Appendix B

Fan thickness noise model

The fan thickness noise is generated by the volume displacement of the blades (Tylor and Sofrin 1962). An analytical representation of the thickness component of the fan noise radiating in an unbounded space is derived in this appendix. The fan noise is generated, as described in section 3.2.1 and in Figure 3.3, by N evenly spaced (circumferentially) spinning line sources.

The thickness component of the noise is obtained by solving the following inhomogeneous wave equation in an unbounded space (Dowling 1983)

$$[\frac{1}{\tilde{c}^2}\frac{\partial^2}{\partial \tilde{t}^2} - \frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}(\tilde{r}\frac{\partial}{\partial \tilde{r}}) - \frac{1}{\tilde{r}^2}\frac{\partial^2}{\partial \psi^2} - \frac{\partial^2}{\partial \tilde{z}^2}]\tilde{p}_i(\tilde{r}, \psi, \tilde{z}, \tilde{t}) =$$

$$\tilde{\rho}_0 \tilde{v}_0 \frac{\partial^2}{\partial \tilde{t}^2} \frac{H(\tilde{r}_l - \tilde{r})}{\tilde{r}} \delta(\tilde{z} - \tilde{V}\tilde{t}) \sum_{m_d=-\infty}^{\infty} \delta(\tilde{\Omega}\tilde{t} - \psi - \frac{2\pi m_d}{N}) . \qquad \text{(B.1)}$$

The right hand side of Eq. (B.1) is a monopole term and models the volume displacement of the fan blades. $\tilde{\gamma}_0$ represents the volume occupied by one fan blade.

Repeating the resolution procedure described in section 3.2.1, Eq. (B.1) first undergoes a series of manipulations in order to be expressed under a form that has a known solution. Thus, Eq. (B.1) is nondimensionalized, yielding

$$[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) - \frac{1}{r^2}\frac{\partial^2}{\partial \psi^2} - \frac{\partial^2}{\partial z^2}]\, p_i(r, \psi, z, t) =$$

$$v_0\,\frac{H(r_l - r)}{r}\;\frac{\partial^2}{\partial t^2}\delta(z - Vt)\sum_{m_d=-\infty}^{\infty}\delta(t - \psi - \frac{2\pi m_d}{N}), \tag{B.2}$$

and then expressed in the moving and stretched reference frame defined by Eq. (3.17) and Eq. (3.18), yielding

$$[\frac{1}{c^2}\left(\frac{\partial}{\partial t} - \frac{V}{\beta}\frac{\partial}{\partial Z}\right)^2 - \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) - \frac{1}{r^2}\frac{\partial^2}{\partial \psi^2} - \frac{1}{\beta^2}\frac{\partial^2}{\partial Z^2}]p_i(r,\psi,Z,t) =$$

$$v_0\,\frac{H(r_l - r)}{r}\left(\frac{\partial}{\partial t} - \frac{V}{\beta}\frac{\partial}{\partial Z}\right)^2\frac{\delta(Z)}{\beta}\sum_{m_d=-\infty}^{\infty}\delta(t - \psi - \frac{2\pi m_d}{N}). \tag{B.3}$$

Replacing $\sum_{m_d=-\infty}^{\infty}\delta(t - \psi - \frac{2\pi m_d}{N})$ by its Fourier series defined in Eq. (3.23), and then, noting that the solution of the resulting equation can be written as a sum of spinning modes, yield

$$[\frac{1}{c^2}\left(i\,n_h N - \frac{V}{\beta}\frac{\partial}{\partial Z}\right)^2 - \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{n_h^2 N^2}{r^2} - \frac{1}{\beta^2}\frac{\partial^2}{\partial Z^2}]\,Q_i^{n_h}(r,Z) =$$

$$-\frac{v_0 N}{2\pi\beta}\;\frac{H(r_l - r)}{r}\;e^{i\kappa M Z}\left(i\,n_h N - \frac{V}{\beta}\frac{\partial}{\partial Z}\right)^2\delta(Z) \tag{B.4}$$

where $Q_i(r,Z)$ is defined by Eq. (3.36) and Eq. (3.25).

Referring to section 3.2.1, Eq. (B.4) can be rewritten as

$$\left[\frac{\partial^2}{\partial Z} + \kappa^2 + \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) - \frac{n_h^2 N^2}{r^2}\right]Q_i^{n_h}(r,Z) =$$

$$-\frac{v_0 N}{2\pi\beta}\;\frac{H(r_l - r)}{r}\;e^{i\kappa M Z}\left(i\,n_h N - \frac{V}{\beta}\frac{\partial}{\partial Z}\right)^2\delta(Z). \tag{B.5}$$

