

### 5.3 Other Features of the Lower Bounding Problem.

In this section, we present some other features of the lower bounding problem such as the procedures that we have developed for computing various information required to generate some of the constraints of  $RLT(\Omega)$ , and the strategies for updating these values at every intermediate node of the branch-and-bound algorithm. We also discuss the construction of the aforementioned surrogate constraint. First, we present an algorithm that determines the extreme points of the convex hull of the existing facilities of EDLAP problem, and constructs the inequalities that define its facets. The facets are used in constructing constraint (5.25k) of Problem  $RLT(\Omega)$ . Using this algorithm, we solve an illustrative example that demonstrates its steps. Second, we discuss the scheme that we employ for computing the value of  $\bar{\pi}$  and the flow  $\bar{w}$  that are used to formulate constraint (5.25e), and the procedure that is used to update this constraint once it is introduced.

#### 5.3.1 Algorithmic Approach for Obtaining the Extreme Points and the Facetial Inequalities of the Convex Hull of the Existing Facility Locations.

Let  $m$  be the number of the existing facilities, and let  $X$  be the set of the existing facility locations, i.e.,  $X = \{ z_j \equiv (x_j, y_j) \equiv (a_j, b_j), \text{ for } j = 1, \dots, m \}$ . (Note that in this section, for notational convenience,  $z_j \equiv (x_j, y_j) \equiv (a_j, b_j)$  is used to refer to the existing facility location, and is not related to the coordinates of the new facilities.)

Define

$$z_1^{\text{Base}} = \underset{z \in X}{\text{arglexmax}} \{y, -x\},$$

$$z_1^{\text{Next}} = \underset{z \in X}{\text{arglexmax}} \{-x, y\},$$

$$z_2^{\text{Base}} = \underset{z \in X}{\text{arglexmax}} \{-x, -y\},$$

$$z_2^{\text{Next}} = \underset{z \in X}{\text{arglexmax}} \{-y, -x\},$$

$$\begin{aligned}
z_3^{\text{Base}} &= \underset{z \in X}{\operatorname{arglexmax}} \{-y, x\}, \\
z_3^{\text{Next}} &= \underset{z \in X}{\operatorname{arglexmax}} \{x, -y\}, \\
z_4^{\text{Base}} &= \underset{z \in X}{\operatorname{arglexmax}} \{x, y\}, \\
z_4^{\text{Next}} &= \underset{z \in X}{\operatorname{arglexmax}} \{y, x\}.
\end{aligned} \tag{5.45}$$

For convenience, we will denote  $z_k^{\text{Base}}$  by  $z_k^{\text{B}}$  and  $z_k^{\text{Next}}$  by  $z_k^{\text{N}}$  for  $k = 1, \dots, 4$ . The locations of  $z_k^{\text{B}}$  and  $z_k^{\text{N}}$  for  $k = 1, \dots, 4$  are depicted in Figure 3 below.

