Solution Representation and Identification for Singular Neutral Functional Differential Equations

Graciela M. Cerezo

Dissertation submitted to the Faculty of the Virginia Polytechnic Institute and State University in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Mathematics

Terry L. Herdman, Chair
John A. Burns
Jeffrey Borggaard
David L. Russell
Eugene M. Cliff

December 6, 1996
Blacksburg, Virginia

Keywords: Neutral Functional Differential Equations, Integral Equations,
Parameter Identification, Collocation Method.

Solution Representation and Identification for Singular Neutral Functional Differential Equations

Graciela M. Cerezo

(ABSTRACT)

The solutions for a class of Neutral Functional Differential Equations (NFDE) with weakly singular kernels are studied. Using singular expansion techniques, a representation of the solution of the NFDE is obtained by studing an associated Volterra Integral Equation. We study the Collocation Method as a projection method for the approximation of solutions for Volterra Integral Equations. Particulary, the possibility of achieving higher order approximations is discussed. Special attention is given to the choice of the projection space and its relation to the smoothness of the approximated solution. Finally, we study the identification problem for a parameter appearing in the weakly singular operator of the NFDE.

Acknowledgements

First and very specially, I want to thank Dr. Terry L. Herdman for all the guidance and support that he gave me during all these years. From the very begining, his dedication to the project, his concern for my well being and his confidence in me and my work gave me the energy and inspiration to work through the difficulties.

I also want to thank Dr. John A. Burns, for encouraging me to move to this country and start this project. Without his kind and constant support I would have never been brave enought to venture so far from home.

Special thanks, to the members of my committee for their friendly, supporting and encouraging ways during all these years. Particular thanks to Dr. Jeffrey Borggaard for carefully reading this manuscript and making lots of useful suggestions.

My most sincere thanks to Melissa Chase, program support technician of ICAM, for being always there to help as a real friend.

Finally, I thank my parents form being my very best friends during all these years. Without them and their constant care, love and encouragment none of this would have ever been possible.

To my parents.

Contents

1 Introduction and Notation **1**

1.1 Introduction 1

1.2 Notation 3

2 A Representation For The Solution **4**

2.1 Preliminaries 4

2.1.1 Classification of Volterra Integral Equations 4

2.1.2 Existence and Uniqueness of Solutions 5

2.2 Well-Posedness Results 9

2.2.1 Well-Posedness Results for $L = 0$ 9

2.2.2 Well-Posedness Results for the General Case 12

2.3 NFDEs as Volterra Integral Equations 15

2.4 Representation of the Solution 19

2.5 Conclusions 45

3 The Approximation Scheme **47**

3.1 Projection Methods and the Collocation Method 48
3.2 Non Polynomial Spline Collocation 50
3.3 Solving for $L = 0$ 50
3.4 Solving for $L \neq 0$ 58
3.5 Conclusions and Open Problems 62
3.6 Numerical Examples 64
3.6.1 Non-perturbed Case 64
3.6.2 Perturbed Case 69
3.6.3 A Special Case 73

4 The Identification Problem **77**

4.1 Introduction 77
4.2 The Identification Problem 78
4.2.1 Statement of the Minimization Problem 79
4.2.2 Differentiability of the Solution wrt the Parameter 82
4.3 The Approximated Minimization Problems 84
4.3.1 Convergence of the Identification Scheme 85
4.4 Conclusions 86
4.5 Numerical Examples 87

List of Figures

3.1 Non-perturbed case, $L = 0$ 65

3.2 Non-perturbed case, $L \neq 0$ 66

3.3 Non-perturbed case, $L \neq 0$ 67

3.4 Perturbed case, constant perturbation 70

3.5 Perturbed case, linear perturbation 71

3.6 Perturbed case, linear perturbation 72

3.7 Discontinuous case 74

3.8 Discontinuous case 75

3.9 Discontinuous case 76

4.1 Smooth initial condition 88

4.2 Smooth initial condition 90

4.3 Discontinuous initial condition 91

4.4 Discontinuous initial condition 92

4.5 Discontinuous initial condition 93

Chapter 1

Introduction and Notation

1.1 Introduction

For a mathematical model to be reliable it is important that all parameters involved are correct. This makes parameter identification a necessary and important field of research. In this work we consider the problem of parameter identification for a certain class of singular neutral functional differential equations (SNFDE) that can be reformulated as Volterra equations with weakly singular kernels. The weak singularity appearing in the kernel of the operators studied here makes identification a difficult task. The Initial Value Problem (IVP) we study has the following structutre:

$$\frac{dDx_t}{dt} = Lx_t + g(t), \quad 0 \le t \le \infty, \tag{1.1.1}$$

$$x_0(s) = \varphi(s), \quad s \in [-1, 0] \tag{1.1.2}$$

where $x_t(s) = x(t+s)$ for $s \in [-1, 0]$, $t \ge 0$ and $g \in C$ is a known function. The right hand side L operator is given by

$$L\varphi = a\varphi(0) + b\varphi(-1) + \int_{-1}^{0} h(s)\varphi(s)\, ds, \tag{1.1.3}$$

where a and b are constants and h is a continuous function. The linear operator $D \in C^m[-1,0]$ is given by

$$D\varphi = \int_{-1}^{0} \varphi(s)k(s,\alpha)\,ds \qquad (1.1.4)$$

where $k(\cdot,\alpha) \in L^1(-1,0)$ is a non-negative, nondecreasing function on $[-1,0]$ which is weakly singular at $s = 0$. In order to solve the parameter identification problem we require the kernel function k to be continuous and differentiable with respect to $\alpha \in (0,1)$ and $\frac{d}{d\alpha}k(\cdot,\alpha) \in L^1(-1,0)$. Definition (1.1.4) for the D operator includes a large class of weakly singular integral operators. However, for practical applications we are interested in a D operator with a kernel function k of the form

$$k(s,\alpha) = c(s)(-s)^{-\alpha} + p(s), \quad s \in [-1,0] \qquad (1.1.5)$$

where $c, p \in C^m[-1,0]$.

Although the goal of this work is to obtain efficient numerical methods for the identification problem, we will first concentrate on the development of approximation schemes for the solution of IVP (1.1.1)-(1.1.2). This step is a basic requirement for our work since any attempt of solving the identification problem will need an approximation to the solution of the associated IVP.

In chapter 2, we analyze the structure of the solution of IVP (1.1.1)-(1.1.2) and we rewrite this IVP as a Volterra integral equation. Using the classical theory of Volterra integral equations we derive a representation of the solution for the original IVP as a smooth part plus a non-smooth part. The importance of such a descomposition is not only to help us understand the structure of the exact solutions but also to give us insight for the development of numerical schemes with high order accurancy. The development of such numerical schemes will be addressed in chapter 3. It is interesting to note that the representation of the solution of IVP (1.1.1)-(1.1.2) as having a non-smooth part plus a smooth part can be viewed as a generalization for the exact solution representation given in [13] for IVP (1.1.1)-(1.1.2) with $L = 0$, $g = 0$ and the kernel function $k(s,\alpha) = (-s)^{-\alpha}$.

In chapter 3, we study the approximation problem via the collocation method. The use of collocation techniques for solving Volterra equations with weak singularities in their kernels have been extensively studied [8], [7], [17], [29]. In this work, we choose non-polynomial spline collocation as a projection method for our numerical approximation. This choice is suggested by the representation of the solution derived in chapter 2 and gives us high-order convergence results. At the end of chapter 3 we present several numerical examples for IVP (1.1.1)-(1.1.2) with a kernel function of the form (1.1.5) for different choices of the functions c and p.

We present the parameter identification problem in chapter 4. We study identification for the parameter α appearing in the weakly singular kernel of the D operator in (1.1.4). Using well-posedness of the IVP (1.1.1)-(1.1.2), we study differentiability of the solution with respect to the parameter α in (1.1.4) and establish the convergence of the identification scheme. Finally we present some numerical results. It is interesting to note that using the approximation scheme developed in chapter 3 we can solve the parameter identification problem for the parameter α in (1.1.4) not only for the case of the IVP (1.1.1)-(1.1.2) with consistent initial data, i.e.

$$Dx_t = \int_0^t Lx_\tau \, d\tau + \int_0^t g(\tau) \, d\tau + \eta \tag{1.1.6}$$

$$x_0(s) = \varphi(s), \quad s \in [-1, 0]. \tag{1.1.7}$$

where $\eta = D\varphi$, but also for the case of inconsistent initial data (i.e. $D\varphi \neq \eta$) even though this case is not covered by the theory we present here. We include several examples showing the accuracy of our numerical scheme for both approximation of the solution and parameter identification.

1.2 Notation

The notation we used is fairly standard. Throughout the paper, the real number $p \in [1, +\infty)$ and the delay $r \in (0, +\infty)$ are assumed fixed. For $1 \leq p \leq +\infty$, and a close interval $I = [a, b] \subset R$, the symbol $L^p(I) = L^p(I, R)$ will denote the customary Lebesgue spaces of real valued functions on I which are integrable when raised to the *pth* power. The usual Banach space $C(I, R)$ of continuous real valued functions on I will be denoted by $C(I)$. Whenever $I = [0, T]$ for a fixed real number T we will simply write L^p and C for $L^p(I)$ and $C(I)$, respectively. We use $\| \cdot \|_X$ to denote the norm on the normed linear space X. However, we will use the symbol $\| \cdot \|$ to denote any one of several norms when it is clear from the context which norm is intended. If $x : [-r, a) \to R$ for some $0 \leq a \leq \infty$, then we define $x_t : [-r, a) \to R$ for $0 \leq a$ by $x_t(s) = x(t + s)$.

Chapter 2

A Representation For The Solution

In this chapter we study the structure and representation of the solutions of IVP (1.1.1)-(1.1.2) via an associated Volterra integral equation. For the sake of completeness, in the first section of this chapter we will summerize some basic results on Volterra integral equations of both the first and second kind. These are classical results that we will use through out the chapter and can be found in [7].

2.1 Preliminaries

2.1.1 Classification of Volterra Integral Equations

An integral equation is a functional equation in which the unknown function appears under one or several integral signs. An integral equation, for the unknown function y, of the form

$$y(t) = f(t) + \int_0^t k(t, s, y(s))ds, \quad t \in I = [0, T] \tag{2.1.1}$$

is called a nonlinear Volterra integral equation of the second kind. Note that f and k, the kernel of the integral equation, are given real valued functions. The function f is known as the initial function or forcing term of the equation since it gives the initial condition $f(0) = y(0)$.

If the unknown function occurs only under the integral sign, we have a nonlinear Volterra integral equation of the first kind, i.e.

$$\int_0^t k(t, s, y(s))ds = f(t), \quad t \in I. \tag{2.1.2}$$

A Volterra integral equation is said to be linear if its kernel has the following form

$$k(t, s, y) = k(t, s)y. \tag{2.1.3}$$

In many applications we find difference kernels of the form

$$k(t, s) = k(t - s).$$

Equations with difference kernels are referred to as convolution equations. When the kernel function is the product of a smooth function and a weakly singular (i.e., an unbounded but integrable) function,

$$k(t, s, y(s)) = (t - s)^{-\alpha}\gamma(t, s, y(s)) \quad 0 < \alpha < 1 \tag{2.1.4}$$

with γ smooth, the equation is called an integral equation of Abel type, since first kind equations with kernels of the form (2.1.4) were first studied by Abel.

In the next section we discuss existence and uniqueness of solutions for linear Volterra integral equations of both the first and the second kind.

2.1.2 Existence and Uniqueness of Solutions

Volterra Linear Integral Equations of the Second Kind

Let $I = [0, T]$ and $S = [0, T] \times [0, T]$. Consider the integral equation

$$y(t) = f(t) + \int_0^t k(t, s)y(s)\, ds, \quad t \in I, \tag{2.1.5}$$

and assume that the kernel k and the forcing function f are real valued and continuous on S and I, respectively.

To analyze existence and uniqueness of the solutions of (2.1.5) we need to define the so-called iterative kernels and the corresponding resolvent kernel. This definition will be of fundamental importance in the development of the solution representation for IVP (2.1.1)-(2.1.2) that we derive in section 3 of this chapter.

Definition 2.1.1 *Let $k_1(t,s) = k(t,s)$ and set*

$$k_n(t,s) = \int_s^t k_1(t,\psi)k_{n-1}(\psi,s)d\psi, \quad n \geq 2. \tag{2.1.6}$$

The functions $k_n, n \geq 1$ are called the iterated kernels associated with the given kernel k in (2.1.5).

The introduction of the iterated kernels of k is motivated by the Picard method for constructing successive approximations $\{y_n(t), n \geq 1\}$ to the exact solution of (2.1.5) by means of

$$y_n(t) = f(t) + \int_0^t k(t,s)y_{n-1}(s)ds, \qquad y_0(t) = f(t), \quad t \in I. \tag{2.1.7}$$

The iterate y_n can be expressed in terms of the given function f and the iterated kernels $k_1, \ldots, k_n$, namely

$$y_n(t) = f(t) + \int_0^t \sum_{m=1}^{n} k_m(t,s)f(s)ds, \quad n \geq 1. \tag{2.1.8}$$

Using (2.1.6) it is clear that if the kernel k in (2.1.5) satisfies $\|k(t,s)\| \leq M$ for all $(t,s) \in S$ we will have

$$\|k_n(t,s)\| \leq M^n \frac{(t-s)^{n-1}}{(n-1)!} \leq M^n \frac{T^{n-1}}{(n-1!)} \tag{2.1.9}$$

for all $n \geq 1$ and $(t, s) \in S$.