Finally, noting that based on the property of the delta function described in Eq. (3.37),

$$e^{i\kappa M Z}\,\frac{\partial}{\partial Z}\delta(Z) = \frac{\partial}{\partial Z}\delta(Z) + i\,\kappa M\,\delta(Z), \tag{b.6}$$

and that

$$e^{i\kappa M Z}\,\frac{\partial^2}{\partial Z^2}\delta(Z) = \frac{\partial^2}{\partial Z^2}\delta(Z) - 2\,i\,\kappa M\frac{\partial}{\partial Z}\delta(Z) - \kappa^2 M^2\,\delta(Z)\,, \tag{B.7}$$

Eq. (B.5) can be expressed as

$$\left[\frac{\partial^2}{\partial Z} + \kappa^2 + \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) - \frac{n_h^2 N^2}{r^2}\right] Q_i^{n_h}(r,Z) =$$

$$-\frac{v_0 N}{2\pi\,\beta}\;\frac{H(r_l - r)}{r}\;\left[A''\frac{\partial^2}{\partial Z^2} + A'\frac{\partial}{\partial Z} + A\right]\delta(Z) \tag{B.8}$$

where

$$A'' = \frac{V^2}{\beta^2}\,, \tag{B.9}$$

$$A' = -\frac{2\,i\,V\left(n_h N\beta + \kappa M V\right)}{\beta}\quad , \tag{B.10}$$

and

$$A = -\left(\frac{\kappa^2 M^2}{\beta^2} + \frac{2\;n_h N V\kappa M}{\beta} + n_h^2 N^2\right). \tag{B.11}$$

The Green's function corresponding to the wave operator of the left hand side of Eq. (B.8) is known and is defined in Eq. (3.39). Therefore, applying the Green's function technique, the solution of Eq. (B.8) can be expressed as

$$Q_i^{n_h}(r,Z) = \frac{-\gamma_0 N}{2\pi\beta}\;\int\limits_{r'=0}^{\infty}\;\int\limits_{Z'=\infty}^{\infty}\;G_{n_h}(r,r',Z-Z')$$

$$\frac{H(r_l - r')}{r'}\;\left[A''\frac{\partial^2}{\partial Z'^2} + A'\frac{\partial}{\partial Z'} + A\right]\delta(Z')\;r'\,dZ'\,dr' \tag{B.12}$$

where A'', A' and A are defined by Eq. (B.9), Eq. (B.10) and Eq. (B.11).

Finally, using integration by parts and applying the properties of the step and Delta functions, Eq. (B.12) yields

$$Q_i^{n_h}(r,Z) = \frac{-\gamma_0 N}{4\pi^2\beta}\;\int\limits_{r'=0}^{\infty}\{\,A''\int\limits_{0}^{\pi}\;C_1(r',\psi')\,C_2(r',\psi')\,d\psi'$$

$$-A'\int_0^{\pi} C_1(r',\psi')\,C_3(r',\psi')\,d\psi' + A\int_0^{\pi} C_1(r',\psi')\,d\psi' \,\}\,dr' \tag{B.13}$$

where C_1, C_2 and C_3 are defined as

$$C_1(r',\psi') = \cos(n_h N\psi')\frac{e^{-i\kappa R}}{R}, \tag{B.14}$$

$$C_2(r',\psi') = \frac{-\kappa^2 R^2 + 3i\kappa R + 3}{R^4}\,Z^2 - \frac{1 + i\kappa R}{R^2}, \tag{B.15}$$

and

$$C_3(r',\psi') = \frac{1 + i\kappa R}{R^2}\,Z\,. \tag{B.16}$$

This solution can be expressed in terms of the incident pressure using Eq. (3.43).

The derivative with respect to r of Eq. (A.13) is given by

$$\begin{aligned}\frac{\partial Q_i^{n_h}}{\partial r}(r,Z) = \frac{-\gamma_0 N}{4\pi^2\beta}\int_{r'=0}^{\infty}\{\,A''\int_0^{\pi} C_7(r',\psi')\,C_4(r',\psi')\,d\psi' \\ + A'\int_0^{\pi} C_7(r',\psi')\,C_5(r',\psi')\,d\psi' + A\int_0^{\pi} C_7(r',\psi')\,C_6(r',\psi')\,d\psi' \,\}\,dr'\end{aligned} \tag{B.17}$$

where

$$C_4(r',\psi') = \frac{i\kappa^3 R^3 - 6\kappa^2 R^2 - 4i\kappa R - 15}{R^5}\,Z^2 + \frac{-\kappa^2 R^2 + 3i\kappa R + 3}{R^3}, \tag{B.18}$$

$$C_5(r',\psi') = \frac{-\kappa^2 R^2 + 3i\kappa R + 3}{R^3}\,Z, \tag{B.19}$$

$$C_6(r',\psi') = \frac{-i\kappa R - 1}{R}\,Z, \tag{B.20}$$

and

$$C_7(r',\psi') = (1 - r'\cos\psi')\cos(n_h N\psi')\frac{e^{-i\kappa R}}{R^2}\,. \tag{B.21}$$

The analytical expressions given by Eq. (B.13) and Eq. (B.17) are the elements needed to obtain the scattered field using the procedure described in Chapter 2.