APPENDIX A

DERIVATION OF JOINT FAILURE DENSITIES

In this Appendix, we present the derivation of the example failure models as shown in Chapter 3. Assume that the time and use to failure are related by the function $u = g(t)$ and the stochastic nature of this relationship can be represented by treating one or two of the parameters of $g(t)$ as random variables. We consider the following four example forms:

(i) $g(t) = at + b$,

(ii) $g(t) = at^2 + bt + g$,

(iii) $g(t) = at^n$, and

(iv) $g(t) = \left(e^{at} - 1\right)/\left(e^{at} + b\right)$.

In each case, we introduce randomness into the function by treating the parameter a as a random variable having distribution $p_a(\cdot)$. Using $p_a(\cdot)$ and the transformation of variables, we construct the marginal probability distribution on usage. That is,

$$f_U(u) = f_{g(t)}(u) = \left|\frac{da(u)}{du}\right| p_a\left(a(u)\right). \tag{A.1}$$

Once the marginal distribution on usage is obtained, we then construct the joint failure density using the conditioning relation, i.e.,

$$f_{T,U}(t,u) = f_{T|U}(t)\, f_U(u)\,. \tag{A.2}$$

In Eq. (A.2), the conditional density $f_{T|U}(t)$ is obtained by using the well-known relationship between a density and its hazard function:

$$f_{T|U}(t) = z_{T|U}(t)\exp\left\{-\int_0^t z_{T|U}(x)dx\right\} = z_{T|g(t)}(t)\exp\left\{-\int_0^t z_{T|g(t)}(x)dx\right\}. \qquad \text{(A.3)}$$

We assume that the conditional bivariate hazard function on age given usage may be stated as:

$$z_{T|U}(t|u) = l(t) + h(u) \qquad \text{(A.4)}$$

so that the definitions of the functions $l(t)$, $h(u)$, and $g(t)$ determine the conditional hazard and ultimately the bivariate life distribution. In here, in order to focus on the functions $g(t)$, we assume that $l(t)$ and $h(u)$ are simple linear functions, i.e., $l(t)$ = lt and $h(u)$ = hu. Thus, we assume

$$z_{T|U}(t|u) = lt + hu. \qquad \text{(A.5)}$$

Under this modeling format, we may obtain the bivariate life distributions corresponding to forms (i), (ii), (iii), and (iv), respectively, as follows:

$$f_{T,U}(t,u) = \frac{lt + hu}{t}\exp\left\{-\frac{h(u+b)}{2}t - \frac{l}{2}t^2\right\}p_a\left(\frac{u-b}{t}\right), \qquad \text{(A.6) (Eq. (3.7))}$$

$$f_{T,U}(t,u) = \frac{lt + hu}{t^2}\exp\left\{-\frac{h(u+2g)}{3}t - \frac{3l + hb}{6}t^2\right\}p_a\left(\frac{u-bt-g}{t^2}\right), \qquad \text{(A.7) (Eq. (3.8))}$$

$$f_{T,U}(t,u)=\frac{lt+hu}{t^n}\exp\left\{-\frac{hu}{n+1}t-\frac{l}{2}t^2\right\}p_a\left(\frac{u}{t^n}\right), \qquad \text{(A.8) (Eq. (3.9))}$$

and

$$f_{T,U}(t,u)=\frac{(1+b)(lt+hu)}{t(1-u)(1+bu)}\exp\left\{-\frac{h}{b}t-\frac{l}{2}t^2-\left(\frac{h\dfrac{b+1}{b}}{\ln\dfrac{1+bu}{1-u}}\ln\frac{1}{1-u}\right)t\right\}p_a\left(\frac{1}{t}\ln\frac{1+bu}{1-u}\right).$$

(A.9) (Eq. (3.10))

A.1 Derivation of Eq. (A.6)

For case (i), $g(t) = at + b$, solving for a yields:

$$a(u)=\frac{u-b}{t} \text{ and } \frac{da(u)}{du}=\frac{1}{t},$$

so:

$$f_U(u)=\frac{1}{t}p_a\left(\frac{u-b}{t}\right). \qquad \text{(A.10)}$$

The conditional bivariate hazard function on age given usage can be constructed as:

$$z_{T|U}(x)=lx+h\left(\left(\frac{u-b}{t}\right)x+b\right) \qquad \text{(A.11)}$$

Substituting Eq. (A.11) into Eq.(A.3), we obtain

$$f_{T|U}(t)=(lt+hu)\exp\left\{-\int_0^t\left[lx+h\left(\left(\frac{u-b}{t}\right)x+b\right)\right]dx\right\}$$

$$= (lt + hu)\exp\left\{-\frac{h(u+b)}{2}t - \frac{l}{2}t^2\right\} \qquad \text{(A.12)}$$