Thus the series

$$\sum_{m=1}^{\infty} k_m(t,s) = \lim_{n\to\infty} \sum_{m=1}^{n} k_m(t,s) \tag{2.1.10}$$

converges absolutely and uniformly on S to a continuous function

$$R(t,s) = \sum_{m=1}^{\infty} k_m(t,s) \tag{2.1.11}$$

called the resolvent kernel of the given kernel k. Then, the function y given by

$$y(t) = f(t) + \int_0^t R(t,s)f(s)ds, \quad t \in I, \tag{2.1.12}$$

is a continuous solution of (2.1.5) on the given interval I.

Uniqueness of the solution is established by employing (2.1.11) together with (2.1.6) to obtain

$$\begin{aligned} R(t,s) &= k_1(t,s) + \sum_{m=2}^{\infty} k_m(t,s) = k(t,s) + \sum_{m=2}^{\infty} \int_s^t k(t,\tau)k_{m-1}(\tau,s)\,d\tau \\ &= k(t,s) + \int_s^t k(t,\tau)R(\tau,s)\,d\tau, \quad (t,s) \in S \end{aligned} \tag{2.1.13}$$

and in a similar fashion to obtain

$$R(t,s) = k(t,s) + \int_s^t R(t,\tau)k(\tau,s)\,d\tau, \quad (t,s) \in S. \tag{2.1.14}$$

Suppose that equation (2.1.5) possesses two solutions $y, z \in C(I)$. That is, assume that in addition to the solution given by (2.1.12) we have z such that

$$z(\tau) = f(\tau) + \int_0^\tau k(\tau, s) z(s)\, ds, \quad \tau \in I. \tag{2.1.15}$$

If we multiply this identity by $R(t,\tau)$ and integrate with respect to τ over $[0,t]$ we obtain

$$\begin{aligned}
\int_0^t R(t,\tau)z(\tau)\, d\tau &= \int_0^t R(t,\tau)f(\tau)\, d\tau + \int_0^t \int_0^\tau R(t,\tau)k(\tau,s)z(s)\, ds\, d\tau \\
&= \int_0^t R(t,\tau)f(\tau)\, d\tau + \int_0^t z(s) \int_s^t R(t,\tau)k(\tau,s)\, d\tau\, ds \\
&= \int_0^t R(t,\tau)f(\tau)\, d\tau + \int_0^t z(s)[R(t,s) - k(t,s)]\, ds \\
&= \int_0^t R(t,s)z(s)\, ds + (y(t) - f(t)) - (z(t) - f(t)), \quad t \in I.
\end{aligned} \tag{2.1.16}$$

Then $0 = y(t) - z(t)$ for all $t \in I$.

We summarize the results presented in this section in the following theorem.

Theorem 2.1.2 *Let the functions f and k characterizing the integral equation* (2.1.1) *be continuous on I and S respectively. Then this equation has a unique solution $y \in C(I)$ given by*

$$y(t) = f(t) + \int_0^t R(t,s)f(s)ds, \quad t \in I \tag{2.1.17}$$

where $R \in C(S)$ is the resolvent kernel associated with the given kernel k. The resolvent kernel satisfies the identities

$$R(t,s) = k(t,s) + \int_s^t k(t,\psi)R(\psi,s)d\psi$$

and

$$R(t,s) = k(t,s) + \int_s^t R(t,\psi)k(\psi,s)d\psi$$

for all $(t,s) \in S$.

For a complete proof of this theorem see [8].

Linear Volterra Integral Equations of the First Kind

The existence of a unique solution $y \in C(I)$ of the first kind integral equation (2.1.1) is assured under certain conditions on k and f.

Theorem 2.1.3 *Let k and f satisfy*

a) $f \in C^1(I)$ with $f(0) = 0$, this assures the continuity of the solution at $t = 0$;

b) $k \in C(S)$, $\frac{\partial k}{\partial t} \in C(S)$;

c) $k(t,t) \neq 0$ for all $t \in I$.

Under these conditions, the first kind equation (2.1.1) has a unique solution $y \in C(I)$.

Under these conditions the first kind equation (2.1.1) can be transform into an integral equation of the second kind for which Theorem 2.1.2 can be applied. For a proof of this theorem see [8].

2.2 Well-Posedness Results

In this section we present well-posedness results for IVP (1.1.1)-(1.1.2). These results are due to Ito, Kappel and Turi and can be found in [27]. We present these results here to guarantee existence and uniqueness of the solution of the IVP we are studing.

2.2.1 Well-Posedness Results for $L = 0$

First we study IVP (1.1.1)-(1.1.2) with $L = 0$ in both the homogenous case (i.e. $g = 0$) and the nonhomogeneous case (i.e. $g \neq 0$).

Consider the operator A defined by

$$dom A = \left\{ \varphi \in C^1(-1,0) \mid \int_{-r}^{0} k(s)\partial\varphi(s)\, ds = 0 \right\}, \tag{2.2.18}$$

$$A\varphi = \partial\varphi, \quad \varphi \in domA. \tag{2.2.19}$$

Here $\partial\varphi$ denotes the derivative $\dot{\varphi}$. Following Ito, Kappel and Turi in [27], we know that the operator (2.2.18) verifies all the conditions of the Lumer-Philips theorem on $C(-1,0)$ (see Lemmas 2.1 and 2.3 of [27]). The operator A with $dom(A)$ given by (2.2.18) is the infinitesimal generator of a contraction semigroup on $C(-1,0)$. This is given by the following theorem.

Theorem 2.2.4 *(see [27], Theorem 2.4) Consider IVP* (1.1.1)-(1.1.2) *with* $L = 0$ *and* $g = 0$*, then the operator* A *defined by* (2.2.18)-(2.2.19) *generates a contraction semigroup* $S(t)$*,* $t \geq 0$*, on* $C(-1,0)$ *satisfying*

$$S(t)\varphi(s) = S(t+s)\varphi(0), \quad t \geq 0, \quad -r \leq s \leq 0, \tag{2.2.20}$$

i.e., $S(\cdot)$ *is a translation. Moreover, the function* $x(t)$ *defined by* $x(t) = (S(t)\varphi)(0)$ *for* $t \geq 0$ *and by* $x(t) = \varphi(t)$ *for* $-1 \leq s \leq 0$ *is the unique continuous solution of equation* (1.1.1) *with* $g = 0$ *and* $L = 0$.

For the nonhomogenuous case (i.e. $g \neq 0$) we have the following result.

Theorem 2.2.5 *(see [27], Theorem 2.10) Let* $T > 0$*,* $g \in C(0,T)$ *and* $\varphi \in C(-1,0)$ *be given. Then, for any* $w > 0$*,*

$$\int_0^t S(t-s)\psi_w g(s)\,ds \in domA, \quad t \in [0,T], \tag{2.2.21}$$

where ψ_w *has the form* $\psi_w(s) = -\Delta_0(w)^{-1}e^{ws}$*, and the function* $\Delta_0(w)$ *is given by*

$$\Delta_0(\lambda) = \lambda \int_{-1}^0 e^{\lambda s}k(s)\,ds \quad \lambda \in C. \tag{2.2.22}$$

Equation (1.1.1) *with* $L = 0$ *has a unique solution* $x \in C(-1,T)$*, which, for any* $w > 0$*, is given by*

$$x(t+\cdot) = S(t)\varphi + (A - wI)\int_0^t S(t-s)\psi_w g(s)\,ds, \quad 0 \leq t \leq T. \tag{2.2.23}$$

Moreover, this solution satisfies the estimate

$$\|x(t)\| \leq e^{w(t-r)} max(\|\varphi\|_{C(-r,0)}, \Delta_0(w)^{-1}\|g\|_{C(0,T)}), \quad 0 \leq t \leq T. \tag{2.2.24}$$

In case $\varphi = 0$*, there exists a continuous, non-negative, monotonically increasing function* $\gamma : [0, \infty) \to R$ *with* $\gamma(0) = 0$ *such that, for all* $g \in C(0,T)$*,* $T > 0$*,*

$$\|x(t)\| \leq \gamma(t)\|f\|_{C(0,t)}, \quad 0 \leq t \leq T. \tag{2.2.25}$$

Remark 2.2.6 In many practical applications the D operator given by equation (1.1.4) will have the form

$$D\varphi = \int_{-1}^{0} \varphi(s)(-s)^{-\alpha}\, ds \tag{2.2.26}$$

with $\alpha \in (0,1)$. Note that in this case D is bounded on C for all $\alpha \in (0,1)$ and is bounded on L^p if $p > \frac{1}{1-\alpha}$. However, D is unbounded, but densily defined on L^p when $p \leq \frac{1}{1-\alpha}$. In chapter 4 we present numerical results for the identification problem for the parameter α appearing in the kernel of the D operator in equation (2.2.26). These kernels arise in applications to aerodynamics and the special case of $\alpha = 1/2$ is of particular interest (see [12]). Here we present a theorem given by Burns, Herdman and Stech in [13] where this special case is analyzed. For simplicity, the case $L = 0$ was considered. That is, we will consider the following IVP:

$$\frac{d}{dt} Dx_t = 0, \quad t > 0, \quad x_0 = \varphi. \tag{2.2.27}$$

where the operator D is given by equation (2.2.26).

Theorem 2.2.7 (see Theorem 4.1 in [13]) *i) For each* $\varphi \in C$ *the IVP*

$$Dx_t = D\varphi, \qquad t \geq 0, \qquad x_0 = \varphi \tag{2.2.28}$$

has a unique continuous solution $x(\cdot, \varphi)$ *on* $[0, \infty)$*. The family of operators* $T(t)\varphi = x_t(\cdot, \varphi), \quad t \geq 0$ *defines a* C_0 *semigroup on* C*.*

ii) If $p < 1/(1-\alpha)$ then for each $(\eta, \varphi) \in R \times L^p$ the initial value problem

$$Dx_t = \eta, \quad t \geq 0, \quad x_0 = \varphi, \tag{2.2.29}$$

has a unique solution $x(\cdot, \varphi)$ defined a.e. on $[0, +\infty)$. Moreover, the family of operators $S(t)(\eta, \varphi) = (Dx_t, x_t) = (\eta, x_t(\cdot, \varphi)), \quad t \geq 0$ defines a C_0 semigroup on $R \times L^p$.

iii) If $p \geq 1/(1-\alpha)$ then (2.2.29) has a unique solution for (η, φ) in a dense subset of $R \times L^p$. However, the family of operators $S(t)$ defined above fails to define a C_0 semigroup on $R \times L^p$.

Remark 2.2.8 Theorem 2.2.7 establishes the well posedness of IVP (1.1.1)-(1.1.2) with $L = 0$ and $g = 0$, where the D operator is of the form (2.2.26) for initial data belonging to $R \times L^p(-1, 0)$ with $p \in (1, (1-\alpha)^{-1})$ only. Note that the case $\alpha = 1/2$, (that is, the case of the aerodynamic model developed in [12], rules out the Hilbert space $R \times L^2(-1, 0)$. However, this problem was proved to be well posed in a weighted L^2 space using as a weighting function the kernel of the difference operator D, see [15], [26], [27]. In this work, we will not be interested in the weighted L^2 spaces since our main goal is the identification of the parameter appearing in the kernel of the D operator given by equation (1.1.4). In the following, we focus our attention on part i) of the above theorem since well posedness on the state space $C(-1, 0)$ will allow us to study the identification of the parameter α in equation (2.2.26).

2.2.2 Well-Posedness Results for the General Case

In this section we consider a general form of IVP (1.1.1)-(1.1.2) for both the homogenuous case (i.e. $g = 0$) and the nonhomogenuous case (i.e. $g \neq 0$). That is, we consider the general problem given by

$$\frac{d}{dt} \int_{-1}^{0} k(s) x(t+s)\, ds = Lx_t + g(t), \quad t \geq 0, \tag{2.2.30}$$

$$x(s) = \psi(s), \quad -1 \leq s \leq 0, \tag{2.2.31}$$

where the operator L is given by

$$L(s) = \int_{-1}^{0} \psi(s) d\mu(s), \quad \psi \in C(-1,0), \tag{2.2.32}$$

with μ a function of bounded variation on $[-1,0]$.

Remark 2.2.9 Taking the measure μ to be

$$\mu(t) = \int_{0}^{t} h(s)\, ds \quad t \in (-1,0),$$

where h is a continuous function and $\mu(0) = -a$, $\mu(-1) = b + \int_0^{-1} h(s)\, ds$, IVP (1.1.1)-(1.1.2) becomes a particular case of the more general IVP (2.2.30)- (2.2.31).

