply that* $\{\mathbf{p}^t\}_{t=1}^{\infty}$ *is convergent and vice versa.*

Theorem 2.1 states that in general, if both $\mathbf{p}^t$ and $\mathbf{T}^t$ change over time, then we cannot guarantee the equivalence of convergence between the two. Note that Theorem 2.1 does not impose any restrictions on $\mathbf{T}^t$. On the other hand, the next theorem shows that with the distance-based updating rule, conformism emerges.

Theorem 2.2. *For* $n > 2$*, with updating rules defined,* $\{\mathbf{p}^t\}_{t=1}^{\infty}$ *is conforming for all row-stochastic* $\mathbf{T}^0$ *and all* $\mathbf{p}^0$*.*

Theorem 2.2 applies to societies with more than 2 members. The case where $n = 1$ is trivial. For $n = 2$, the opinion vector $\mathbf{p}^t$ is *not* confirming if $\mathbf{T}^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Because in this case the 2 agents will be switching opinions and flip-flopping between the 2 initial opinions. For the case where $n > 2$, please refer to Appendix B for proof of convergence.[8]

Intuitively, with the distance-based updating rule, equity in influence emerges as agents approach to closer opinions. Theorem 2.2 states that conformism occurs, which means that the tentative new weights w_{ij}^t all reach the same upper bound value $\frac{1}{\underline{d}}$. Then after normalization, all $\mathbf{T}_{ij}^{t+1}$ are indeed equal.

Theorem 2.3. *For* $n > 2$*, with the updating rules defined,* $\{\mathbf{T}^t\}_{t=1}^{\infty}$ *is convergent for all row-stochastic* $\mathbf{T}^0$ *and all* $\mathbf{p}^0$*. Namely,* $\lim_{t\to\infty} \mathbf{T}^t = \mathbf{T}^*$ *where* $\mathbf{T}_{ii}^* = 0, \mathbf{T}_{ij}^* = \dfrac{1}{n-1}$*, for all* i *and* $j \neq i$*.*

Theorem 2.3 claims that with the distance-based redistribution rule for the influence matrix and weighted average approach for opinion updating, the society converges to a

[8]I appreciate Dr. Hans Haller's contribution to the proof of Theorem 2.2.

uniform state that agents hold the same belief; and they also place the same weight on all the other agents. Moreover, with conformism, the weighted average of opinions of each agent remain p^*. Consequently the weights remains the evenly distributed pattern, which means it has indeed reached a steady state. Besides, the updating rule presented in equation (2.5) implies that the final influence matrix converges to have zero diagonal elements.

2.3 Interaction Effects: An Experiment

2.3.1 Persistent Agents

Obviously the updating rules that define the information exchange among agents have significant impact on the final outcomes. I introduce a minority group of so-called "persistent agents" to examine the effects that the updating rules have. Those agents are "persistent" in the sense that they insist on the initial weights they assign to others, or their own opinions, or both. We may see them as individuals who do not have access to others' opinions, or do not want to make any changes. There are three different types of persistent agents.

Type I persistent agents do not update their influence weights. However, type I persistent agents follow the opinions updating rules and take weighted average of others' opinions. Type II persistent agents insist on their initial opinions. Although they would redistribute influence, those weights practically have no impact on their opinions. The third type of persistent agents are called "double-persistent" agents, in that they update neither influence nor opinions. Note that different types of persistent agents do not co-exist.

Furthermore, to better illustrate the learning effects, the society is assumed to be polarized, i.e. at $t = 0$, instead of having arbitrary initial opinions, non-persistent and persistent agents hold 2 different opinions, and the members in each category share the same initial opinion.

2.3.2 Modified Interaction Dynamics and Convergence

In this extended model, we have a finite set $N = \{1, 2, \ldots, n\}$ of agents, $\sigma \geq 1$ of which are persistent and do not fully interact with others. Since the persistent agents are a minority in the society, we assume that $\frac{\sigma}{n} \leq 10\%$. Without loss of generality, we arrange the order of the agents in such a way that number 1 to $n - \sigma$ agents are non-persistent and the rest σ are persistent.

At $t = 0$, $p^0_{\overline{s}} = p^0_1$ for all $\overline{s} \in \overline{S}$;[9] $p^0_s = p^0_2 \neq p^0_1$ for all $s \in S$. And an arbitrary positive row-stochastic $n \times n$ influence matrix $\mathbf{T}^0$ shows the initial weight assignment.[10]

For $t > 0$, there are 3 variations of the interaction dynamics in this model, respectively to the 3 different types of persistent agents.

Type I persistent Agents

We have $S_1 \subset N$, $|S_1| = \sigma_1$. For $t > 0$, $\mathbf{T}^t$ and $\mathbf{p}^t$ are updated based on following rules.

i) For $\overline{s} \in \overline{S_1}$, the updating rules for $\mathbf{T}^t$ take the same form as equation (2.5) and equation (2.6):

$$\mathbf{T}^{t+1}_{\overline{s}\overline{s}} = \frac{\mathbf{T}^t_{\overline{s}\overline{s}}}{\mathbf{T}^t_{\overline{s}\overline{s}} + \sum_{j \in N_{-\overline{s}}} w^t_{\overline{s}j}}, \tag{2.5'}$$

$$\mathbf{T}^{t+1}_{\overline{s}j} = \frac{w^t_{\overline{s}j}}{\mathbf{T}^t_{\overline{s}\overline{s}} + \sum_{j \in N_{-\overline{s}}} w^t_{\overline{s}j}} \text{ for all } j \neq \overline{s}, \tag{2.6'}$$

where $w^t_{\overline{s}j} = \frac{1}{d^t_{\overline{s}j}} = \frac{1}{\max(\underline{d}, |p^t_{\overline{s}} - p^t_j|)}$.

ii) For $s \in S_1$ we have:

$$\mathbf{T}^t_{sj} = \mathbf{T}^0_{sj}, \text{ for all } j. \tag{2.8}$$

And opinion updating is the same as that in the basic model indicated by equation

[9] Here for a set $S, \overline{S} = N - S$.

[10] Note that for this variation, the initial influence matrix is assumed to be positive instead of non-negative. Because for one special case where $\mathbf{T}^0_{s\overline{s}} = 0$ for all $s \in S_1$ and $\overline{s} \in \overline{S_1}$, the persistent agents are essentially of type II instead of type I.

(2.7), i.e. $\mathbf{p}^t = \mathbf{T}^t\mathbf{p}^{t-1}$ for $t > 0$.

Theorem 2.4. *For societies with type I persistent agents, $\{\mathbf{p}^t\}_{t=1}^{\infty}$ is conforming, i.e. $\lim_{t\to\infty} \mathbf{p}^t = (p^*, \ldots, p^*)^{\mathrm{T}}$. Also, $\{\mathbf{T}^t\}_{t=1}^{\infty}$ is convergent, for all non-persistent agents $\overline{s} \in \overline{S_1}$, for all $s' \neq \overline{s}$, $\lim_{t\to\infty} \mathbf{T}^t_{\overline{s}\overline{s}} = 0$ and $\lim_{t\to\infty} \mathbf{T}^t_{\overline{s}s'} = \dfrac{1}{n-1}$.*

Proposition 2.5. *Consider the updating process with type I persistent agents, during each round $t > 0$, the change of opinion for a non-persistent agent is less than $\underline{d}$, i.e. $|p^{t+1}_{\overline{s}} - p^t_{\overline{s}}| < \underline{d}$ for all $\overline{s} \in \overline{S}$.*

That is, similar to the basic model, the initial polarized opinions in a society with type I persistent agents conform to the same value, which results in equal distribution of influence weights. In addition, Proposition 2.5 states that the change in non-persistent agent during each round is bounded by $\underline{d}$. Recall that $\underline{d}$ is used to prevent taking inverse of zero-distance when applying equation (2.6); it is assumed to be a small number. Simulation results show that the final opinion in this case is closer to non-persistent agents' initial opinion. More on this issue and proofs of Theorem 2.4 and Proposition 2.5 can be found in Appendix B.

Type II persistent Agents

We have $S_2 \subset N$, $|S_2| = \sigma_2$. For $t > 0$, $\mathbf{T}^t$ and $\mathbf{p}^t$ are updated based on following rules.

For all $i \in N$, the updating process of $\mathbf{T}^t$ is the same as it is in the basic model, i.e. we apply equation (2.5) and equation (2.6):

$$\mathbf{T}^{t+1}_{ii} = \frac{\mathbf{T}^t_{ii}}{\mathbf{T}^t_{ii} + \sum_{j\in N_{-i}} w^t_{ij}}, \text{ for all } i, \tag{2.5}$$

$$\mathbf{T}^{t+1}_{ij} = \frac{w^t_{ij}}{\mathbf{T}^t_{ii} + \sum_{j\in N_{-i}} w^t_{ij}} \text{ for all } j \neq i, \tag{2.6}$$

$$\text{where } w^t_{ij} = \frac{1}{d^t_{ij}} = \frac{1}{\max(\underline{d}, |p^t_i - p^t_j|)}.$$

Then, agents in S_2 do not update their opinions, i.e.

i) For $\overline{s} \in \overline{S_2}$, we use equation (2.7):

$$p_{\overline{s}}^t = \sum_{j=1}^{n} \mathbf{T}_{\overline{s}j}^t \cdot p_j^{t-1}. \tag{2.7'}$$

ii) For $s \in S_2$, persistent agents insist on their initial opinions:

$$p_s^t = p_s^0. \tag{2.9}$$

Theorem 2.6. *For societies with type II persistent agents, $\{\mathbf{p}^t\}_{t=1}^{\infty}$ is conforming to persistent agents' initial opinion, i.e. $\lim_{t\to\infty} \mathbf{p}^t = (p_s^0, \ldots, p_s^0)^{\mathrm{T}}$. Also, $\{\mathbf{T}^t\}_{t=1}^{\infty}$ is convergent, $\lim_{t\to\infty} \mathbf{T}_{ii}^t = 0$ and $\lim_{t\to\infty} \mathbf{T}_{ij}^t = \dfrac{1}{n-1}$, for all $i \in N$, for all $j \neq i$.*

Theorem 2.6 states that type II persistent agents are able to drive the whole society to converge to their initial opinions, by not changing their views. This shows the dominant influential power that a minority group have and reflects the interesting fact that when people are constantly exposed to something, then they are highly likely to believe what they have been told, even though they used to have a completely different point of view.

Double-persistent Agents

We have $S_3 \subset N$, $|S_3| = \sigma_3$. For $t > 0$, $\mathbf{T}^t$ and $\mathbf{p}^t$ are updated based on following rules.

i) For $\overline{s} \in \overline{S_3}$, we use equation (2.5) and equation (2.6) to update $\mathbf{T}^t$:

$$\mathbf{T}_{\overline{s}\overline{s}}^{t+1} = \frac{\mathbf{T}_{\overline{s}\overline{s}}^t}{\mathbf{T}_{\overline{s}\overline{s}}^t + \sum_{j \in N_{-\overline{s}}} w_{\overline{s}j}^t}, \tag{2.5'}$$

$$\mathbf{T}_{\overline{s}j}^{t+1} = \frac{w_{\overline{s}j}^t}{\mathbf{T}_{\overline{s}\overline{s}}^t + \sum_{j \in N_{-\overline{s}}} w_{\overline{s}j}^t} \text{ for all } j \neq \overline{s}, \tag{2.6'}$$

where $w_{\overline{s}j}^t = \dfrac{1}{d_{\overline{s}j}^t} = \dfrac{1}{\max(\underline{d}, |p_{\overline{s}}^t - p_j^t|)}$.

Updating rule of opinions follows equation (2.7):

$$p_{\overline{s}}^t = \sum_{j=1}^{n} \mathbf{T}_{\overline{s}j}^t \cdot p_j^{t-1}. \tag{2.7'}$$

ii) For $s \in S_3$, double-persistent agents do not update influence weights or opinions, i.e. we apply equation (2.8) and equation (2.9):

$$\mathbf{T}_{sj}^t = \mathbf{T}_{sj}^0, \text{ for all } j; \text{ and } p_s^t = p_s^0.$$

Theorem 2.7. *For societies with double-persistent agents, $\{\mathbf{p}^t\}_{t=1}^{\infty}$ is conforming to persistent agents' initial opinion, i.e. $\lim_{t\to\infty} \mathbf{p}^t = (p_s^0, \ldots, p_s^0)^{\mathrm{T}}$. Also, $\{\mathbf{T}^t\}_{t=1}^{\infty}$ is convergent, $\lim_{t\to\infty} \mathbf{T}_{\overline{s}\overline{s}}^t = 0$ and $\lim_{t\to\infty} \mathbf{T}_{\overline{s}s'}^t = \dfrac{1}{n-1}$, for all $\overline{s} \in \overline{S_1}$, for all $s' \neq \overline{s}$.*

This variation is a combination of the other two. The notable statement made here is that the persistence on initial opinions has more impact on the final outcome than the persistence on initial influence weight does. So if persistent agent insist on their initial opinion, then no matter whether they update their influence weights or not, the final opinion conforms to their initial view.

In summary, the existence of type I persistent agents does not show significant impact on the outcome of the learning. Simulation evidence will be presented in the next section. On the other hand, type II and double-persistent agents are essentially the same and they show influence power that can "brainwash" the normal agents. One of the assumptions we have is that all persistent agents have the same opinion. It would be interesting to relax this assumption and see if persistent agent with different opinions will lead the population into diversity. This question will be explored in the next chapter.

2.4 Simulation

Theorem 2.2 implies that the learning converges to a steady state. When running simulations of the updating process, we need to have find a way to detect the steady state and terminate the simulation. Convergence of both opinions and the influence matrix indicates that the learning is stable. Also, I have found that opinions $\{\mathbf{p}^t\}_{t=1}^{\infty}$ converge significantly faster than the influence matrix $\{\mathbf{T}^t\}_{t=1}^{\infty}$. Thus, the criterion for deciding the termination is based on the convergence of the influence matrix. In practice, the concept of Cauchy formula is utilized. The change in influence weights is illustrated by the norm of the difference between the two influence matrices that are Δ periods apart. Define $\Delta\mathbf{T}^t = \mathbf{T}^t - \mathbf{T}^{t-\Delta}$; then the society is considered to have reached its steady state when $\parallel \Delta\mathbf{T}^t \parallel < \epsilon^t$, i.e., the change in the matrix is adequately small. In other words, $\mathbf{T}^t$ is stable.

Initially, we set a small positive number ϵ^0. Then after every $2n$ rounds at time t, if $\parallel \Delta\mathbf{T}^t \parallel < \epsilon^t$, convergence is detected and outcomes are recorded. Otherwise, set $\epsilon^{t+2n} = \epsilon^t \cdot \delta$, where $0 < \delta < 1$ and run the updating for another $2n$ rounds before we check again.

Simulations have been run with different parameter values. As ϵ^0 will practically decide when to stop a simulation, it is designed to be correlated with the matrix size n. It is decided that $\epsilon^0 = \dfrac{1}{100n}$ for the simulations.[11] The value of δ has shown no impacts on the outcomes and is set to 0.8 during practice. The bound $\underline{d} = 10^{-6}$ determines the upper bound for interim weights. [12]

2.4.1 Basic Model

Recall the complete interaction dynamics discussed in Section 2. At $t = 0$ agents are endorsed with arbitrary initial influence assignments and for the simulation, initial opinions are randomly drawn from 0 to 10. Then agents update their opinions and redistribute the weights during each period.

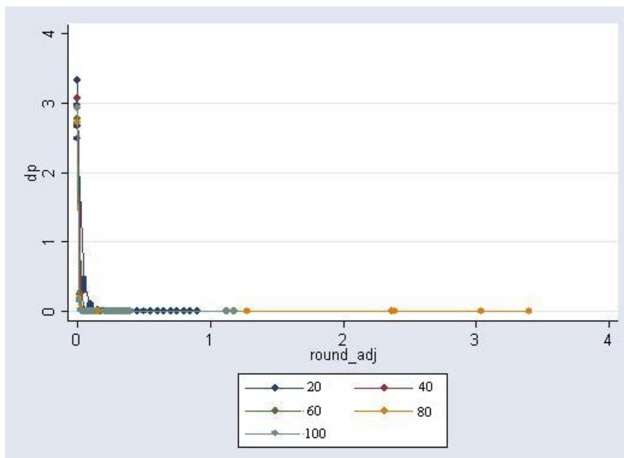
All the simulation results show consensus as Theorem 2.2 implies. The influence matrix $\mathbf{T}^t$ also converges as Theorem 2.3 states. While the time it takes to reach the steady state varies, the final patterns of influence distribution are consistently uniform. Namely, each agent places the minimum required positive weight on himself and divide the rest weights equally on all the other agents.

---

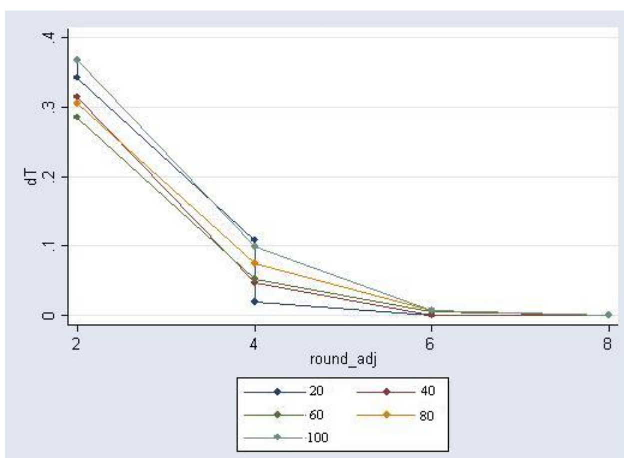
[11] Tested values include $\frac{1}{2n}$, $\frac{1}{100n}$, and $\frac{1}{400n}$. The speed of convergence differs with these values while the final outcomes are not affected.

[12] Results have constantly shown that the patterns and numerical values of equilibrium opinions and $\mathbf{T}^t$ matrix remain the same as when $1/\underline{d}$ takes larger values ranging from 10^3 to 10^6. The value of $\underline{d}$ only matters when it comes to the speed of convergence, i.e. the lower the "cut-off point" $\underline{d}$ is, the faster the society will reach the steady state.

Figure 2.2 shows the convergence paths of opinions of 5 different society sizes: 20, 40, 60, 80, and 100. The x-axis shows total rounds adjusted to size, i.e. $x = \frac{t}{n}$. In subfigure (a), the y-axis shows the variance of $p_1^t, \ldots, p_n^t$, which converges to 0 within very short time. The y-axis in subfigure (b) indicates $\parallel \Delta \mathbf{T}^t \parallel$ which is computed every $2n$ rounds. We observe a drop to zero in that value as well, which implies convergence of the influence matrix. Moreover, convergence of both opinions and the influence matrix shows significant robustness, in that the initial conditions have been completely arbitrary during each simulation, yet convergence emerges every time with the same patterns. Society size does not matter, either.



(a) Opinions



(b) Influence weights

Figure 2.2: Convergence paths of the basic model

From the graphs we also observe that the speed of convergence of opinions is much faster than that of the influence matrix. Thus the algorithm choice of determining the steady state and terminating simulation based on convergence of influence is a good choice.

2.4.2 Persistent Experiment

In this experiment, the initial opinions of normal non-persistent agents are assigned to be 0, while those of persistent agents (all three types) are assigned to be 10; so that it'd be easier and clearer to see the interaction effects.

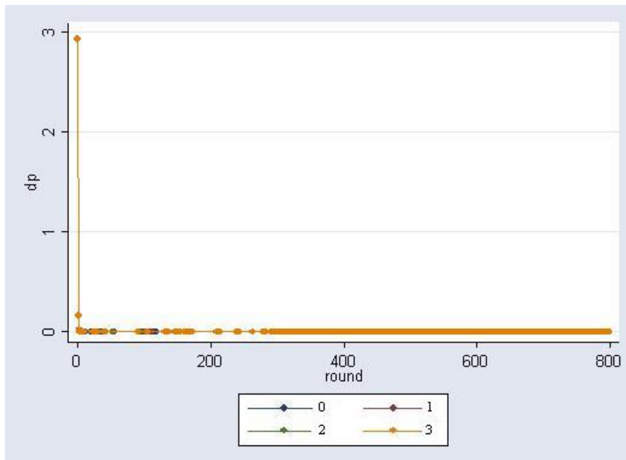
During each round of the simulation, a control group is also conducted. We introduce so-called "placebo" agents instead of persistent agents in the control group, and the initial opinions follow the 0-10 assignment rule. However, the placebo agents in a control group behave the same as the normal agents: they update both influence and opinions.

Again, we have 5 society sizes: 20, 40, 60, 80, and 100. Additionally, the persistent population proportion is randomly drawn from (0, 10%) every time. Note that for smaller societies, the arbitrary proportion could result in no persistent agents. Thus it is imposed that during each round of the simulation, if the arbitrary proportion is too small to generate any persistent agents, we will still have one.

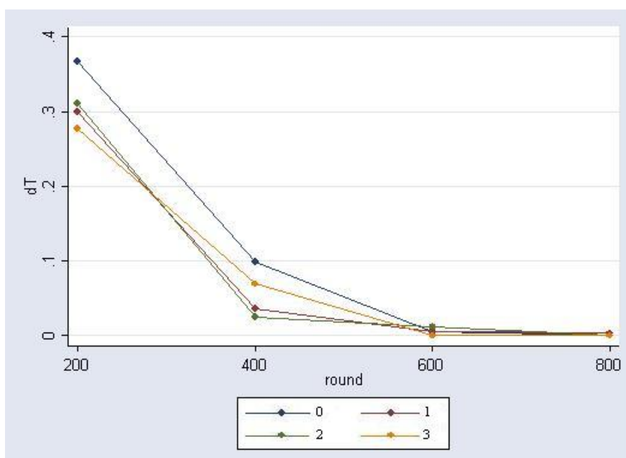
Both opinions and the influence matrix converge as stated in the theorems. And the convergence paths for all three types of persistent agents are very similar to those of the basic model as shown in Figure 2.3. The society showing in Figure 2.3 has a size of 100 agents.[13] In the graph, numbers 0 to 3 are used to tag different updating rules. Namely, rule 0 shows the control group with placebo agents and essentially represents the basic model. Whereas rule 1 to 3 represent updating rules with type I, II, and double-persistent agents, respectively.

The focus of the experiment is the outcome of final opinions. As shown in Table 2.1, the key of interaction effects is the opinion updating. Recall that persistent agents have initial opinions of 10 and normal ones have 0, it clearly shows that persistent agents could "drag" the final opinions to converge closer to 10, if they stick to their initial

[13] We observe convergence with all society sizes.



(a) Opinions



(b) Influence weights

Figure 2.3: Convergence paths of models with different updating rules

opinions. On the other hand, keeping the initial influence distribution does not have much impact. Hence the final opinions of control group does not differ much from that of type I persistent treatment group; while type II persistent and double-persistent treatment groups share similar patterns. Also, it is noticeable that for control group and type I persistent treatment group, the final opinion is much closer to non-persistent agents' initial opinion, which is 0. This observation is discussed in the proof of Proposition 2.5 in Appendix B.

Agent type	**Mean**	**Std. Dev.**	**Min**	**Max**
Control	0.5718571	0.3479276	0.0876295	1.447859
Type I persistent	0.5914877	0.3380458	0.0882642	1.483018
Type II persistent	7.384943	1.823661	2.305136	9.999651
Double-persistent	7.356037	1.799642	3.221011	9.999463

Table 2.1: Summary of final opinions by different agent types

It appears in Table 2.1 that with type II persistent and double-persistent agents, the final opinion does not always equal to 10 as Theorem 2.6 and 2.7 state. This discrepancy is caused by the actual parameter settings of the simulation. More on this issue will be discussed in Appendix B.