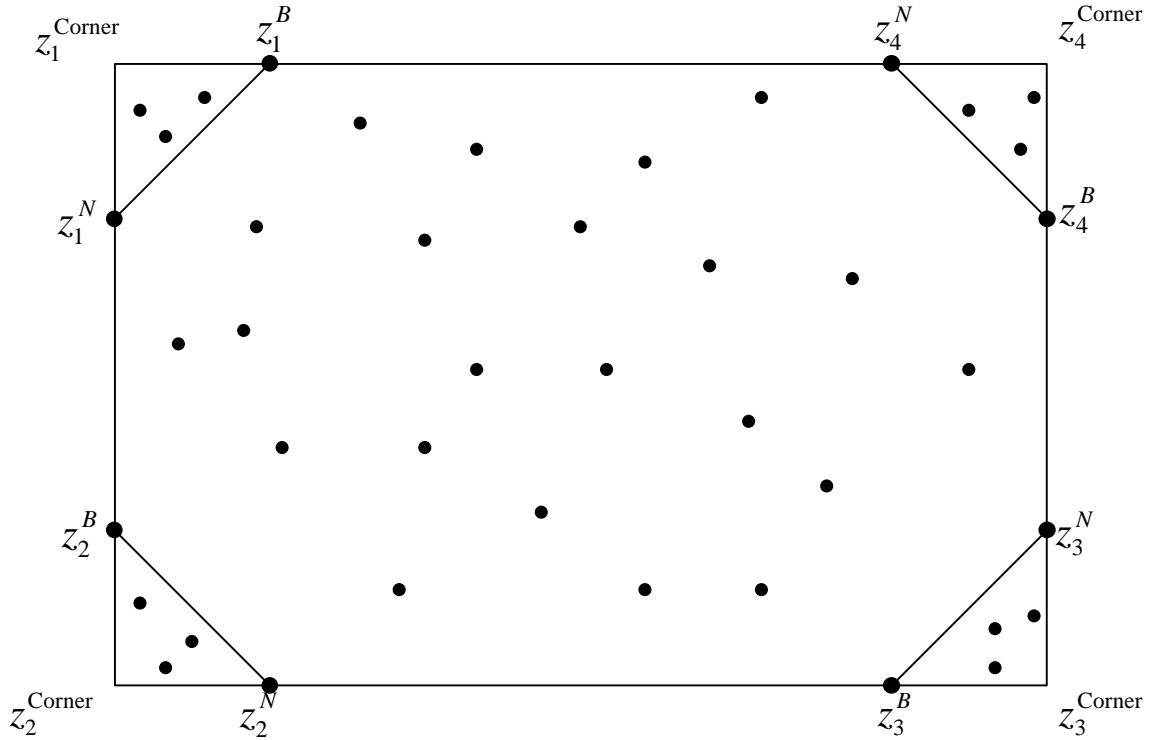
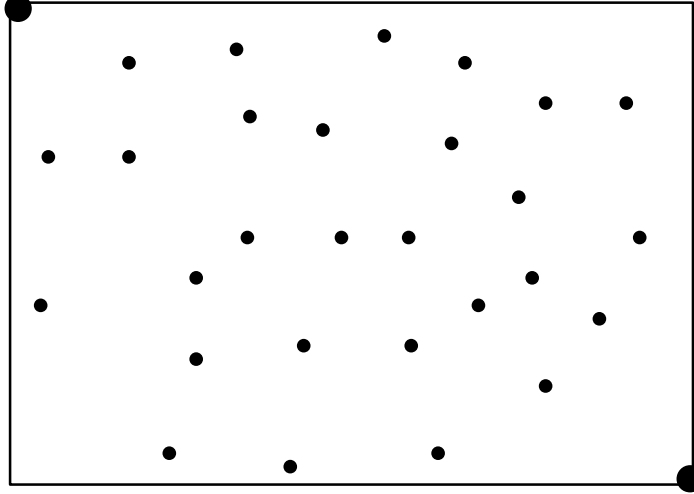


Figure 3. Depiction of notation for the convex hull constraints.

Also, let the corners of the rectangle bounded by  $\min \{x: z \in X\} \leq x \leq \max \{x: z \in X\}$  and by  $\min \{y: z \in X\} \leq y \leq \max \{y: z \in X\}$  be designated as in Figure 3 by  $z_k^{\text{Corner}} \equiv z_k^{\text{C}}$  for  $k = 1, \dots, 4$ . Note that if the volume of the rectangle is zero (bounds on either  $x$  or  $y$  coincide), then  $\operatorname{Conv}(X)$  is trivially determined. Hence, suppose that this is not the case. Note that we, could however, have coincident Base and Next points, as shown in Figure 4, for example.

$$z_1^B = z_1^N = z_2^B = z_4^N$$



$$z_2^N = z_3^B = z_3^N = z_4^B$$

Figure 4. Coincident Base and Next points.

The schema of the algorithm is given by the flowchart depicted in Figure 6. Let  $X_k$  for  $k = 1, \dots, 4$  be a subset of  $X$  defined as follows:

$$X_k = \begin{cases} z_j \in X \text{ s.t. } x_1^N < x_j < x_1^B \text{ and } y_1^N < y_j < y_1^B & \text{if } K=1 \text{ and } z_1^B \neq z_1^N \\ z_j \in X \text{ s.t. } x_2^B < x_j < x_2^N \text{ and } y_2^N < y_j < y_2^B & \text{if } K=2 \text{ and } z_2^B \neq z_2^N \\ z_j \in X \text{ s.t. } x_3^B < x_j < x_3^N \text{ and } y_3^B < y_j < y_3^N & \text{if } K=3 \text{ and } z_3^B \neq z_3^N \\ z_j \in X \text{ s.t. } x_4^N < x_j < x_4^B \text{ and } y_4^B < y_j < y_4^N & \text{if } K=4 \text{ and } z_4^B \neq z_4^N \\ \Phi & \text{if not given as above} \end{cases}$$

Now, for any  $k$  and for any subset  $\bar{X} \subset X_k \subseteq X$ , given a current point  $z_p \in X_k$  the angle  $\alpha_i$  to a point  $z_i \in \bar{X}$  is measured as the anticlockwise sweep of the vertical upward ray through  $z_p$  to the direction  $z_i - z_p$ , as shown in Figure 5.