We will first consider the homogenuous case, i.e. $g = 0$. Following Ito, Kappel and Turi in [27] we consider the operator A_L given by

$$DomA_L = \{\varphi \in C^1(-1,0) | \int_{-1}^{0} k(s)\partial\varphi(s)\, ds = \int_{-1}^{0} \varphi(s) d\mu(s)\}, \tag{2.2.33}$$

$$A_L\varphi = \partial\varphi, \quad \varphi \in DomA_L. \tag{2.2.34}$$

We define the subspace Z_w of $C(-1,0)$ and the operator $B_w : C(-1,0) \to Z_w$ by

$$Z_w = span(\psi_w), \tag{2.2.35}$$

$$B_w\varphi = L(\varphi)\psi_w, \quad \varphi \in C(-1,0). \tag{2.2.36}$$

where ψ_w is defined in Theorem 2.2.5. The norm on Z_w is defined by $\|\alpha\psi_w\|_{Z_w} = |\alpha|, \quad \alpha \in R$. Note that Z_w is continuously embedded in $C(-1,0)$ and B_w is continuous.

Remark 2.2.10 The operator A_L is a multiplicative perturbation of A, see Lemma 2.11 in [27].

The following two theorems will guarantee the well posedness of the general IVP (1.1.1)-(1.1.2).

Theorem 2.2.11 *(See [27], Theorem 2.13) The operator A_L is the infinitesimal generator of a shift semigroup S_L. The spectrum of A_L is all point spectrum and is given by*

$$\sigma(A) = \{\lambda \in C | \Delta(\lambda) = 0\}, \tag{2.2.37}$$

where

$$\Delta(\lambda) = \lambda \int_{-1}^{0} e^{\lambda s} k(s)\, ds - \int_{-1}^{0} e^{\lambda s}\, d\mu(s), \quad \lambda \in C. \tag{2.2.38}$$

Moreover, for any $\varphi \in C(-1,0)$ *, the abstract Cauchy problem* (2.2.30)-(2.2.31) *with* $g = 0$ *has a unique solution* $x \in C(-1,\infty)$ *and*

$$x_t = S_L(t)\varphi, \quad t \geq 0. \tag{2.2.39}$$

Theorem 2.2.12 *Let* $T > 0$*,* $g \in C(0,T)$ *and* $\varphi \in C(-1,0)$ *be given. For* $w \in \rho(A_L) \cap R^+$ *we set*

$$\psi_{w,L}(\theta) = -\frac{1}{\Delta_L(w)} e^{w\theta}, \quad -1 \leq \theta \leq 0, \tag{2.2.40}$$

where Δ_L *is defined by* (2.2.38)*. Then the following statements are true:*

a) Equation (2.2.30)-(2.2.31) *has a unique solution* $x \in C(-1,T)$*.*

b) There exists an $w_0 \geq 0$ *such that for any* $w > w_0$

$$(A_L - wI) \int_0^t S_L(t-s)\psi_{w,L} g(s)\, ds \in domA, \quad 0 \leq t \leq T \tag{2.2.41}$$

where domA is given by equation (2.2.18)*, and*

$$x_t = S_L(t)\psi + (A_L - wI) \int_0^t S_L(t-s)\psi_{w,L} g(s)\, ds, \quad 0 \leq t \leq T. \tag{2.2.42}$$

c) The solution of the general IVP (2.2.30)-(2.2.31) *depends continuously on* ψ *and* g*. In particular, for any* $w > w_0$ *we have the estimate*

$$\|x\|_{C(-1,T)} \leq e^{w(T+1)} \max(\|\varphi\|_{C(-1,0)}, \frac{1}{\gamma_{w,L}} \|g\|_{C(0,T)}), \tag{2.2.43}$$

where

$$\gamma_{w,L} = w \int_{-1}^{0} e^{w\theta} k(\theta)\, d\theta - \int_{-1}^{0} |d\mu(\theta)| \tag{2.2.44}$$

($w_0 \geq 0$ is determined by $\gamma_{w,L} > 0$ for $w > w_0$).

2.3 NFDEs as Volterra Integral Equations

In this section we establish a relationship between IVP (1.1.1)- (1.1.2) and a Volterra integral equation. The goal of working with an associate Volterra equation is to derive a representation for the exact solution of IVP (1.1.1)-(1.1.2). To achive this goal we will consider a particular form for the kernel function $k(\cdot, \alpha)$ of the D operator in equation (1.1.4). We will devote our attention to “solving” IVP (1.1.1)-(1.1.2) on $[0, 1]$ only. Our methods can be extended to any interval $[0, T], T \geq 0$, in an obvious way.

Consider IVP (1.1.1)-(1.1.2) with $t \in [0, 1]$. Let the kernel function k of the D operator in equation (1.1.4) be of the form

$$k(s, \alpha) = c(s)(-s)^{-\alpha} + p(s), \quad s \in [-1, 0] \tag{2.3.45}$$

where c and p are smooth functions. Let the right hand side function $g \in C^m$. That is we will consider the problem

$$\frac{dDx_t}{dt} = Lx_t + g(t), \quad 0 \leq t \leq 1, \tag{2.3.46}$$

with initial data

$$x_0(s) = \varphi(s), \quad s \in [-1, 0], \tag{2.3.47}$$

where $x_t(s) = x(t + s)$ for $s \in [-1, 0], \;\; t \geq 0$, $g \in C^m$ is a known function and the linear operators D and L have the following representation for $\varphi \in C[-1, 0]$

$$Dx_t = \int_{-1}^{0} [c(s)(-s)^{-\alpha} + p(s)] x(t + s)\, ds, \tag{2.3.48}$$

where $c, p \in C^m[-1, 0]$. The right hand side L operator has the following form

$$L\varphi = a\varphi(0) + b\varphi(-1) + \int_{-1}^{0} h(s)\varphi(s)\, ds. \tag{2.3.49}$$

We let β denote the usual beta function. The following theorem will establish a relationship between IVP (2.3.46)-(2.3.47) and a Volterra integral equation of the second kind.

Theorem 2.3.13 *Assume $c(0) \neq 0$. Then the solution x of IVP* (2.3.46)-(2.3.47) *satisfies the following Volterra equation of the second kind.*

$$x(t) - \int_0^t [(t-s)^{\alpha-1} K(s-t) + M(s-t)] x(s)\, ds = f(t) \tag{2.3.50}$$

for $t \in [0, 1]$ where K is defined on $[-1, 0]$ by

$$\begin{aligned} K(s) &= \frac{1}{c(0)\beta(1-\alpha,\alpha)}[-a + \int_0^1 (1-u)^{\alpha-1}\{\alpha[p(us) - \int_{us}^0 h(w)\, dw] \\ &+ us[\dot{p}(us) + h(us)]\}]\, du, \end{aligned} \tag{2.3.51}$$

M is defined on $[-1, 0]$ by

$$M(s) = \frac{1}{c(0)\beta(1-\alpha,\alpha)} \int_0^1 (1-u)^{\alpha-1} u^{1-\alpha} \dot{c}(-us)\, du, \tag{2.3.52}$$

and the right hand side function f is defined by

$$f(t) = \frac{1}{c(0)\beta(1-\alpha,\alpha)} \frac{d}{dt} \int_0^t (t-s)^{\alpha-1} F(s)\, ds, \quad t \in [0, 1] \tag{2.3.53}$$

where the continuous function F is defined on $[0, 1]$ by

$$\begin{aligned} F(t) &= D\varphi + b\int_0^t \varphi(s-1)\,ds + \int_0^t g(s)\,ds \\ &+ \int_0^t \int_{-1}^{-s} h(u)\varphi(s+u)\,du\,ds - \int_t^1 [c(-s)s^{-\alpha} + p(-s)]\varphi(t-s)\,ds. \end{aligned} \tag{2.3.54}$$

Proof: Integrating (2.3.46) over the interval $[0, t]$ we have that

$$\begin{aligned} \int_0^t \frac{d}{ds} Dx_s\,ds &= Dx_t - D\varphi \\ &= a\int_0^t x(s)\,ds + b\int_0^t \varphi(s-1)\,ds + \int_0^t \int_{-1}^0 h(u)x(s+u)\,du\,ds + \int_0^t g(s)\,ds. \end{aligned} \tag{2.3.55}$$

In order to collect all terms involving x together on the right hand side we note that for $0 \leq s \leq t$, $[-1, 0] = [-1, -s] \cup [-s, 0]$, $[0, 1] = [0, t] \cup [t, 1]$ and employ (2.3.46) and (2.3.48) with the change of variables $s = -\tau$ to obtain

$$\int_0^t [c(-\tau)\tau^{-\alpha} + p(-\tau)]x(t-\tau)\,d\tau - a\int_0^t x(s)\,ds - \int_0^t \int_{-s}^0 h(u)x(s+u)\,du\,ds = F(t)$$

with F given by (2.3.54). For the third term on the left hand side of the above identity we first change the outer integration variable from s to τ, make a change of variables $s = \tau + u$ and change the order of integration. This, together with a change of variables $s = t - \tau$ in the first term on the left hand side yields

$$\int_0^t [c(s-t)(t-s)^{-\alpha} + p(s-t) - a - \int_s^t h(s-\tau)\,d\tau]x(s)\,ds = F(t). \tag{2.3.56}$$

For the convenience of notation, we define the function $H \in C^1[-1, 0]$ by

$$H(s) = \int_s^0 h(u)\,du \tag{2.3.57}$$

and use τ as the integration variable to obtain

$$\int_0^t [c(\tau - t)(t-\tau)^{-\alpha} + p(\tau - t) - a - H(\tau - t)]x(\tau)\, d\tau = F(t). \tag{2.3.58}$$

We evaluate the above expresion at s, multiply by $(t-s)^{\alpha-1}$ and integrate over $[0,t]$ to obtain

$$\int_0^t (t-s)^{\alpha-1} \int_0^s [c(\tau - s)(s-\tau)^{-\alpha} + p(\tau - s) - a - H(\tau - s)]x(\tau)\, d\tau\, ds = \int_0^t (t-s)^{\alpha-1} F(s)\, ds$$

which after a change in the order of integration becomes

$$\int_0^t x(\tau) \int_\tau^t (t-s)^{\alpha-1}[c(\tau - s)(s-\tau)^{-\alpha} + p(\tau - s) - a - H(\tau - s)]\, ds\, d\tau = \int_0^t (t-s)^{\alpha-1} F(s)\, ds.$$

The inner integral involving the constant a can be evaluated to obtain

$$\begin{aligned} \frac{-a}{\alpha}\int_0^t x(\tau)(t-\tau)^\alpha\, d\tau \quad &+ \quad \int_0^t x(\tau)\int_\tau^t (t-s)^{\alpha-1}[c(\tau - s)(s-\tau)^{-\alpha} \\ &+ \quad p(\tau - s) - H(\tau - s)]\, ds\, d\tau = \int_0^t (t-s)^{\alpha-1}F(s)\, ds. \end{aligned}$$

The change of variables $u = (s-\tau)/(t-\tau)$ yields

$$\begin{aligned} \frac{-a}{\alpha}\int_0^t x(\tau)(t-\tau)^\alpha\, d\tau \quad &+ \quad \int_0^t x(\tau)\int_0^1 (1-u)^{\alpha-1}u^{-\alpha}c(u(\tau - t))\, du\, d\tau \\ &+ \quad \int_0^t x(\tau)(t-\tau)^\alpha \int_0^1 (1-u)^{\alpha-1}[p(u(\tau-t)) - H(u(\tau - t))]\, du\, d\tau \\ &= \quad \int_0^t (t-s)^{\alpha-1}F(s)\, ds. \end{aligned} \tag{2.3.59}$$

We define continuous functions q and m on $[-1,0]$ by

$$q(s) = \int_0^1 (1-u)^{\alpha-1}[p(us) - H(us)]\, du - \frac{a}{\alpha}, \tag{2.3.60}$$

$$m(s) = \int_0^1 (1-u)^{\alpha-1}u^{-\alpha}c(us)\, du, \tag{2.3.61}$$

respectively. Note that $\dot{q}$ and $\dot{m}$ exist on $[-1, 0]$ and are given by

$$\dot{q}(s) = \int_0^1 (1-u)^{\alpha-1}[\dot{p}(us) - \dot{H}(us)]u\,du' \tag{2.3.62}$$

$$\dot{m}(s) = \int_0^1 (1-u)^{\alpha-1}u^{1-\alpha}\dot{c}(us)\,du, \tag{2.3.63}$$

respectively. Equation (2.3.59) can be simplified to

$$\int_0^t x(\tau)[(t-\tau)^\alpha q(\tau - t) + m(\tau - t)]\,d\tau = \int_0^t (t-s)^{\alpha-1}F(s)\,ds. \tag{2.3.64}$$

Our next step in deriving the desired Volterra equation is to differentiate the above equality with respect to t on $[0, 1]$ which gives

$$x(t) - \int_0^t [(t-\tau)^{\alpha-1}K(\tau - t) + M(\tau - t)]x(\tau)\,d\tau = f(t) \tag{2.3.65}$$

where K and M are defined by (2.3.51) and (2.3.52), respectively. The proof is complete.

2.4 Representation of the Solution

In this section we derive the resolvent kernel for the Volterra integral equation given above and we study the behavior of the right hand side function f given in (2.3.50). This leads us to a representation of the exact solution of IVP (2.3.46)-(2.3.47).

The following lemma characterizes the behavior of the right hand side function f in (2.3.50).

