3.3 A Lagrangian Dual Approach

In this section, we demonstrate that EMFLP is precisely a Lagrangian dual corresponding to a smooth convex programming problem, and hence, we derive a second equivalent, differentiable reformulation for EMFLP. As we shall see, this recovers the dual formulation of Francis and Cabot (1972). As a preliminary, consider the following result related to optimizing a linear function over a unit ball.

Lemma 4. *Consider the convex program*

$$\underset{(z_1, z_2)}{\text{Maximize}} \ \{c_1 z_1 + c_2 z_2 : \ z_1^2 + z_2^2 \le 1\}. \tag{3.17a}$$

Then, the optimal objective function value for (3.17a) *is given by*

$$\sqrt{c_1^2 + c_2^2}\,. \tag{3.17b}$$

Proof. The necessary and sufficient optimality conditions for (3.17) assert that (note that Slater's constraint qualification holds true, for example — see Bazaraa *et al.*, 1993)

$$2z_1\lambda = c_1,\ 2z_2\lambda = c_2,\ \lambda \ge 0,\ z_1^2 + z_2^2 \le 1 \text{ and } \lambda(z_1^2 + z_2^2 - 1) = 0. \tag{3.18}$$

If $c_1 = c_2 = 0$, then (3.17b) obviously holds true. Otherwise, λ must be positive in (3.18), and so, $z_1 = c_1 / 2\lambda$, $z_2 = c_2 / 2\lambda$, and $z_1^2 + z_2^2 = 1$ yields $\lambda = \sqrt{c_1^2 + c_2^2} / 2$, and $c_1 z_1 + c_2 z_2 = (c_1^2 + c_2^2) / 2\lambda = \sqrt{c_1^2 + c_2^2}$ at optimality. This completes the proof.

Using Lemma 4, we now derive an equivalent differentiable reformulation of (3.1). Note that in view of (3.17), we may rewrite (3.1) as

$$\text{Minimize} \sum\sum_{(i,j)\in A_{NE}} w_{ij} \begin{bmatrix} \text{maximum}\{(a_j - x_i)z_{1ij} + (b_j - y_i)z_{2ij}\} \\ \text{subject to} \quad z_{1ij}^2 + z_{2ij}^2 \le 1 \end{bmatrix} +$$

$$\sum\sum_{(k,\ell)\in A_{NN}} \nu_{k\ell} \begin{bmatrix} \text{maximum}\{(x_\ell - x_k)\zeta_{3k\ell} + (y_\ell - y_k)\zeta_{4k\ell}\} \\ \text{subject to} \quad \zeta_{3k\ell}^2 + \zeta_{4k\ell}^2 \le 1 \end{bmatrix}.$$

Putting the above separable problems together, we get that this overall problem is equivalent to

Minimize $f(x, y)$

where $f(x, y)$ = maximum $\sum\sum_{(i,j)\in A_{NE}} w_{ij}(a_j\zeta_{1ij} + b_j\zeta_{2ij})$

$$- \sum_i x_i\left[\sum_j w_{ij}\zeta_{1ij} + \sum_{\ell>i} \nu_{i\ell}\zeta_{3i\ell} - \sum_{\ell<i} \nu_{\ell i}\zeta_{3\ell i}\right]$$

$$- \sum_i y_i\left[\sum_j w_{ij}\zeta_{2ij} + \sum_{\ell>i} \nu\zeta_{4i\ell} - \sum_{\ell<i} \nu_{\ell i}\zeta_{4\ell i}\right] \quad (3.19a)$$

subject to

$$\zeta_{1ij}^2 + \zeta_{2ij}^2 \leq 1 \;\; \forall\, (i, j) \in A_{NE} \quad (3.19b)$$

$$\zeta_{3k\ell}^2 + \zeta_{4k\ell}^2 \leq 1 \;\; \forall\, (k, \ell) \in A_{NN}\,. \quad (3.19c)$$

Note that this is precisely the Lagrangian dual to the following problem, where the constraints (3.20b) and (3.20c) are dualized using Lagrange multipliers x_i, $i = 1, \ldots, n$, and y_i, $i = 1, \ldots, n$, respectively, in order to formulate the Lagrangian dual subproblem (3.19).

$$\text{DEMFLP: Maximize} \sum\sum_{(i,j)\in A_{NE}} w_{ij}(a_j\zeta_{1ij} + b_j\zeta_{2ij}) \quad (3.20a)$$

$$\text{subject to } \sum_j w_{ij}\zeta_{1ij} + \sum_{\ell>i} \nu_{i\ell}\zeta_{3i\ell} - \sum_{\ell<i} \nu_{\ell i}\zeta_{3\ell i} = 0 \;\; \forall\; i = 1, \ldots, n \quad (3.20b)$$

$$\sum_j w_{ij}\zeta_{2ij} + \sum_{\ell>i} \nu_{i\ell}\zeta_{4i\ell} - \sum_{\ell<i} \nu_{\ell i}\zeta_{4\ell i} = 0 \;\; \forall\; i = 1, \ldots, n \quad (3.20c)$$

$$\zeta_{1ij}^2 + \zeta_{2ij}^2 \leq 1 \;\; \forall\, (i, j) \in A_{NE} \quad (3.20d)$$

$$\zeta_{3k\ell}^2 + \zeta_{4k\ell}^2 \leq 1 \;\; \forall\, (k, \ell) \in A_{NN}\,. \quad (3.20e)$$

This observation leads to the following result.