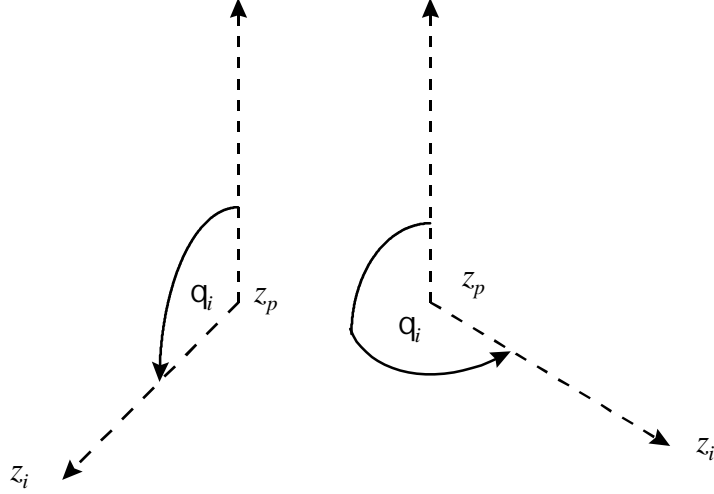


Figure 5. Depiction of the angle  $q_i$ .

Given any choice of  $k$  and  $z_p \in X_k$ , the algorithm will need to compute this angle  $q_i$  only for those  $z_i \equiv (x_i, y_i) \in X_k$  for which

$$\begin{cases} x_i < x_p \text{ and } y_i < y_p & \text{if } k = 1 \\ x_i > x_p \text{ and } y_i < y_p & \text{if } k = 2 \\ x_i > x_p \text{ and } y_i > y_p & \text{if } k = 3 \\ x_i < x_p \text{ and } y_i > y_p & \text{if } k = 4. \end{cases}$$

Accordingly,  $q_i$  is given as follows:

$$q_i = \begin{cases} \frac{\rho}{2} + \tan^{-1}\left(\frac{y_p - y_i}{x_p - x_i}\right) & \text{if } k = 1 \\ \frac{3\rho}{2} - \tan^{-1}\left(\frac{y_p - y_i}{x_i - x_p}\right) & \text{if } k = 2 \\ \frac{3\rho}{2} + \tan^{-1}\left(\frac{y_i - y_p}{x_i - x_p}\right) & \text{if } k = 3 \\ \tan^{-1}\left(\frac{x_p - x_i}{y_i - y_p}\right) & \text{if } k = 4. \end{cases}$$