Lemma 2.4.14 *Assume that $c, \varphi \in C^{m+1}[-1, 0]$ for some positive integer m. Let*

$$w_0(t) = \int_t^1 c(-s)s^{-\alpha}\varphi(t-s)\,ds. \tag{2.4.66}$$

Then w_0 can be represented as follows

$$w_0(t) = \sum_{j=1}^{m} t^{j-\alpha} g_j(t) + v_{m+1}(t), \quad 0 \leq t \leq 1 \tag{2.4.67}$$

where $v_{m+1} \in C^{m+1}[0,1]$, $g_j \in C^{m+1}[0,1]$, $j = 1, \cdots, m$.

Proof:

Define the operator $I : L^1[0,1] \to C[0,1]$ as follows: for $v \in L^1[0,1]$,

$$(Iv)(t) = \int_0^t v(s)\, ds. \tag{2.4.68}$$

Also define

$$w_n(t) = \int_t^1 c(-s) s^{-\alpha} \varphi^{(n)}(t-s)\, ds \quad n = 0, \cdots, m+1 \tag{2.4.69}$$

where $\varphi^{(0)} = \varphi$ and $\varphi^{(n)}$ denotes the nth-derivative of φ, and let

$$u(t) = c(-t)t^{-\alpha} \qquad t \geq 0. \tag{2.4.70}$$

We first prove that w_0 can be represented as follows

$$\begin{aligned} w_0(t) &= \sum_{j=0}^{n-1} (I^j w_j(0))(t) \\ &- \sum_{j=1}^{n} \varphi^{j-1}(0)(I^j u)(t) - (I^n w_n)(t), \quad n = 1, \cdots, m+1 \end{aligned} \tag{2.4.71}$$

where $I^0 v(t) = v(t)$ and $I^j v(t) = I(I^{j-1} v(t))$, $j \geq 1$.

The proof is by induction. Differentiating w_0 we obtain

$$\begin{aligned} \dot{w}_0(t) &= -\varphi(0)c(-t)t^{-\alpha} - \int_t^1 c(-s)s^{-\alpha}\dot{\varphi}(t-s)\,ds \\ &= -\varphi(0)u(t) - w_1(t). \end{aligned}$$

Integrating the above equality we have that

$$w_0(t) = w_0(0) - \varphi(0)(Iu)(t) - (Iw_1)(t). \tag{2.4.72}$$

Therefore (2.4.71) is true for $n = 1$. Assume that (2.4.71) is true for $n = k \leq m$. Similar to the above arguement, we have that

$$w_k(t) = w_k(0) - \varphi^k(0)(Iu)(t) - (Iw_{k+1})(t). \tag{2.4.73}$$

Substituting $w_n(n = k)$, in (2.4.71) and using the representation (2.4.73) for w_k, we have

$$\begin{aligned} w_0(t) &= \sum_{j=0}^{k-1}(I^j w_j(0))(t) - \sum_{j=1}^{k}\varphi^{(j-1)}(0)(I^j u)(t) \\ &+ (I^k w_k(0))(t) - \varphi^k(0)(I^{k+1}u)(t) - (I^{k+1}w_{k+1})(t) \\ &= \sum_{j=0}^{k}(I^j w_j(0))(t) - \sum_{j=1}^{k+1}\varphi^{(j-1)}(0)(I^j u) - (I^{k+1}w_{k+1})(t). \end{aligned}$$

This proves (2.4.71). To prove (2.4.67), we consider the three terms on the right hand side of (2.4.71). For the first term, it is easy to show that

$$(I^j w_j(0))(t) = \frac{w_j(0)}{j!}t^j, \quad j = 1, \cdots, m. \tag{2.4.74}$$

So the first term is actually a polynomial of degree m. For the second term, letting $j = 1$, and using the change of variables $s = \tau t$ we have

$$
\begin{aligned}
(Iu)(t) &= \int_0^t c(-s)s^{-\alpha}\,ds \\
&= t^{1-\alpha}\int_0^1 c(-\tau t)\tau^{-\alpha}\,d\tau.
\end{aligned}
$$

Setting

$$\overline{g}_1(t) = \int_0^1 c(-\tau t)\tau^{-\alpha}\,d\tau,$$

we have

$$(Iu)(t) = t^{1-\alpha}\overline{g}_1(t).$$

Since $c \in C^{m+1}[0,1]$, $\overline{g}_1 \in C^{m+1}[0,1]$. Similarly we have by induction that

$$(I^j u)(t) = t^{j-\alpha}\overline{g}_j(t) \tag{2.4.75}$$

where $\overline{g}_j \in C^{m+1}$ is given by

$$\overline{g}_j(t) = \int_0^1 \overline{g}_{j-1}(t\tau)\tau^{j-1-\alpha}\,d\tau, \qquad j = 1,\cdots,m.$$

It follows that

$$\sum_{j=1}^{m} \varphi^{j-1}(0)(I^j u)(t) = \sum_{j=1}^{m} \varphi^{j-1}(0)t^{j-\alpha}\overline{g}_j(t). \tag{2.4.76}$$

Let

$$g_j(t) = \varphi^{j-1}(0)\overline{g}_j(t), \tag{2.4.77}$$

and $v_{m+1} = I^{m+1}w_{m+1}$. Since $w_{m+1} \in C[0,1]$, we have that

$$I^{m+1}w_{m+1} \in C^{m+1}[0,1], \tag{2.4.78}$$

and (2.4.67) is a direct consequence of (2.4.71). The lemma is proved.

In the following Lemma we give a representation for the right hand side function f in (2.3.53).

Lemma 2.4.15 *Consider IVP (2.3.46)-(2.3.47). Assume that $g \in C^m[0,1]$ and $p, \varphi, h, c \in C^{m+1}[0,1]$. Then the right hand side function f in (2.3.50) can be represented as*

$$f(t) = \sum_{j=1}^{m} t^{j-1}q_j(t) + t^\alpha q(t), \quad 0 \le t \le 1, \tag{2.4.79}$$

where the functions $q, q_j \in C^m[0,1]$, $j = 1, \ldots, m$ are given by

$$\begin{aligned} q_j(t) &= \frac{1}{c(0)\beta(\alpha, 1-\alpha)} j \int_0^1 (1-s)^{\alpha-1}s^{j-\alpha}g_j(st)\,ds + t\int_0^1 (1-s)^{\alpha-1}s^{j-\alpha+1}\dot{g}_j(st)\,ds. \\ q(t) &= \frac{1}{c(0)\beta(\alpha, 1-\alpha)} \int_0^1 (1-s)^{\alpha-1}\dot{g}_{m+1}(ts)\,ds, \end{aligned} \tag{2.4.80}$$

with g_j, $j = 1, \cdots, m$ given by (2.4.77). The function g_{m+1} is given by

$$\begin{aligned} g_{m+1}(t) &= D\varphi + b\int_0^t \varphi(s-1)\,ds + \int_0^t g(s)\,ds \\ &+ \int_0^t \int_{-1}^{-s} h(u)\varphi(u+s)\,du\,ds - \int_t^1 p(-s)\varphi(t-s)\,ds + v_{m+1}(t) \end{aligned} \tag{2.4.81}$$

and $v_{m+1} \in C^{m+1}$ given by the previous lemma.

Proof: It is easy to see that $F(0) = 0$ for F given by (2.3.54). Using the assumption that $g \in C^m[0,1]$ and $p, \varphi,\, h \in C^{m+1}[0,1]$, we have

$$\int_0^t \varphi(s-1)\,ds, \quad \int_0^t g(s)\,ds, \quad \int_0^t \int_{-1}^{-s} h(u)\varphi(s+u)\,du\,ds,$$
$$\int_t^1 [c(-\rho)\rho^{-\alpha} + p(-\rho)]\varphi(t-\rho)\,d\rho \in C^{m+1}[0,1]. \tag{2.4.82}$$

Thus, as a direct consequence of Lemma (2.4.14) and equation (2.3.54), we have that

$$F(t) = \sum_{j=1}^{m} t^{j-\alpha} g_j(t) + g_{m+1}(t), \quad 0 \le t \le 1, \tag{2.4.83}$$

where $g_j \in C^{m+1}[0,1]$, $j = 1, \ldots, m+1$ and g_{m+1} are given by equations (2.4.81) and (2.4.77) respectively.

Note that from equation (2.4.83) we have that $F(0) = g_{m+1}(0)$. Moreover, from equation (2.3.54) we know that $F(0) = 0$, thus $g_{m+1}(0) = 0$.

Replacing equation (2.4.83) in (2.3.53), we have

$$\begin{aligned}
f(t) &= \frac{1}{c(0)\beta(1-\alpha,\alpha)} \frac{d}{dt} \int_0^t (t-s)^{\alpha-1} F(s)ds \\
&= \frac{1}{c(0)\beta(1-\alpha,\alpha)} \frac{d}{dt} \int_0^t (t-s)^{\alpha-1} \left(\sum_{j=1}^{m} g_j(s) s^{j-\alpha} + g_{m+1}(s) \right) ds \\
&= \frac{1}{c(0)\beta(1-\alpha,\alpha)} \frac{d}{dt} \left(\int_0^t (t-s)^{\alpha-1} \sum_{j=1}^{m} g_j(s) s^{j-\alpha} ds + \int_0^t (t-s)^{\alpha-1} g_{m+1}(s) ds \right) \\
&= \frac{1}{c(0)\beta(1-\alpha,\alpha)} \left[\sum_{j=1}^{m} \frac{d}{dt} \int_0^t (t-s)^{\alpha-1} g_j(s) s^{j-\alpha} ds + \frac{d}{dt} \int_0^t (t-s)^{\alpha-1} g_{m+1}(s) ds, \right].
\end{aligned} \tag{2.4.84}$$

For the first term of (2.4.84), we use the change of variables $s = \tau t$ to obtain

$$\sum_{j=1}^{m} \frac{d}{dt} \int_0^t (t-s)^{\alpha-1} g_j(s) s^{j-\alpha}\, ds = \sum_{j=1}^{m} \frac{d}{dt} t^j \int_0^1 (1-\tau)^{\alpha-1} \tau^{j-\alpha} g_j(\tau t)\, d\tau. \tag{2.4.85}$$

Integrating by parts, we rewrite the second term of (2.4.84) as:

$$\begin{aligned}
&\frac{d}{dt}\int_0^t (t-s)^{\alpha-1} g_{m+1}(s)\, ds \\
&= \frac{d}{dt}[-\frac{(t-s)^\alpha}{\alpha} g_{m+1}(s)|_0^t + \int_0^t \frac{(t-s)^\alpha}{\alpha} \dot{g}_{m+1}(s)\, ds] \\
&= \frac{d}{dt}\int_0^t \frac{(t-s)^\alpha}{\alpha} \dot{g}_{m+1}(s)\, ds \\
&= \int_0^t (t-s)^{\alpha-1} \dot{g}_{m+1}(s)\, ds.
\end{aligned}$$

Letting $\tau = s$ in (2.4.85), noting the above identity and recalling (2.4.84) we have the following representation for f

$$\begin{aligned}
f(t) &= \frac{1}{c(0)\beta(\alpha, 1-\alpha)}[\sum_{j=1}^{m} \frac{d}{dt} t^j \int_0^1 (1-s)^{\alpha-1} s^{j-\alpha} g_j(st)\, ds \\
&+ \int_0^t (t-s)^{\alpha-1} \dot{g}_{m+1}(s)\, ds] \\
&= \frac{1}{c(0)\beta(\alpha, 1-\alpha)}[\sum_{j=1}^{m}[j t^{j-1} \int_0^1 (1-s)^{\alpha-1} s^{j-\alpha} g_j(st)\, ds \\
&+ t^j \int_0^1 (1-s)^{\alpha-1} s^{j-\alpha+1} \dot{g}_j(st)\, ds] \\
&+ \int_0^t (t-s)^{\alpha-1} \dot{g}_{m+1}(s)\, ds].
\end{aligned} \tag{2.4.86}$$

For the last term of (2.4.86) we change variables $s = \tau t$ and then set $s = \tau$ to get

$$\int_0^t (t-s)^{\alpha-1} \dot{g}_{m+1}(s)\, ds = t^\alpha \int_0^1 (1-s)^{\alpha-1} \dot{g}_{m+1}(st)\, ds.$$

Substituting this equation into (2.4.86), we can write

$$f(t) = \sum_{j=1}^{m} t^{j-1} q_j(t) + t^{\alpha} q(t)$$

where

$$\begin{aligned} q_j(t) &= \frac{1}{c(0)\beta(\alpha, 1-\alpha)} j \int_0^1 (1-s)^{\alpha-1} s^{j-\alpha} g_j(st)\, ds + t \int_0^1 (1-s)^{\alpha-1} s^{j-\alpha+1} \dot{g}_j(st)\, ds, \quad j = 1, \cdots, m \\ q(t) &= \frac{1}{c(0)\beta(\alpha, 1-\alpha)} \int_0^1 (1-s)^{\alpha-1} \dot{g}_{m+1}(st)\, ds, \end{aligned}$$

$q_j, q \in C^m$, $j = 1, \ldots, m$, and g_{m+1} are given by (2.4.77) and (2.4.81) respectively. The proof is complete.

