ChapterV

Global Optimization Approach for the Capacitated Euclidean Location-Allocation Problem

In this chapter, we present a procedure that seeks the global minimum for the capacitated Euclidean location-allocation problem which is defined as follows. Given the fixed location of m existing facilities or customers on a continuous plane and their associated demands, we wish to determine the location of n new facilities or sources with known capacities, and the allocation of its supply, in order to satisfy the demand requirements of customers at a minimum total cost. The decisions are where to locate the n sources and how much shipment to send from each source to each customer. The costs are directly proportional to the quantities shipped and the Euclidean distance over which this shipment occurs. For convenience, we assume that the total supply is equal to the total demand. This problem can be mathematically stated as follows.

EDLAP: Minimize $f(x,w) \equiv \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}\, w_{ij}\, \{(x_i - a_j)^2 + (y_i - b_j)^2\}^{1/2}$

$$\text{subject to} \quad \sum_{j=1}^{m} w_{ij} = s_i \qquad i = 1,\ldots,n$$

$$\sum_{i=1}^{n} w_{ij} = d_j \qquad j = 1,\ldots,m \tag{5.1}$$

$$w_{ij} \geq 0 \qquad i = 1,\ldots,n, \;\; j = 1,\ldots,m.$$

As we mentioned earlier, as recognized by Cooper (1972), the objective function is nonconvex which results in a multiple local minima. Moreover, an optimal flow solution occurs at an extreme point of the set of the feasible solutions W, where $W = \{w \equiv w_{ij} : w$ satisfies the transportation constraints$\}$, while the optimal locations of the sources lie in the convex hull of the destinations' locations. Another important property of the EDLAP is that its objective function is nondifferentiable at the points where the location of any source coincides with the location of any destination. Consequently, this difficulty precludes the use of a gradient-based algorithm for solving this problem. We develop a procedure based on performing an implicit enumeration (branch-and-bound) of the vertices of W. For deriving lower bounds on node subproblems, we employ the Reformulation-Linearization Technique (RLT) developed by Sherali and Tuncbilek (1992), by suitably designing a specialized enhanced variant of this procedure for our problem. This method operates in two phases, namely, the Reformulation Phase and the Linearization Phase. In the Reformulation Phase, new nonlinear constraints are generated by considering variable factors (of the form [variable - its lower bound] and [upper bound - the corresponding variable]) which are used to multiply constraints, along with other suitable factors that are designed to multiply various specific constraints. These product constraints constitute the principal structure of the reformulated problem that lends strength to the relaxation, and its design for our particular problem is the main step of applying the RLT procedure. In the Linearization Phase, the objective function and the newly generated constraints are linearized using a suitable variable substitution strategy. This process transforms the representation of the nonconvex EDLAP from the original defining space into a higher dimensional space associated with a lower bounding (**largely linear**) convex program henceforth referred to as RLT(Ω), based on a hyper rectangle or that bounds the flows. The problem RLT(Ω) approximates the closure convex hull of feasible solutions to the nonconvex EDLAP, when its objective function is accommodated into the constraints. In order to compute a lower bound on the node subproblem, we consider the maximum of the RLT relaxation based lower bound, obtained via a Lagrangian dual formulation of this problem, and another readily computed lower bound derived via a projected location-space viewpoint: Accordingly, an appropriate partitioning strategy is developed for the branch-and-bound scheme to induce global convergence to an optimal solution. Suitable heuristic local search steps are interspersed within this process in order to actively seek good quality feasible solutions (or upper bounds on the problem). The developed algorithm is described in detail below, and has been applied to solve several test problems.

The remainder of this chapter is organized as follows. Section 5.1, presents the development of the lower bounding problem RLT(Ω). Section 5.2 discuses the design of a Lagrangian relaxation scheme for solving problem RLT(Ω). Section 5.3 addresses some other related strategies that are employed in solving this lower bounding problem. Section 5.4 provides certain theoretical results pertaining to the lower bounding problem and the overall convergence process. Section 5.5 details the various components of the proposed implicit branch-and-bound algorithm, and finally Section 5.6 presents the computational results obtained using several test problems.