Theorem 4. *Consider the differentiable convex program* (3.20). *Denote x and y as the Lagrange multiplier vectors associated with constraints* (3.20b) *and* (3.20c), *respectively. Then the corresponding Lagrangian dual problem*

LD: *Minimize* $\{f(x,y)\}$, *where* $f(x,y)$ *is given* by (3.19) (3.21)

is precisely equivalent to EMFLP. Moreover, DEMFLP has an optimum, and achieves the same optimal value as EMFLP. Furthermore, $(\bar{x},\bar{y})$ *solves EMFLP if and only if it corresponds to an optimal set of KKT Lagrange multipliers associated with the constraints* (3.20b) *and* (3.20c*).*

Proof. The equivalence of LD to EMFLP follows from the foregoing derivation of (3.19) and (3.20) based on Lemma 4. Furthermore, since DEMFLP seeks the maximum of a linear function over a compact, nonempty set, by Weierstrass' theorem (see Bazaraa *et al*., 1993), it has an optimum. Moreover, given an optimum $(\bar{x},\bar{y})$ to EMFLP, since this is equivalently an optimum to LD from above, there exists a zero subgradient of *f* at $(\bar{x},\bar{y})$. But by Theorem 6.3.4 in Bazaraa *et al*. (1993), the terms within $[\cdot]$ in (3.19a) at optimality in (3.19) characterizes the components of any subgradient of *f*. Hence, the solution to (3.19) that yields such a zero subgradient is feasible to DEMFLP, and by (3.19a), produces the same objective value of $f(\bar{x},\bar{y})$. Therefore, by Lagrangian duality, this solution is an optimum to DEMFLP and yields a zero duality gap. Furthermore, since Slater's constraint qualification holds for DEMFLP (the equality constraints are homogeneous and linear, the inequality constraint expressions are convex, and $z \equiv 0$ is a feasible solution that lies in the interior of the inequality constrained region — see Bazaraa *et al*., 1993), this optimum is a KKT solution, and since it is also an optimum and a KKT solution for (3.19) with $(x,y) \equiv (\bar{x},\bar{y})$ from above, we have that $\bar{x}$ and $\bar{y}$ are optimal Lagrange multipliers associated with (3.20b) and (3.20c), respectively.

Conversely, given an optimum to DEMFLP, since this is a KKT solution as shown above, there exist KKT Lagrange multipliers $\bar{x}$ and $\bar{y}$ associated with the constraints (3.20b) and (3.20c), respectively, in particular. By the convexity of DEMFLP (see Theorem 6.2.6 in

Bazaraa *et al.*, 1993), $(\bar{x}, \bar{y})$ therefore solves LD in (3.21), i.e., it solves EMFLP, yielding no duality gap. This completes the proof.

A few comments are in order at this point. First, observe that as evident from the proof of Theorem 4, feasibility to (3.20) essentially characterizes a zero subgradient for EMFLP. Examining (3.7) and (3.8), for each $(i, j) \in A_{NE}$, z_{1ij} and z_{2ij} respectively correspond to $(\bar{x}_i - a_j)/\bar{a}_{ij}$ and $(\bar{y}_i - b_j)/\bar{a}_{ij}$ with (3.20d) holding as an equality if $\bar{a}_{ij} > 0$. Similarly, for each $(k, \ell) \in A_{NN}$, $z_{3k\ell}$ and $z_{4k\ell}$ respectively relate to $(\bar{x}_k - \bar{x}_\ell)/\bar{b}_{k\ell}$ and $(\bar{y}_k - \bar{y}_\ell)/\bar{b}_{k\ell}$ if $\bar{b}_{k\ell} \neq 0$, and are arbitrary quantities satisfying (3.20e) if $\bar{b}_{k\ell} = 0$. Second, note that DEMFLP is precisely the dual to EMFLP as previously developed by Francis and Cabot (1972) using Sinha's (1966) duality results. (This can be seen by using the linear transformation $Y_{ij} = w_{ij}(z_{1ij}, z_{2ij})$ $\forall (i, j) \in A_{NE}$ and $T_{k\ell} = n_{k\ell}(z_{3k\ell}, z_{4k\ell})$ $\forall (k, \ell) \in A_{NN}$ in the notation of Francis and Cabot.) Besides being simple and direct, our derivation of this dual also exhibits connections with Lagrangian duality, and therefore brings to bear the accompanying rich theory of Lagrangian duality to the pair of problems EMFLP and DEMFLP. In particular, the various properties of these primal and dual problems and the complementarity conditions derived by Francis and Cabot are all a direct consequence of this Lagrangian duality relationship. Furthermore, as stated in Theorem 4, any ("*dual adequate*") nonlinear programming algorithm for smooth, convex optimization that is guaranteed to converge to a KKT solution and that also produces (an estimate of) Lagrange multipliers at optimality, would automatically yield a (near) optimum for EMFLP. The generalized reduced gradient method, successive linear and quadratic programming algorithms, and various penalty function methods (see Bazaraa *et al*., 1993, for example), all fall within this class of dual adequate algorithms.