Substituting Eqs. (A.10) and (A.12) into Eq. (A.2), we derive

$$f_{T,U}(t,u) = \frac{lt + hu}{t}\exp\left\{-\frac{h(u+b)}{2}t - \frac{l}{2}t^2\right\} p_a\left(\frac{u-b}{t}\right).$$

A.2 Derivation of Eq. (A.7)

For case (ii), $g(t) = at^2 + bt + g$, solving for a yields:

$$a(u) = \frac{u - bt - g}{t^2} \text{ and } \frac{da(u)}{du} = \frac{1}{t^2},$$

so:

$$f_U(u) = \frac{1}{t^2} p_a\left(\frac{u - bt - g}{t^2}\right). \qquad \text{(A.13)}$$

The conditional bivariate hazard function on age given usage can be constructed as:

$$z_{T|U}(x) = lx + h\left(\left(\frac{u - bt - g}{t^2}\right)x^2 + bx + g\right) \qquad \text{(A.14)}$$

Substituting Eq. (A.14) into Eq.(A.3), we obtain

$$f_{T|U}(t) = \left(l t + hu\right)\exp\left\{-\int_0^t \left[l x + h\left(\left(\frac{u - bt - g}{t^2}\right)x^2 + bx + g\right)\right]dx\right\}$$

$$= \left(l t + hu\right)\exp\left\{-\frac{h\left(u + 2g\right)}{3}t - \frac{3l + hb}{6}t^2\right\} \qquad \text{(A.15)}$$

Substituting Eqs. (A.13) and (A.15) into Eq. (A.2), we derive

$$f_{T,U}(t,u) = \frac{l t + hu}{t^2}\exp\left\{-\frac{h(u + 2g)}{3}t - \frac{3l + hb}{6}t^2\right\}p_a\left(\frac{u - bt - g}{t^2}\right).$$

A.3 Derivation of Eq. (3.9)

For case (ii), $g(t) = at^n$, solving for a yields:

$$a(u) = \frac{u}{t^n} \text{ and } \frac{da(u)}{du} = \frac{1}{t^n},$$

so:

$$f_U(u) = \frac{1}{t^n}p_a\left(\frac{u}{t^n}\right). \qquad \text{(A.16)}$$

The conditional bivariate hazard function on age given usage can be constructed as:

$$z_{T|U}(x) = l x + h\left(\left(\frac{u}{t^n}\right)x^n\right) \qquad \text{(A.17)}$$

Substituting Eq. (A.17) into Eq.(A.3), we obtain

$$f_{T|U}(t) = (lt + hu)\exp\left\{-\int_0^t \left[lx + h\left(\frac{u}{t^n}\right)x^n \right] dx\right\}$$
$$= (lt + hu)\exp\left\{-\frac{hu}{n+1}t - \frac{l}{2}t^2\right\} \tag{A.18}$$

Substituting Eqs. (A.16) and (A.18) into Eq. (A.2), we derive

$$f_{T,U}(t,u) = \frac{lt + hu}{t^n}\exp\left\{-\frac{hu}{n+1}t - \frac{l}{2}t^2\right\} p_a\left(\frac{u}{t^n}\right).$$

A.4 Derivation of Eq. (3.10)

For case (iv), $g(t) = \left(e^{at} - 1\right)/\left(e^{at} + b\right)$, solving for a yields:

$$a(u) = \frac{1}{t}\ln\left(\frac{1 + bu}{1 - u}\right) \text{ and } \frac{da(u)}{du} = \frac{1 + b}{t(1 - u)(1 + bu)},$$

so:

$$f_U(u) = \frac{1 + b}{t(1 - u)(1 + bu)} p_a\left(\frac{1}{t}\ln\left(\frac{1 + bu}{1 - u}\right)\right). \tag{A.19}$$

The conditional bivariate hazard function on age given usage can be constructed as:

$$z_{T|U}(x) = lx + h\left(\frac{\exp\left[\frac{x}{t}\ln\left(\frac{1 + bu}{1 - u}\right)\right] - 1}{\exp\left[\frac{x}{t}\ln\left(\frac{1 + bu}{1 - u}\right) + b\right]}\right). \tag{A.20}$$

Substituting Eq. (A.20) into Eq.(A.3), we obtain

$$f_{T|U}(t) = (lt + hu)\exp\left\{ -\int_0^t \left[lx + h\left(\frac{\exp\left[\frac{x}{t}\ln\left(\frac{1+bu}{1-u}\right)\right] - 1}{\exp\left[\frac{x}{t}\ln\left(\frac{1+bu}{1-u}\right)\right] + b} \right) \right] dx \right\}$$

$$= (lt + hu)\exp\left\{ -\int_0^t \left[lx + h\left(\frac{\left(\frac{1+bu}{1-u}\right)^{\frac{x}{t}} - 1}{\left(\frac{1+bu}{1-u}\right)^{\frac{x}{t}} + b} \right) \right] dx \right\}$$