2.5 Conclusion

I introduced a social learning framework where agents update not only their opinions but also the influence weights they place on others. The basic assumptions used for the interaction dynamics are that 1) a non-negative row-stochastic matrix indicates the influence weights that agents place on each other; 2) during each period, an agent takes the weighted average of others' opinions as his updated opinion for the next period; they also redistribute the weights on others, based on the rule that weights are proportional to the closeness of opinions. It has been shown that with the basic learning model, we observe convergence of both opinions and the influence matrix. Moreover, opinions are conforming, and influence weights show an even distribution.

Two findings are quite interesting. First, although one pays the least attention to the farthest opinion holders, we do not observe islands of different opinions. Rather, in the steady state we have consensus: everyone holds the same opinion. Secondly, Lorenz (2005) argued that a sequence of weight matrices needs to satisfy three sufficient conditions to reach consensus, including positive diagonal elements and symmetric zero elements. However, in the steady state, $\mathbf{T}_{ii}^t$ converges to zero for all i. Moreover, with the proof of Theorem 2.2, we can see that even if $\mathbf{T}_{ii}^t = 0$ for all i for all t, consensus will be reached as long as $\mathbf{T}_{ij}^t > \underline{\epsilon}$ for all $j \neq i$ for all t, where $\underline{\epsilon}$ is an arbitrary positive constant. I propose a general theorem on conditions of consensus in the context of naive learning in Pan (2009b).

An experiment with persistent agents is conducted to concentrate on the different effects that opinion updating and influence updating have on the final outcomes. The findings are that if a group of agents keep their weight assignments unchanged, the final outcome is unlikely to be significantly affected. However, the final opinion of the whole society could be noticeably altered if at least 1 agent is persistent on his initial opinion.[14]

The persistent experiment shows the tremendous influence possessed by a minority group. It is argued in Watts and Dodds (2007) that the key is not the power of a minority to be influential, but rather how influenceable the majority is. Watts' argument suggests an interesting angle to look into the results where we have persistent agents. That is, the non-persistent agents (majority) always assign a (small) positive weight on persistent agents, and take the weighted average of all opinions to update their own. This indeed makes normal agents very influencable.

I believe these findings can be used for meaningful applications, such as the field experiment on consumer behavior (Leider, Möbius, Rosenblat, and Do, 2007) and politics (DeMarzo, Vayanos, and Zwiebel, 2003). Network structure and social positions are not fully discussed in this chapter. Whereas they also impose greatly on the path of social learning (Bala and Goyal, 2001; Friedkin and Johnsen, 1997). Also, interaction network and information network are not always the same and should be separated (Durieu and

[14] As shown in the proofs of Theorem 2.4 to 2.7 and the follow-up discussion presented in Appendix B, the number of persistent agents does not affect the final outcome, as long as $\sigma \geq 1$.

Solal, 2003; Barona, Durieu, Haller, and Solal, 2002). In the following chapters, I incorporate network structures and game theory analysis into my model (Pan and Gilles, 2009; Pan, 2009a).

Chapter 3

Naive Learning and Game Play in a Dual Social Network Framework [1]

In the previous chapter we consider learning in a fixed population of individuals that update both their opinions and the weights they assign to others. The learning process leads to consensus where all individuals have the same opinion. Such a steady state of is found on social behavior in community and family settings (Case and Katz, 1991; Evans, Oates, and Schwab, 1992; Borjas, 1995; Glaeser, Sacerdote, and Scheinkman, 1996; Katz, Kling, and Liebman, 2001) as well as on investment decisions by individuals in social interaction situations (Duflo and Saez, 2002; Madrian and Shea, 2001). Studies in the latter class of literature suggest that information dissemination is often used to maximize one's utility or payoffs. Many models in social learning also integrate utility functions to examine the efficiency of the learning process (Bala and Goyal, 2001; Banerjee and Fudenberg, 2004; Gale and Kariv, 2003). In this chapter I add utility functions to the learning model presented in Chapter 2, which represents the bridge between information exchange and decision-making processes.

Another phenomena observed in activities in social networks is that we interact with a selected group of individuals; whereas we collect information from a much broader source. French (1956) already argued that whom influences us should be separated from whom

[1] This chapter is based on Pan and Gilles (2009).

we listen to. The spatial game presented in Barona, Durieu, Haller, and Solal (2002) and Durieu and Solal (2003) also shed some light on the difference between interaction and influence, in that players in their models play games with close neighbors and make decisions based on a sample of the whole population. In this chapter I present a dual social network framework: a social influence network that determines the strategy that individuals take in their interaction with other individuals and an interaction network that describes which individuals interact with each other and play the given coordination game. I explicitly separate these two correlated yet different networks.

Namely, in this chapter we have a population of individuals that engage with each other in binary value-generating interaction, in particular through the coordination of strategic behavior. Here, this interaction is modeled as a simple 2×2 coordination game. I assume that individuals select some mixed strategy in this 2×2 game that they play with all individuals whom they interact with. In other words, a player only plays the coordination game with a selected group of opponents. One's selection is modeled by the connections she has in the interaction network. Second, I assume that individuals determine their game-theoretic strategy in this coordination game through a naive learning process in the sense of French (1956) and DeGroot (1974). The influence network captures the learning process and is distinguished from the interaction network. Thus, individuals that influence our strategy are not necessarily the same individuals that we interact with.

In the dynamic naive learning process each individual updates all three constituting elements of her social environment: (1) her interaction neighborhood of individuals with whom she plays the given coordination games; (2) the influence weights she puts on observations of strategies used by other individuals in the given population; and (3) the employed mixed strategy in the coordination game.

Another key assumption is that game play is costly and that in each time period the execution of the given coordination game results into the charge of a given interaction cost $c > 0$. This implies that in each time period each individual evaluates her interaction neighborhood and that she will only interact with those individuals which interaction resulted in a payoff larger than the given interaction cost. This describes the severance of existing links between individuals in every time period.

New interaction links are considered as follows. In every time period a randomly drawn pair of individuals is asked whether they want to form a link between them. Such a link is indeed formed if both individuals expect a return on their interaction that exceeds the given interaction cost c. This updating process of the social interaction network is akin to the evolutionary models of game play under social network formation considered in, e.g., Jackson and Watts (2002a) and Goyal and Vega-Redondo (2005).

Next, every individual updates her influence neighborhood, in particular the influence weights that she puts on the other individuals in the population. I assume that each individual adapts these weights according to the success of the individual in her interactions with other players in the population. The influence weight is determined as the normalized returns of an individual from the play of the coordination game in her interaction with other members of the population. Thus, there is a clear correlation between the interaction and influence networks in that a player observes her neighbors in the interaction network when updating her influence weights.

Following the updating process of the two networks, in each time period every individual updates her mixed strategy used in the coordination game that she plays with each individual that she interacts with. As indicated, every individual uses the naive updating process considered in the previous chapter (DeGroot, 1974; Golub and Jackson, 2007; Pan, 2008). This implies that each individual uses a weighted average of the mixed strategies of the individuals that influence her, given the influence weights that this individual puts on other individuals. I assume that individuals can observe each other's mixed strategies due to the structure of the French-DeGroot model of naive learning, which presupposes full observation in the influence network.

At this stage, consider a game where players can choose any real number x between 0 and 1 to place on one strategy and $1 - x$ on the other.[2] Then all players take actions simultaneously; at the same time they will also see what others have chosen. This is not really equivalent to observation of mixed strategy, of course. But this game shares the

[2]It may be interpreted as betting with fixed endowment that allows players to arbitrarily divide their endowment (instead of having a minimum unit, such as one cent or 1/100 of a dollar). Note that players are not betting on the outcome, which will make it a different game. Rather, they are deciding on how much to place on one of the two strategies and collect payoffs according to opponents' action.

same underlying mathematical structure as what we propose. In other words, although the naive learning process implies the availability of information from the whole population, the hypothesis that players can observe everybody's *mixed* strategy is a very strong assumption. However, we may consider additional mechanism that makes it feasible and keep the mathematical model. One direction for further discussion on this problem is the relevation principle and mechanism design (Dasgupta, Hammond, and Maskin, 1979; Myerson, 1979; Haller, 1992). We may also interpret this with a naive Bayesian learning model (Eichberger, Haller, and Milne, 1993).

I show that if interaction costs are sufficiently small, the naive learning process considered here converges to a stable state in which there is full conformity or consensus among the individuals in all aspects: all individuals' mixed strategies converge to the same limit; all individuals interact with all other members of the population; and all individuals have equal influence on the other members of the population. Sensitivity analysis shows that if interaction costs increase this state of full conformity can no longer be expected as the limit state of the naive learning process. In fact, I show that convergence breaks down complete beyond the given threshold value of the interaction costs.

Next, I consider the influence of persistent individuals in the naive learning process considered. Here, an individual is *persistent* if she does not adapt her initially assigned mixed strategy in the given coordination game, describing the interaction between these individuals. Also, unlike the uniform persistent agents that we have in the previous chapter, persistent players in this model may have different mixed strategies.

I show that the population still converges to a stable state of full conformity, but in which the mixed strategy used is fully determined by the used strategies by the group of persistent individuals. In the case that all persistent individuals have the same initial strategy, the complete population will learn to that initially assigned strategy. If persistent individuals have different initial strategies, the other individuals in the population will learn toward a weighted average of these initial strategies. Sensitivity analysis shows that these convergence patterns break down at the indicated threshold value of the interaction costs: for higher interaction costs there is no longer guaranteed that the naive learning process converges.

The chapter is structured as follows. The next section introduces the formal setup of the model and the naive learning process. Section 3 analyzes the standard setting, while Section 4 discusses the influence of persistent individuals in this framework. Finally, Section 5 debates future directions of research. All proofs are relegated to Appendix C.

3.1 Interactive social networks

We have a finite set of players $N = \{1, 2, \ldots, n\}$ who interact with their neighbors in a standard social network. Their interaction is assumed to consist of the playing of a given coordination game. Standard hypotheses usually impose that this interaction is based on some form of rational behavior. Here we explicitly restrict these players' behavior to be bounded and instead represents the collection of information of strategies used by other players in this social system. Thus, players are not concerned with finding optimal strategic behavior but instead observe and replicate the behavior of other players in the system.

Players obtain their information from observing other players in a second network environment, being represented by an interaction structure as introduced in Golub and Jackson (2007). That is, players observe all other players' actions and assign a certain weight to the various players. Based on these weights each player then determines her own mixed strategy in the coordination games she plays with other players in the given interaction network. The updating process of these strategies is based on observed selections by the other players only.

This section formally introduces the various structures in our model. We introduce two networks that illustrate the connection and information dissemination among players.

3.1.1 Social Interaction

The players in set N interact with each other according to a standard social network formulation based on, e.g., Jackson (2004). A neighborhood structure is used to represent the social interaction network $\mathbf{G}^t$ on N at time $t \in \mathbb{N}$; the neighborhood structure at time

t is represented by an $n \times n$ *interaction* matrix $\mathbf{G}^t$, where

$$\mathbf{G}_{ij}^t = \begin{cases} 1 & \text{if } i, j \text{ are connected,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Define $L_i^t = \{j \in N \mid \mathbf{G}_{ij}^t = 1\}$ as the set of player i's neighbors at time t, i.e., the players with whom i is connected with. Technically, we assume that each player is always connected with herself, i.e., $\mathbf{G}_{ii}^t = 1$, for all $i \in N$, for all t.

Connection is assumed to be consent-based, which means that permission from both players should be obtained if a link is formed between them. On the other hand, a single player can always sever any link under her control. Thus, $\mathbf{G}_{ij}^t = \mathbf{G}_{ji}^t$ for all $i, j \in N$, for all t. Or in other words, the interaction matrix $\mathbf{G}^t$ is symmetric.

Denote g^t as the interaction network defined through the interaction matrix $\mathbf{G}^t$ given by

$$g^t = \{ij \mid \mathbf{G}_{ij}^t = \mathbf{G}_{ji}^t = 1\}. \tag{3.2}$$

Note that $ii \in g^t$ for every player $i \in N$ and every time period $t \in \mathbb{N}$. At the extreme we use $g_0 = \varnothing$ to denote the empty network and $g_N = \{ij \mid i, j \in N\}$ to denote the complete network consisting of all possible links between arbitrary pairs of players.

Furthermore, the process of adding and deleting a link between players i and j at time t can be written as $g^t + ij$ and $g^t - ij$, respectively.

An player $i \in N$ only interacts with her neighbors $j \in L_i^t$ at time $t \in \mathbb{N}$. The interaction process between each pair of players is modeled as a 2×2 symmetric coordination game shown in Table 3.1.

	A	B
A	a, a	0, 0
B	0, 0	1, 1

Table 3.1: The 2×2 coordination game played between linked players

Players may choose from two actions, A and B. Both players both receive zero payoffs if they choose different actions. Otherwise, if both choose B, each of them will get a payoff of one unit. Finally, if both choose A, each will receive a payoff of a units, where it is assumed that $a \geqslant 1$. Thus, we have 2 pure strategy Nash equilibria: (A, A) and (B, B). Since we assume that $a \geqslant 1$, (A, A) is the Pareto optimal equilibrium. Moreover, (A, A) is the risk-dominant equilibrium as defined in Harsanyi and Selten (1988), which is the pure strategy equilibrium with a larger basin of attraction than (B, B).[3] Although from a strategic point of view, A is in all respects a superior convention, later we see the social learning process does not necessarily settle on a convention to play A.

All players' actions at time t are represented by an n-dimensional *mixed strategy* vector $\mathbf{p}^t = (p_1^t, \ldots, p_n^t)^{\mathrm{T}} \in [0, 1]^n$, where $p_i^t \in [0, 1]$ is the probability that player i chooses A. The strategy vector is time-dependent as players modify their actions over time based on the information they collect.

3.1.2 Information Dissemination

In this model, information dissemination is separated from actual game play based on the social network g^t introduced above. Players observe actions chosen by other players in a different social communication structure. I model this communication structure in a similar fashion as the communication network introduced in Pan (2008). In such a structure the players observe all other players arbitrarily and put a certain weight on every other player to form an opinion of what action to select from the strategy set $\{A, B\}$ at time $t \in \mathbb{N}$.

Formally, for every $t \in \mathbb{N}$ we have an $n \times n$ nonnegative matrix $\mathbf{T}^t$ which is referred to as the *influence* matrix at time t. For all $i, j \in N$, the number $\mathbf{T}_{ij}^t \in [0, 1]$ indicates the weight that player i places on player j's strategic choice at time t and a higher weight indicates that one player weighs the other more on her strategy choice. Thus, the influence matrix captures the information collection process at time t. It is assumed that players determine the $\mathbf{T}^t$-weighted average of the mixed strategies collected from all other players.

[3] In other words, A is the strategy that is a best response to the largest set of beliefs over possible plays of the opponent. Specifically, playing A is a player's best response if the fraction of her opponents who play A is greater than or equal to $\frac{1}{a+1}$ and less than $\frac{1}{2}$.

Based on this computed $\mathbf{T}^t$-weighted average at time t, each player determines her own mixed strategy to be executed for time $t+1$.

Same as the basic model in Chapter 2, for every $t \in \mathbb{N}$ the influence matrix $\mathbf{T}^t$ is row-stochastic, i.e.,

$$\sum_{j=1}^{n} \mathbf{T}_{ij}^t = 1 \text{ and } \mathbf{T}_{ij}^t \geqslant 0, \text{ for all } i, j \in N, \text{ for all } t \in \mathbb{N}. \tag{3.3}$$

Unlike the interaction matrix $\mathbf{G}^t$, $\mathbf{T}^t$ may be asymmetric, so that $\mathbf{T}_{ij}^t \neq \mathbf{T}_{ji}^t$ for some i, j. Moreover, one's information collection is not restricted to one's neighbors. That is, for some i, j, $\mathbf{T}_{ij}^t > 0$ while $\mathbf{G}_{ij}^t = 0$. On the other hand, the two matrices are correlated through a dynamic updating process, as will be discussed in the next subsection.

3.1.3 The Dynamic Updating Process

In the social learning process that we consider here, the key assumption in the redistribution of influence is that an player tends to place more weight on others whose actions do or might lead to higher payoffs to that player. Another important assumption is that a player consult her neighbors before updating the influence weights that she places on others.

At $t = 0$ the process starts with an initial interaction structure $\mathbf{G}^0$ and an initial influence matrix $\mathbf{T}^0$. The initial interaction structure is assumed to be the autarkic network given by

$$\mathbf{G}_{ii}^0 = 1, \text{ for all } i \in N \text{ and } \mathbf{G}_{ij}^0 = 0 \text{ for all } i \neq j. \tag{3.4}$$

This initialization hypothesis is a reasonable assumption and has consequences for the updating process of the influence network, which will be discussed later in this section.

Second, in the initialization of the social learning process players initially have arbitrary strategies and an arbitrary influence distribution. That is,

$$p_i^0 \in [0,1], \text{ for all } i \in N. \tag{3.5}$$

$$\sum_{j=1}^{n} \mathbf{T}_{ij}^0 = 1 \text{ for all } i \in N; \quad \mathbf{T}_{ij}^0 \in [0,1], \text{ for all } i,j \in N. \tag{3.6}$$

Note that although $\mathbf{G}_{ii}^0 = 1$ for all i, the case that $\mathbf{T}_{ii}^0 = 0$ is not excluded. Thus, it is possible that one assigns zero weight on oneself during the initialization period even though one plays the coordination game with oneself according to the hypotheses made so far. On the other hand, once i gets the chance to update her influence weights during t, we have that $\mathbf{T}_{ii}^t > 0$ due to the formulation of the updating process. The dynamic updating process of influence matrix will be discussed next.

Updating Interaction Network

First, interaction is costly. This leads to the introduction of a common interaction cost $c \geqslant 0$. I assume that both the initiation and the maintenance of a link between two players imposes the same cost c on both interacting parties. This implies that, when a link is initiated, both players pay the common interaction cost c. Also, each player pays the common interaction cost c for the maintenance of every existing link $ih \in g^t$ with $h \in L_i^t$ during each time period $t \in \mathbb{N}$. The assumption that both initiation and maintenance costs are represented by the same cost parameter c is a simplification, but quite acceptable within the context of the described interaction process.

I emphasize that I assume each player $i \in N$ has no costs of interacting with himself. Hence, the common cost c is not imposed on the self-referential interaction $ii \in g^t$, $t \in \mathbb{N}$.

During each time period $t \in \mathbb{N}$, two players $i \in N$ and $j \in N$ are selected randomly to update their interaction network. Given the payoffs defined by the 2×2 coordination game shown in Table 3.1, at time t player i receives a payoff of π_{ij}^t by interacting with player j, where

$$\pi_{ij}^t = ap_i^t p_j^t + (1-p_i^t)(1-p_j^t). \tag{3.7}$$

Note that due to the symmetric nature of the coordination game, each pair of players $i, j \in N$ receives identical payoffs from such interaction, $\pi_{ij}^t = \pi_{ji}^t$, for all $i, j \in N$.

During the updating process in period t, the randomly selected players i and j are assumed to consider the formation of interaction link ij, if the two are not yet connected, i.e., if $ij \notin g^{t-1}$. Otherwise, if the link between i and j has already been formed, i.e., $ij \in g^{t-1}$, then it will be severed if the common payoff π_{ij}^t earned by both players i and j from the existing link is less than its maintenance cost c. No changes will be made otherwise.

Here I assume that players can observe each other's mixed strategies from the previous round of play through the influence network, i.e., if $\mathbf{T}_{ij}^t > 0$. This hypothesis implies that past mixed strategies can be observed fully. This kind of public randomization requirement conforms with the framework of the French-DeGroot naive learning process, i.e., the formulation of the updated strategy as the weighted average of *all* p_i^t.

In summary, if players $i, j \in N$ are selected to update their interaction network in period t, the updating rule for the interaction network $\mathbf{G}^t$ is given by

$$\mathbf{G}_{hk}^t = \mathbf{G}_{hk}^{t-1} \text{ for all } (h, k) \neq (i, j) \tag{3.8}$$

and

$$\mathbf{G}_{ij}^t = \begin{cases} 1 & \text{if } \pi_{ij}^{t-1} \geqslant c \\ 0 & \text{if } \pi_{ij}^{t-1} < c \end{cases} \tag{3.9}$$

Updating Influence Weights

A player collects information about all other players through observation of actual game play with her neighbors in the interaction network $\mathbf{G}^t$. I model the updating of the influence matrix $\mathbf{T}^t$ to be based on the obtained payoffs in the game play in $\mathbf{G}^t$. Indeed, updating the influence matrix $\mathbf{T}^t$ in this fashion increases the potential for more beneficial links and higher payoffs, while a player's neighbors act as effective filters. Thus, when an player decides how much influence weight to place on another player, she calculates the total payoffs that her neighbors could obtain from interacting with that player, given all players' past actions.

We recall that the influence matrix is always row-stochastic. Thus, each player redistributes her influence weight assignment proportionally according to the total payoffs and then normalizes the weights to make sure that the row sum equals to 1. This implies that the redistribution of influence follows the rules below:

$$\mathbf{T}^t_{ij} = \frac{w^t_{ij}}{\sum_{k=1}^{n} w^t_{ik}}, \text{ for all } i, j \in N \text{ and } t \in \mathbb{N}, \tag{3.10}$$

where

$$w^t_{ij} = \sum_{l \in L^t_i} \mathbf{G}^t_{lj} \pi^{t-1}_{lj}.$$

Recall that the formation of interaction network g^t takes into account the payoffs and costs associated with links. Consider the weight $\mathbf{T}^t_{ij}$ assigned by i to j. If j's action does not guarantee a sufficiently high payoff, j would not be connected with any of i's neighbors. That is, $\mathbf{G}^t_{hj} = 0$ for all $h \in L^t_i$. Consequently, $w^t_{ij} = 0$, which results to zero weight $\mathbf{T}^t_{ij} = 0$. Note that if $L^t_i = \varnothing$, we have a problem when applying the equations above in that all weights w^t_{ij} are 0 and the sum of the i-th row of the influence matrix $\mathbf{T}^t$ does not add up to 1. This problem is prevented by the assumption that each player is connected with herself during round 0 and stays connected with herself during any subsequent period $t \in \mathbb{N}$ since it is assumed that player i has no costs related to her self-referential interaction $\mathbf{G}^t_{ii} = 1$ or $ii \in g^t$.

Updating Strategies

Next, all players modify their strategies during the second part of period t. The action updating rule is that a player takes the weighted average of all players' strategies in forming her own new strategy, i.e.,

$$p_i^t = \sum_{j \in N} \mathbf{T}_{ij}^t p_j^{t-1} \qquad \text{for all } i \in N, t > 0. \tag{3.11}$$

So the updating process for all players can be conveniently written as:

$$\mathbf{p}^t = \mathbf{T}^t \mathbf{p}^{t-1}. \tag{3.12}$$

As pointed out, players whose actions do not result in sufficiently high payoffs get zero weights in the influence matrix. So their actions and information from them are not taken into account when player i updates her mixed strategy p_i^t. In other words, each player actually takes the weighted average among the beneficial or potentially beneficial actions during the updating process. The assumption that we imposed on the updating process of interaction network is also applied here, i.e. players can observe each other's previous mixed strategies or expected payoffs with the naive learning mechanism.

The dynamic updating process is summarized in Figure 3.1 below.