5.1 A Reformulation Convexification / Linearization Lower Bounding Approach

This section presents the development of a Reformulation Convexification / Linearization Technique (RLT) based lower bounding scheme for problem EDLAP. The RLT method is comprised of two phases: the Reformulation Phase and the Convexification / Linearization Phase. These are discussed in turn below.

5.1.1 The Reformulation Phase

For the purpose of generating the reformulated problem, we first consider the following equivalent restatement of EDLAP.

$$\text{Minimize } \sum_{i}\sum_{j} c_{ij}\ w_{ij}\ a_{ij} \qquad (5.2.a)$$

$$\text{subject to } a_{ij}\ " \{(x_i - a_j)^2 + (y_i - b_j)^2\}^{1/2} \qquad \forall (i, j) \qquad (5.2.b)$$

$$w \in W \qquad (5.2.c)$$

where $W = \{w\!: w" (w_{ij})$ satisfies the transportation constraints, and $l_{ij} \le w_{ij} \le u_{ij}$, where l_{ij} and u_{ij} are some current valid bounds on the w- variables}. These bounds are initialized as

$l_{ij} \equiv 0$, and u_{ij} = min $\{s_i, d_j\}$ $\forall(i, j)$. Note that the branch-and-bound algorithm will change the bounds on w as it progresses.

Additional linear/ convex constraints will now be developed and added as below in order to tighten the ensuing relaxation.

5.1.1. A. Facetial Inequalities

Let Z be the convex hull of the existing facility locations. Since the optimal location lies in Z, we impose the valid constraints

$$z_i = (x_i, y_i) \in Z \qquad \forall \; i = 1, \ldots, n, \tag{5.3}$$

where Z is defined by a set of K facetial inequalities of the form

$Y_{1k}\, x + Y_{2k}\, y \le Y_{0k}$ for $k = 1, 2, ..., K$.

5.1.1. B. Bounds on the Variable α_{ij}

1 - Upper Bounds on the Variables α_{ij}

Convex hull based upper bound $u_{a_{ij}}$

Since a_{ij} represents the Euclidean distance between source i and destination j, we can impose

$$0 \le \alpha_{ij} \le u_{\alpha_{ij}} \tag{5.4}$$

where $u_{a_{ij}} = \underset{z \in vert(Z)}{maximum} \left\| z - (a_j, b_j) \right\| \quad \forall i$ for each existing facility j. This is valid since the new facility locations must lie in Z, and since the problem

$max\{\left\| z - (a_j, b_j) \right\| : \quad z \in Z\}$ achieves its maximum at a vertex of Z because its objective function is convex.

2- Lower Bounds on the Variables α_{ij}

a. $l_{¥}$ based lower bound

Since $\| \|_2 \ge \| \|_\infty$, we can generate the valid lower bounds

$\alpha_{ij} \geq \max\{|x_i - a_j|, |y_i - b_j|\} \quad \forall(i, j)$. (5.5).

The following is the linearized representation of this constraint.

$\alpha_{ij} \geq x_i - a_j$, $\alpha_{ij} \geq a_j - x_i$, $\alpha_{ij} \geq y_i - b_j$, $\alpha_{ij} \geq b_j - y_i \quad \forall(i, j)$. (5.6)

b. Lower bounds based on the relationship between l_1 and l_2 norms

Using the property of the l_2 and the l_1 norms that

$\sqrt{2}\ \|\nu\|_2 \geq \|\nu\|_1$ for any vector $\nu \in R^2$, the following lower bounding set of constraints can be defined:

$$\sqrt{2}\ \alpha_{ij} \geq |x_i - a_j| + |y_i - b_j| \quad \forall(i, j). \tag{5.7}$$