Let $x = \left(\frac{1+bu}{1-u}\right)^{\frac{1}{t}}$ then

$$f_{T|U}(t) = (lt + hu)\exp\left\{ -\int_0^t \left[lx + h\left(\frac{x^x - 1}{x^x + b} \right) \right] dx \right\}$$

$$= (lt + hu)\exp\left\{ \left[-\frac{l}{2}x^2 + \frac{hx}{b} - \frac{h(1+b)\ln(x^x + b)}{b\ln x} \right]_0^t \right\}$$

$$= (lt + hu)\exp\left\{ -\frac{l}{2}t^2 + \frac{ht}{b} - \frac{h(1+b)\ln(x^t + b)}{b\ln x} + \frac{(h + bh)\ln(1+b)}{b\ln x} \right\}$$

$$= (lt + hu)\exp\left\{ -\frac{l}{2}t^2 + \frac{h}{b}t - \left(\frac{h\left(\frac{b+1}{b}\right)}{\ln\left(\frac{1+bu}{1-u}\right)} \ln\frac{1}{1-u} \right) t \right\}$$

(A.21)

Substituting Eqs. (A.21) and (A.19) into Eq. (A.2), we derive

$$f_{T,U}(t,u)=\frac{(1+b)(lt+hu)}{t(1-u)(1+bu)}\exp\left\{-\frac{l}{2}t^2+\frac{h}{b}t-\left(\frac{h\frac{b+1}{b}}{\ln\frac{1+bu}{1-u}}\ln\frac{1}{1-u}\right)t\right\}p_a\left(\frac{1}{t}\ln\frac{1+bu}{1-u}\right).$$

APPENDIX B

THE BIVARIATE DIRAC DELTA FUNCTION

In this Appendix, we introduce the bivariate *Dirac Delta* function that is used in Chapter 6 to model the longevity of the single-unit system in the PM cycle of the ARPM model. We begin with the discussion of the univariate Dirac Delta function and its properties. Then, we extend the univariate function to two dimensions and present the results for the bivarite Dirac Delta function.

B.1 Univariate Dirac Delta Function

Consider the following function:

$$F_e(t) = \begin{cases} 1/e, & T \le t \le T + e, e > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{B.1}$$

Figure B.1.1 shows the plot of $F_e(t)$. Geometrically, as $e \to 0$, $F_e(t) = 1/e \to \infty$. Then, we have the following integral:

$$\int_0^\infty F_e(t)dt = 1. \tag{B.2}$$

Let $d(t)$ be the limiting function of $F_e(t)$, as $e \to 0$. $d(t)$ is called univariate *Dirac Delta* (or *Unit Impulse*) function for $F_e(t)$ (see Spiegel [1965]).

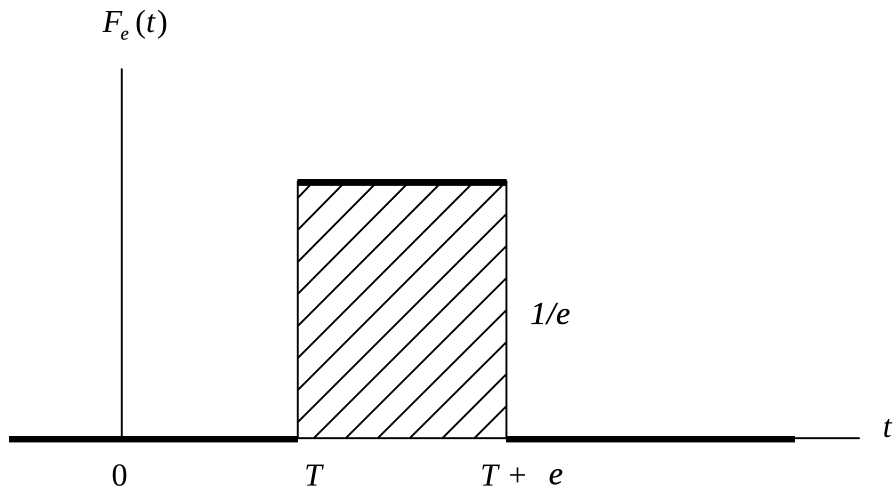

Figure B.1.1 Plot of $F_e(t)$.