We now apply the general theory on Volterra equations to equation (2.3.50). Consider the equation

$$x(t) - \int_0^t [(t-s)^{\alpha-1} K(s-t) + M(s-t)] x(s)\, ds = f(t), \quad t \in [0, 1].$$

Remark 2.4.16 Note that the functions K and M defined by (2.3.51) and (2.3.52) are continuous on $S = [0, 1] \times [0, 1]$. Thus, there will be constants M_1 and M_2 such that $\|K\| \leq M_1$ and $\|M\| \leq M_2$.

Let $G(t, s, \alpha) = (t-s)^{\alpha-1}[K(s-t) + (t-s)^{1-\alpha} M(s-t)]$. That is, we write $G(t, s, \alpha) = (t-s)^{\alpha-1} KM(t, s, \alpha)$ where $KM(t, s, \alpha) = K(s-t) + (t-s)^{1-\alpha} M(s-t)$ is such that $\|KM\| \leq C = M_1 + T^{1-\alpha} M_2$ for $(t, s) \in S$ and $T \geq 1$. Setting $K_1(t, s, \alpha) = G(t, s, \alpha)$ and following the classical theory for iterative kernels (see [8]) we write the iterative kernels associated with the given kernel G as

$$K_n(t, s, \alpha) = \int_s^t K_1(t, \tau, \alpha) K_{n-1}(\tau, s, \alpha)\, d\tau, \quad (t, s) \in S.$$

Again, following the classical theory on Volterra equations we have the following result,

Lemma 2.4.17 *The iterative kernels K_n can be rewritten as*

$$K_n(t,s,\alpha) = (t-s)^{\alpha-1}\Phi_n(t,s,\alpha), \tag{2.4.87}$$

where

$$\Phi_n(t,s,\alpha) = (t-s)^{(n-1)\alpha}\varphi_n(t,s,\alpha), \tag{2.4.88}$$

and

$$\varphi_n(t,s,\alpha) = \int_0^1 \frac{KM(t,s+(t-s)u,\alpha)\varphi_{n-1}(s+(t-s)u,s,\alpha)}{u^{1-(n-1)\alpha}(1-u)^{1-\alpha}}\,du, \tag{2.4.89}$$

where $\varphi_1(t,s,\alpha) = KM(t,s,\alpha)$.

Note that K_n given by (2.4.87) is continuous on $S = [0,T]\times[0,T]$ whenever $n \geq 1/\alpha$.

We now satisfy the conditions of Theorem 1.3.2 in [8]. For the sake of completeness we will state the theorem for our particular case.

Theorem 2.4.18 *Suppose that the functions f and KM are continuous on $I = [0,T]$ and $S = [0,T]\times[0,T]$ respectively, and let $\alpha \in (0,1)$. Then the integral equation*

$$x(t) - \int_0^t (t-s)^{\alpha-1} KM(t,s,\alpha)\,ds = f(t)$$

has a unique solution $y \in C(I)$ given by

$$x(t) = f(t) + \int_0^t R(t,s,\alpha) f(s)\,ds, \quad t \in I,$$

where the resolvent kernel $R(t,s,\alpha)$ has the form

$$R(t,s,\alpha) = (t-s)^{-\alpha} Q(t,s,\alpha), \quad (t,s) \in S,$$

with $Q \in C(S)$ for each $\alpha \in (0,1)$ and Q given by

$$Q(t,s,\alpha) = \lim_{n\to\infty} \sum_{m=1}^{n} \Phi_m(t,s,\alpha)$$

where Φ_m is given by (2.4.88).

In the next Lemma we rewrite the function $\varphi_n, n \geq 2$ given by (2.4.89) as the product of a smooth function and a non-smooth term.

Lemma 2.4.19 *The function $\varphi_n, n \geq 2$ given by equation* (2.4.89) *can be represented as*

$$\begin{aligned}\varphi_n(t,s,\alpha) \quad &= \quad \sum_{k_n=1}^{2} (t-s)^{(k_n-1)(1-\alpha)} \sum_{k_{n-1}=1}^{2} (t-s)^{(k_{n-1}-1)(1-\alpha)} \\ &\quad \cdots \sum_{k_2=1}^{2} (t-s)^{(k_2-1)(1-\alpha)} I^n I^{n-1} \ldots I^3 I_{k_2}^{2,0} \varphi_1(t,s,\alpha). \qquad (2.4.90)\end{aligned}$$

where the operators I^j, $j = 3, \ldots, n$ are given by

$$I^n \varphi \quad = \quad I_{k_n}^{n, \sum_{i=n-1}^{2} (k_i - 1)(1-\alpha)} \varphi \qquad (2.4.91)$$

and the operators $I_k^{n,m}$, $k = 1, 2$, $n \geq 2$, $m \geq 0$ are given by

$$I_1^{n,m} \varphi(t,s,\alpha) \quad = \quad \int_0^1 \frac{K((t-s)(1-u))u^m}{u^{1-(n-1)\alpha}(1-u)^{1-\alpha}} \varphi(s+(t-s)u, s, \alpha)\, du, \qquad (2.4.92)$$

$$I_2^{n,m} \varphi(t,s,\alpha) \quad = \quad \int_0^1 \frac{M((t-s)(1-u))u^m}{u^{1-(n-1)\alpha}(1-u)^{1-\alpha}} \varphi(s+(t-s)u, s, \alpha)\, du, \qquad (2.4.93)$$

with the functions K and M given by (2.3.51) *and* (2.3.52) *respectively.*

Proof: Let us first give motivation for identity (2.4.90). We will formaly derive equation (2.4.90) and define several operators for convinience to the presentation.

Using that $KM(t,s,\alpha) = K(s-t) + (t-s)^{1-\alpha}M(s-t)$ together with equation (2.4.89) we can rewrite φ_n as

$$\begin{aligned}
\varphi_n(t,s,\alpha) &= \int_0^1 \frac{K((t-s)(1-u))\varphi_{n-1}(s+(t-s)u,s,\alpha)}{u^{1-(n-1)\alpha}(1-u)^{1-\alpha}}\,du \\
&+ (t-s)^{1-\alpha}\int_0^1 \frac{M((t-s)(1-u))\varphi_{n-1}(s+(t-s)u,s,\alpha)}{u^{1-(n-1)\alpha}(1-u)^{1-\alpha}}\,du \\
&= I_1^{n,0}\varphi_{n-1}(t,s,\alpha) + (t-s)^{1-\alpha}I_2^{n,0}\varphi_{n-1}(t,s,\alpha) \\
&= \sum_{k=1}^{2}(t-s)^{(k-1)(1-\alpha)}I_k^{n,0}\varphi_{n-1}(t,s,\alpha). \qquad (2.4.94)
\end{aligned}$$

This equation starts a recursive prosess that will lead us to equation (2.4.90). First notice that for any function f we can write

$$\begin{aligned}
I_1^{n,m}[f(t,s,\alpha)(t-s)^j] &= \int_0^1 \frac{K((t-s)(1-u))u^m}{u^{1-(n-1)\alpha}(1-u)^{1-\alpha}}f(s+(t-s)u,s,\alpha)((t-s)u)^j\,du \\
&= (t-s)^j I_1^{n,m+j}f(t,s,\alpha) \qquad (2.4.95) \\
I_2^{n,m}[f(t,s,\alpha)(t-s)^j] &= \int_0^1 \frac{M((t-s)(1-u))u^m}{u^{1-(n-1)\alpha}(1-u)^{1-\alpha}}f(s+(t-s)u,s,\alpha)((t-s)u)^j\,du \\
&= (t-s)^j I_2^{n,m+j}f(t,s,\alpha) \qquad (2.4.96)
\end{aligned}$$

for $k = 1, 2$, $n \geq 2$, $m \geq 0$, $j \geq 1$.

Using equations (2.4.95), (2.4.96) and (2.4.94) for φ_{n-1}, we rewrite φ_n as

$$\begin{aligned}
\varphi_n(t,s,\alpha) &= \sum_{k_n=1}^{2}(t-s)^{(k_n-1)(1-\alpha)}I_{k_n}^{n,0}[\varphi_{n-1}(t,s,\alpha)] \\
&= \sum_{k_n=1}^{2}(t-s)^{(k_n-1)(1-\alpha)}I_{k_n}^{n,0}[\sum_{k_{n-1}=1}^{2}(t-s)^{(k_{n-1}-1)(1-\alpha)}I_{k_{n-1}}^{n-1,0}\varphi_{n-2}(t,s,\alpha)]
\end{aligned}$$

$$= \sum_{k_n=1}^{2}(t-s)^{(k_n-1)(1-\alpha)}\sum_{k_{n-1}=1}^{2} I_{k_n}^{n,0}[(t-s)^{(k_{n-1}-1)(1-\alpha)} I_{k_{n-1}}^{n-1,0}\varphi_{n-2}(t,s,\alpha)]. \tag{2.4.97}$$

Applying equations (2.4.95)and (2.4.96) at $I_{k_n}^{n,0}[(t-s)^{(k_{n-1}-1)(1-\alpha)} I_{k_{n-1}}^{n-1,0}\varphi_{n-2}(t,s,\alpha)]$ in the above equation we have

$$\begin{aligned} I_{k_n}^{n,0} & [\quad (t-s)^{(k_{n-1}-1)(1-\alpha)} I_{k_{n-1}}^{n-1,0}\varphi_{n-2}(t,s,\alpha)] \\ & = \quad (t-s)^{(k_{n-1}-1)(1-\alpha)} I_{k_n}^{n,(k_{n-1}-1)(1-\alpha)}[I_{k_{n-1}}^{n-1,0}\varphi_{n-2}(t,s,\alpha)]. \end{aligned} \tag{2.4.98}$$

This together with equation (2.4.97) allow us to rewrite φ_n as

$$\begin{aligned} \varphi_n(t,s,\alpha) &= \sum_{k_n=1}^{2}(t-s)^{(k_n-1)(1-\alpha)}\sum_{k_{n-1}=1}^{2} \\ & \quad (t-s)^{(k_{n-1}-1)(1-\alpha)} I_{k_n}^{n,(k_{n-1}-1)(1-\alpha)} I_{k_{n-1}}^{n-1,0}\varphi_{n-2}(t,s,\alpha). \end{aligned} \tag{2.4.99}$$

Using formula (2.4.94) for φ_{n-2}, the term $I_{k_{n-1}}^{n-1,0}\varphi_{n-2}(t,s,\alpha)$ becomes

$$\begin{aligned} I_{k_{n-1}}^{n-1,0}\varphi_{n-2}(t,s,\alpha) &= I_{k_{n-1}}^{n-1,0}[\sum_{k_{n-2}=1}^{2}(t-s)^{(k_{n-2}-1)(1-\alpha)} I_{k_{n-2}}^{n-2,0}\varphi_{n-3}(t,s,\alpha)] \\ &= \sum_{k_{n-2}=1}^{2} I_{k_{n-1}}^{n-1,0}[(t-s)^{(k_{n-2}-1)(1-\alpha)} I_{k_{n-2}}^{n-2,0}\varphi_{n-3}(t,s,\alpha)] \\ &= \sum_{k_{n-2}=1}^{2}(t-s)^{(k_{n-2}-1)(1-\alpha)} I_{k_{n-1}}^{n-1,(k_{n-2}-1)(1-\alpha)}[I_{k_{n-2}}^{n-2,0}\varphi_{n-3}(t,s,\alpha)]. \end{aligned} \tag{2.4.100}$$

Equation (2.4.99) now can be represented as

$$\varphi_n(t,s,\alpha) = \sum_{k_n=1}^{2}(t-s)^{(k_n-1)(1-\alpha)}$$

$$\sum_{k_{n-1}=1}^{2}(t-s)^{(k_{n-1}-1)(1-\alpha)}I_{k_n}^{n,(k_{n-1}-1)(1-\alpha)}$$
$$\sum_{k_{n-2}=1}^{2}(t-s)^{(k_{n-2}-1)(1-\alpha)}I_{k_{n-1}}^{n-1,(k_{n-2}-1)(1-\alpha)}[I_{k_{n-2}}^{n-2,0}\varphi_{n-3}(t,s,\alpha)], \tag{2.4.101}$$

or by using equations (2.4.95) and (2.4.96) we have

$$\begin{aligned}\varphi_n(t,s,\alpha) &= \sum_{k_n=1}^{2}(t-s)^{(k_n-1)(1-\alpha)}\sum_{k_{n-1}=1}^{2}(t-s)^{(k_{n-1}-1)(1-\alpha)} \\ &\quad \sum_{k_{n-2}=1}^{2}I_{k_n}^{n,(k_{n-1}-1)(1-\alpha)}[(t-s)^{(k_{n-2}-1)(1-\alpha)}I_{k_{n-1}}^{n-1,(k_{n-2}-1)(1-\alpha)}I_{k_{n-2}}^{n-2,0}\varphi_{n-3}(t,s,\alpha)]. \\ &= \sum_{k_n=1}^{2}(t-s)^{(k_n-1)(1-\alpha)}\sum_{k_{n-1}=1}^{2}(t-s)^{(k_{n-1}-1)(1-\alpha)} \\ &\quad \sum_{k_{n-2}=1}^{2}(t-s)^{(k_{n-2}-1)(1-\alpha)}I_{k_n}^{n,(k_{n-2}-1)(1-\alpha)+(k_{n-1}-1)(1-\alpha)} \\ &\quad I_{k_{n-1}}^{n-1,(k_{n-2}-1)(1-\alpha)}[I_{k_{n-2}}^{n-2,0}\varphi_{n-3}(t,s,\alpha)].\end{aligned} \tag{2.4.102}$$

Repeating this process $n-1$ times we get

$$\begin{aligned}\varphi_n(t,s,\alpha) &= \sum_{k_n=1}^{2}(t-s)^{(k_n-1)(1-\alpha)}\sum_{k_{n-1}=1}^{2}(t-s)^{(k_{n-1}-1)(1-\alpha)} \\ &\quad \cdots\sum_{k_2=1}^{2}(t-s)^{(k_2-1)(1-\alpha)}I^nI^{n-1}\dots I^3I_{k_2}^{2,0}\varphi_1(t,s,\alpha),\end{aligned} \tag{2.4.103}$$

where the operators I^j, $j=3,\dots,n$ are given by

$$I^n\varphi = I_{k_n}^{n,\sum_{i=n-1}^{2}(k_i-1)(1-\alpha)}\varphi. \tag{2.4.104}$$

We prove this result by induction on n.