Example 3. To illustrate, consider the problem of Example 2 for which recall that the optimum solution was not a KKT point for REMFLP. Solving this instance using the formulation DEMFLP using MINOS 5.2 yields an optimal objective function value of 3.4142 ($\approx 2 + \sqrt{2}$) over 23 iterations starting from the squared Euclidean distance problem's solution, consuming 0.14 cpu seconds on IBM 3090 computer, and yielded optimal Lagrange multipliers of $\bar{x} = 0$ and $\bar{y} = 0$ (to within machine tolerance) with respect to the constraints (3.20b) and (3.20c), respectively. This corresponds to an optimum for Problem EMFLP by Theorem 3.

3.4 Computational Experience

In this section, we present some computational experience in solving EMFLP via the differentiable formulations REMFLP, and DEMFLP, using a collection of standard four standard test problems (TP) from the literature. Table 1 gives the sizes and the sources of all of these test.

Table 1: Size and Source of Test Problems.

TP	***n***	***m***	**Source**
1	2	5	Eyster *et al* (1973)
2	2	3	Francis *et al*. (1991)
3	5	3	Francis *et al*. (1991
4	9	5	Calamai and Charalambous (1980)

Results for the HAP procedure of Eyster *et al.* (1973) are also provided to identify cases where nondifferentiability at optimality might exist and cause ill-conditioning effects for this procedure. In these experiments we used an IBM 3090 computer. The HAP procedure was coded in FORTRAN, and runs were made using $\varepsilon = 10^{-4}$, and with a termination tolerance of 10^{-6} between consecutive objective function values, as prescribed by Eyster *et al.* For solving REMFLP and DEMFLP, we used GAMS-MINOS 5.2 with the default optimality tolerance of 10^{-6}. Bounds on all variables were provided based on the rectangle enclosing the existing facility location for REMFLP, and using implied bounds of [-1, 1] on all variables in DEMFLP.

Table 2 provides the results obtained for the four test problems. For all the runs, HAP and REMFLP were initialized at the readily computed optimal solution for the underlying squared Euclidean distance location problem by ignoring the interaction between the new facilities.

Table 2. Results on Test Problems From the Literature.

TP	**HAP**			**REMFLP**			**DEMFLP**		
	f_{best}	**Iters**	**cpu sec**	f_{best}	**Iters**	**cpu sec**	f_{best}	**Iters**	**cpu sec**
1	67.239	33	.009	67.681	24	0.190	67.239	110	0.160
2	172.273	2655	0.510	172.256	20	0.180	172.256	9	0.160
3	40.042	24	0.017	39.000	5	0.220	39.000	6	0.190
4	201.872	5425	12.629	201.872	7	.440	201.872	312	0.320

Observe that both the primal and dual differentiable formulations perform quite efficiently with respect to cpu time. Due to the advanced-start solution that can be easily computed in the primal space, REMFLP takes fewer iterations than does DEMFLP, although DEMFLP is still slightly faster than REMFLP because it has fewer constraints. Except for Test 3 for which HAP did not converge sufficiently close to an optimum, all procedures were quite comparable with respect to the accuracy of solutions produced at termination. However, the differentiable formulations appear to be relatively faster as problem size increases. Of course, their main advantages is that they are solvable using standard differentiable optimization software, without posing ill-conditioning effects. Some additional test runs are provided in table 6 of the next chapter.

In conclusion, we mention that as an alternative to solving DEMFLP and recovering the solution to EMFLP via the optimal Lagrange multipliers obtained with respect to constraints (3.20b) and (3.20c) as in Theorem 4, one could instead solve its Lagrangian dual given by (3.19) and (3.21) to directly obtain an optimum $(\bar{x}, \bar{y})$ for EMFLP. Here, the subgradients of f are characterized by Lemma 1 or by the solution to (3.19), and in the presence of nondifferentiability, low-norm subgradients that might tend to yield descent directions could be generated via (3.19). Specialized line-searches could also be developed and used in conjunction with this subgradient optimization scheme. Research along these lines is presented in the next chapter.