We designate the univariate Dirac Delta function as $d(t-T)$ that has the following properties:

(i) $\int_0^{\infty} d(t-T)dt = 1$, (B.3)

(ii) $\int_0^{\infty} d(t-T)f(t)dt = f(T)$, for any continuous function $f(t)$. (B.4)

Note that $d(t-T)$ has a singularity of infinite value at $t = T$ and is equal to zero at all other values of t. Property (i) gives the Dirac Delta function the characteristic of a probability density function. Property (ii) helps in taking Laplace transform of the Dirac Delta function. The Laplace transform of $d(t-T)$ is given by

$$L_{s,v}\{d(t-T)\} = \int_0^\infty d(t-T)e^{-st}dt = e^{-sT} \quad \text{(B.5)}$$

B.2 Bivariate Dirac Delta Function

We now construct the bivariate Dirac Delta function with analogical properties as in univariate case. Consider the following function:

$$F_e(t,u) = \begin{cases} 1/e, & T \le t \le T+e, U \le u \le U+e, e > 0, \\ 0, & \text{otherwise.} \end{cases} \quad \text{(B.6)}$$

Figure B.2.1 shows the plot of $F_e(t,u)$. Geometrically, as $e \to 0$, $F_e(t,u) = 1/e \to \infty$. Then, we have the following integral:

$$\int_0^\infty \int_0^\infty F_e(t,u)dtdu = 1. \quad \text{(B.7)}$$

Let $d(t,u)$ be the limiting function of $F_e(t,u)$, as $e \to 0$. $d(t,u)$ is called bivariate *Dirac Delta* (or *Unit Impulse*) function for $F_e(t,u)$. We designate the univariate Dirac Delta function as $d(t-T)$ that has the following properties:

The bivariate Dirac Delta function for this PM cycle, $d(t-T, u-U)$, has the following properties:

(i) $\int_0^\infty \int_0^\infty d(t-T, u-U)dtdu = 1$ (B.8)

(ii) $\int_0^\infty \int_0^\infty d(t-T, u-U)f(t,u)dtdu = f(T,U)$ for any continuous function f (t,u). (B.9)

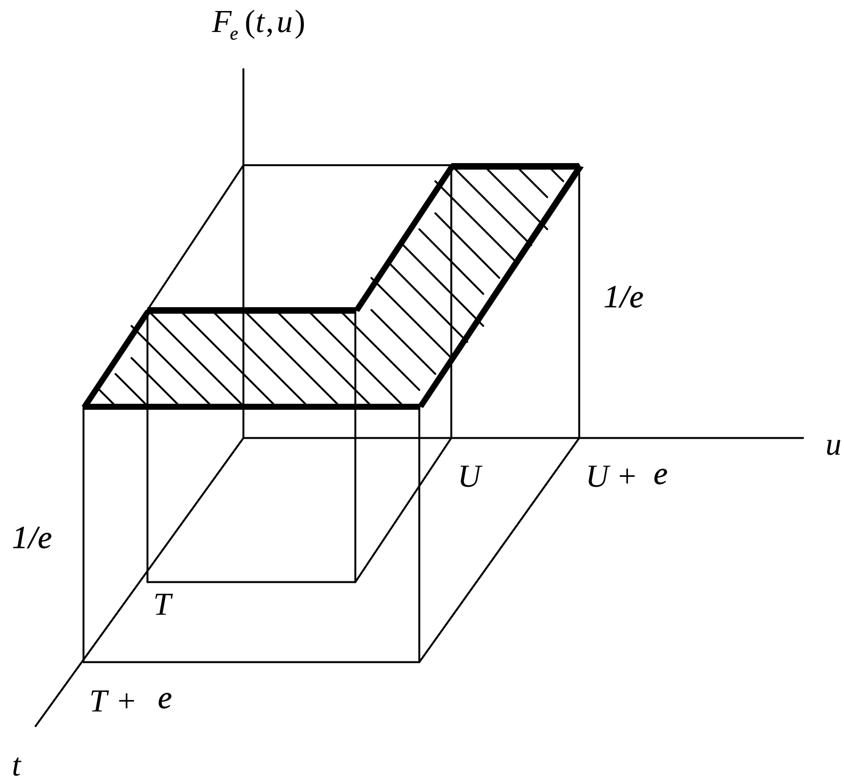

Figure B.2.1 Plot of $F_e(t,u)$.

Property (i) gives the bivariate Dirac Delta function the characteristic of a bivariate probability density function. Property (ii) helps in taking Laplace transform of the bivariate Dirac Delta function. The Laplace transform of $d\left(t-T,u-U\right)$ is

$$L_{s,v}\{d\left(t-T,u-U\right)\}=\int_0^\infty\int_0^\infty d\left(t-T,u-U\right)e^{-st-vu}dtdu=e^{-sT-vU} \quad . \tag{B.10}$$