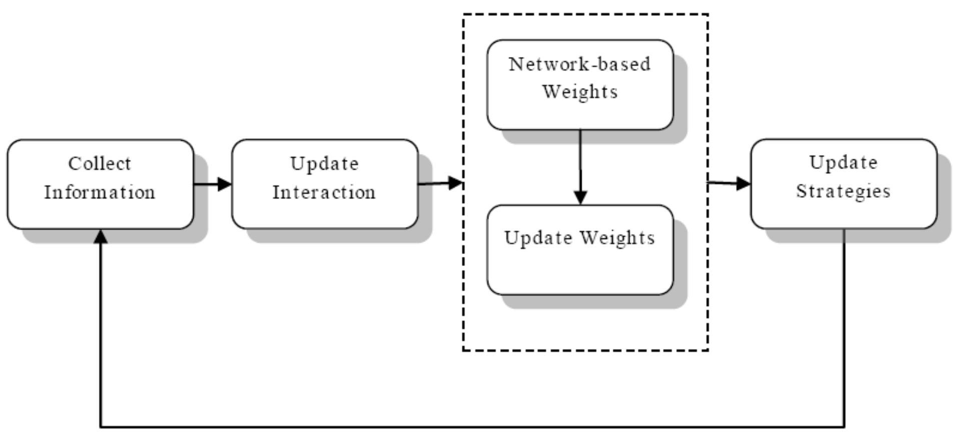


Figure 3.1: Updating process in the dual network framework

Finally, each player $i \in N$ plays the given coordination game with her neighbors $j \in L_i^t$ in the interaction structure $\mathbf{G}^t$ and collects payoffs for the period t, with subtraction of interaction costs for all active links (except the one with himself) in period t:

$$\pi_i^t = \sum_{h \in L_i^t} \pi_{ih}^t - (\#L_i^t - 1)c = \sum_{h \in L_i^t} \left[a p_i^t p_h^t + (1 - p_i^t)(1 - p_h^t) \right] - (\#L_i^t - 1)c, \qquad (3.13)$$

where $\#L_i^t$ is the number of i's neighbors during time period t.[4]

3.2 Convergence of Behavior

The social learning process introduced in the previous section clearly is founded on a boundedly rational form of replicator dynamics. In this case the determination of one's strategy is more subtle and involves the weighing of mixed strategies selected by the players that one emulates. It is clear that in such a setting it is unlikely that behavior will converge to one of the pure Nash equilibrium strategies, which is indeed confirmed by our results in the analysis of the dynamic updating process proposed here.

In the process described above, players update their neighborhood structure, influence weight assignment, and strategies independently. Also, players are myopic in that they do not consider the implications of updating to the future. However, they base their decisions on the success of the mixed strategies adopted by the players that they observe. This is conducted through the updating mechanism of the influence weights put on the various observations.

I conduct a full analysis of the learning process developed in the previous section in two stages. First, I identify a lower bound $\underline{\pi}$ on the payoffs that are obtained in the dynamic process of game play in the population N. Second, I show the full convergence of the social learning process to a common mixed strategy if the interaction cost parameter c is less than the identified lower bound $\underline{\pi}$.

In order to state my main results, denote $\underline{p}^0 = \min\{p_1^0, \dots, p_n^0\}$ as the minimum of all initial probabilities of playing pure strategy A in the given game and, similarly, let $\overline{p}^0 = \max\{p_1^0, \dots, p_n^0\}$ be the maximum of these initial probabilities.

[4]Note that, whereas the payoff from the self-referential interaction ii is taken into account here, there are no costs related to that interaction.

Proposition 3.1. *Let*

$$\underline{\pi} = \min\left\{ \frac{a}{a+1}\,,\, a\underline{p}^0\overline{p}^0 + \left(1-\underline{p}^0\right)\left(1-\overline{p}^0\right) \right\} \geqslant 0. \tag{3.14}$$

Then $\underline{\pi}$ *is a lower bound for the set of all payoffs given by*

$$\left\{\pi_{ij}^t \mid i,j \in N \text{ and } t \in \mathbb{N}\right\}.$$

A proof of Proposition 3.1 can be found in Appendix C.

My first main result states that the dynamic process described by the sequence of mixed strategies $\mathbf{p}^t$ is convergent and, moreover, all players' mixed strategies converge to the same strategy. That is, the described social learning process results into conformism. Essentially, p^* acts as a convention in the society, even though this convention is not necessarily one of the two pure strategy Nash equilibria in the given game or even the mixed strategy Nash equilibrium.

Theorem 3.2. *If* $c \leqslant \underline{\pi}$, *then the updating process converges to a situation in which there emerge a fully connected interaction network, evenly distributed weights, and all players choose the same mixed strategy* $p^* \in [0,1]$, *i.e., for every* $\epsilon > 0$ *there exists some* $T^* > 0$ *such that for all* $t > T^*$ *it holds that* $\mathbf{G}_{ij}^t = 1$ *as well as* $|\mathbf{T}_{ij}^t - \frac{1}{n}| < \epsilon$ *for all* $i,j \in N$ *and* $|p_i^t - p^*| < \epsilon$ *for all players* $i \in N$.

Again we refer to appendix for a proof of this assertion. Theorem 3.2 states that if the formation of interaction links is low enough—or nearly costless—the interaction structure converges to a fully connected network. Also, the long run emerging strategies show conformism through communication.

One important implication is that that the society converges to the stable state that is not a Nash equilibrium unless we have extreme initial conditions where $\mathbf{p}^0 = (0,\ldots,0)^{\mathrm{T}}$ or $\mathbf{p}^0 = (1,\ldots,1)^{\mathrm{T}}$. Especially, even if some of the players have 0 or 1 initially, they will lose that best response choice through communication with others.

Sensitivity Analysis

In this subsection I examine sensitivity of changing costs. During computer simulations I set $a = 2$. Cost c ranges from 0 to 2 and takes 0.05 increment during each round. Initial strategy vector $\mathbf{p}^0$ and influence matrix $\mathbf{T}^0$ are arbitrary, the interaction network $\mathbf{G}^0$ is autarkic in the sense that $\mathbf{G}^0_{ii} = 1$ and $\mathbf{G}^0_{ij} = 0$ for $j \neq i$. The society size has been set to 20, 40, 60, 80, and 100. Recall that during each round $t \in \mathbb{N}$, a link ij and an player i get chosen randomly to update the interaction and influence networks. Thus each combination of society size, cost, and initial conditions (the arbitrary $\mathbf{p}^0$ and $\mathbf{T}^0$) is used to run 3 times, in order to examine outcome patterns with that kind of randomness in updating.

The change in society size does not show any significant effect in the final outcome. Thus in this subsection I am only showing the results with $n = 20$. Results are recorded when the learning process researches a steady state.[5] The x-axis shows value of cost c. For the subfigure (a) in Figure 3.2, the y-axis indicates the standard deviation among all strategies at the steady state, i.e. $\sqrt{\sum_{i\in N}(p_i^t - \frac{1}{n}\sum_{i\in N} p_i^t)^2}$. In subfigure (b) in Figure 3.2 the y-axis shows the mean value of all strategies at the steady state, i.e. $\frac{1}{n}\sum_{i\in N} p_i^t$.

When standard deviation equals to 0 the strategies conform, and $p_i^t = p^*$ which also equals to the mean value for all i. In subfigure (a) we see that standard deviation is 0 when cost is relatively low. Correspondingly, in these cases the mean values shown in subfigure (b) fall into a narrow range (represented by a thin horizontal bar) for each combination of initial conditions.[6] This observed conformism coincides with the empirical studies on the influence of social networks, which show indeed that individuals' decision-making process, opinions, and behavior patterns are affected by their (social) neighbors. For instance, colleagues show similarity in investment patterns (Duflo and Saez (2002)); and behaviors of neighborhood peers appear to substantially affect youth behaviors (Case

[5]Similar to Pan (2008), the program determines a steady state when $\| \Delta\mathbf{T}^t \| < \frac{1}{100n}$, where$\| \Delta\mathbf{T}^t \|$ is the norm of $\Delta\mathbf{T}^t = \mathbf{T}^t - \mathbf{T}^{t-2n}$.

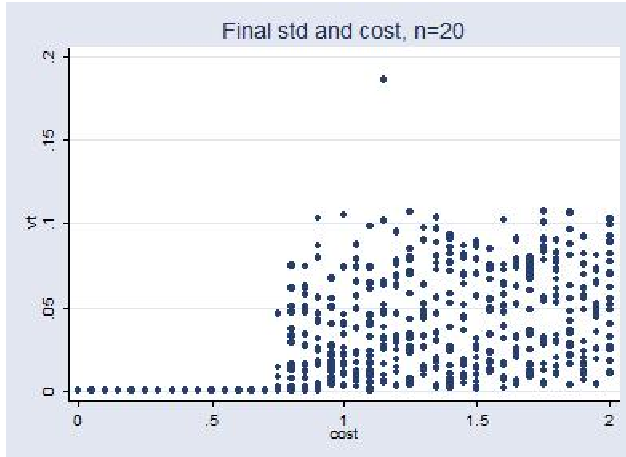
[6]Note that we cannot determine the exact value of final strategy p^*, because it depends on the order of the link and agent chosen in this round, as well as the initial conditions.

and Katz (1991)).

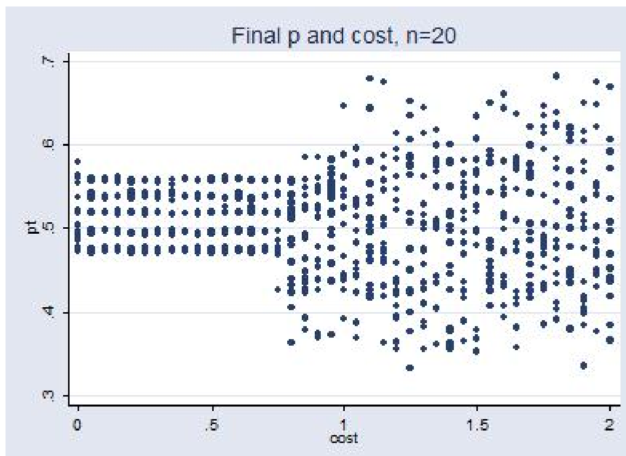
On the other hand, when cost exceed $\underline{\pi}$ the outcomes are quite random. The standard deviation could be anywhere between 0 and 0.1 including 0. In one case the standard deviation is 0.2, which suggests spread out distribution patterns of strategies. Also, the mean values form a cloud and exhibits obvious randomness as well. In these cases, the outcome of updating process depends on the order of the players chosen during each round and the initial conditions. That is, during each period, if a link ij will be formed or severed depends on the opinions of i and j, which are randomly chosen. The resulting interaction network might be fully or near fully connected if the two randomly chosen players happen to have strategies that lead to high payoffs. Otherwise the network would be sparsely connected and we have closed groups, where an player only interacts with members in her group and members in each group conform to the same strategy. Another noticeable finding is that convergence occurs much faster with small closed groups than with a well-connected network. This observation agrees to the argument on speed of convergence in DeMarzo, Vayanos, and Zwiebel (2003). Especially, in the case where c is higher than $\overline{\pi}^0$, which is the highest payoff that any 2 players' opinions generate, then we have an isolated interaction network where no one interacts with anybody else and $\mathbf{T}^t_{ii} = 1$, $\mathbf{T}^t_{ij} = 0$ for all i and $j \neq i$.[7]

To conclude, when cost exceed $\underline{\pi}$ the networks and strategy vector still converge. But the questions of whether $\mathbf{p}^t$ conforms, the value of final strategies, and the patterns of interaction and influence networks are all indeterminate as outcomes are random and depends on the order of chosen links, players, as well as the initial conditions.

[7]Note that we cannot claim that $p^t_i = p^0_i$ in this case. Because during each round only one player gets the chance to update her influence weights $\mathbf{T}^t_{ij}$, after which all players update their strategies. So those who did not update influence weights use their initial weight assignments and still take weighted average of all strategies to form their own for the next period. The learning process reaches the steady state when all players update their influence weights, which takes at least n periods of updating.



(a) Standard deviation



(b) Mean value

Figure 3.2: Cost sensitivity analysis of basic model

3.3 Persistent Players

Based on the persistent player model introduced in Pan (2008), I extend the basic model to include persistent players. A player is called "persistent" if she does not change her initially assigned mixed strategy over time. In other words, a persistent player will not update her strategy and will only incompletely update her interaction structure.

The set of persistent players is introduced as the subset $S \subset N$, where we usually assume that $1 \leqslant |S| < n$. A persistent player is usually denoted by $s \in S$. Now, every persistent player $s \in S$ is characterized by the hypothesis that she does not update her mixed strategy, i.e., if the persistent player s is endowed with an initial strategy $p_s^0 \in [0, 1]$, then $p_s^t = p_s^0$ for all $t \in \mathbb{N}$. However, I assume that every persistent player updates her interaction network L_i^t as well as her influence weights $\{T_{ij}^t \mid j \in N\}$ according to the dynamic updating process introduced in the previous section. Of course, a persistent player does not use her influence weights to update her strategy in every time period as do the other (non-persistent) players in the population.

The main insight is that the introduction of persistent players into the population alters the outcome of the social learning process substantially. First consider the introduction of persistent players in the population with a common persistent strategy denoted by $p_\alpha \in [0, 1]$. Such a case models a *uniform* group of persistent players in the population.

Theorem 3.3. *Consider a situation in which* $c \leqslant \underline{\pi}$ *and there exists a set of persistent players* $S \subset N$ *such that* $|S| \geqslant 1$ *and all* $s \in S$ *have a common persistent strategy given by* $p_s^0 = p_s^t = p_\alpha \in [0, 1]$ *for all* $t \in \mathbb{N}$. *Then the social learning process converges to a fully connected interaction network, evenly distributed weights, and all players' strategies converge to* p_α, *i.e., for all* $\epsilon > 0$ *there exists some* $T^* > 0$ *such that for all* $t > T^*$ *it holds that* $\mathbf{G}_{ij}^t = 1$ *as well as* $|\mathbf{T}_{ij}^t - \frac{1}{n}| < \epsilon$ *for all* $i, j \in N$ *and* $|p_i^t - p_\alpha| < \epsilon$ *for all non-persistent players* $i \notin S$.

Theorem 3.3 indicates that persistent players possess a form of widespread and strong influence in determining all players' strategic choices. Namely, the final strategy of all players equals to persistent players' initial strategy p_α. So, when $p_\alpha = 1$ or $p_\alpha = 0$, the social learning process converges to the Nash equilibrium outcomes (A, A) and (B, B), respectively. Otherwise the strategy vector of the whole society reaches a steady state given by $\{p_\alpha \mid i \in N\}$ that is not necessarily a Nash equilibrium, but conforms with the initial strategy of all persistent players in the population. Also, in this case where the persistent players have uniform initial (persistent) strategies, the total number of them $|S|$ only affects the speed of convergence, not the final outcome.

If we have multiple persistent players that persistently stick to different strategies, the social learning process converges to a convex combination of the persistent strategies adhered to by members of the group of persistent players. In this case, the final strategy of normal players are significantly influenced by persistent players' initial strategies.

Theorem 3.4. *Consider a situation in which $c \leqslant \underline{\pi}$ and the subset of persistent players $S \subset N$ is characterized by $2 \leqslant |S| \leqslant n-1$ such that there are $s_i, s_j \in S$ with $p^t_{s_i} = p^0_{s_i}$ and $p^t_{s_j} = p^0_{s_j}$ for all $t \in \mathbb{N}$, where $p^0_{s_i} \neq p^0_{s_j}$. Then the social learning process converges to a fully connected interaction network, evenly distributed weights, and all players' strategies converge to some $p_\beta \in [0,1]$, i.e., for all $\epsilon > 0$ there exists some $T^* > 0$ such that for all $t > T^*$ it holds that $\mathbf{G}^t_{ij} = 1$ as well as $|\mathbf{T}^t_{ij} - \frac{1}{n}| < \epsilon$ for all $i, j \in N$ and $|p^t_i - p_\beta| < \epsilon$ for all non-persistent players $i \notin S$.*

The assertion of Theorem 3.4 leaves open the issue where exactly the social learning process will lead the non-persistent players with regard to the mixed strategy that they will adhere to. Proposition 3.5 below partially solves this issue and states upper and lower bounds on the mixed strategy to which the non-persistent players will lean to. Corollary 3.6 presents a formula to calculate the actual mixed strategy that is determined by diverse persistent players' strategies.

Proposition 3.5. *Consider the situation as stated in Theorem 2. If there are $m = |S|$ diverse persistent players such that $\#\{p^0_s \mid s \in S\} = m$, then in the social learning process, there exists some $T' > 0$ such that for all $t > T'$ it holds that $\underline{p}_s \leqslant p^t_i \leqslant \overline{p}_s$ for every player $i \in N$, where $\underline{p}_s = \min_{s \in S} p^0_s$ and $\overline{p}_s = \max_{s \in S} p^0_s$.*

Corollary 3.6. *Consider the situation with m diverse persistent players stated in Proposition 3.5, then all non-persistent players' strategies converge to p_β that satisfies the following properties*

$$\sum_{s \in S} (\mathbf{T}^*_{\cdot s} p^0_s) = p_\beta \sum_{s \in S} \mathbf{T}^*_{\cdot s}, \tag{3.15}$$

$$\frac{\mathbf{T}^*_{\cdot s_i}}{\mathbf{T}^*_{\cdot s_j}} = \frac{x\left[(a+1)p^0_{s_i} - 1\right] + n\left(1 - p^0_{s_i}\right)}{x\left[(a+1)p^0_{s_j} - 1\right] + n\left(1 - p^0_{s_j}\right)}, \text{ for all } s_i, s_j \in S, \tag{3.16}$$

where $\mathbf{T}^*$ is the limit influence matrix resulting from the social learning process and

$$x = (n - |S|)p_\beta + \sum_{s \in S} p^0_s.$$

Corollary 3.6 follows immediately in view of the common strategy updating rule

$$p_i^t = \sum_{j \in N} \mathbf{T}_{ij}^t p_j^{t-1},$$

and that

$$\lim_{t\to\infty} \mathbf{T}_{ik}^t = \lim_{t\to\infty} \mathbf{T}_{jk}^t = \frac{x\left[(a+1)p_k^* - 1\right] + n\left(1 - p_k^*\right)}{(a+1)x^2 - 2nx + n^2},$$

where

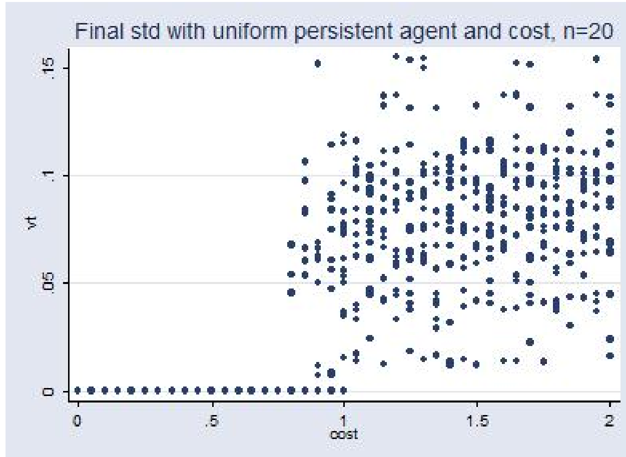
$$x = \sum_{i \in N} p_i^*.$$

Proof of other theorems and propositions can be found in Appendix C.

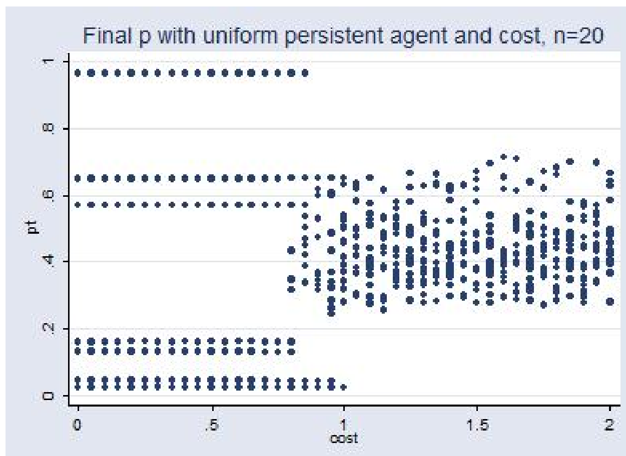
Sensitivity Analysis

We use the same settings as the basic model, i.e., $a = 2$ and c ranges from 0 to 2 in 0.05 increments. Persistent player(s)' initial strategies are also arbitrary. Each combination of society size, cost, initial conditions, and persistent player(s)' initial strategy(s) is used to run 3 times, as regard to the randomness in choosing a link ij and an player i during each updating period of interaction and influence networks.

Similar to the basic model, the society size does not affect the final outcomes. Thus for both uniform and diverse persistent players, we are only showing the cases where $n = 20$ with the x-axis showing value of cost and y-axis showing standard deviation and mean value of the final strategies. Also, we do not have more than 1 persistent players with the same strategy since the number of persistent players with the same strategy does not affect the final outcome, whether we have uniform or diverse persistent players. Figure 3.3 illustrates the case where we have 1 persistent player which represents the uniform persistent model. In Figure 3.4 we have 3 diverse persistent players, i.e. persistent players with 3 different initial strategies. Simulation results have shown that learning with different number of diverse persistent strategies exhibits similar outcome patterns.



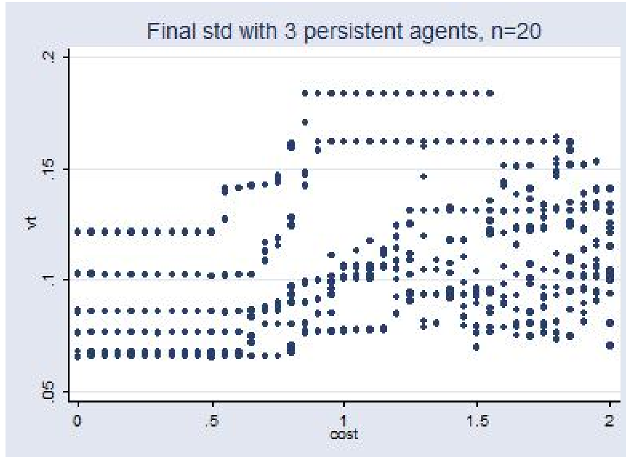
(a) Standard deviation



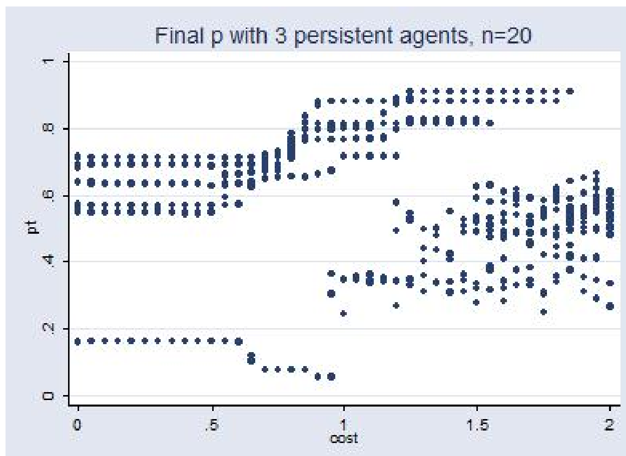
(b) Mean value

Figure 3.3: Cost sensitivity analysis with uniform persistent players

Again, when cost is low, we observe conformism, i.e. the mean value of p_i^t now takes one value and forms a line instead of a thin bar in the basic model (Figure 3.2). That is because with uniform persistent players s, we know that $\lim_{t\to\infty} p_i^t = p_s^0$ for all i; and with diverse persistent players p^* is also determined by $p_{s_i}^0$ by Corollary 3.6. Whereas in the basic model, p^* depends on the order of the link and agent chosen in this round, as well as the initial conditions. Note that the standard deviation is never 0 when we have diverse persistent players. The reason is that, by Proposition 3.5, we know that p^* for non-persistent players is in the range of $(\underline{p}_s, \overline{p}_s)$. Thus the standard deviation is at least $\sqrt{(\overline{p}_s - p^*)^2 + (\underline{p}_s - p^*)^2} > 0$. In this case, a constant standard deviation and mean value indicate conformism.