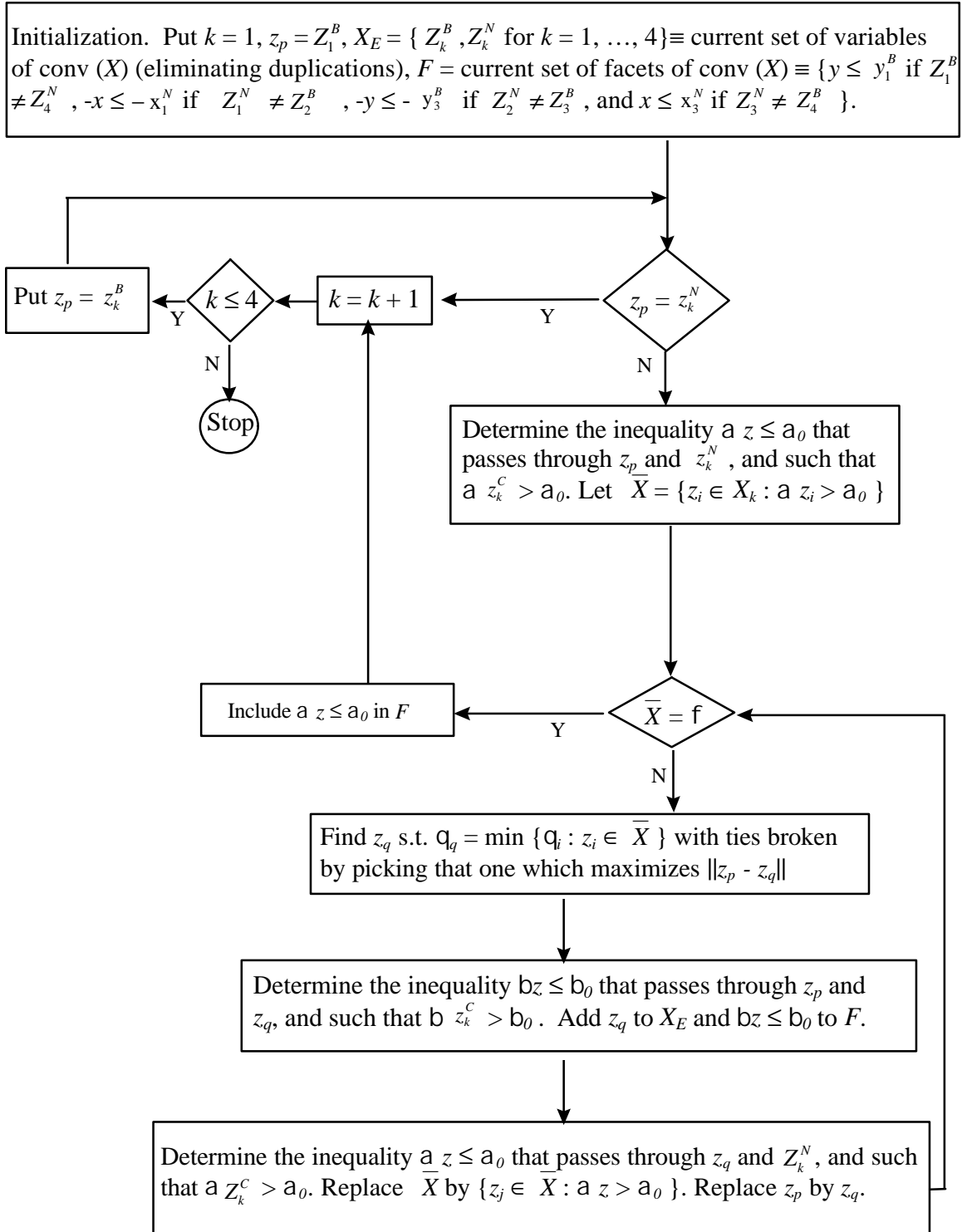


Figure 6. Flowchart for determining the convex hull of existing facility locations.

**Example 5.1**

Consider the existing facility locations exhibited in Figure 7.

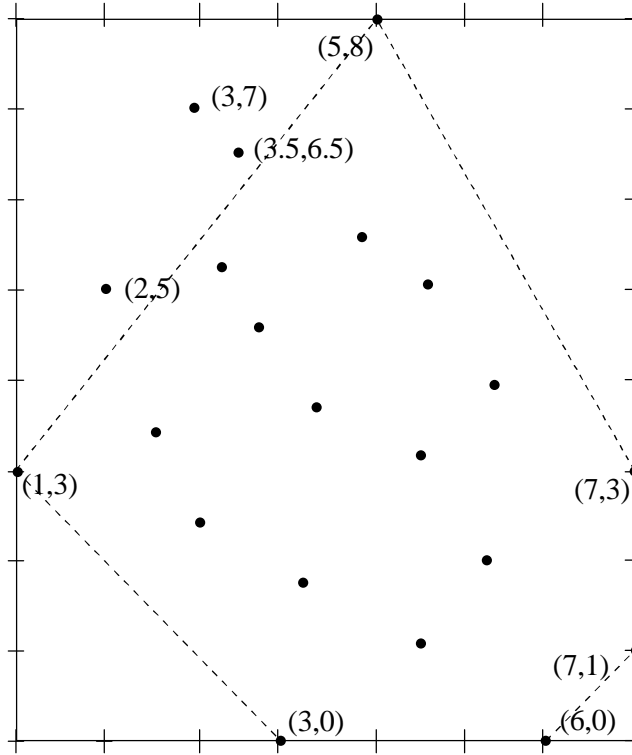


Figure 7. Depiction of the locations of  $z_1^B, z_1^N, z_2^N, z_3^B, z_3^N$ , and  $z_4^B$  of example 5.1.

Let the coordinates of the locations of the existing facilities 1 - 9 be (5, 8), (1, 3), (3,0), (6,0), (7,1), (7,3), (3,7), (3.5, 6.5), (2, 5), respectively, with other existing facilities as shown by dots in Figure 7. For this problem, we have  $z_1^B = z_4^N = (5,8)$ ,  $z_1^N = z_2^B = (1,3)$ ,  $z_2^N = (3,0)$ ,  $z_3^B = (6,0)$ ,  $z_3^N = (7,1)$ , and  $z_4^B = (7,3)$ .

By employing our developed algorithm, we initialize at  $k = 1$  with  $z_p = z_1^B$  and  $X_E = \{(5,8), (1,3), (3,0), (6,0), (7,1), (7,3)\}$ . Since,  $z_1^B = z_4^N$  and  $z_1^N = z_2^B$ , our initial set  $F = \{-y \leq 0, x \leq 7\}$ .

Now since  $z_p \neq z_1^N$ , then  $y - 1.25x \leq 1.75$  determines the inequality that passes through  $z_p$  and  $z_1^N$ . Note that at  $z_1^{Corner} = (1, 8)$ , a  $z_1^{Corner} = 6.75 > 1.75$ . Since  $\bar{x} = \{(3,7), (3.5, 6.5), (2,5)\} \neq f$ , we compute  $q_1, q_2$  and  $q_3$ . These are equal to 2.03, 2.35, and 2.35,