Using equation (2.4.94) and equation (2.4.103) for φ_{n-1} we have that

$$\begin{aligned}
\varphi_n(t,s,\alpha) &= \sum_{k_n=1}^{2}(t-s)^{(k_n-1)(1-\alpha)} I_k^{n,0}\varphi_{n-1}(t,s,\alpha) \\
&= \sum_{k_n=1}^{2}(t-s)^{(k_n-1)(1-\alpha)} I_k^{n,0} \sum_{k_{n-1}=1}^{2}(t-s)^{(k_{n-1}-1)(1-\alpha)} \\
&\quad \sum_{k_{n-2}=1}^{2}(t-s)^{(k_{n-2}-1)(1-\alpha)} \\
&\quad \cdots \sum_{k_2=1}^{2}(t-s)^{(k_2-1)(1-\alpha)} I^{n-1}\dots I^3 I_{k_2}^{2,0}\varphi_1(t,s,\alpha) \\
&= \sum_{k_n=1}^{2}(t-s)^{(k_n-1)(1-\alpha)} \sum_{k_{n-1}=1}^{2}(t-s)^{(k_{n-1}-1)(1-\alpha)} I_k^{n,(k_{n-1}-1)(1-\alpha)} \\
&\quad \sum_{k_{n-2}=1}^{2}(t-s)^{(k_{n-2}-1)(1-\alpha)} \\
&\quad \cdots \sum_{k_2=1}^{2}(t-s)^{(k_2-1)(1-\alpha)} I^{n-1}\dots I^3 I_{k_2}^{2,0}\varphi_1(t,s,\alpha) \\
&= \sum_{k_n=1}^{2}(t-s)^{(k_n-1)(1-\alpha)} \sum_{k_{n-1}=1}^{2}(t-s)^{(k_{n-1}-1)(1-\alpha)} \sum_{k_{n-2}=1}^{2}(t-s)^{(k_{n-2}-1)(1-\alpha)} \\
&\quad I_k^{n,(k_{n-1}-1)(1-\alpha)+(k_{n-2}-1)(1-\alpha)} \dots \sum_{k_2=1}^{2}(t-s)^{(k_2-1)(1-\alpha)} I^{n-1}\dots I^3 \\
&= \sum_{k_n=1}^{2}(t-s)^{(k_n-1)(1-\alpha)} \sum_{k_{n-1}=1}^{2}(t-s)^{(k_{n-1}-1)(1-\alpha)} \\
&\quad \sum_{k_{n-2}=1}^{2}(t-s)^{(k_{n-2}-1)(1-\alpha)} \\
&\quad \dots \sum_{k_2=1}^{2}(t-s)^{(k_2-1)(1-\alpha)} I_k^{n,\sum_{i=n-1}^{2}(k_i-1)(1-\alpha)} I^{n-1}\dots I^3 \\
&= \sum_{k_n=1}^{2}(t-s)^{(k_n-1)(1-\alpha)} \sum_{k_{n-1}=1}^{2}(t-s)^{(k_{n-1}-1)(1-\alpha)}
\end{aligned}$$

$$\sum_{k_{n-2}=1}^{2}(t-s)^{(k_{n-2}-1)(1-\alpha)}$$
$$\dots\sum_{k_2=1}^{2}(t-s)^{(k_2-1)(1-\alpha)}I^nI^{n-1}\dots I^3. \quad (2.4.105)$$

This proves the lemma.

Lemma 2.4.20 *The function φ_n can be rewritten as*

$$\varphi_n(t,s,\alpha)=F_1(t,s,\alpha)+(t-s)^{1-\alpha}F_2(t,s,\alpha). \quad (2.4.106)$$

where

$$\begin{aligned} F_1(t,s,\alpha) &= \sum_{k_n=1}^{2}(t-s)^{(k_n-1)(1-\alpha)}\sum_{k_{n-1}=1}^{2}(t-s)^{(k_{n-1}-1)(1-\alpha)}\sum_{k_{n-2}=1}^{2} \\ &\quad (t-s)^{(k_{n-2}-1)(1-\alpha)}\dots\sum_{k_2=1}^{2}(t-s)^{(k_2-1)(1-\alpha)}I^nI^{n-1}\dots I^3\bar{I}_{k_2}^{1-\alpha}K(s-t), \\ F_2(t,s,\alpha) &= \sum_{k_n=1}^{2}(t-s)^{k_n(1-\alpha)}\sum_{k_{n-1}=1}^{2}(t-s)^{k_{n-1}(1-\alpha)}\sum_{k_{n-2}=1}^{2} \\ &\quad (t-s)^{k_{n-2}(1-\alpha)}\dots\sum_{k_2=1}^{2}(t-s)^{k_2(1-\alpha)}I^nI^{n-1}\dots I^3\bar{I}_{k_2}^{0}M(s-t) \end{aligned} \quad (2.4.107)$$

and the operators $\bar{I}_k^m, k=1,2, m\in R$ are defined by

$$\bar{I}_1^m f(t) = \int_0^1 \frac{K((t-s)(1-u))f(tu)}{u^m(1-u)^{1-\alpha}}\,du \quad (2.4.108)$$
$$\bar{I}_2^m f(t) = \int_0^1 \frac{M((t-s)(1-u))f(tu)}{u^m(1-u)^{1-\alpha}}\,du. \quad (2.4.109)$$

Proof: Using that $\varphi_1(s+(t-s)u,s,\alpha) = K((s-t)u) + ((t-s)u)^{1-\alpha}M((s-t)u)$, the operators $I_k^{2,0}, k=1,2$ defined in Lemma 2.4.20 and equations (2.4.108), (2.4.109) we have

$$\begin{aligned}
I_1^{2,0}\varphi_1(t,s,\alpha) &= \int_0^1 \frac{K((t-s)(1-u))K((s-t)u)}{u^{1-\alpha}(1-u)^{1-\alpha}}\,du \\
&+ (t-s)^{1-\alpha}\int_0^1 \frac{K((t-s)(1-u))M((s-t)u)}{(1-u)^{1-\alpha}}\,du \\
&= \bar{I}_1^{1-\alpha}K(s-t) + (t-s)^{1-\alpha}\bar{I}_1^0 M(s-t) \qquad (2.4.110)\\
I_2^{2,0}\varphi_1(t,s,\alpha) &= \int_0^1 \frac{M((t-s)(1-u))K((s-t)u)}{u^{1-\alpha}(1-u)^{1-\alpha}}\,du \\
&+ (t-s)^{1-\alpha}\int_0^1 \frac{M((t-s)(1-u))M((s-t)u)}{(1-u)^{1-\alpha}}\,du \\
&= \bar{I}_2^{1-\alpha}K(s-t) + (t-s)^{1-\alpha}\bar{I}_2^0 M(s-t). \qquad (2.4.111)
\end{aligned}$$

Thus

$$I_{k_2}^{2,0}\varphi_1(t,s,\alpha) = \bar{I}_{k_2}^{1-\alpha}K(s-t) + (t-s)^{1-\alpha}\bar{I}_{k_2}^0 M(s-t) \qquad k=1,2. \qquad (2.4.112)$$

Using this expression in equation (2.4.90), we get

$$\begin{aligned}
\varphi_n(t,s,\alpha) &= \sum_{k_n=1}^{2}(t-s)^{(k_n-1)(1-\alpha)}\sum_{k_{n-1}=1}^{2}(t-s)^{(k_{n-1}-1)(1-\alpha)} \\
&\dots\sum_{k_2=1}^{2}(t-s)^{(k_2-1)(1-\alpha)}I^n I^{n-1}\dots I^3[\bar{I}_{k_2}^{1-\alpha}K(s-t) + (t-s)^{1-\alpha}\bar{I}_{k_2}^0 M(s-t)],
\end{aligned} \qquad (2.4.113)$$

or equivalently

$$\varphi_n(t,s,\alpha) = \sum_{k_n=1}^{2}(t-s)^{(k_n-1)(1-\alpha)}\sum_{k_{n-1}=1}^{2}(t-s)^{(k_{n-1}-1)(1-\alpha)}$$

$$
\begin{aligned}
& \dots \sum_{k_2=1}^{2} (t-s)^{(k_2-1)(1-\alpha)} I^n I^{n-1} \dots I^3 \bar{I}_{k_2}^{1-\alpha} K(s-t) \\
+ & \sum_{k_n=1}^{2} (t-s)^{(k_n-1)(1-\alpha)} \sum_{k_{n-1}=1}^{2} (t-s)^{(k_{n-1}-1)(1-\alpha)} \\
& \dots \sum_{k_2=1}^{2} (t-s)^{(k_2-1)(1-\alpha)} I^n I^{n-1} \dots I^3 (t-s)^{1-\alpha} \bar{I}_{k_2}^{0} M(s-t) \\
= & F_1(t,s,\alpha) + (t-s)^{1-\alpha} F_2(t,s,\alpha),
\end{aligned}
\tag{2.4.114}
$$

where

$$
\begin{aligned}
F_1(t,s,\alpha) = & \sum_{k_n=1}^{2} (t-s)^{(k_n-1)(1-\alpha)} \sum_{k_{n-1}=1}^{2} (t-s)^{(k_{n-1}-1)(1-\alpha)} \\
& \dots \sum_{k_2=1}^{2} (t-s)^{(k_2-1)(1-\alpha)} I^n I^{n-1} \dots I^3 \bar{I}_{k_2}^{1-\alpha} K(s-t),
\end{aligned}
\tag{2.4.115}
$$

and

$$
\begin{aligned}
F_2(t,s,\alpha) = & \sum_{k_n=1}^{2} (t-s)^{k_n(1-\alpha)} \sum_{k_{n-1}=1}^{2} (t-s)^{k_{n-1}(1-\alpha)} \\
& \dots \sum_{k_2=1}^{2} (t-s)^{k_2(1-\alpha)} I^n I^{n-1} \dots I^3 \bar{I}_{k_2}^{0} M(s-t).
\end{aligned}
\tag{2.4.116}
$$

Now we have all the elements to present the main result in this chapter. We will use the above result together with Lemma 2.4.15 to get the desired representation of the solution of IVP (2.3.46)-(2.3.47).

Theorem 2.4.21 *Let x be the solution of IVP* (2.3.46)-(2.3.47). *Then x can be represented as*

$$
\begin{aligned}
x(t) &= \sum_{j=1}^{m} t^{j-1} q_j(t) + t^{\alpha} q(t) + \sum_{n=1}^{\infty}\sum_{j=1}^{m} t^{n\alpha-1+j} p_{j,n}(t) \\
&+ \sum_{n=1}^{\infty} t^{(n+1)\alpha} p_n(t). \qquad (2.4.117)
\end{aligned}
$$

where $p_{j,n}(t), j = 1, \ldots, m, n \geq 0$ *are given by*

$$
\begin{aligned}
p_{j,n}(t) &= \int_0^1 (1-\tau)^{n\alpha-1} \tau^{j-1} \varphi_n(t, \tau t, \alpha) q_j(\tau t)\, d\tau, \qquad j = 1, \cdots, m, \quad n \geq 1, \\
p_n(t) &= \int_0^1 (1-\tau)^{n\alpha-1} \varphi_n(t, \tau t, \alpha) \tau^{\alpha} q(t\tau)\, d\tau, \qquad n \geq 1,
\end{aligned}
\qquad (2.4.118)
$$

and q*,* $q_j, j = 1, \cdots, m$ *are given by equations* (2.4.80).