(a) Standard deviation



(b) Mean value

Figure 3.4: Cost sensitivity analysis with 3 diverse persistent players

When cost exceeds $\underline{\pi}$, the outcomes are random. The standard deviation suggest an indeterminate strategy distribution, which implies indeterminate interaction and influence network structures. The outcome is determined by the order of link and agent chosen in this round, as well as the initial conditions. Same as the basic model, with higher cost we may have close groups with conformism in each group, and diverse strategies across groups. When cost is higher than $\overline{\pi}^0$, agents choose to stay isolated.

3.4 Conclusion

The social learning model discussed in this chapter has a dynamic double-layer network structure. Namely, players play a coordination game in an interaction network; they also update their strategies based on an influence network. Previous works with similar settings assume that both networks are exogenous and/or time-invariant; whereas in this framework they are endogenous and change over time.

I examine the convergence of the two networks and players' strategies as well as the patterns of the final outcomes. The main argument is that when cost is low, the interaction network converges to be fully-connected, influence network converges to exhibit even distribution patterns, and players' strategies conform to the same value. This kind of conformism coincides with empirical findings; it also has implications on various issues such as politics and marketing (DeMarzo, Vayanos, and Zwiebel (2003)). However, the steady state is usually not a Nash equilibrium.

A group of persistent players are introduced in an extended game. They interact with normal players under the rules defined, but do not change their strategies. Persistent player may be unified or diverse, in the sense that their strategies may or may not be the same within the persistent group. It is proven that with the existence of persistent players, normal player still show conformism. I present the formula for computering the value of the final strategy of normal players, which is significantly influenced by persistent players' strategies. A similar approach was proposed in Pan (2008). However, persistent players in that model are assumed to be unified.

Jackson and Watts (1999) discussed the case where strategies are pure and cost is zero; in this chapter we adopt mixed strategies and give a lower bound of cost that is non-negative. When cost is higher than the lower bound, simulation results show that the outcomes are random and cannot be determined or fully characterized. In both frameworks, the resulting interaction network is completely connected. In next chapter we look into different network structures, such as star, scale-free, and small world; I also present an endogenous network formation algorithm that generates networks with certain characteristics.

Chapter 4

Neighborhood Structure and Learning in Evolving Social Networks [1]

We communicate with others all the time. Dinner conversations help us stay in touch with our family; we also often seek for advice from them. Discussions with colleagues play an important role in getting our job done. If all but one agree on a proposal, then that one person may want to reconsider his vote. A friend that you trust recommends a financial column to you, then chances are you will read it and buy or sell your stocks based on what the author says. We share information with people that we are connected with; the information we spread and collect will be most likely used to make a decision. Therefore, social networks are central to understanding many social-economic questions and phenomena.

Granovetter (1973) has pointed out the importance of information dissemination in social networks a few decades ago. In the past several years, the study of network theory has attracted more and more attention from researchers in different disciplines including physics, biology, sociology, and economics. The work of Granovetter can indeed be integrated into a broader framework of diffusion in networks: signals, knowledge, opinions, disease, etc. In any of these cases, the key is how the network structure bears on the process of diffusion (Amaral, Díaz-Guilera, Moreira, Goldberger, and Lipsitz, 2004;

[1] This chapter is based on Pan (2009a).

Bikhchandani, Hirshleifer, and Welch, 1998; Chwe, 2000; Kauffman, 1993; Sporns, 2002).

This chapter first investigates the effects of different network structures on a learning process. I carry on the social learning model presented in previous chapters, in which agents continuously update their opinions by taking weighted averages of those of others'.

Connections in a social network carry different strength and meanings, which may change over time. Similarly, the weights that one places on others' opinions tend to change as well. The influence updating process in Chapter 2 is based on the distance between the opinions of two agents. With an imposed neighborhood structure, in Chapter 3 I introduce a network-based updating rule for influence weights. In this chapter, agents redistribute influence weights using a network-distance combination algorithm which takes into account both proximity of opinions and impact of neighbors.

The neighborhood structure is assumed to be exogenous. Network types that do or do not involve randomness are examined. Theorem 2.2 implies that the opinion updating process converges to a state of conformity, i.e. all agents converge to have the same opinion. In this chapter, the correlation between network structure and convergence speed is one of the main questions discussed. Another question is the influence of hub nodes on the final opinion, which is also shown to be significantly related to the network structure.

Next, this study moves on to an endogenous network formation model. One of the pioneer works on network formation is Erdős and Rényi (1959), which builds a random network based on probabilities of creating links. However, with the discovery of interesting statistical features that complex networks have (Barabási, 2003), the Erdős-Rényi model falls short to fit those features. Researchers have been trying to explain how those properties emerge over the evolution of complex networks. Jackson (2006) did a comprehensive review on different models of network formation with an emphasis on social networks. Two famous models are used in this chapter: the Watts-Strogatz model that constructs networks with small-world characteristics (Watts and Strogatz, 1998) and the Barabási-Albert mechanism resulting into scale-free networks (Barabási and Albert, 1999).

The formation model presented in this chapter has a growing population and in which the new agents build connections based on both proximity of opinions and impact of neighbors. The algorithm takes two stages. First, a new agent meets a selected group of potential neighbors. He then forms links with those whose opinions are close enough to his. Second, the new agent meets another selected group of potential connections, who are drawn from the neighbors of the first group. The same rule applies when the new agent forms links with agents from this group.

This algorithm is similar to the one proposed by Jackson and Rogers (2007), who used a network-based two-stage model and attempted to capture topological features of different networks by adjusting probabilities of getting connected to candidates that one meets. With the distance-based rule in this model, resulting networks clearly show that hub nodes' opinions are closer to the mean value of all opinions.

One major finding is that networks with hub nodes lead to significantly faster convergence speed. Whereas under wheel networks, the most distracting structure, it can take a lot of rounds of updating to reach conformity even for a small-size community. Also, under scale-free networks, the hub agents' opinions have a clear impact on the final opinion that all agents converge to. The pattern is linear and positive.

On the endogenous network formation, results from simulation show a clear power-law degree distribution that fits empirical data very well. Given the distribution of all agents' opinions, it is also observed that agents with opinions closer to the mean value are most likely to become hub nodes in the resulting network. This can be explained by a transition of the distance-based rule to a probability format.

The organization of the chapter is as below. The basic social learning model is presented in Section 2. In the following section, networks that do not involve randomness (empty, star, cycle, wheel, and full) and those that do (small-world and scale-free) are imposed. Simulation outcomes show the effects that different structures have on the learning process. Section 4 focuses on the endogenous network formation and the resulting networks. The last section concludes the study and discusses extensions for future research.

4.1 Model

4.1.1 Agents, Opinions, and Influence

A finite set N={1, 2, ..., n} of agents interact and share *opinions*, which are represented by a n×1 vector $\mathbf{p}^t = (p_1^t, p_2^t, \ldots, p_n^t)^{\mathrm{T}} \in \mathbb{R}^n$, where p_i^t is agent i's opinion at time t. It is assumed that $0 \leq p_i^t \leq 1$ for all i and t.[2]

The neighborhood structure $\mathbf{G}^t$ on N at time $t \in \mathbb{N}$ is represented by an $n \times n$ *neighborhood* matrix $\mathbf{G}^t$, where

$$\mathbf{G}_{ij}^t = \begin{cases} 1 & \text{if } i, j \text{ are connected,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Define $L_i^t = \{j \in N \mid \mathbf{G}_{ij}^t = 1\}$ as set of the agents with whom i is connected with, which we may refer to as agent i's neighbors. In this model the neighborhood structure is assumed to be undirected. Thus, the matrix $\mathbf{G}^t$ is symmetric. Also, self-loop is not allowed. That is,

$$\mathbf{G}_{ij}^t = \mathbf{G}_{ji}^t, \quad \mathbf{G}_{ii}^t = 0 \text{ for all } i, t. \tag{4.2}$$

A n×n nonnegative matrix $\mathbf{T}$ is referred to as the *influence* matrix. $\mathbf{T}^t$ captures the interaction patterns at time t, i.e., for all $i, j \in N$, the weights $\mathbf{T}_{ij}^t \in [0, 1]$ indicates the influence that agent i places on agent j's opinion at time t. The influence matrix is row-stochastic, i.e.

$$\sum_{j=1}^{n} \mathbf{T}_{ij}^t = 1 \text{ and } \mathbf{T}_{ij}^t \geq 0, \text{ for all } i, j \in N, \text{ for all } t. \tag{4.3}$$

Moreover, $\mathbf{T}^t$ may be asymmetric, so that $\mathbf{T}_{ij}^t \neq \mathbf{T}_{ji}^t$ for some i, j.

[2]Here p_i^t is scalar; whereas in DeMarzo, Vayanos, and Zwiebel (2003), each agent has a vector of opinions and does multiple estimates, which are interpreted as multidimensional opinions. Their paper also shows that the multidimensional opinions are convergent and thus can be represented on a unidimensional scale.

4.1.2 Updating Processes

At $t = 0$, both opinions and influence weights exhibit a uniform distribution. That is, each agent i is endowed with an initial opinion p_i^0 that is evenly distributed between 0 and 1 and an equal weight on all agents:

$$p_i^0 = (i-1) * \frac{1}{n-1} \text{ for all } i \in N \tag{4.4}$$

$$\mathbf{T}_{ij}^0 = \frac{1}{n} \text{ for all } i, j \in N; \tag{4.5}$$

The initial neighborhood structure is assumed to be exogenous and static, i.e., $G^t = G^0$ for all $t > 0$. More on the network structure will be discussed in the next section.

Updating Influence Weights

Here I impose a so-called network-distance combination updating process to modify influence weights. The first step is to calculate the interim weights based on network structure, which is based on the idea from Pan and Gilles (2009). Namely, when an agent i decides on the weight to place on agent j for the next period, he first consults to his neighbors and sees how much weight they currently place on j in total:

$$\eta_{ij}^t = \sum_{k \in L_i^t} \mathbf{T}_{kj}^t \quad \text{for all } i, j \in N, j \neq i.$$

Next, the agent assigns distance-based weights on his neighbors. We follow the rule in Pan (2008), which assumes an inverse relationship between closeness and weight: an agent assigns higher weights on those whose opinions are closer to his own during the current period.

$$w_{ij}^t = \frac{1}{d_{ij}^t} = \frac{1}{\max(\underline{d}, |p_i^t - p_j^t|)} \quad \text{for all } i, j \in N, j \neq i,$$

where $\underline{d}$ is a small positive number that prevents taking inverse over 0 when $p_i^t = p_j^t$.

Finally, the agent applies the distance-based weights on the interim network-based weights. Recall that $\mathbf{T}^t$ is a row-stochastic matrix, one needs to normalizes the weights. The resulting matrix is the influence distribution for the next period.

$$\mathbf{T}_{ij}^{t+1} = \frac{\varrho_{ij}^t}{\mathbf{T}_{ii}^t + \sum_{k \in N_{-i}} \varrho_{ik}^t}, \text{ where } \varrho_{ii}^t = \mathbf{T}_{ii}^t, \varrho_{ij}^t = \eta_{ij}^t w_{ij}^t. \tag{4.6}$$

Note that one does not apply the network- and distance-based methods on himself, since self-loop is not allowed and the distance between an agent and himself is always 0. The weight that an agent places on himself is changed indirectly via the normalization process.

Updating Opinions

Then, for $t > 0$, each agent takes a weighted average of others' current opinions in forming his own for the next period. That is, we have the opinions updating rule as:

$$\mathbf{p}^{t+1} = \mathbf{T}^{t+1}\mathbf{p}^t \qquad \text{for } t > 0. \tag{4.7}$$

This kind of naive updating rule is rooted in DeGroot (1974) and recently adopted in Golub and Jackson (2007) and Pan (2008). As shown in Pan (2008), learning processes in DeMarzo, Vayanos, and Zwiebel (2003) and Friedkin and Johnsen (1997) can be incorporated into this framework as well.

A period ends after both the influence matrix and opinion vector are properly updated. The updating process repeats during each period (Figure 4.1).

4.2 Convergence with Exogenous Networks

By Theorem 2 in Pan (2008), we know that the opinions are conforming, i.e., all agents converge to hold the same opinion p^*. The focus of this study is on the convergence speed

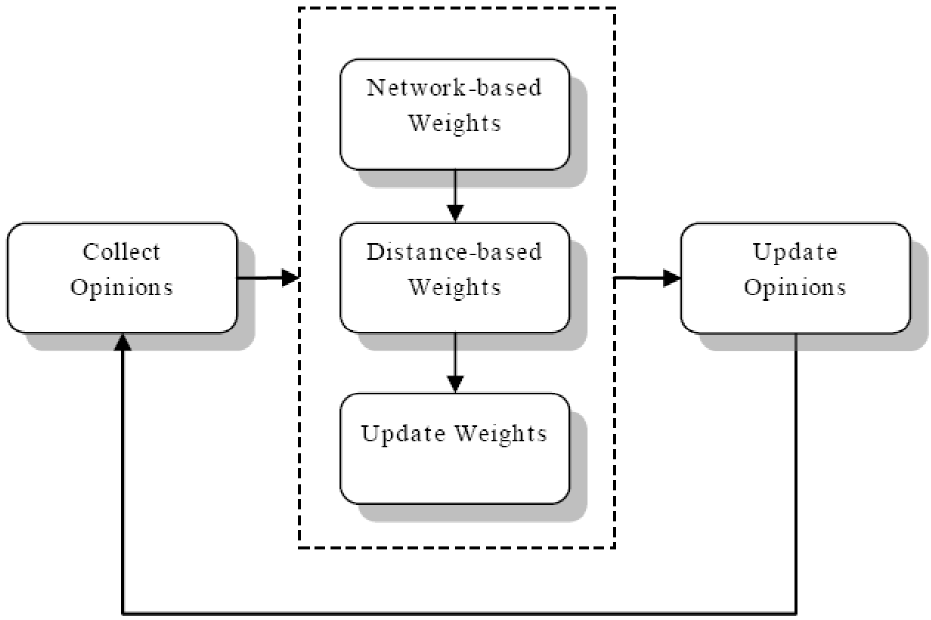


Figure 4.1: Updating process with exogenous network structures

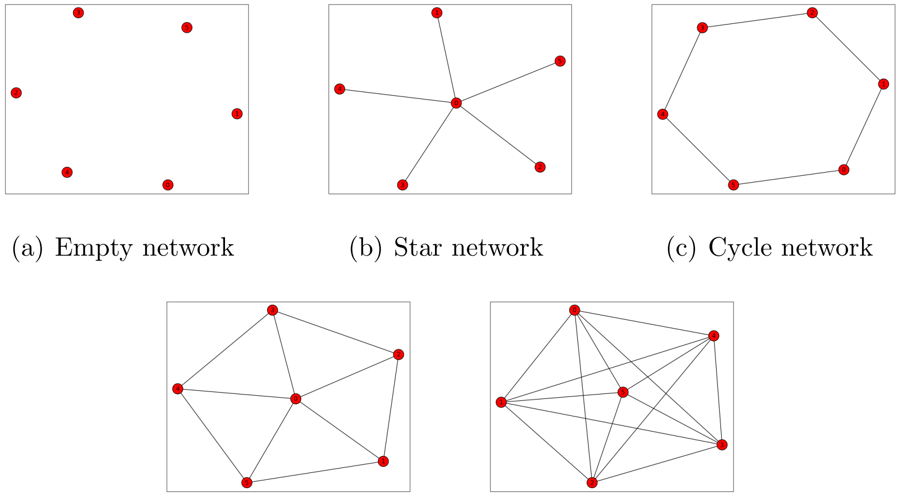
and final opinion with an exogenous network structure $\mathbf{G}^t = \mathbf{G}$ for all t.[3]

4.2.1 Basic Networks

First, consider a few network structures that do not involve randomness. Namely, the empty network, the star network, the cycle network, the wheel network, and the complete network. Figure 4.2 below illustrates these structures among a society of 6 agents.

Simulations are implemented with the combinations of these 5 different network structures and 4 different society sizes: 20, 60, 80, and 100. Note that the construction of the star and wheel networks requires the assignment of a center node. In this study, agent 1 (whose initial opinion is 0) is set as the center. Once conformism is detected, the total number of periods of updating it takes to reach that state and the final opinion that agents converge to are recorded. Since the increment between the initial opinions of two agents of consecutive order is $\frac{1}{n-1}$, during simulations the cut-off distance that prevents taking reverse of zero $\underline{d} = \frac{1}{100n}$.

[3]When executing simulations, convergence is assumed if the standard deviation among $p_1^t, \ldots, p_n^t$ is less than 10^{-6}.



(a) Empty network (b) Star network (c) Cycle network

(d) Wheel network (e) Complete network

Figure 4.2: Networks with no randomness

Table 4.1 shows the summary of convergence speed, indicated by the total number of updating periods. Figure 4.3 shows the standard deviation of opinions over time, with $n = 20$. It is noticeable that star networks result in the fastest convergence speed; whereas it takes numerous periods of updating for agents in a cycle network to reach conformism. Besides, empty and complete networks have roughly the same convergence speed and both are on the low side. Convergence speed with wheel networks ranks in the middle and is significantly faster than that with cycles.

	Empty	Star	Cycle	Wheel	Complete
20	27	10	368	32	14
60	34	11	35509	252	26
80	35	11	62643	271	26
100	59	12	183593	1493	29

Table 4.1: Convergence speed with different network structures and sizes

For the first finding on the fastest and slowest networks, the intuitive explanation is that the center agent attracts others to listen to him more with the most connections that he has. In a sense, the center agent acts as the opinion leader of the group. This also explains why convergence speed with wheel networks are much faster than that with

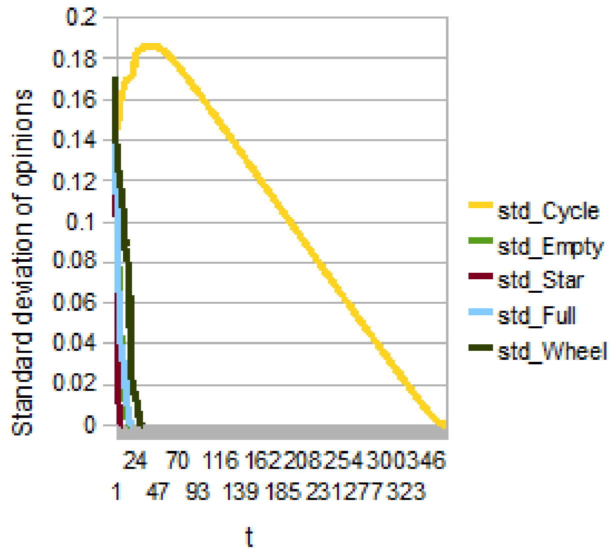


Figure 4.3: Standard deviation of opinions over time (n=20)

cycles, since a wheel network is essentially generated by adding a center node to a cycle network. On the other hand, agents in a cycle network get distracted by their neighbors. Especially, the first and last agents whose initial opinions are the farthest away (which are 0 and 1, respectively) are connected. That kind of distraction leads to the extremely slow convergence.

As for empty and complete networks, the similarity lies in the first step of updating influence weights. Namely, when agent i assigns network-based weights, with empty or complete networks the sum of neighbors' weights are the same for all $j \neq i$. So with the normalization step, for both networks only the distance between agents actually matters in updating influence weights.

Table 4.2 shows the final opinion that all agents end up holding. Recall that initially at $t = 0$, agents' opinions evenly range between 0 and 1. It is interesting to see agents in networks with no center or hub nodes (empty, cycle, and full) converge to the mean value of 0.5. For star and wheel networks, the center agent has an initial opinion of 0. It seems that with star networks, the final opinion gets closer to 0 as size increases. Whereas the opposite trend can be observed for wheel networks.

	Empty	Star	Cycle	Wheel	Complete
20	0.500	0.310	0.500	0.333	0.500
60	0.500	0.228	0.500	0.352	0.500
80	0.500	0.233	0.500	0.359	0.500
100	0.500	0.217	0.500	0.384	0.500

Table 4.2: Final opinion with different network structures and sizes

4.2.2 Stochastic Networks

A stochastic graph is a graph that is generated by some random process. Watts and Strogatz (1998) argued that stochastic networks could be categorized based on two important properties: shortest path length, and cluster coefficient.

A path in a graph is a sequence of nodes such that from each of its nodes there is an edge to the next node in the sequence. The problem of shortest path is to find a path between two nodes such that the sum of the weights of its constituent edges is minimized. For an unweighted graph, the problem is equivalent to finding a path between nodes that has the least total links/edges. The clustering coefficient of a node in a graph quantifies how close the node and its neighbors are to being a clique (complete graph).[4]

Two structures that are often of interest in studies on complex networks are small-world networks and scale-free networks. Both have been shown to capture some of the most noticeable and important characteristics that we observe in many of the real-world networks, including various social networks, the WWW, and the Internet (Albert and Barabási, 2002).

Small-World Network

In terms of graph theory, a small-world network is mainly characterized by a small average shortest path length and a high cluster coefficient. These two properties make it possible to link any two nodes in a small-world network with a relatively short path.

The small-world phenomenon refers to how strangers being linked by a mutual ac-

[4]For an undirected graph, the clustering coefficient for a node v_i that has k_i neighbors is defined as $C_i = \dfrac{2|\{e_{jk}\}|}{k_i(k_i-1)} : v_j, v_k \in L(i), e_{jk} \in E$, where $L(i)$ is the set of v_i's neighbors.

quaintance and is often arguably associated with the famous term of "six degrees of separation" as well as the experiment conducted by Stanley Milgram (Travers and Milgram, 1969) a few decades ago. Nowadays we can still see plenty of applications and similar experiments being carried out as part of pop culture.[5]

A certain category of small-world networks were identified as a class of stochastic graphs by Watts and Strogatz (1998), who also proposed a 2-phase algorithm that generated networks with small-world properties. The first step is to construct a regular ring lattice, a graph with n nodes each connected to k neighbors ($\frac{k}{2}$ on each side). Then the next step is to rewire the existing edges with a probability θ. That is, we replace a node in an edge with another node that is chosen with a uniform probability.

Thus for a fixed size n, it is of interest to test how the 2 key parameters k and θ change the network structure and the path of updating process. During simulations, k takes value from {2, 3, 4, 5} and θ is randomly drawn from 0 and 1.