Writing (5.7) as four equivalent inequalities, we obtain

$$\sqrt{2}\ \alpha_{ij} \geq (x_i - a_j + y_i - b_j),\ \sqrt{2}\ \alpha_{ij} \geq x_i - a_j + b_j - y_i,$$
$$\sqrt{2}\ \alpha_{ij} \geq a_j - x_i + y_i - b_j,\ \sqrt{2}\ \alpha_{ij} \geq a_j - x_i + b_j - y_i \quad \forall(i, j). \tag{5.8}$$

<u>Remark 5.1</u> Note that (5.5) and (5.7) can be expressed jointly via seven inequalities instead of the eight inequalities suggested in (5.6) and (5.8) above. This can be done via additional variables α_{1ij} and $\alpha_{2ij}\ \forall(i, j)$ as follows:

$$\alpha_{1ij} \geq x_i - a_j,\ \alpha_{1ij} \geq a_j - x_i,\ \alpha_{2ij} \geq y_i - b_j,\ \alpha_{2ij} \geq b_j - y_i,$$
$$\alpha_{ij} \geq \alpha_{1ij},\ \alpha_{ij} \geq \alpha_{2ij},\ \sqrt{2}\ \alpha_{ij} \geq \alpha_{1ij} + \alpha_{2ij}.\ \forall(i, j). \tag{5.9}$$

However, since we will be generating additional valid inequalities through products with (5.6) and (5.8), we do not introduce these new variables, and we use the eight inequalities stated in (5.6) and (5.8).ð

<u>Remark 5.2</u> Also, note that the relationships (5.6) and (5.8) insure that the minimum feasible α_{ij} to these constraints is equal to the value given by (5.2.b) along the dotted lines shown in Figure 1 below for each (i, j).

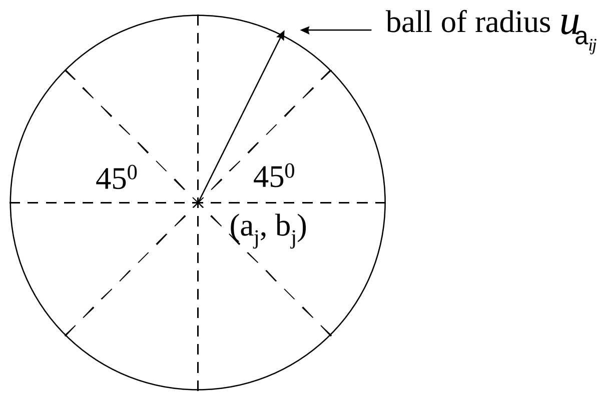

Figure 1. Rib centered at (a_j, b_j).

Hence, in essence, we enforce (5.2b) via (5.6) and (5.8) along this skeletal rib centered at (a_j, b_j). We could, in addition, generate additional supports for the convex function at some additional points $(\bar{x}_i, \bar{y}_i) \neq (a_j, b_j)$ at which this function is differentiable. The following is the inequality that represents this tangential support constraint

$$a_{ij} \geq g_i(\bar{x}_i, \bar{y}_i) + (x_i - \bar{x}_i, y_i - \bar{y}_i)\, \nabla g_i(\bar{x}_i, \bar{y}_i), \forall (i, j) \tag{5.10}$$

where $g_i(x_i, y_i) = \{(x_i - a_j)^2 + (y_i - b_j)^2\}^{1/2}$, and $(\bar{x}_i, \bar{y}_i) \neq (a_j, b_j)$, $\forall (i, j)$.

The point $(\bar{x}_i, \bar{y}_i)$ can be selected as the incumbent location decisions (with the coordinates being suitably perturbed for nondifferentiable terms in order to generate additional constraints of the type (5.6) and 5.8). However, we will for now just generate (5.6) and (5.8), and suggest this strategy for future research. ð