Proof: Using Theorem 2.4.18 we write

$$
\begin{aligned}
x(t) &= f(t) + \int_0^t R(t, s, \alpha) f(s)\, ds \\
&= f(t) + \int_0^t (t-s)^{\alpha-1} Q(t, s, \alpha) f(s)\, ds \\
&= f(t) + \int_0^t (t-s)^{\alpha-1} \sum_{n=1}^{\infty} (t-s)^{(n-1)\alpha} \varphi_n(t, s, \alpha) f(s)\, ds.
\end{aligned}
$$

Using the representation of f given by Lemma 2.4.15 we can rewrite the integral term of the above equation as

$$
\begin{aligned}
& \int_0^t (t-s)^{\alpha-1} \sum_{n=1}^{\infty} (t-s)^{(n-1)\alpha} \varphi_n(t, s, \alpha) [\sum_{j=1}^{m} s^{j-1} q_j(s) + s^{\alpha} q(s)]\, ds \\
= & \int_0^t (t-s)^{\alpha-1} \sum_{n=1}^{\infty} (t-s)^{(n-1)\alpha} \varphi_n(t, s, \alpha) \sum_{j=1}^{m} s^{j-1} q_j(s)\, ds
\end{aligned}
$$

$$\begin{aligned}
&+ \int_0^t (t-s)^{\alpha-1} \sum_{n=1}^{\infty} (t-s)^{(n-1)\alpha} \varphi_n(t,s,\alpha) s^{\alpha} q(s)\, ds \\
&= \sum_{n=1}^{\infty} \sum_{j=1}^{m} \int_0^t (t-s)^{\alpha-1+(n-1)\alpha} s^{j-1} \varphi_n(t,s,\alpha) q_j(s)\, ds \\
&+ \sum_{n=1}^{\infty} \int_0^t (t-s)^{\alpha-1+(n-1)\alpha} \varphi_n(t,s,\alpha) s^{\alpha} q(s)\, ds \\
&= \sum_{n=1}^{\infty} \sum_{j=1}^{m} \int_0^t (t-s)^{n\alpha-1} s^{j-1} \varphi_n(t,s,\alpha) q_j(s)\, ds,
\end{aligned}$$

where the functions $q_j, j = 1, \ldots, m, g_{m+1}$ and q are given by Lemma 2.4.15.

Using the change of variables $s = \tau t$ we rewrite the above equation as

$$\begin{aligned}
\sum_{n=1}^{\infty} \sum_{j=1}^{m} t^{n\alpha-1+j} \int_0^1 (1 &- \tau)^{n\alpha-1} \tau^{j-1} \varphi_n(t,\tau t,\alpha) q_j(\tau t)\, d\tau \\
&+ \sum_{n=1}^{\infty} t^{n\alpha} \int_0^1 (1-\tau)^{n\alpha-1} \varphi_n(t,\tau t,\alpha) (t\tau)^{\alpha-1} t\tau q(\tau t)\, d\tau \\
&= \sum_{n=1}^{\infty} \sum_{j=1}^{m} t^{n\alpha-1+j} \int_0^1 (1-\tau)^{n\alpha-1} \tau^{j-1} \varphi_n(t,\tau t,\alpha) q_j(\tau t)\, d\tau \\
&+ \sum_{n=1}^{\infty} t^{(n+1)\alpha} \int_0^1 (1-\tau)^{n\alpha-1} \varphi_n(t,\tau t,\alpha) \tau^{\alpha} q(\tau t)]\, d\tau \\
&= \sum_{n=1}^{\infty} \sum_{j=1}^{m} t^{n\alpha-1+j} p_{j,n}(t) + \sum_{n=1}^{\infty} t^{(n+1)\alpha} p_n(t),
\end{aligned}$$

where $p_{j,n}$ are given by

$$\begin{aligned}
p_{j,n}(t) &= \int_0^1 (1-\tau)^{n\alpha-1} \tau^{j-1} \varphi_n(t,\tau t,\alpha) q_j(\tau t)\, d\tau, \qquad j = 1, \ldots, m, \quad n \geq 1, \\
p_n(t) &= \int_0^1 (1-\tau)^{n\alpha-1} \varphi_n(t,\tau t,\alpha) \tau^{\alpha} q(t\tau)\, d\tau, \qquad n \geq 1.
\end{aligned}$$

Replacing this in the first equation and using the representation for f given by Lemma 2.4.15 we have

$$\begin{aligned} x(t) &= \sum_{j=1}^{m} t^{j-1} q_j(t) + t^{\alpha} q(t) + \sum_{n=1}^{\infty}\sum_{j=1}^{m} t^{n\alpha-1+j} p_{j,n}(t) \\ &+ \sum_{n=1}^{\infty} t^{(n+1)\alpha} p_n(t). \end{aligned}$$

This completes the proof.

Using Lemma (2.4.20) we can rewrite the representation of the solution x in the following way.

Theorem 2.4.22 *The function x can be represented as*

$$\begin{aligned} x(t) &= \sum_{j=1}^{m} t^{j-1} q_j(t) + t^{\alpha} q(t) + \sum_{n=1}^{\infty}\sum_{j=1}^{m} t^{n\alpha-1+j} \sum_{k_n=1}^{2} t^{(k_n-1)(1-\alpha)} \sum_{k_{n-1}=1}^{2} t^{(k_{n-1}-1)(1-\alpha)} \\ &\quad \cdots \sum_{k_2=1}^{2} t^{(k_2-1)(1-\alpha)} P^1_{n,j,k_i} + t^{1-\alpha} \sum_{k_n=1}^{2} t^{k_n(1-\alpha)} \sum_{k_{n-1}=1}^{2} t^{k_{n-1}(1-\alpha)} \\ &\quad \cdots \sum_{k_2=1}^{2} t^{k_2(1-\alpha)} P^2_{n,j,k_i} \\ &+ \sum_{n=1}^{\infty} t^{(n+1)\alpha} \sum_{k_n=1}^{2} t^{(k_n-1)(1-\alpha)} \sum_{k_{n-1}=1}^{2} t^{(k_{n-1}-1)(1-\alpha)} \\ &\quad \cdots \sum_{k_2=1}^{2} t^{(k_2-1)(1-\alpha)} P^3_{n,j,k_i} \\ &+ t^{1-\alpha} \sum_{k_n=1}^{2} t^{k_n(1-\alpha)} \sum_{k_{n-1}=1}^{2} t^{k_{n-1}(1-\alpha)} \\ &\quad \cdots \sum_{k_2=1}^{2} t^{k_2(1-\alpha)} P^4_{n,j,k_i} \end{aligned} \tag{2.4.119}$$

where $P^1_{n,j,k_i}, P^2_{n,j,k_i}, P^3_{n,j,k_i}, P^4_{n,j,k_i} \in C^m$, $i = 1, \cdots, m$ are given by

$$
\begin{aligned}
P^1_{n,j,k_i} &= \int_0^1 (1-\tau)^{n\alpha-1+\sum_{i=2}^{n}(k_i-1)(1-\alpha)}\tau^{j-1} \\
&\quad I^n I^{n-1}\dots I^3 \bar{I}^{1-\alpha}_{k_2} K(t(\tau-1))q_j(\tau t)\,d\tau, \\
P^2_{n,j,k_i} &= \int_0^1 (1-\tau)^{(n-1)\alpha+\sum i=2^n k_i(1-\alpha)}\tau^{j-1} \\
&\quad I^n I^{n-1}\dots I^3 \bar{I}^{0}_{k_2} M(t(\tau-1))q_j(\tau t)\,d\tau, \\
P^3_{n,j,k_i} &= \int_0^1 (1-\tau)^{n\alpha-1+\sum_{i=2}^{n}(k_i-1)(1-\alpha)}\tau^{\alpha} \\
&\quad I^n I^{n-1}\dots I^3 \bar{I}^{1-\alpha}_{k_2} K(t(\tau-1))q_j(\tau t)\,d\tau, \\
P^4_{n,j,k_i} &= \int_0^1 (1-\tau)^{(n-1)\alpha+\sum i=2^n k_i(1-\alpha)}\tau^{\alpha} \\
&\quad I^n I^{n-1}\dots I^3 \bar{I}^{0}_{k_2} M(t(\tau-1))q_j(\tau t)\,d\tau.
\end{aligned}
$$

Proof: Replacing φ_n appearing in (2.4.118) with the representation given in (2.4.106) we have

$$
\begin{aligned}
p_{j,n}(t) &= \int_0^1 (1-\tau)^{n\alpha-1}\tau^{j-1}[F_1(t,\tau t,\alpha)+(t(1-\tau))^{1-\alpha}F_2(t,\tau t,\alpha)]q_j(\tau t)\,d\tau \\
&= \int_0^1 (1-\tau)^{n\alpha-1}\tau^{j-1}[F_1(t,\tau t,\alpha)q_j(\tau t)\,d\tau \\
&+ t^{1-\alpha}\int_0^1 (1-\tau)^{n\alpha-1}\tau^{j-1}(1-\tau)^{1-\alpha}F_2(t,\tau t,\alpha)]q_j(\tau t)\,d\tau \\
&\qquad j=1,\cdots,m, \quad n\geq 1,
\end{aligned}
\tag{2.4.120}
$$

and

$$
\begin{aligned}
p_n(t) &= \int_0^1 (1-\tau)^{n\alpha-1}[F_1(t,\tau t,\alpha)+(t(1-\tau))^{1-\alpha}F_2(t,\tau t,\alpha)]\tau^{\alpha}q(t\tau)\,d\tau. \\
&= \int_0^1 (1-\tau)^{n\alpha-1}[F_1(t,\tau t,\alpha)\tau^{\alpha}q(t\tau)\,d\tau \\
&+ t^{1-\alpha}\int_0^1 (1-\tau)^{n\alpha-1}(1-\tau)^{1-\alpha}F_2(t,\tau t,\alpha)]\tau^{\alpha}q(t\tau)\,d\tau \qquad n\geq 1.
\end{aligned}
\tag{2.4.121}
$$

For $F_1(t,\tau t,\alpha)$ and $F_2(t,\tau t,\alpha)$ we have

$$F_1(t,\tau t,\alpha) = \sum_{k_n=1}^{2}(t(1-\tau))^{(k_n-1)(1-\alpha)}\sum_{k_{n-1}=1}^{2}(t(1-\tau))^{(k_{n-1}-1)(1-\alpha)} \dots \sum_{k_2=1}^{2}(t(1-\tau))^{(k_2-1)(1-\alpha)} I^n I^{n-1}\dots I^3 \bar{I}_{k_2}^{1-\alpha} K(t(\tau-1)) \tag{2.4.122}$$

$$F_2(t,\tau t,\alpha) = \sum_{k_n=1}^{2}(t(1-\tau))^{k_n(1-\alpha)}\sum_{k_{n-1}=1}^{2}(t(1-\tau))^{k_{n-1}(1-\alpha)} \dots \sum_{k_2=1}^{2}(t(1-\tau))^{k_2(1-\alpha)} I^n I^{n-1}\dots I^3 \bar{I}_{k_2}^{0} M(t(\tau-1)). \tag{2.4.123}$$

Substituting this into equation (2.4.121) we have

$$\begin{aligned} p_{j,n}(t) &= \int_0^1 (1-\tau)^{n\alpha-1}\tau^{j-1}F_1(t,\tau t,\alpha)q_j(\tau t)\,d\tau \\ &+ t^{1-\alpha}\int_0^1 (1-\tau)^{n\alpha-1}\tau^{j-1}(1-\tau)^{1-\alpha}F_2(t,\tau t,\alpha)q_j(\tau t)\,d\tau \\ &= \sum_{k_n=1}^{2} t^{(k_n-1)(1-\alpha)}\sum_{k_{n-1}=1}^{2} t^{(k_{n-1}-1)(1-\alpha)} \\ &\dots \sum_{k_2=1}^{2} t^{(k_2-1)(1-\alpha)}\int_0^1 (1-\tau)^{n\alpha-1+\sum_{i=2}^{n}(k_i-1)(1-\alpha)}\tau^{j-1} \\ &I^n I^{n-1}\dots I^3\bar{I}_{k_2}^{1-\alpha}K(t(\tau-1))q_j(\tau t)\,d\tau \\ &+ t^{1-\alpha}\sum_{k_n=1}^{2} t^{k_n(1-\alpha)}\sum_{k_{n-1}=1}^{2} t^{k_{n-1}(1-\alpha)} \\ &\dots \sum_{k_2=1}^{2} t^{k_2(1-\alpha)}\int_0^1 (1-\tau)^{(n-1)\alpha+\sum i=2^n k_i(1-\alpha)}\tau^{j-1} \\ &I^n I^{n-1}\dots I^3\bar{I}_{k_2}^{0}M(t(\tau-1))q_j(\tau t)\,d\tau, \qquad j=1,\cdots,m, \quad n\ge 1, \end{aligned} \tag{2.4.124}$$

and

$$p_n(t) = \int_0^1 (1-\tau)^{n\alpha-1}[F_1(t,\tau t,\alpha)+(t(1-\tau))^{1-\alpha}F_2(t,\tau t,\alpha)]\tau^{\alpha}q(t\tau)\,d\tau \qquad n\ge 1,$$

$$
\begin{aligned}
&= \int_0^1 (1-\tau)^{n\alpha-1} F_1(t,\tau t,\alpha)\tau^{\alpha} q(t\tau)\,d\tau \\
&+ t^{1-\alpha}\int_0^1 (1-\tau)^{(n-1)\alpha} F_2(t,\tau t,\alpha)\tau^{\alpha} q(t\tau)\,d\tau \\
&= \sum_{k_n=1}^{2} t^{(k_n-1)(1-\alpha)} \sum_{k_{n-1}=1}^{2} t^{(k_{n-1}-1)(1-\alpha)} \\
&\quad \dots \sum_{k_2=1}^{2} t^{(k_2-1)(1-\alpha)} \int_0^1 (1-\tau)^{n\alpha-1+\sum_{i=2}^{n}(k_i-1)(1-\alpha)}\tau^{\alpha} \\
&\quad I^n I^{n-1}\dots I^3 \bar{I}_{k_2}^{1-\alpha} K(t(\tau-1)) q_j(\tau t)\,d\tau \\
&+ t^{1-\alpha}\sum_{k_n=1}^{2} t^{k_n(1-\alpha)} \sum_{k_{n-1}=1}^{2} t^{k_{n-1}(1-\alpha)} \\
&\quad \dots \sum_{k_2=1}^{2} t^{k_2(1-\alpha)} \int_0^1 (1-\tau)^{(n-1)\alpha+\sum i=2^n k_i(1-\alpha)}\tau^{\alpha} \\
&\quad I^n I^{n-1}\dots I^3 \bar{I}_{k_2}^{0} M(t(\tau-1)) q_j(\tau t)\,d\tau \qquad j=1,\cdots,m, \quad n\ge 1.
\end{aligned}
\tag{2.4.125}
$$