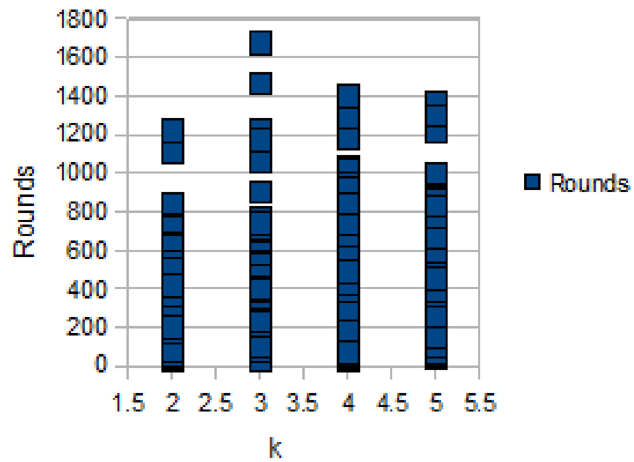
Figure 4.4 shows the convergence speed with $n = 100$ and different values of k and θ. Subfigure (a) shows convergence speed with respect to k, with $\theta \in [0, 1]$. The value of k does not seem to have a significant effect on convergence speed. On the other hand, the relationship between convergence speed (total rounds it takes to reach conformism) and the probability θ looks interesting, as shown in subfigure (b) where $k \in \{2, 3, 4, 5\}$. Exponential regression give a good fitting trend curve:

$$f(\theta) = 750.6 \cdot 0.07^{\theta}, \quad R^2 = 0.72. \tag{4.8}$$

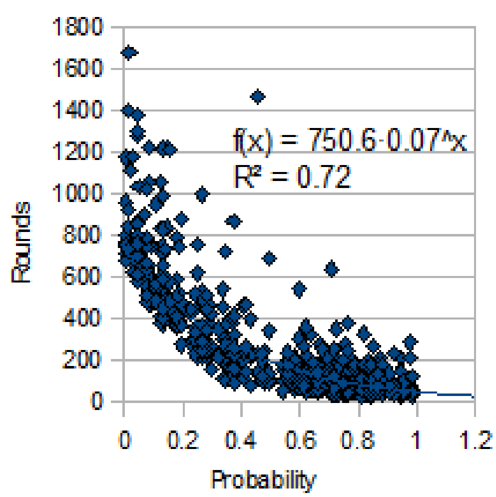
Barabási-Albert Network

A scale-free network is a network whose degree distribution follows a power law (at least asymptotically). In other words, the fraction $P(k)$ of nodes in the network having k connections to other nodes (i.e. with a degree of k) can be approximated with the distribution $P(k) \sim k^{-\gamma}$ where γ is a constant whose value is typically in the range between

[5] For instance, Facebook has a few applications that calculate and illustrate in graphic format the network structure (including shortest path length) among its users.



(a) With respect to k



(b) With respect to θ

Figure 4.4: Convergence speed with small-world networks ($n = 100$)

2 and 3. For instance, Figure 4.5 shows a power law degree distribution with $\gamma = 2.4$, where x-axis indicates the degree and y-axis shows the total number of nodes of certain degree x. The size of the network is 10,000.

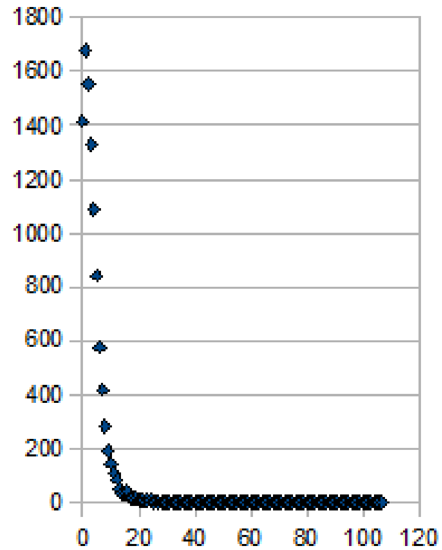


Figure 4.5: Power law degree distribution

As with most disordered networks, such as the small-world network model, the average distance between two vertices in the network is very small relative to a highly ordered network such as a lattice. The clustering coefficient of scale-free networks can vary significantly depending on other topological details. The distribution of clustering coefficient distribution also follows a power law and decreases as the node degree increases.

Many empirical studies have shown the scale-free property in observed complex networks (Barabási, 2003). Scale-free networks are also related to concepts and phenomena such as the "80-20 law" and "rich-get-richer". That is, 80% of the connections are possessed by 20% of the nodes. And nodes with high degree are more likely to get connected with even more nodes. Those hub nodes in scale-free networks enhance robustness of the network. Namely, failures occur at random and the likelihood that a hub would be affected is almost negligible since the vast majority of nodes are those with small degree. Even if such event occurs, the network will not lose its connectedness, which is guaranteed by the remaining hubs.

One of the most widely used algorithm in creating a scale-free network is the Barabási-Albert model which emphasizes the two crucial ingredients of scale-free network: growth and preferential attachment. That is, the number of nodes grows over time and the nodes with higher degrees are more likely to get connected with newcomers (Barabási and Albert, 1999).

For the simulations, at the initial stage of network construction [6] we have m_0 agents with a star structure; then during each stage a new node is added to the graph and $m = m_0 - 1$ edges are formed based on preferential attachment. Figure 4.6 shows the convergence speed with 100-agent scale-free networks with respect to different values of m: 2, 5, 10, and 20. The total rounds of updating it takes to conform appear to be decreasing as m increases. With a larger m, the center of the initial star is almost definite to have the most links at the end. Then, similar to star and wheel networks, the presence of degree-dominating nodes leads to faster convergence speed.[7]

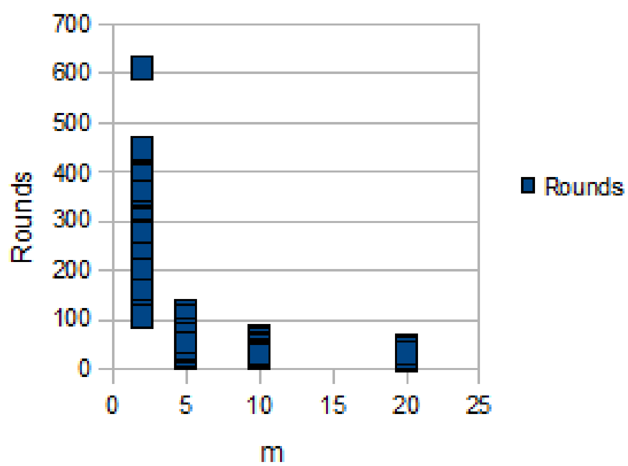


Figure 4.6: Convergence speed with Barabási-Albert networks ($n = 100$)

In Appendix D I show findings on the correlation between hub agents' initial opinions and the final opinion that all agents converge to have.

[6]Please note that this should be distinguished from the initial period of the updating process ($t = 0$). We construct the network first, then impose it onto the agents at $t = 0$.

[7]Note that here I use the term "Barabási-Albert network" instead of "scale-free networks". The reason is the learning process is among 100 agents, which makes the network size too small to be referred to as scale-free. However, with the Barabási-Albert algorithm, the degree distribution of the networks exhibit power-law patterns.

4.3 Endogenous Network Formation

In this section, I consider a model in which agents form an endogenous neighborhood rather than accommodating to an exogenous network structure. The main idea stems from the network-distance combination updating process of influence weights that is discussed earlier. Here the network is growing, we have new agents joining the community, one at a time. When the newcomer decides on whom to connect with, he takes two steps:

- Out of the existing agents, m_r are selected with a uniform probability. We call these m_r agents initial contacts. The new agent i will form a link with an agent j in this pool of initial contact, if the distance between their opinions is close enough. Namely, if $|p_i - p_j| < d_r$.

- Next, the initial contacts (whether connected to i or not, based on the distance criteria) introduce the new agent to their neighbors, i.e. the agents to whom they are connected with. From these neighbors of initial contacts, m_n are chosen with a uniform probability. They can be seen as the (network-based) secondary contacts. Following a similar rule, the new agent will form a link with an agent k among the m_n if their opinions are close enough: $|p_i - p_k| < d_n$.

Note that the network is undirected, which means that once a new agent decides to initiate a link with an existing agent, the latter will certainly accept the invitation. This can also be interpreted as a consent between the two agents, since the distance-based rule is symmetric.

The links that a new agent forms with initial and secondary contacts are the same, therefore it would be natural to assume that he does not use double standards towards the two groups of potential neighbors, i.e., $d_r = d_n = d$. It also eliminates one degree of freedom. In simulations, m_r and m_n are set to be equal as well. Variations will be discussed later.

This model is very similar to the one presented in Jackson and Rogers (2007), which is based on probabilities of getting connected to initial and secondary contacts. With this algorithm, the randomness of the graph is modeled by agents' opinions, which also reflect the heterogeneity of agents.

The effects of this kind of heterogeneity is the main focus of this section. Simulation outcomes were analyzed to find out the correlation between one's opinion and his degree. First, consider the case where the opinions of agents follow a normal distribution: $p_N \sim$ N(0, 1).[8] For $m_r = m_n = 5$ and a total size of 10,000, d takes 3 different values: 0.1, 0.3, and 0.5. From Figure 4.7 we see that nodes with opinions around 0 are significantly more likely to become hubs. Also, the plots all take a bell shape, just like the normal distribution. Another noticeable feature is that the highest degree increases in d.

Actually, it doesn't take 10,000 nodes to show the bell shape and the positive relationship between the highest degree and d. I found that for all values of d, these two attributes are observed with 1,000 nodes. And trend is that the lower d is, the fewer nodes we need to have the bell shape. In Figure 4.8 I show the plots with 500 nodes.

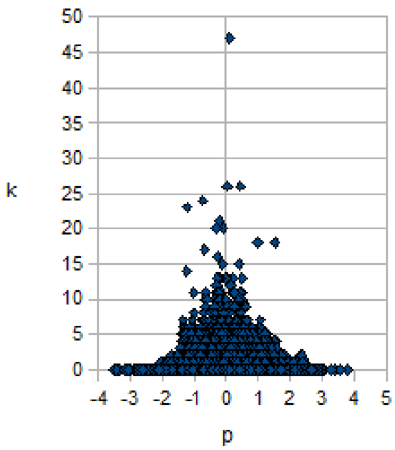
To explain these findings, let's transform the distance-based rules into probabilities. With a normal distribution, for an agent with opinion p_i, the probability that he will be connected can be written as:

$$\vartheta(p_i, d) = \int_{p_i-d}^{p_i+d} f(p)dp = F(p_i + d) - F(p_i - d), \tag{4.9}$$

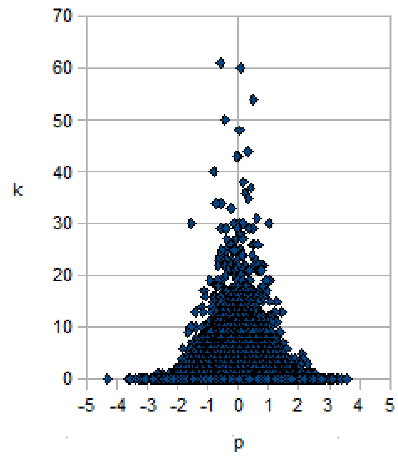
where $F(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{x} \exp(-\frac{u^2}{2})du$ is the cumulative distribution function (cdf) of a normal distribution. Figure 4.9 shows $\vartheta(p_i, d)$ with d = 0.1, 0.3, and 0.5, where the x axis indicates one's opinion and y axis indicates the probability of getting connected with that opinion. They all take the bell shape and get taller as d increases. This implies an increasing probability of nodes closer to the center getting connected, which consequently leads to the increasing degree of those hubs.

Figure 4.10 shows that the resulting networks exhibit power-law degree distributions with $\gamma = 2.48 \pm 0.2$ (x axis indicates degrees of nodes and y axis shows the total number of nodes with a certain degree). The resulting γ fits various empirical data very well. Namely, the WWW has γ_{out} = 2.45 (Albert, Jeong, and Barabási, 1999), the Internet

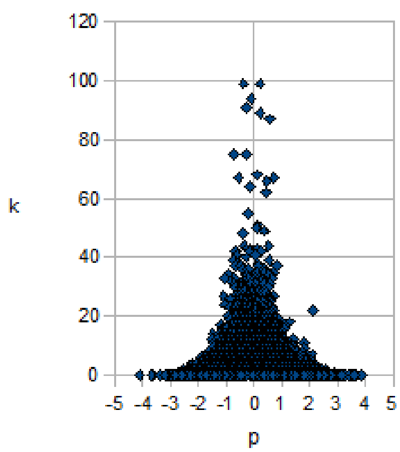
[8]Note that $p \in [0, 1]$ is assumed earlier in the basic social learning model. Here the focus is on network formation and we assume a different range and distribution of opinions.



(a) $d = 0.1$

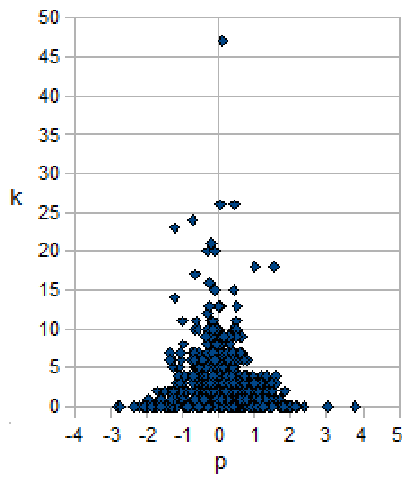


(b) $d = 0.3$

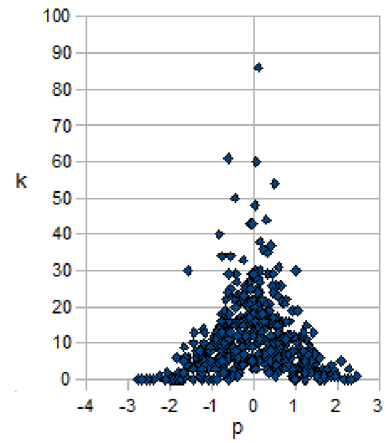


(c) $d = 0.5$

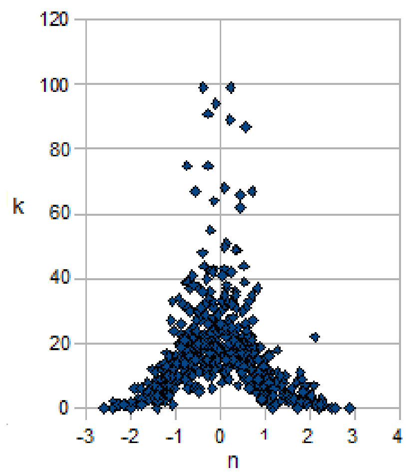
Figure 4.7: Correlation between opinions and degrees ($p_N \sim \mathrm{N}(0,\,1)$)



(a) $d = 0.1$



(b) $d = 0.3$



(c) $d = 0.5$

Figure 4.8: Correlation between opinions and degrees ($n = 500$)

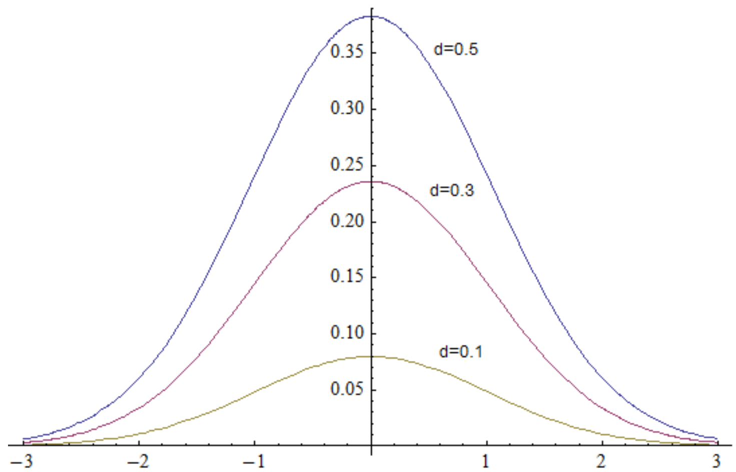


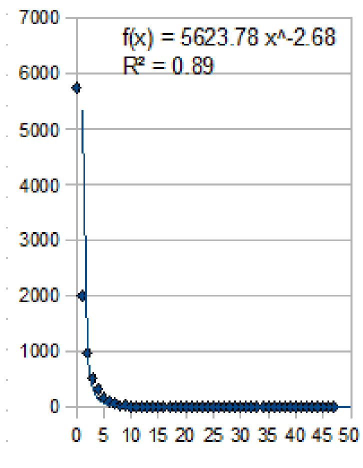
Figure 4.9: Probability of getting connected ($p_N \sim$ N(0, 1))

router network has $\gamma = 2.48$ (Faloutsos, Faloutsos, and Faloutsos, 1999), the network of movie stars has $\gamma = 2.3$ (Barabási and Albert, 1999), and collaboration among math scholars shows a γ of 2.5 (Barabási, Jeong, Neda, Ravasza, Schubert, and Vicsek, 2002).

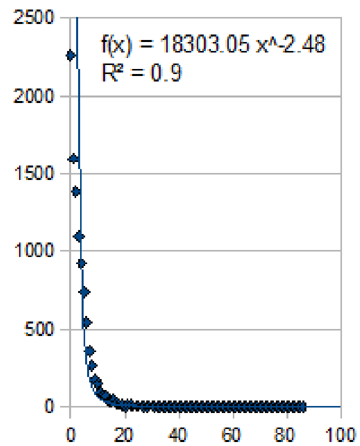
The findings are consistent with work by Barabási and Albert (1999), who argued that the two key ingredients of power-low degree distribution are 1) growth and 2) preferential attachment. Here the preferential attachment is captured by the second phase, where a newcomer is introduced to a pool of secondary contacts by initial contacts. That is, if one is connected with many, that he is more likely to get introduced to the newcomer as an initial contact's neighbor and therefore more likely to build a link with the newcomer than someone who has the same opinion but a lower degree.

Another aspect of (the basic) preferential attachment is the early-node advantage. Namely, the longer one has been in the network, the more likely he will accumulate more links and obtain a higher degree. My formation algorithm shows such phenomena as well. Nodes are numbered by the order of joining the network and we may interpret the order as age, i.e., nodes labeled with lower numbers are older. In Figure 4.11 we see that for all values of d, older nodes have significantly higher degrees than the younger ones.

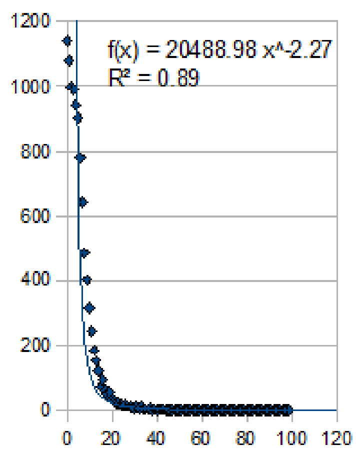
In Appendix D I present results with a uniform distribution of opinions. We observe power-law distribution and a different hub pattern corresponding to a different probability function $\vartheta(p_i, d)$.



(a) $d = 0.1$

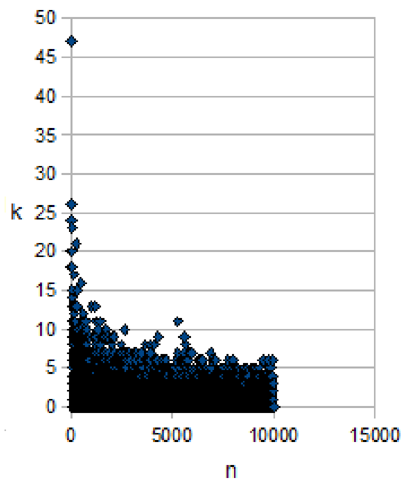


(b) $d = 0.3$

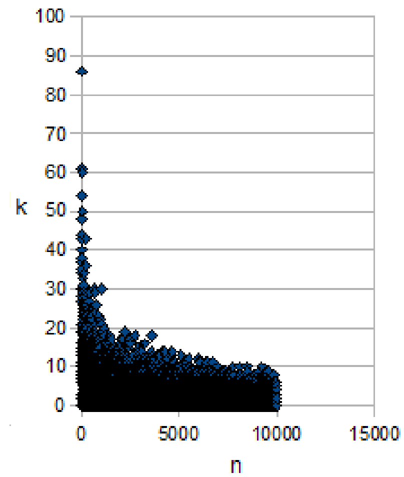


(c) $d = 0.5$

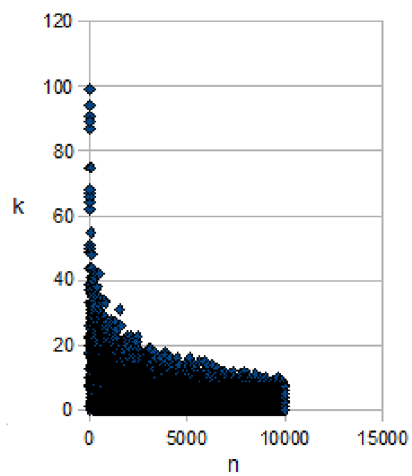
Figure 4.10: Degree distribution ($p_N \sim$ N(0, 1))



(a) $d = 0.1$



(b) $d = 0.3$



(c) $d = 0.5$

Figure 4.11: Correlation between age and degree ($p_N \sim \mathrm{N}(0,\, 1)$)

4.4 Conclusion

I present a model of information dissemination in a framework of interactive networks. Agents update not only their opinions but also the influence weights that they place on others. Two key elements in the interaction and learning process are network and distance. Agents refer to their neighbors when determining weights on other members in the community; they also tend to give higher weights on those whose opinions are closer to theirs. The updating process starts with uniform opinions and weights, and converges to conformity of opinions among agents. With different network structures, it is shown that the convergence speed is faster when there are hub nodes in the network.

The study continues to investigation into endogenous network formation with a growing population. A new agent meets a group agents (initial contacts) at random and form links with those that share close enough opinions. Then the new agent gets introduced to a group of randomly selected neighbors of the initial contacts (secondary contacts), and repeats the distance-based link formation. The network-based process of the algorithm contributes to a power-law degree distribution, in that agents with higher degrees are more likely to be chosen to meet the new agents. And the distance-based rule makes agents whose opinions are closer to the mean more likely to become hubs.

One possible extension to the network formation algorithm is randomness in connection decisions. Two variations could be considered. The first is that with a probability η, the new agent does the opposite of the distance-based decision. The intuition is that although people often make friends with others who share a similar point of view, we sometimes get connected with those who do not or don't connect with someone that has a close opinion. It may be interesting to combine this kind of randomness with a Bernoulli distribution, which translates into the binary opinions people hold over a lot of issues: yes or no, left or right, liberal or conservative. Higher clustering should be expected since agents form links with those from the same team; while the randomness implies that a mule and an elephant don't have to be enemies. The second is that new agents indeed use "double standards" when meeting initial and secondary contacts in that they may form links with secondary contacts with pure probability, not distance-based criteria. Especially, for friends of friends, we may get along with them just because we don't want to give our friends a hard time.

The motivation for these extensions is that the resulting networks from the present algorithm may leave some agents unconnected and thus do not have a small average path length. Another extension is a larger pool of secondary contacts, corresponding to the growing network (Jackson and Rogers, 2007). With randomness of connection and more candidates for connection, the probability of agents getting connected increases without harming other features of the algorithm, which may reduce the average shortest path length.

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Appendix A

In this appendix, I review some of the basic notations and concepts of network theory. The fundamental units of a network are a set of items called vertices or sometimes nodes, with connections between them, called edges. Edges are also referred to as bonds (physics), links (computer science), or ties (sociology). An edge is directed if it runs in only one direction; and undirected if it runs in both directions. A graph is directed if all of its edges are directed. Often the directedness of links is of interest when discussing information and benefit flows in a network.

The number of edges connected to a node is called its degree. It is a basic property of a node. A directed graph has both an in-degree and an out-degree for each node, which are the numbers of in-coming and out-going edges respectively. The degree distribution, which denotes the statistical probability characteristics of degrees of all the nodes in a network, is often of interest in network researches.

A graphical illustration consisting of nodes and links can of course be used to demonstrate a network. In the case where we have a directed network, the edges are indicated by lines with arrows. A network with n nodes can also be presented by a $n \times n$ matrix. Namely, we can construct a matrix g such that $g_{ij} = 1$ if i, j are connected and $g_{ij} = 0$ otherwise. The matrix g can be asymmetric to present a directed network, i.e., there exists some i and j such that $g_{ij} = 1$ and $g_{ji} = 0$. In addition, g does not have to take on values of 0 and 1 only. When elements g_{ij} have other values, the matrix illustrates a network with heterogenous relationships between its members, such as weights or allocation of resources that members place on each connection. For instance, naive social learning models use a row-stochastic matrix that has positive elements and the sum of the elements of each row equal to 1.

A path in a graph is a sequence of nodes such that from each of its nodes there is an edge to the next node in the sequence. The problem of shortest path is to find a path between two nodes such that the sum of the weights of its constituent edges is minimized. For an unweighted graph, the problem is equivalent to finding a path between nodes that has the least total nodes. The minimum number of paths needed to connect two agents is also called the distance between them. When two agents can not be reached, the distance is infinity. The diameter of a graph is the maximal distance between any pair of its nodes, i.e. the maximal edges it takes to connect two nodes.

The clustering coefficient of a node in a graph quantifies how close the node and its neighbors are to being a clique (complete graph). For an undirected graph, the clustering coefficient for a node v_i that has k_i neighbors is defined as:

$$C_i = \frac{2|\{e_{jk}|e_{ij} \in E, v_j, v_k \in L(i)\}|}{k_i(k_i - 1)}, \tag{A.1}$$

where $L(i)$ is the set of v_i's neighbors.

Appendix B

This appendix contains proofs and discussions of theorems presented in Chapter 2 (basic learning model and extension with persistent agents).

Proof of Theorem 2.1

For the first part, suppose we have a sequence $\{\mathbf{T}^t\}_{t=1}^{\infty}$ that

$$\mathbf{T}^t = \overline{\mathbf{T}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ for all } t.$$

Then $\{\mathbf{T}^t\}_{t=1}^{\infty}$ is obviously convergent since $\mathbf{T}^t$ is time-invariant.

However, $\displaystyle\prod_{\tau=t}^{1} \mathbf{T}^{\tau} = \overline{\mathbf{T}}^{(t)} = \begin{cases} \overline{\mathbf{T}} & \text{if } t \text{ is odd,} \\ \mathbf{I} & \text{if } t \text{ is even.} \end{cases}$

That is, $\mathbf{p}^t = \displaystyle\prod_{\tau=t}^{1} \mathbf{T}^{\tau}\mathbf{p}^0$ is not convergent, although $\mathbf{T}^t$ is.

Next, another counter example that shows the second part of the theorem. Suppose we have a sequence $\{\mathbf{T}^t\}_{t=1}^{\infty}$ that $\mathbf{T}^t = \begin{cases} \overline{\mathbf{T}} & \text{if } t \text{ is odd,} \\ \mathbf{I} & \text{if } t \text{ is even.} \end{cases}$ where $\overline{\mathbf{T}} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

$\mathbf{T}^t$ is not convergent as it is switching between two matrices.

On the other hand, $\prod_{\tau=t}^{1} \mathbf{T}^{\tau} \rightarrow \begin{pmatrix} 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \end{pmatrix}$. Hence we have,

$$\lim_{t\to\infty} \mathbf{p}^t = \lim_{t\to\infty} \prod_{\tau=t}^{1} \mathbf{T}^{\tau}\mathbf{p}^0 = (p^*, p^*, p^*)^{\mathrm{T}}, \text{ where } p^* = \frac{2}{5}p_1^0 + \frac{2}{5}p_2^0 + \frac{1}{5}p_3^0.$$

That is, in this case $\mathbf{p}^t$ converges whereas $\mathbf{T}^t$ does not.

Proof of Theorem 2.2

First, with Theorem 3.1 in Seneta (1981),

$$\max_{i,j} |p_i^{t+1} - p_j^{t+1}| \leq \mu_t(\mathbf{T})\{\max_{i,j} |p_i^t - p_j^t|\}, \tag{B.1}$$

where $\mu_t(\mathbf{T}) = \frac{1}{2}\max_{i,j}\sum_{k=1}^{n} |\mathbf{T}_{ik}^t - \mathbf{T}_{jk}^t|$.

Next, consider $\mu_{t+1}(\mathbf{T}) = \frac{1}{2}\max_{i,j}\sum_{k=1}^{n} |\mathbf{T}_{ik}^{t+1} - \mathbf{T}_{jk}^{t+1}|$. Denote $\delta_{ijk}^{t+1} = \mathbf{T}_{ik}^{t+1} - \mathbf{T}_{jk}^{t+1}$. Let $B(i)$ denote indices that $\delta_{ijk}^{t+1} \geq 0$ and $B(j)$ denote indices that $\delta_{ijk}^{t+1} \leq 0$. Since $\mathbf{T}^t$ is row-stochastic, $\sum_{B(i)} \delta_{ijk}^{t+1} = \sum_{B(j)} \delta_{ijk}^{t+1}$. That is,

$$\mu_{t+1}(\mathbf{T}) = \frac{1}{2} * 2 \max_{i,j} \sum_{B(i)} \delta_{ijk}^{t+1} = \max_{i,j} \sum_{B(i)} \delta_{ijk}^{t+1}. \tag{B.2}$$

For $n > 2$, it must be true that either $|B(i)| \geq 2$ or $|B(j)| \geq 2$. Without loss of generality, assume that $|B(j)| \geq 2$. By definition, for $l \in B(j), \mathbf{T}_{il}^{t+1} \geq \mathbf{T}_{jl}^{t+1}$. Since $|B(i)| \geq 2$, there exists $l \in B(i)$, s.t. $l \neq j$. Choose such an l.

Denote $\overline{p} = \max_{i\in N} |p_i^0|$. Since $|p_i^0| < \infty$ for all i, $\overline{p}$ exists. Also, $\mathbf{p}^t = \mathbf{T}^t\mathbf{p}^{t-1}$ and $\mathbf{T}^t$ is row-stochastic. Thus we have $|p_t^t - p_j^t| < 2\overline{p}$, for all i, j, t.

Hence, $d_{il}^t \in [\underline{d}, \underline{d}+2\overline{p}]$, for all i, t. That is, $w_{il}^t \in [\frac{1}{\underline{d}+2\overline{p}}, \frac{1}{\underline{d}}]$. Therefore,

$$\mathbf{T}_{jl}^{t+1} = \frac{w_{jl}^t}{\mathbf{t}_{jj}^t + \sum_{i \in N_{-j}} w_{ji}^t} > \frac{\frac{1}{\underline{d}+2\overline{p}}}{1 + \frac{n-2}{\underline{d}} + \frac{1}{\underline{d}+2\overline{p}}}. \tag{B.3}$$

Denote $\frac{\frac{1}{\underline{d}+2\overline{p}}}{1 + \frac{n-2}{\underline{d}} + \frac{1}{\underline{d}+2\overline{p}}}$ as δ_c. Then, $\mu_t(\mathbf{T}) = \max\limits_{i,j} \sum\limits_{B(i)} \delta_{ijl}^t < 1 - \delta_c$. That is, μ_t is less than and bounded away from 1. Thus we have $|p_i^{t+1} - p_j^{t+1}| \to 0$ since $\max\limits_{i,j} |p_i^{t+1} - p_j^{t+1}| \leq \prod\limits_{\tau=1}^{t} \mu_\tau(\mathbf{T}) \max\limits_{i,j} |p_i^0 - p_j^0|$.

Next, show that $\mathbf{p}^t$ is conforming. Denote for each t,

$$\underline{p}^t = \min\{p_1^t, \ldots, p_n^t\}, \quad \overline{p}^t = \max\{p_1^t, \ldots, p_n^t\}. \tag{B.4}$$

Thus $p_i^t \in [\underline{p}^t, \overline{p}^t]$ for all i. Moreover, $p_i^{t+1} \in [\underline{p}^t, \overline{p}^t]$ for all i since p_i^{t+1} is a convex combination of p_i^t. That is, $\{\mathbf{p}^t\}_{t=0}^\infty$ is a sequence in a compact set $[\underline{p}^0, \overline{p}^0]^n$.

Therefore, $\{\mathbf{p}^t\}_{t=0}^\infty$ has a subsequence that is convergent. Suppose that the subsequence is convergent to $\mathbf{p}^*$. If $\mathbf{p}^t$ does not converge to $\mathbf{p}^*$, then there exists another subsequence of $\{\mathbf{p}^t\}_{t=0}^\infty$ that is convergent to $\mathbf{p}^{**} \neq \mathbf{p}^*$.

We have shown that $p_i^t - p_j^t \to 0$ for all i, j, so $\mathbf{p}^* = (p', \ldots, p')$ and $\mathbf{p}^{**} = (p'', \ldots, p'')$. Also, $\overline{p}^t - \underline{p}^t \to 0$. For $\epsilon = |p' - p''|$, by definition, there exists t', s.t. for $t > t', \overline{p}^t - \underline{p}^t < \epsilon$, i.e. $|p' - p''| < \epsilon$ since $p', p'' \in [\underline{p}^t, \overline{p}^t]$. Therefore, $p' = p''$. That is, $\mathbf{p}^t$ is conforming, $\lim\limits_{t \to \infty} \mathbf{p}^t = (p^*, \ldots, p^*)^{\mathrm{T}}$.

Proof of Theorem 2.3

By Theorem 2.2, under the conditions indicated, $\mathbf{p}^t$ is conforming, i.e. there exists p^* s.t. for $\epsilon = \frac{1}{2}\underline{d}$, there exists $t^* > 0$ s.t. $|p_i^t - p^*| \leq \epsilon$ for $t > t^*$, for all $i \in N$.

Hence for $t > t^*$, $|p_i^t - p_j^t| \leq 2\epsilon = \underline{d}$, for all i, j. By the given updating rule, for all

$j \neq i$, $d_{ij}^t = \underline{d}$. That is, $w_{ij}^t = \dfrac{1}{\underline{d}}$, for all $j \neq i$. Thus for all i and $j, k \neq i$,

$$\mathbf{T}_{ij}^{t+1} = \mathbf{T}_{ik}^{t+1} = \frac{1/\underline{d}}{\mathbf{T}_{ii}^t + (n-1)/\underline{d}}, \quad \mathbf{T}_{ii}^{t+1} = \frac{\mathbf{T}_{ii}^t}{\mathbf{T}_{ii}^t + (n-1)/\underline{d}}. \tag{B.5}$$

Then for all $t > t^*$ we have

$$\frac{\mathbf{T}_{ii}^{t+1}}{\mathbf{T}_{ii}^t} = \frac{1}{\mathbf{T}_{ii}^t + (n-1)/\underline{d}} < 1. \tag{B.6}$$

$$\Rightarrow \mathbf{T}_{ii}^t \to 0 \Rightarrow \mathbf{T}_{ij}^t \to \frac{1}{n-1} \quad \text{for all } i \text{ and } j \neq i.$$

That is, $\lim_{t\to\infty} \mathbf{T}^t = \mathbf{T}^*$ where $\mathbf{T}_{ii}^* = 0, \mathbf{T}_{ij}^* = \dfrac{1}{n-1}$, for all i and $j \neq i$.

Proof of Theorem 2.4

Recall Theorem 3.1 in Seneta (1981) that is used to prove Theorem 2.2. In this case, define

$$\delta_s = \min_{s\in S, j\in N} \{\delta_c, \mathbf{T}_{sj}^0\}. \tag{B.7}$$

Then we have

$$\mu_t(\mathbf{T}) = \frac{1}{2} \max_{\overline{s},s} \sum_{i=1}^n |\mathbf{T}_{\overline{s}i}^t - \mathbf{T}_{si}^t| < 1 - \delta_s \tag{B.8}$$

i.e., $\mu_t(\mathbf{T})$ is less than and bounded away from 1. Thus follow the same proof as we have for Theorem 2.2 and 2.3, we have that $\lim_{t\to\infty} p_i^t = p^*$ for all i and for $\overline{s} \in \overline{S_1}$, for all $s' \neq \overline{s}$, $\lim_{t\to\infty} \mathbf{T}_{\overline{s}\overline{s}}^t = 0$ and $\lim_{t\to\infty} \mathbf{T}_{\overline{s}s'}^t = \dfrac{1}{n-1}$.

Proof of Proposition 2.5

For $t > 0$,

$$R_{\overline{s}}^t = [\underbrace{\overline{\mathbf{T}}_{\overline{s}}^t, \dots, \overline{\mathbf{T}}_{\overline{s}}^t, \mathbf{T}_{\overline{s}\overline{s}}^t, \overline{\mathbf{T}}_{\overline{s}}^t, \dots, \overline{\mathbf{T}}_{\overline{s}}^t}_{n-\sigma_1}, \underbrace{\underline{\mathbf{T}}_{\overline{s}s_1}^t, \dots, \underline{\mathbf{T}}_{\overline{s}s_m}^t}_{\sigma_1}], \text{ for } \overline{s} \in \overline{S_1},$$
$$R_s^t = R_s^0 \text{ for } s \in S_1,$$

where R_i^t is the i-th row of $\mathbf{T}^t$ for all $i \in N$ and

$$\overline{\mathbf{T}}_{\overline{s}}^t = \frac{\frac{1}{\underline{d}}}{\mathbf{T}_{\overline{s}\overline{s}}^{t-1} + \frac{n-\sigma_1}{\underline{d}} + \sum_{s \in S_1} \frac{1}{|p_{\overline{s}}^{t-1} - p_s^{t-1}|}},$$
$$\underline{\mathbf{T}}_{\overline{s}s_j}^t = \frac{\frac{1}{|p_{\overline{s}}^{t-1} - p_{s_j}^{t-1}|}}{\mathbf{T}_{\overline{s}\overline{s}}^{t-1} + \frac{n-\sigma_1}{\underline{d}} + \sum_{s \in S_1} \frac{1}{|p_{\overline{s}}^{t-1} - p_{s_j}^{t-1}|}}.$$

Note that non-persistent agents always have the same opinion and influence weights, i.e. $p_i^t = p_j^t = p_{\overline{s}}^t$ and $\mathbf{T}_{ik}^t = \mathbf{T}_{jk}^t$ for all $i, j \notin S$ for all t.[9] Denote the maximum difference in persistent and non-persistent agents' opinions $\Delta p^t = \max_{s \in S} |p_{\overline{s}}^t - p_s^t|$. Then we have

$$\begin{aligned}\underline{\mathbf{T}}_{\overline{s}s_j}^t &= \frac{\frac{1}{|p_{\overline{s}}^{t-1} - p_s^{t-1}|}}{\mathbf{T}_{\overline{s}\overline{s}}^{t-1} + \frac{n-\sigma_1}{\underline{d}} + \sum_{s \in S_1} \frac{1}{|p_1^{t-1} - p_s^{t-1}|}} \leq \frac{\frac{1}{|p_{\overline{s}}^{t-1} - p_s^{t-1}|}}{\mathbf{T}_{\overline{s}\overline{s}}^{t-1} + \frac{n-\sigma_1}{\underline{d}} + \frac{\sigma_1}{\Delta p^t}} \\ &\leq \left(\frac{1}{|p_{\overline{s}}^{t-1} - p_s^{t-1}|}\right) / \left(\frac{n-\sigma_1}{\underline{d}} + \frac{\sigma_1}{\Delta p^t}\right).\end{aligned}$$

Denote the change in non-persistent agents' opinion over each period $\Delta p_{\overline{s}}^t = |p_{\overline{s}}^t - p_{\overline{s}}^{t-1}|$, then with equation (2.7):

$$\Delta p_{\overline{s}}^t \leq \sum_{s \in S} \underline{\mathbf{T}}_{\overline{s}}^t |p_s^{t-1} - p_{\overline{s}}^{t-1}| \leq \frac{\sigma_1 \underline{d} \Delta p^t}{n \Delta p^t - \sigma_1 \Delta p^t + \sigma_1 \underline{d}}. \tag{B.9}$$

[9] The reasoning is as follows: initially non-persistent agents have the same opinion, which leads to the same interim weights w_{ij}^t and eventually results in the same weight distribution. Since non-persistent agents take weighted average of all opinions in updating their own, the same weight distribution implies that all non-persistent agents share the same opinion in the next period. Then again that same opinion results in the same weight assignment, and so on and so forth.

Compare that with $\underline{d}$, we need to determine the sign of

$$\frac{\sigma_1 \underline{d} \Delta p^t}{\underline{d}} - (n\Delta p^t - \sigma_1 \Delta p^t + \sigma_1 \underline{d}). \tag{B.10}$$

The formula equals to:

$$\begin{aligned}
&\sigma_1 \Delta p^t - (n\Delta p^t - \sigma_1 \Delta p^t + \sigma_1 \underline{d}) \\
=\ &2\sigma_1 \Delta p^t - n\Delta p^t - \underline{d}\Delta p^t - \sigma_1 \underline{d} \\
=\ &(2\sigma_1 - n)\Delta p^t - \sigma_1 \underline{d}
\end{aligned}$$

Since $\sigma_1 \leq n \cdot 10\%$, $2\sigma_1 - n < 0$.[10] Besides, $\underline{d} > 0, \Delta p^t > 0,$ and $\sigma_1 \geq 0$,

$$\begin{aligned}
&\Rightarrow (2\sigma_1 - n)\Delta p^t - \sigma_1 \underline{d} < 0, \\
&\Rightarrow \frac{\sigma_1 \underline{d} \Delta p^t}{\underline{d}} - (n\Delta p^t - \sigma_1 \Delta p^t + \sigma_1 \underline{d}) < 0, \\
&\Rightarrow |p^1_{\bar{s}_1} - p^1_{\bar{s}_2}| \leq \frac{\sigma_1 \underline{d} \Delta p^t}{n\Delta p^t - \sigma_1 \Delta p^t + \sigma_1 \underline{d}} < \underline{d}.
\end{aligned}$$

That is, after each round of updating, the change in non-persistent agents' opinion is less than $\underline{d}$. On the other hand, from period 0 to period 1, the change in the opinion of a persistent agent s_i is $|p^0_s - p^0_{\bar{s}}| \sum_{\bar{s} \in \bar{S}} \mathbf{T}^t_{s_i \bar{s}}$, which is often much larger than $\underline{d}$ during simulations. Also, note that persistent agents have different opinions during the early period due to the difference in their initial influence weights. And for $t \leq 1$, the change in the opinion of s_i is $\sum_{\bar{s} \in \bar{S}} \mathbf{T}^t_{s_i \bar{s}}(p^t_{\bar{s}} - p^{t-1}_{\bar{s}}) + \sum_{s \in S} \mathbf{T}^t_{s_i s}(p^t_s - p^{t-1}_s)$, which is often larger than $\underline{d}$ as well. Thus the simulation results show that the final opinion is closer to $p^0_{\bar{s}}$ than it is to p^0_s.

[10] Actually, as long as we have less than 50% of persistent agents, this inequality and the following inequalities are always valid.

Proof of Theorem 2.6

With type II persistent agents, the updating process can be rewritten as

$$\mathbf{p}^t = \widetilde{\mathbf{T}}^t \mathbf{p}^{t-1}, \tag{B.11}$$

$$\text{where } \widetilde{\mathbf{T}}^t = \begin{pmatrix} \mathbf{T}^t_{11} & \dots & \mathbf{T}^t_{1j} & \dots & \dots & \mathbf{T}^t_{1n} \\ \vdots & & \vdots & & & \vdots \\ \mathbf{T}^t_{n-\sigma_2,1} & \dots & \mathbf{T}^t_{n-\sigma_2,j} & \dots & \dots & \mathbf{T}^t_{n-\sigma_2,n} \\ 0 & \dots & 0 & \frac{1}{\sigma_2} & \dots & \frac{1}{\sigma_2} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & \frac{1}{\sigma_2} & \dots & \frac{1}{\sigma_2} \end{pmatrix} \tag{B.12}$$

Then

$$\mu_t(\widetilde{\mathbf{T}}) = \frac{1}{2} \max_{\bar{s},s} \sum_{i=1}^{n} |\widetilde{\mathbf{T}}^t_{\bar{s}i} - \widetilde{\mathbf{T}}^t_{si}| < 1 - \delta_c.$$

is still true. Thus the difference of opinions converges to zero.

Next, we need to prove that $p_i^t \to p_s^0$. Suppose that $p_i^t \to p^*$ and $p^* \neq p_s^0$. Then for $\epsilon^* = \frac{1}{2}|p_s^0 - p^*|$, since for all $s \in S, p_s^t = p_s^0, \forall t$, $|p_s^0 - p^*| > \epsilon^*$ for all t. However, it is assumed that $p_i^t \to p^*$, which implies that $\forall \epsilon > 0$, there exists $t > 0$, s.t. $|p_i^t - p^*| < \epsilon$ for all i. Thus we have a contradiction, which means that $p^* = p_s^0$. In other words, $\lim_{t\to\infty} p_i^t = p_s^0, \forall i$.

Then similar to the basic model, with conforming opinions, $\lim\limits_{t\to\infty} \mathbf{T}^t_{ii} = 0$ and $\lim\limits_{t\to\infty} \mathbf{T}^t_{ij} = \dfrac{1}{n-1}$, for all $i \in N$, for all $j \neq i$.

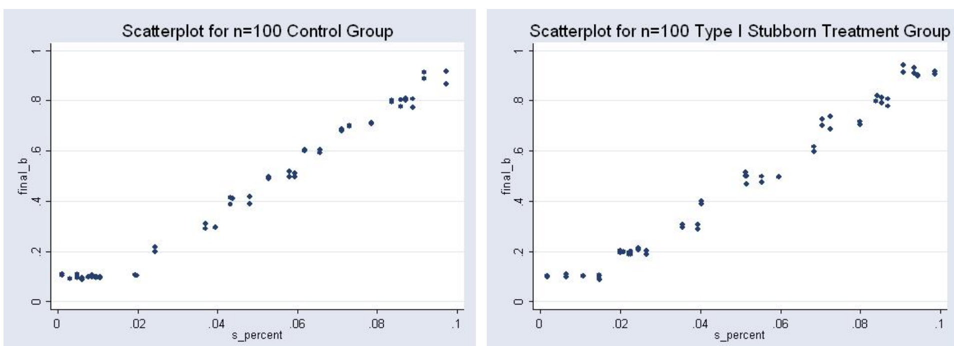
Proof of Theorem 2.7

In this case we can apply the same opinion updating process as that with type II persistent agents, as shown in equation (B.11) and equation (B.12). The statement as regard to the convergence of both opinions and the influence matrix follows if we mimic the proof of

Theorem 2.6 (omitted).

More on Simulation of Persistent Experiment

In this subsection we discuss the simulation results of updating rules with persistent agents. It seems interesting that linear patterns in the cases of control type I persistent groups is observed. Whereas for type II and double-persistent treatment groups, the linear relationship is much less noticeable. Since smaller groups are forced to have at least one persistent agent would result in bias. Scatter plots for 100-agent societies more clearly show the relationship between persistent percentage and final opinion (Figure B.1 and Figure B.2). Recall the proof of Proposition 2.5 on the change in persistent agents' opinions, the linear pattern is valid. That is, when we have fewer type I persistent agents, their opinions change more than they would if we had more persistent agents.

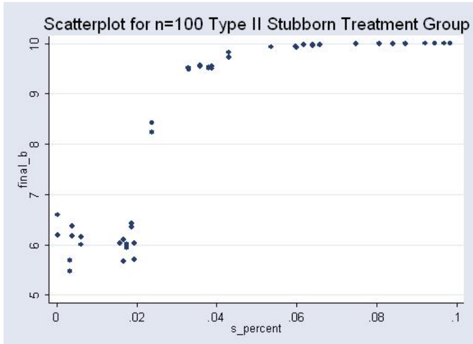


(a) Control group, n=100 (b) Type I persistent treatment, n=100

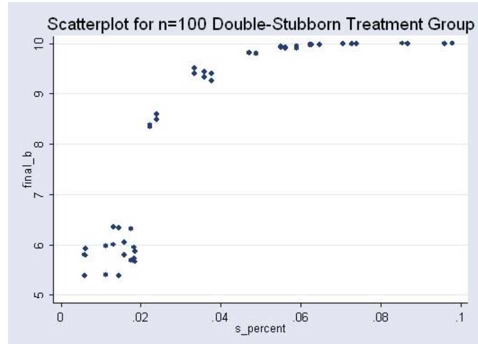
Figure B.1: Final opinions and persistent proportion, n=100 A

In Figure B.2, when $s \geq 2$, the final opinions "jump" to close to 10. This leads to the question: the theory says that the number of persistent agents does not matter; but is there a threshold in the persistent percentage that determines the final outcome?

Actually, these observed results, and the discrepancy that final opinion does not always equal to 10 (shown in Table 2.1), are caused by the parameter settings of the simulations. Especially, the simulation will be terminated once $\| \Delta \mathbf{T}^t \| < \epsilon^t$, where $\epsilon^t = \epsilon^{t-1} \cdot \delta = 0.8\epsilon^{t-1}$ and $\epsilon^0 = \dfrac{1}{100n}$. Besides, $p_s^0 = 10$, $p_{\bar{s}}^0 = 0$, $\underline{d} = 10^{-6}$, and $10 \leq n \leq 100$.



(a) Type II persistent treatment, n=100



(b) Double-persistent treatment, n=100

Figure B.2: Final opinions and persistent proportion, n=100 B

For example, suppose we have 1 type II persistent agent. Denote $\Delta p^t = |p^t_{\overline{s}} - p^t_s|$. Then $\Delta p^0 = 10$ and at $t = 1$, the $\overline{s}$-th row of $\mathbf{T}^t$

$$
\begin{aligned}
R^1_{\overline{s}} &= (\underbrace{\overline{\mathbf{T}}^1, \ldots, \overline{\mathbf{T}}^1, \mathbf{T}^1_{\overline{ss}}, \overline{\mathbf{T}}^1, \ldots, \overline{\mathbf{T}}^1}_{n-1}, \underline{\mathbf{T}}^1), \\
\text{where } \overline{\mathbf{T}}^1 &= \frac{10^6}{\mathbf{T}^0_{\overline{ss}} + (n-2)10^6 + (\Delta p^0)^{-1}}, \\
\underline{\mathbf{T}}^1 &= \frac{(\Delta p^0)^{-1}}{\mathbf{T}^0_{\overline{ss}} + (n-2)10^6 + (\Delta p^0)^{-1}}, \\
\mathbf{T}^1_{\overline{ss}} &= \frac{\mathbf{T}^0_{\overline{ss}}}{\mathbf{T}^0_{\overline{ss}} + (n-2)10^6 + (\Delta p^0)^{-1}}.
\end{aligned}
$$

So $p^1_{\overline{s}} = R^1_{\overline{s}} \times \mathbf{p}^0 = \dfrac{1}{\mathbf{T}^0_{\overline{ss}} + (n-2)10^6 + 10^{-1}}$ and $p^1_s = 10$.

Define $\delta p = |\Delta p^1 - \Delta p^0| = \dfrac{1}{\mathbf{T}^0_{\overline{ss}} + (n-2)10^6 + 10^{-1}} < \dfrac{1}{(n-2)10^6}$. Then for the next period,

$$
\begin{aligned}
&\Delta\overline{\mathbf{T}} = \delta p \cdot \overline{f}(10), \quad \Delta\underline{\mathbf{T}} = \delta p \cdot \underline{f}(10), \quad \Delta\mathbf{T}_{\overline{ss}} = \delta p \cdot \tilde{f}(10). \\
&\text{where } \overline{f}(\Delta p^0) = \frac{d\overline{\mathbf{T}}^1}{d(\Delta p^0)}, \quad \underline{f}(\Delta p^0) = \frac{d\underline{\mathbf{T}}^1}{d(\Delta p^0)}, \quad \tilde{f}(\Delta p^0) = \frac{d\mathbf{T}^1_{\overline{ss}}}{d(\Delta p^0)}.
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \overline{f}(10) &= 10^4(\delta p)^2, \quad \underline{f}(10) = 10^{-3}(\delta p)^2 - 10^{-2}\delta p, \\
\text{and } \tilde{f}(10) &= \mathbf{T}^1_{\overline{ss}} \cdot 10^{-2}(\delta p)^2 \leq 10^{-2}(\delta p)^2.
\end{aligned}
$$

Thus the norms of the difference of $\mathbf{T}^t$ during these 2 periods take the form:

$$
\begin{aligned}
\|\Delta\mathbf{T}\| &= (n-1)[(n-2)(\Delta\overline{\mathbf{T}})^2 + (\Delta\underline{\mathbf{T}})^2] + \sum_{\bar{s}=1}^{n-1}(\Delta\mathbf{T}_{\bar{s}\bar{s}})^2 \\
&\leq (n-1)n[10^4(\delta p)^3]^2 < n(n-1)10^{-28}\frac{1}{(n-2)^6}
\end{aligned}
$$

$$
\Rightarrow \quad \parallel \Delta\mathbf{T} \parallel << \frac{1}{1000n} < \epsilon^1.
$$

In other words, it only takes 2 rounds of updating to meet the termination condition of the simulation. When the program exits the simulation, agents' opinions have not really reached the steady state yet.

In conclusion, it is not the "threshold" on the number of persistent agents that determines the final outcome, but the parameter settings during the simulation that cause the observed discrepancy. That is, number of persistent agents does not matter. In order to test the claim, I have executed the simulation with only 1 or 2 persistent agents and without the automatic termination procedure. The results showed that the society's opinion does conform to 10, which supports the theoretical prediction.

Appendix C

In this appendix I show proofs of theorems and propositions presented in Chapter 3 (learning and game play in a dual network framework).

Proof of Proposition 3.1

Denote for each t,

$$\underline{p}^t = \min\{p_1^t, \ldots, p_n^t\}, \quad \overline{p}^t = \max\{p_1^t, \ldots, p_n^t\}.$$

Thus $p_i^t \in [\underline{p}^t, \overline{p}^t]$ for all i. Moreover, $p_i^{t+1} \in [\underline{p}^t, \overline{p}^t]$ for all i since p_i^{t+1} is a convex combination of p_i^t. That is, $\{\mathbf{p}^t\}_{t=0}^{\infty}$ is a sequence in a compact set $[\underline{p}^0, \overline{p}^0]^n$.

Recall that $\pi_{ij}^t = ap_i^t p_j^t + (1-p_i^t)(1-p_j^t)$. We have $\frac{\partial \pi_{ij}^t}{\partial p_j^t} = (a+1)p_i^t - 1$. So π_{ij}^t increases with p_j^t when $p_i^t \geqslant \dfrac{1}{a+1}$ and decreases with p_j^t when $p_i^t < \dfrac{1}{a+1}$.

1. $\frac{1}{a+1} \leqslant \underline{p}^0 \leqslant \overline{p}^0$.
   In this case, define $\pi_{min}^1 = a(\underline{p}^0)^2 + (1-\underline{p}^0)^2$. Then for all i, t, $p_i^t \geqslant \underline{p} \geqslant \dfrac{1}{a+1}$, $\frac{\partial \pi_{ij}^t}{\partial p_i^t} \geqslant 0$. Thus, $\pi_{ij}^t \geqslant \pi_{min}^1$, for all i, j, t.

2. $\underline{p}^0 \leqslant \overline{p}^0 < \frac{1}{a+1}$.
   In this case, define $\pi_{min}^2 = a(\overline{p}^0)^2 + (1-\overline{p}^0)^2$. Then for all i, t, $p_i^t \leqslant \overline{p} < \dfrac{1}{a+1}$, $\frac{\partial \pi_{ij}^t}{\partial p_i^t} < 0$. Thus, $\pi_{ij}^t \geqslant \pi_{min}^2$, for all i, j, t.

3. $\underline{p}^0 < \frac{1}{a+1} \leqslant \overline{p}^0$.

In this case, define $\pi^3_{min} = a\underline{p}^0\overline{p}^0 + (1-\underline{p}^0)(1-\overline{p}^0)$. For arbitrary i, j, t, without loss of generality, assume that $p_i^t \leqslant p_j^t$. We have 2 possible scenarios in this case.

(i) $p_j^t < \dfrac{1}{a+1} \leqslant \overline{p}^0$. Then since $\frac{\partial \pi^t_{ij}}{\partial p_i^t} < 0$, $\overline{p}^0 \geqslant p_i^t$,

$$\pi^t_{ij} \geqslant ap_j^t\overline{p}^0 + (1-p_j^t)(1-\overline{p}^0).$$

And $\frac{\partial \pi^t_{ij}}{\partial \overline{p}^0} > 0$, $\underline{p}^0 \leqslant p_j^t$, so

$$ap_j^t\overline{p}^0 + (1-p_j^t)(1-\overline{p}^0) \geqslant a\underline{p}^0\overline{p}^0 + (1-\underline{p}^0)(1-\overline{p}^0) = \pi^3_{min},$$

which implies that $\pi^t_{ij} \geqslant \pi^3_{min}$

(ii) $\dfrac{1}{a+1} \leqslant p_j^t \leqslant \overline{p}^0$. Then similar to the previous case,

$$\pi^t_{ij} \geqslant ap_j^t\underline{p}^0 + (1-p_j^t)(1-\underline{p}^0) \geqslant a\underline{p}^0\overline{p}^0 + (1-\underline{p}^0)(1-\overline{p}^0) = \pi^3_{min}.$$

That is, in all cases, we can find a π_{min} such that $\pi^t_{ij} \geqslant \pi_{min}$ for all i, j, t. Especially, in the first 2 cases, $\pi_{min} = a\rho^2 + (1-\rho)^2 \geqslant \frac{a}{a+1}$, for $0 \leqslant \rho \leqslant 1$. So $\pi^t_{ij} \geqslant \frac{a}{a+1}$ in these 2 cases. Otherwise, $\pi^t_{ij} \geqslant a\underline{p}^0\overline{p}^0 + (1-\underline{p}^0)(1-\overline{p}^0)$. Recall that $\underline{\pi} = \min\{\frac{a}{a+1},\ k\underline{p}^0\overline{p}^0 + (1-\underline{p}^0)(1-\overline{p}^0)\}$, then it holds that $\pi^t_{ij} \geqslant \underline{\pi}$ for all i, j, t in all cases. Since $0 \leqslant \underline{p}^0 \leqslant \overline{p}^0 \leqslant 1$, $a\underline{p}^0\overline{p}^0 + (1-\underline{p}^0)(1-\overline{p}^0) \geqslant 0$. Also $\frac{a}{a+1} > 0$. Thus $\underline{\pi} \geqslant 0$; the equality holds only when $\underline{p}^0 = 0$ and $\overline{p}^0 = 1$.

Preliminaries to Proof of Theorem 3.2

Lemma 1. *If $c \leqslant \underline{\pi}$, then with the updating process the interaction network g^t converges to be fully connected, i.e., $g^t \to g_N$ as $t \to \infty$.*

Proof. From Proposition 3.1 it follows that $\pi^t_{ij} \geqslant \underline{\pi} \geqslant 0$ for all i, j, t. Furthermore, every pair $i, j \in N$ is selected randomly, and, thus, in the long term each pair is selected with probability 1. So for $c \leqslant \underline{\pi}$, $\pi^t_{ij} \geqslant c$ is always true and any randomly chosen link ij at time

t is always formed if $ij \notin g^{t-1}$ or stays formed if $ij \in g^{t-1}$. Thus, $g^t \to g_N$ in probability. Hence, there exists some $\bar{t} > 0$, such that for $t > \bar{t}$ we have $g^t = g^N$, where g^N denotes the fully connected interaction network among N. □

Lemma 2. *If $c \leqslant \underline{\pi}$ and there are $i, j \in N$ such that $p_i^0 \neq p_j^0$, then with the updating process defined for the basic model, there exists $\hat{t} > 0$, s.t. for all $t > \hat{t}, 0 < p_k^t < 1$ for all $k \in N$.*

Proof. First, show that the conditions stated in the assertion indicate that the elements of $\mathbf{p}^0$ cannot be all 0s or all 1s.

As shown in the proof of Proposition 3.1, $\{\mathbf{p}^t\}_{t=0}^{\infty}$ is a sequence in a compact set $[\underline{p}^0, \overline{p}^0]^n$. So if $0 < p_i^0 < 1$ for all i, then the assertion of Lemma 2 is true.

Next, consider the case where there exists $\gamma \in N$ such that p_γ^0 is either 0 or 1. Then in order to have $p_\gamma^t = p_\gamma^0$, it must hold that $\mathbf{T}_{\gamma j}^t > 0$ if $p_\gamma^0 = p_j^0$ and $\mathbf{T}_{\gamma j}^t = 0$ otherwise. Suppose that $\mathbf{T}^0$ satisfies that condition. Recall that

$$\mathbf{T}_{ij}^t = \frac{w_{ij}^t}{\sum_{k=1}^n w_{ik}^t}, \text{ for all } j \in N, t > 0, \tag{C.1}$$

$$\text{where } w_{ij}^t = \sum_{l \in L_i^t} v_{lj}^t \pi_{lj}^{t-1}, \quad v_{lj}^t = \begin{cases} 1 & \text{if } lj \in g^t, \\ 0 & \text{otherwise.} \end{cases}$$

Consider two players γ, j such that $p_\gamma^0 \neq p_j^0$. With Proposition 3.1, we have that $\pi_{ij}^t \geqslant \underline{\pi}$ for all i, j, t. So a link between any 2 players is always beneficial and will be formed when chosen. Thus there exists $\hat{t}_\gamma$, s.t. for $t > \hat{t}_\gamma$, $j \in L(\gamma)^t$ and $j \in L(j)^t$. $\pi_{jj}^t = k(p_j^t)^2 + (1 - p_j^t)^2 > 0$, which means that $w_{\gamma j}^t > 0$. Thus, $\mathbf{T}_{\gamma j}^t > 0$ even though $p_\gamma^0 \neq p_j^0$. In other words, player γ cannot remain his initial strategy. We can repeat this process for all $\{\lambda \mid \lambda \in N, p_\lambda^0 = 0 \text{ or } 1\}$. Thus $\exists \hat{t} = \max_\lambda \hat{t}_\lambda$, s.t. $\forall t > \hat{t}, 0 < p_k^t < 1$ for all $k \in N$. □

Proof of Theorem 3.2

With Lemma 1, we know that $\mathbf{G}^t$ converges to a fully connected network.

As for the final strategy patterns, first consider the special cases where $\mathbf{p}^0 = (0, \ldots, 0)^{\mathrm{T}}$

and $\mathbf{p}^0 = (1, \ldots, 1)^{\mathrm{T}}$. Obviously, in these cases, strategy vector remain the same. So $p^* = p_i^0$.

Then, for other cases. First, we show that

$$\sum_{k=1}^{n} |\mathbf{T}_{ik}^t - \mathbf{T}_{jk}^t| < 2 - \tilde{\tau}, \tag{C.2}$$

where $\tilde{\tau}$ is a positive number that has a lower bound.

By Lemma 2, we know that there exists $\hat{t} > 0$, s.t. for all $t > \hat{t}$, we have $0 < p_i^t < 1$ for all $i \in N$. Denote

$$\underline{p}^{\hat{t}} = \min_{i \in N} p_i^{\hat{t}} > 0, \quad \overline{p}^{\hat{t}} = \max_{i \in N} p_i^{\hat{t}} < 1. \tag{C.3}$$

Then it holds that for all $t > \hat{t} + 1$, for all i, $p_i^t \in [\underline{p}^{\hat{t}}, \overline{p}^{\hat{t}}]$. Define $\pi_{min}^{\hat{t}} = \min\{\frac{a}{a+1}, a\underline{p}^{\hat{t}}\overline{p}^{\hat{t}} + (1 - \underline{p}^{\hat{t}})(1 - \overline{p}^{\hat{t}})\}$. We have $\pi_{min}^{\hat{t}} > 0$ since $0 < \underline{p}^{\hat{t}} \leqslant \overline{p}^{\hat{t}} < 1$. Then we can mimic the proof of Proposition 3.1 and prove that that $\pi_{ij}^t \geqslant \pi_{min}^{\hat{t}}$, for all i, j and $t > \hat{t}$.[11] Also, $\pi_{ij}^t \leqslant a$ for all i, j, t. Then $\mathbf{T}_{ij}^t > \frac{\pi_{min}^{\hat{t}}}{an^2} > 0$. Denote $\tilde{\tau} = \frac{1}{2} * \frac{\pi_{min}^{\hat{t}}}{an^2} > 0$. Then we have

$$\sum_{k=1}^{n} |\mathbf{T}_{ik}^t - \mathbf{T}_{jk}^t| < 2 - 2\tilde{\tau}. \tag{C.4}$$

Theorem 3.1 in Seneta (1981) states that,

$$\max_{i,j} |p_i^{t+1} - p_j^{t+1}| \leqslant \mu_t(\mathbf{T})\{\max_{i,j}] \mid p_i^t - p_j^t|\}, \tag{C.5}$$

where $\mu_t(\mathbf{T}) = \frac{1}{2} \max_{i,j} \sum_{k=1}^{n} |\mathbf{T}_{ik}^t - \mathbf{T}_{jk}^t|$.

Since $\sum_{k=1}^{n} |\mathbf{T}_{ik}^t - \mathbf{T}_{jk}^t| < 2 - 2\tilde{\tau}$ for $t > \hat{t}$,

$$\mu_t(\mathbf{T}) = \frac{1}{2} \max_{i,j} \sum_{k=1}^{n} |\mathbf{T}_{ik}^t - \mathbf{T}_{jk}^t| < 1 - \tilde{\tau}, \text{ for } t > \hat{t}. \tag{C.6}$$

That is, $\mu_t(\mathbf{T}) \leqslant 1$ for all t and it is strictly less than and bounded away from 1 for $t > \hat{t}$.

[11] Proof omitted here.

Since $\max_{i,j}|p_i^{t+1}-p_j^{t+1}| \leqslant \prod_{\tau=1}^{t}\mu_\tau(\mathbf{T})\max_{i,j}|p_i^0-p_j^0|$, we have $\lim_{t\to\infty}|p_i^{t+1}-p_j^{t+1}|=0$

Next, we show that p_i^t is conforming to the same value p^*. As shown in the proof of Lemma 1, $\{\mathbf{p}^t\}_{t=0}^{\infty}$ is a sequence in a compact set $[\underline{p}^0,\overline{p}^0]^n$. Therefore, $\{\mathbf{p}^t\}_{t=0}^{\infty}$ has a subsequence that is convergent. Suppose that the subsequence is convergent to $\mathbf{p}^*$. If $\mathbf{p}^t$ does not converge to $\mathbf{p}^*$, then there exists another subsequence of $\{\mathbf{p}^t\}_{t=0}^{\infty}$ that is convergent to $\mathbf{p}^{**}\neq\mathbf{p}^*$.

We have shown that $p_i^t-p_j^t\longrightarrow 0$ for all i,j, so $\mathbf{p}^*=(p',\ldots,p')$ and $\mathbf{p}^{**}=(p'',\ldots,p'')$. Also, $\overline{p}^t-\underline{p}^t\longrightarrow 0$. For $\epsilon=|p'-p''|$, by definition, $\exists t'$, s.t. for $t>t', \overline{p}^t-\underline{p}^t<\epsilon$, i.e., $|p'-p''|<\epsilon$ since $p',p''\in[\underline{p}^t,\overline{p}^t]$. Therefore, $p'=p''$.

That is, $\lim_{t\to\infty}p_i^t=p^*,\ \text{for all} i\in N$.

As for the influence matrix $\mathbf{T}^t$. Recall equation (3.10) that

$$\mathbf{T}_{ij}^t=\frac{w_{ij}^t}{\sum_{k=1}^n w_{ik}^t},\ \text{for all } j\in N, t>0,$$

$$\text{where } w_{ij}^t=\sum_{l\in L_i^t}v_{lj}^t\pi_{lj}^{t-1},\quad v_{lj}^t=\begin{cases}1 & \text{if } lj\in g^t,\\ 0 & \text{otherwise.}\end{cases}$$

Then if p_i^t converges to p_i^* for all i, the fully connected network results in such a $\mathbf{T}^t$ that

$$\lim_{t\to\infty}\mathbf{T}_{ij}^t=\mathbf{T}_{ij}^*=\frac{\sum_{k\in N}[ap_k^*p_j^*+(1-p_k^*)(1-p_j^*)]}{\sum_{l\in N}\sum_{k\in N}[ap_k^*p_l^*+(1-p_k^*)(1-p_l^*)]},\forall i,j\in N.$$

The expression of $\mathbf{T}_{ij}^*$ can be simplified as

$$\mathbf{T}_{ij}^*=\frac{x[(a+1)p_j^*-1]+n(1-p_j^*)}{(a+1)x^2-2nx+n^2},\ \text{where } x=\sum_{k\in N}p_k^*.$$

The simplified function shows that all the elements in each column j converges to the same value $\mathbf{T}_{\cdot j}^*=\frac{x[(a+1)p_j^*-1]+n(1-p_j^*)}{(a+1)x^2-2nx+n^2}$ which depends on p_j^*. Since $\lim_{t\to\infty}p_i^t=p^*,\ \text{for all} i\in N$, $\mathbf{T}_{\cdot i}^*=\mathbf{T}_{\cdot j}^*$ for all $i,j\in N$, which implies that $\lim_{t\to\infty}\mathbf{T}_{ij}^t=\frac{1}{n}$ for all i,j.

Proof of Theorem 3.3

With Lemma 1, we know that $\mathbf{G}^t$ converges to a fully connected network.

Denote $m = |S|$ as the total number of persistent players. Rearrange the players in such an order that number 1 to $n - m$ are normal players and the last m players are persistent. Essentially, the only step during updating that differs from the basic model is when players modify their strategies by taking weighted averages. Since persistent players do not change their strategies, their influence weight assignments do not affect their choices or the final outcome. Thus the strategy updating process can be rewritten as:

$$\mathbf{p}^t = \widetilde{\mathbf{T}}^t \mathbf{p}^{t-1}, \tag{C.7}$$

where

$$\widetilde{\mathbf{T}}^t = \begin{pmatrix} \mathbf{T}^t_{11} & \dots & \mathbf{T}^t_{1j} & \dots & \dots & \mathbf{T}^t_{1n} \\ \vdots & & \vdots & & & \vdots \\ \mathbf{T}^t_{n-m,1} & \dots & \mathbf{T}^t_{n-m,j} & \dots & \dots & \mathbf{T}^t_{n-m,n} \\ 0 & \dots & 0 & \frac{1}{m} & \dots & \frac{1}{m} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & \frac{1}{m} & \dots & \frac{1}{m} \end{pmatrix} \tag{C.8}$$

In this case, $\mu_t(\widetilde{\mathbf{T}}) = \frac{1}{2}\max_{i,j}\sum_{s=1}^{n}|\widetilde{\mathbf{T}}^t_{is} - \widetilde{\mathbf{T}}^t_{js}| < 1 - \tilde{\tau}$ still holds. Thus from the proof of Proposition 1, we have that $\mathbf{T}^t_{ij} \to \frac{1}{n}$, and that $p^t_i \to p^*$ for all i.

Next, we need to prove that $p^t_i \to p_\alpha$. Suppose that $p^t_i \to p^*$ and $p^* \neq p_\alpha$. Then for $\epsilon^* = \frac{1}{2}|p_\alpha - p^*|$, since for all $s \in S, p^t_s = p_\alpha, \forall t$, $|p_\alpha - p^*| > \epsilon^*$ for all t. However, it is assumed that $p^t_i \to p^*$, which implies that $\forall \epsilon > 0, \exists t > 0$, s.t. $|p^t_i - p^*| < \epsilon$ for all i. Thus we have a contradiction, which means that $p^* = p_\alpha$. In other words, $\lim_{t\to\infty} p^t_i = p_\alpha, \forall i$.

Then similar to the basic model, with fully connected interaction network and conforming strategies, $\mathbf{T}^t_{ij} \to \frac{1}{n}, \forall i, j$. The influence matrix exhibits equal distribution patterns at the stable state.

Proof of Theorem 3.4

With Lemma 1, we know that $\mathbf{G}^t$ converges to a fully connected network.

Then, similar to proof of Theorem 2, denote $m = |S|$ as the total number of persistent players and rearrange the players in such an order that number 1 to $n-m$ are normal players and the last m players are persistent.

Define

$$\tilde{\mathbf{T}}^t = \begin{pmatrix} \mathbf{T}^t_{11} & \dots & \mathbf{T}^t_{1j} & \dots & \mathbf{T}^t_{1n} \\ \vdots & & \vdots & & \vdots \\ \mathbf{T}^t_{n-m,1} & \dots & \mathbf{T}^t_{n-m,j} & \dots & \mathbf{T}^t_{n-m,n} \\ \underbrace{0,\dots,0}_{n-m-1} & & 0 & 1 & 0,\dots,0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix}. \tag{C.9}$$

That is, a persistent player places weight 1 on himself and 0 on everybody else. Then essentially the updating process of strategy vector with diverse persistent players can be represented as:

$$\mathbf{p}^t = \tilde{\mathbf{T}}^t \mathbf{p}^{t-1}. \tag{C.10}$$

Next, for normal players $i \leqslant n-m$, define

$$\mathbf{C}^t_i = \check{\mathbf{T}}^t \mathbf{C}^{t-1}_i = (\prod_{\theta=1}^{t} \check{\mathbf{T}}^\theta)\mathbf{C}^0_i, \tag{C.11}$$

where $\mathbf{C}^0_i$ is the i-th column of $\mathbf{T}^0$ and

$$\check{\mathbf{T}}^t = \begin{pmatrix} \mathbf{T}^t_{11} & \dots & \mathbf{T}^t_{1,n-m} & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \mathbf{T}^t_{n,1} & \dots & \mathbf{T}^t_{n,n-m} & 0 & \dots & 0 \end{pmatrix}. \tag{C.12}$$

That is, we set the last m columns of the (original) influence matrix to 0. Then the j-th element of $\mathbf{C}_i^t$ indicate the weight that player j places on the player i during time t for $j \leqslant n-m$. It can be interpreted as player i's contribution to $\mathbf{p}^t$ received by j. Note that the last m elements of $\mathbf{C}_i^t$ is always 0, as it is indicated by equation (C.10).

For $\mathbf{C}_i^t = (C_{i1}^t, \ldots, C_{in}^t)^{\mathrm{T}}$, we have

$$\begin{aligned}
|C_{ij}^t - C_{ik}^t| &= |\sum_{l \in N} (\check{\mathbf{T}}_{jl}^t - \check{\mathbf{T}}_{kl}^t) C_l^{t-1}| \\
&\leq |\textstyle\sum_{l=1}^{n-m} (\mathbf{T}_{jl}^t - \mathbf{T}_{kl}^t) C_l^{t-1}| + \textstyle\sum_{l=n-m+1}^{n} |\mathbf{T}_{jl}^t - \mathbf{T}_{kl}^t| (\overline{C}_i^{t-1} - \underline{C}_i^t) \\
&\leqslant \mu_t(\mathbf{T}) \{\max_{j,k} |C_{ij}^{t-1} - C_{ik}^{t-1}|\}.
\end{aligned}$$

That is, $\max_{j,k} |C_{ij}^t - C_{ik}^t| \leqslant \mu_t(\mathbf{T}) \{\max_{j,k} |C_{ij}^{t-1} - C_{ik}^{t-1}|\}$. The trick here is to use the row-stochastic matrix $\mathbf{T}_t$. Then for the last m elements, for j, k such that $\mathbf{T}_{jl}^t - \mathbf{T}_{kl} \geqslant 0$ we replace C_{ij}^{t-1} with $\overline{C}_i^{t-1} = \max_{l \in N} C_{il}^{t-1}$ and C_{ik}^{t-1} with $\underline{C}_i^{t-1} = \min_{l \in N} C_{il}^{t-1}$. Vice versa. This allows us to utilize Seneta's theorem, which requires a row-stochastic matrix ($\check{\mathbf{T}}^t$ is not).

From the proof of Theorem 3.2 we know that $|C_{ij}^t - C_{ik}^t| \to 0$. That is, for players $i, j, k \leqslant n-2$, we have $\lim_{t \to \infty} \mathbf{T}_{ik}^t = \lim_{t \to \infty} \mathbf{T}_{jk}^t = \mathbf{T}_{\cdot k}^*$. That is, the normal players converge to have same weight assignment on other normal players.

Then, for each persistent player s, define

$$\tilde{\mathbf{T}}_s^t = \begin{pmatrix}
\mathbf{T}_{11}^t, \ldots, \mathbf{T}_{1,n-m}^t & 0, \ldots, 0 & \mathbf{T}_{1s}^t & 0, \ldots, 0 \\
\vdots & \vdots & \vdots & \vdots \\
\mathbf{T}_{s-1,1}^t, \ldots, \mathbf{T}_{s-1,n-m}^t & 0, \ldots, 0 & \mathbf{T}_{s-1,s}^t & 0, \ldots, 0 \\
0, \ldots, 0 & 0, \ldots, 0 & 1 & 0, \ldots, 0 \\
\mathbf{T}_{s+1,1}^t, \ldots, \mathbf{T}_{s+1,n-m}^t & 0, \ldots, 0 & \mathbf{T}_{s+1,s}^t & 0, \ldots, 0 \\
\vdots & \vdots & \vdots & \vdots \\
\mathbf{T}_{n,1}^t, \ldots, \mathbf{T}_{n,n-m}^t & 0, \ldots, 0 & \mathbf{T}_{n,s}^t & 0, \ldots, 0
\end{pmatrix}. \tag{C.13}$$

That is, we set the s-th row of the (original) influence matrix to $\mathbf{1}_s$, where the s-th element is 1 and the rest are all 0. Also, the value of the i-th element in the s-th column is $\mathbf{T}_{is}^t$

(except $\mathbf{T}^t_{ss}$) instead of 0. Whereas the other $m-1$ columns numbered after $n-m$ are all 0s.

Then we can now apply equation (C.11) to persistent players. Namely, $\mathbf{C}^t_s = \tilde{\mathbf{T}}^t_s \mathbf{C}^{t-1}_s$. Note here that for each s we have a specified matrix $\tilde{\mathbf{T}}^t_s$, unlike the case with normal players where we use the same matrix for them all.

Next, we use the same technic shown above and have that $|C^t_{sj} - C^t_{sk}| \to 0$, which means that normal players assign the same weight to the same persistent player as well.

Thus, for all $i, j \notin S$, $|p^t_i - p^{t-1}_i| = \sum_{k \in N} (\mathbf{T}^t_{ik} - \mathbf{T}^t_{jk}) p^{t-1}_k \to 0$. That is, $\mathbf{p}^t$ converges and all normal players' strategies conform:

$$\lim_{t\to\infty} p^t_i = \lim_{t\to\infty} p^t_j = p^* \text{ for all } i, j \notin S. \tag{C.14}$$

Also, $\mathbf{G}^t \to \mathbf{G}^N$, which means that $\mathbf{T}^t$ converges to $\mathbf{T}^*$ where

$$\lim_{t\to\infty} \mathbf{T}^t_{ik} = \lim_{t\to\infty} \mathbf{T}^t_{jk} = \frac{x[(a+1)p^*_k - 1] + n(1-p^*_k)}{(a+1)x^2 - 2nx + n^2}, \tag{C.15}$$

where

$$x = \sum_{k \in N} p^*_k, \text{ for all } i, j \notin S, \text{ for all } k \in N.$$

Since $\lim_{t\to\infty} p^t_i = \lim_{t\to\infty} p^t_j$ for $i \notin S$, actually $\lim_{t\to\infty} \mathbf{T}^t_{ij} = \lim_{t\to\infty} \mathbf{T}^t_{ik}$ as well. That is, elements in $(\mathbf{T}^*_{ij})_{(n-m)\times(n-m)}$ all have the same value.

Finally, we let $p_\beta = p^*$, which completes the proof of Theorem 3.4.

Proof of Proposition 3.5

Suppose that $\underline{p}^t < \underline{p}_s$ for all t, that is, the lowest value of persistent players' initial strategies is never a lower bound. From the proof of Theorem 3.2 we know that there exists $\hat{t} > 0$ such that for $t > \hat{t}$, for all i, j $\mathbf{T}^t_{ij} > \tau$, where $\tau = \frac{\pi^{\hat{t}}_{min}}{an^2} \in (0, 1]$ is bounded away from 0. Then we have that:

$$\begin{aligned}
\underline{p}^t &\geq (1-m\tau)\underline{p}^{t-1} + \tau \textstyle\sum_{s\in S} p_s^0 \\
&\geq ((1-m\tau))[(1-m\tau)\underline{p}^{t-2} + \tau \textstyle\sum_{s\in S} p_s^0] + \tau \textstyle\sum_{s\in S} p_s^0 \\
&\vdots \\
&\geq (1-m\tau)^{t-\hat{t}}\underline{p}^{\hat{t}} + \tau(\textstyle\sum_{s\in S} p_s^0)\textstyle\sum_{\eta=0}^{t-\hat{t}}(1-m\tau)^{\eta}.
\end{aligned}$$

We have $\lim_{t\to\infty}(1-m\tau)^{t-\hat{t}}\underline{p}^{\hat{t}} + \tau(\sum_{s\in S} p_s^0)\sum_{\eta=0}^{t-\hat{t}}(1-m\tau)^{\eta} = \frac{1}{m}\sum_{s\in S} p_s^0 > \underline{p}_s$,[12] which is a contradiction to the assumption that $\underline{p}^t < \underline{p}_s$. Thus there exists $t' > 0$ such that $\underline{p}_s$ is a lower bound of p_i^t for all $t > t'$.

Similarly, suppose that p_n^0 is never an upper bound of p_i^t. Then we have that $\overline{p}^t \geq (1-m\tau)^{t-\hat{t}}\overline{p}^{\hat{t}} + \tau(\sum_{s\in S} p_s^0)\sum_{\eta=0}^{t-\hat{t}}(1-m\tau)^{\eta} \to \frac{1}{m}\sum_{s\in S} p_s^0 < \overline{p}_s$, which is a contradiction to the assumption. Thus $\overline{p}_s$ is an upper bound of p_i^t for all $t > t'$.

[12]Here $\frac{1}{m}\sum_{s\in S} p_s^0 \neq \underline{p}_s$ because we have diverse persistent players, which implies that there exists at least one persistent player whose strategy does not equal to $\underline{p}_s$. This reasoning also applies to the statement that $\frac{1}{m}\sum_{s\in S} p_s^0 < \overline{p}_s$ below.

Appendix D

The focus of this appendix is the correlation between the final opinions and the initial opinions of nodes that have the most connections, in the context of the network-distance combination learning model proposed in Chapter 4. I also show simulation findings with $p_N \sim$ Uniform(0, 1).

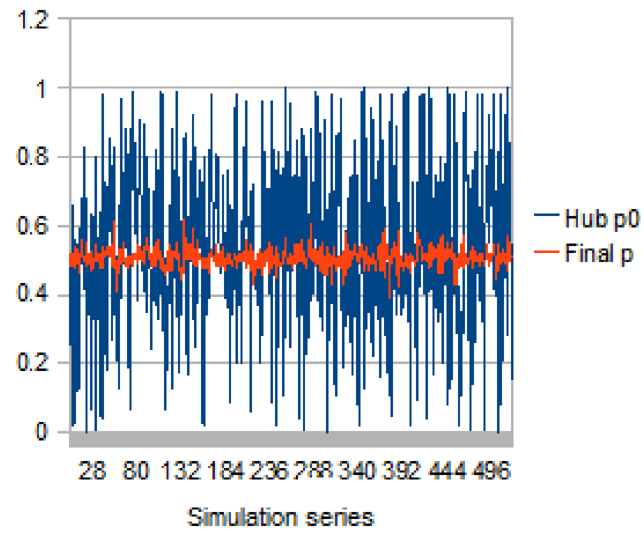
Recall that in a small-world network, strangers from two cliques may be linked with a mutual friend. One question stems from this phenomenon is if there exists a "popular" individual who has the most connections, which contributes to the small average shortest path length. In this section I use the term "hub" loosely and call those individuals with the highest degree hubs. After the neighborhood structure $\mathbf{G}^0$ is constructed, the node with the highest degree and the initial opinion of the represented agent are recorded. In the cases where we have a tie among multiple nodes, the average of their initial opinions is taken as the input.

Table D.1 summarizes of the calculated hub agents' initial opinion and the final opinion that all agents converge to hold. We see that the former is very arbitrary while the latter appears to fall into a narrow range around 0.5.

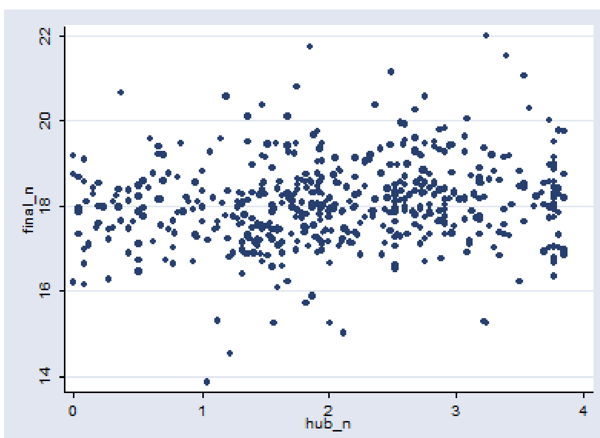
Variable	Mean	Std. Dev.	Min	Max
Hub opinion	0.544	0.260	0	1
Final opinion	0.507	0.028	0.389	0.617

Table D.1: Summary of hub opinion and final opinion

Figure D.1 illustrates the relationship between "hub opinion" and the final opinion that all conform to. The opinion of hub node(s) may take any value[13] and does not show significant impact on final opinions. Since the two variables differ largely in standard deviation, we may normalize the deviation and the scatter plot of normalized data also shows clearly that final opinion is not affected by hub nodes' initial opinions.



(a) Original data



(b) Normalized std

Figure D.1: Hub opinion and final opinion with small-world networks ($n = 100$)

On the other hand, with the power-law pattern of degree distribution, hubs in scale-

[13]The data also show that 1) the emergence of the hub node(s) is independent of k and θ; 2) the degree distribution does not have a power-law pattern.

free have significant degree dominance, in that a hub node's degree is much higher than most of the nodes in the networks. Also, hubs in scale-free networks maintain and enhance the robustness of the networks.

Figure D.2 is the scatter plot of the initial opinion of the hub node (or the average of hub nodes) and the final opinion. Unlike that with small-world networks, which does not show influence of hub node(s) over the final opinion, now with Barabási-Albert networks a clear linear pattern is observed with a high R^2 value of 91%:

$$f(x) = 0.84x + 0.28, \quad R^2 = 0.91. \tag{D.1}$$

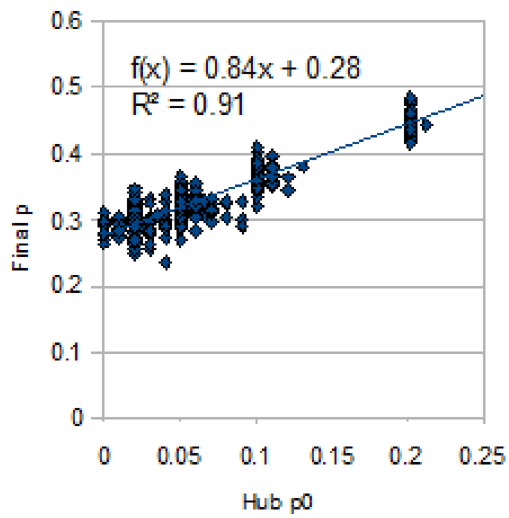
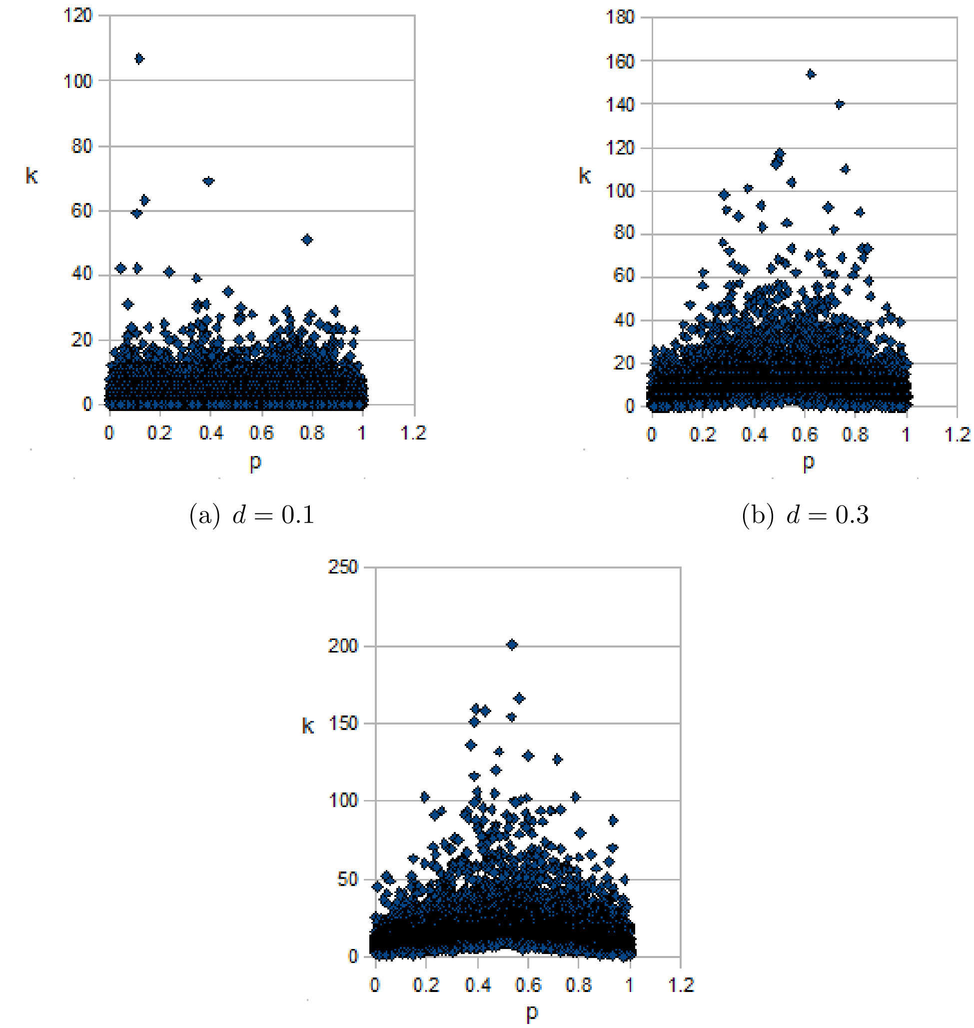


Figure D.2: Hub opinion and final opinion with Barabási-Albert networks ($n = 100$)

The critical factor here is again degree dominance. I examine the degree distribution of the small-world networks generated and used in the simulations. The results show that the gap between degree of "hubs" in the networks and that of a "normal" node is not nearly as big as the gap we observe in the Barabási-Albert networks.

Simulations with opinions drawn from a uniform distribution between (0, 1) were done for comparison. Results show a more rectangle look in the scatter plot of opinions and degree, with the hub nodes having opinions close to 0.5 (Figure D.3). Degree distributions of the resulting networks show power-law patterns with lower γ; and the maximum degree is higher than that with a normal distribution (Figure D.4). These findings can be attributed to a different $\vartheta(p_i, d)$, which is illustrated below (Figure D.5).

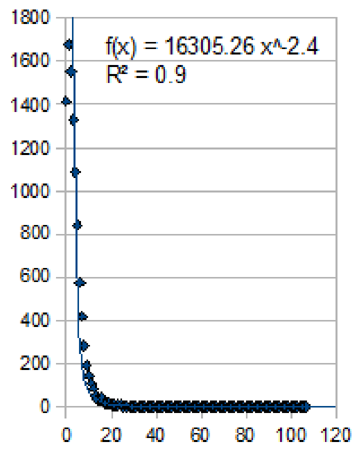


(a) $d = 0.1$

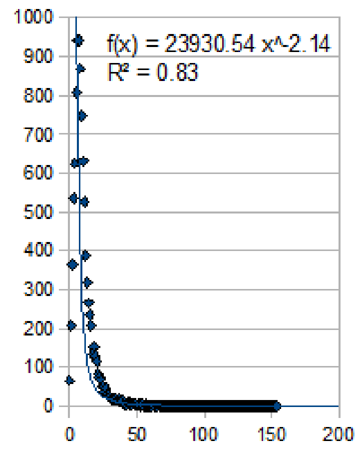
(b) $d = 0.3$

(c) $d = 0.5$

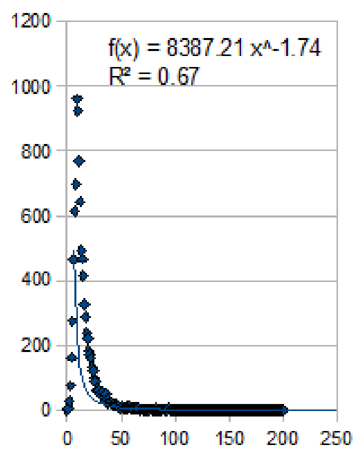
Figure D.3: Correlation between opinion and degree ($p_N \sim$ Uniform(0, 1))



(a) $d = 0.1$



(b) $d = 0.3$



(c) $d = 0.5$

Figure D.4: Degree distribution ($p_N \sim$ Uniform(0, 1))

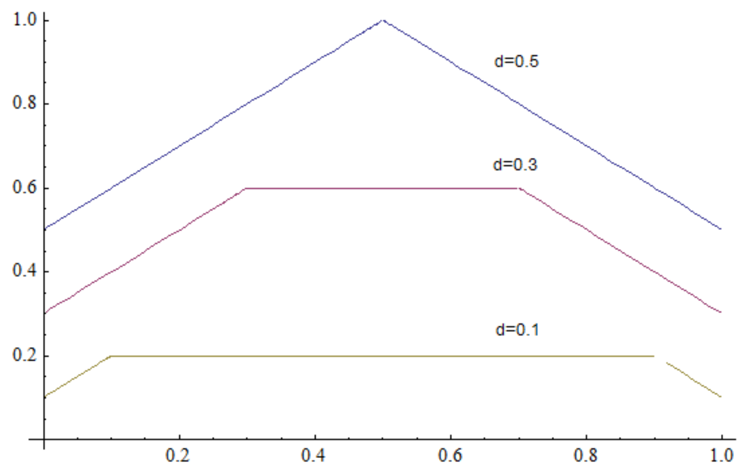


Figure D.5: Probability of getting connected ($p_N \sim$ Uniform(0, 1))