Using this in equation (2.4.119), we have

$$
\begin{aligned}
x(t) &= \sum_{j=1}^{m} t^{j-1}q_j(t) + t^{\alpha}q(t) + \sum_{n=1}^{\infty}\sum_{j=1}^{m} t^{n\alpha-1+j} \sum_{k_n=1}^{2} t^{(k_n-1)(1-\alpha)} \sum_{k_{n-1}=1}^{2} t^{(k_{n-1}-1)(1-\alpha)} \\
&\quad \dots \sum_{k_2=1}^{2} t^{(k_2-1)(1-\alpha)} \int_0^1 (1-\tau)^{n\alpha-1+\sum_{i=2}^{n}(k_i-1)(1-\alpha)}\tau^{j-1} \\
&\quad I^n I^{n-1}\dots I^3 \bar{I}_{k_2}^{1-\alpha} K(t(\tau-1)) q_j(\tau t)\,d\tau \\
&+ t^{1-\alpha}\sum_{k_n=1}^{2} t^{k_n(1-\alpha)} \sum_{k_{n-1}=1}^{2} t^{k_{n-1}(1-\alpha)} \\
&\quad \dots \sum_{k_2=1}^{2} t^{k_2(1-\alpha)} \int_0^1 (1-\tau)^{(n-1)\alpha+\sum_{i=2}^{n} k_i(1-\alpha)}\tau^{j-1} \\
&\quad I^n I^{n-1}\dots I^3 \bar{I}_{k_2}^{0} M(t(\tau-1)) q_j(\tau t)\,d\tau \\
&+ \sum_{n=1}^{\infty} t^{(n+1)\alpha} \sum_{k_n=1}^{2} t^{(k_n-1)(1-\alpha)} \sum_{k_{n-1}=1}^{2} t^{(k_{n-1}-1)(1-\alpha)}
\end{aligned}
$$

$$
\begin{aligned}
&\ldots \sum_{k_2=1}^{2} t^{(k_2-1)(1-\alpha)} \int_0^1 (1-\tau)^{n\alpha-1+\sum_{i=2}^{n}(k_i-1)(1-\alpha)} \tau^\alpha I^n I^{n-1} \ldots I^3 \bar{I}_{k_2}^{1-\alpha} K(t(\tau-1)) q_j(\tau t)\, d\tau \\
+ \quad & t^{1-\alpha} \sum_{k_n=1}^{2} t^{k_n(1-\alpha)} \sum_{k_{n-1}=1}^{2} t^{k_{n-1}(1-\alpha)} \\
&\ldots \sum_{k_2=1}^{2} t^{k_2(1-\alpha)} \int_0^1 (1-\tau)^{(n-1)\alpha+\sum_{i=2}^{n} k_i(1-\alpha)} \tau^\alpha I^n I^{n-1} \ldots I^3 \bar{I}_{k_2}^{0} M(t(\tau-1)) q_j(\tau t)\, d\tau.
\end{aligned}
$$

This proves the theorem.

Remark 2.4.23 In the above theorem we have established a representation of the solution x. That is, we write x as a sum, each term being either a polynomial in t multiplied by a C^m function, t^α multiplied by a C^m function or $t^{k\alpha+j}$, with k, j non-negative integers, multiplied by a C^m function. This will allow us to build a numerical scheme with order of convergence m. We study this problem in chapter 3.

Corollary 2.4.24 *If $\varphi^j(0) = 0, j = 0, \cdots, m$. The representation of x given by* (2.4.117) *can be written as*

$$
x(t) = t^\alpha q(t) + \sum_{n=1}^{\infty} t^{(n+1)\alpha} p_n(t), \tag{2.4.126}
$$

where $q(t)$ is given by (2.4.80) *and p_n is given by* (2.4.118).

Proof: If $\varphi^j(0) = 0$, $j \geq 0$ the functions $g_j(t) = \varphi^{j-1}(0)\overline{g}_j(t)$ in equation (2.4.77) become $g_j(t) = 0$, $j \geq 1$. This gives us $q_j(t) = 0$, $j \geq 1$ in (2.4.80) so both the first term and the third term of $x(t)$ in (2.4.117) vanish.

Remark 2.4.25 Note that for the particular case where the functions $c(s) = 1$ and $p(s) = 0$ in (2.3.45), and the right hand side operator $L = 0$ and $g = 0$, IVP (2.3.46)-(2.3.47) reduces to

$$
\begin{aligned}
\frac{d}{dt} D x_t &= 0 \\
x_0(s) &= \varphi(s), \qquad s \in [-1, 0],
\end{aligned} \tag{2.4.127}
$$

where the operator D is given by

$$D\varphi = \int_{-1}^{0} (-s)^{\alpha} \varphi(s)\, ds, \tag{2.4.128}$$

and the associated Volterra integral equation, given by (2.3.50), reduces to $x(t) = f(t)$, where

$$f(t) = \frac{1}{c(0)\beta(1-\alpha,\alpha)} \frac{d}{dt} \int_0^t (t-s)^{\alpha-1} F(s)\, ds, \quad t \in [0,1], \tag{2.4.129}$$

and the continuous function F is defined on $[0,1]$ by

$$F(t) = D\varphi - \int_t^1 \varphi(t-s) s^{-\alpha}\, ds. \tag{2.4.130}$$

That is, for this particular case, Theorem 2.3.13 leads us to a Volterra integral equation of the second kind with kernel function $KM = 0$.

This case was studied in detail by Burns, Herdman and Stech in [13]. In that paper the solution was found via a classical inversion formula for the Abel equation. For completeness we present that result here

Lemma 2.4.26 (see [13], Lemma 4.3) *Let $f \in L^{\infty}(0,1)$. Then x satisfies*

$$\int_0^t x(s)(t-s)^{-\alpha}\, ds = f(t), \quad t \in (0,1),$$

if and only if

$$\int_0^t x(s)\, ds = \frac{sin\alpha\pi}{\pi} \int_0^t f(s)(t-s)^{\alpha-1}\, ds, \quad t \in [0,1]. \tag{2.4.131}$$

Lemma 2.4.27 (see [13], Theorem 4.2) *Let $\varphi \in C$, $\eta \in R$. Then IVP* (2.4.127) *has the unique integrable solution*

$$\begin{aligned} x(t) &= \frac{\sin\alpha\pi}{\pi}\int_{-1}^{0}\frac{1}{t-s}\left|\frac{t}{s}\right|^{\alpha}\varphi(s)\,ds \\ &\quad + \frac{\sin\alpha\pi}{\pi}\int_{0}^{t}\frac{(t-s)^{\alpha-1}}{(t-s)+1}\varphi(s-1)\,ds + [\eta - D\varphi]t^{\alpha-1}, \quad 0 < t \le s. \end{aligned}$$

Remark 2.4.28 We will only consider the case with consistent initial data, i.e. $D\varphi = \eta$, so the last term of the above representation vanishes.

The above solution was found by differentiating (2.4.131) with respect to t where the function f is given by

$$f(t) = \int_{-1}^{0}[s^{-\alpha} - (t-s)^{-\alpha}]\varphi(s)\,ds + \int_{0}^{t}(t-u-1)^{-\alpha}\varphi(u-1)\,du. \qquad (2.4.132)$$

Using first the change of variables $u = s+1$, and then $\tau = t-u$, we can rewrite g as

$$\begin{aligned} f(t) &= \int_{-1}^{0}s^{-\alpha}\varphi(s)\,ds - \int_{-1}^{0}(t-s)^{-\alpha}\varphi(s)\,ds + \int_{0}^{t}(t-u+1)^{-\alpha}\varphi(u-1)\,du \\ &= D\varphi - \int_{-1}^{0}(t-s)^{-\alpha}\varphi(s)\,ds + \int_{-1}^{t-1}(t-s)^{-\alpha}\varphi(s)\,ds \\ &= D\varphi - \int_{t-1}^{0}(t-s)^{-\alpha}\varphi(s)\,ds \\ &= D\varphi - \int_{t}^{1}\tau^{-\alpha}\varphi(t-\tau)\,d\tau. \end{aligned} \qquad (2.4.133)$$

Thus $f(t) = F(t)$ in (2.4.130). So as expected, the representation given by the above lemma is a particular case of the representation given by Theorem 2.4.21.

2.5 Conclusions

In this chapter we proved that IVP (2.3.46)-(2.3.47) can be view as a Volterra integral equation of the second kind. For the particular case studied in [13], that is for $c(s) = 1$ and $p(s) = 0$ in (2.3.45) and the right hand side operator $L = 0$ and $g = 0$, the second kind Volterra equation is of the form $x(t) = f(t)$ where f is the forcing function, that is, we have a second kind equation with a zero kernel. This case needs to be studied separately from the general case. Numerically, we will treat this problem as a Volterra integral equation of the first kind. For the general case however, we will always study a Volterra equation of the second kind. That is, we can make use of all the classical theory of second kind Volterra equations. In particular, we have the following result

Theorem 2.5.29 (See [8], page 30) *Let* $f_i \in C^m[0,T], i = 1,2, m \geq 1$ *and let* $K \in C^m[0,T] \times [0,T]$. *Then the unique solution of*

$$x(t) = f(t) + \int_0^t (t-s)^{-\alpha} K(t,s) x(s)\, ds, \qquad t \in [0,T], \tag{2.5.134}$$

where

$$f(t) = f_1(t) + t^\beta f_2(t), \quad \beta > 0, \quad \beta \notin N, \tag{2.5.135}$$

is in $C[0,T] \cup C^m(0,T]$ *and has the form*

$$x(t) = f(t) = f_1(t) + t^\beta f_2(t) + \sum_{n=1}^{\infty} \{\Psi_{n,1}(t,\alpha) + t^\beta \Psi_{n,2}(t,\alpha,\beta)\} t^n(1-\alpha), \tag{2.5.136}$$

where

$$\begin{aligned} \Psi_{n,1}(t,\alpha) &= \int_0^1 (1-s)^{n(1-\alpha)-1} \Phi_n(t,st,\alpha) f_1(st)\, ds \\ \Psi_{n,2}(t,\alpha,\beta) &= \int_0^1 (1-s)^{n(1-\alpha)-1} s^\beta \Phi_n(t,st,\alpha) f_2(st)\, ds, \end{aligned} \tag{2.5.137}$$

satisfy $\Psi_{n,i} \in C^m(0,T]$, $i = 1,2, n \leq 1$, *with* Φ_n *defined by*

$$\begin{aligned} \Phi_n(t,s,\alpha) &= \int_0^1 \frac{K(t,s+(t-s)u)\Phi_{n-1}(s+(t-s)u,s,\alpha)}{u^{1-(n-1)(1-\alpha)}(1-u)^{\alpha}}\,du \qquad n \geq 2, \\ \Phi_1(t,s,\alpha) &= K(t,s), \end{aligned}$$

where K is the kernel of equation (2.5.134)*.*

In our case, we consider a slightly more general situation since the second kind equation we need to solve has a non-smooth kernel of the form $KM(t,s,\alpha) = K(s-t)+(t-s)^{1-\alpha}M(s-t)$ where $K \in C^m[-1,0]$ and $M \in C^m[-1,0]$ are given by (2.3.51) and (2.3.52), respectively. We also have that the forcing term f is a non-smooth function that can be rewritten as

$$f(t) = \sum_{j=1}^{m} t^{j-1}q_j(t) + t^{\alpha}q(t), \quad 0 \leq t \leq 1, \tag{2.5.138}$$

where the functions $q, q_j \in C^m[0,1]$, $j = 1, \ldots, m$.

Under these more general conditions we found a representation of the solution for the second kind Volterra equation of the form given by Theorem 2.5.29. Moreover, we proved that this solution is also a solution of IVP (2.3.46)-(2.3.47). This representation of the solution of the second kind Volterra equation will allow us to use classical non-polynomial spline collocation techniques (see [8]) to approximate the solution of IVP (2.3.46)-(2.3.47) with high order convergence rates. We will study this problem in the next chapter.

Chapter 3

The Approximation Scheme

In this chapter we present the approximation scheme used for approximation of solutions for the Volterra equations associated with IVP (2.3.46)-(2.3.47). For simplicity in the presentation of the numerical scheme we first consider IVP (2.3.46)-(2.3.47) where the kernel function of the difference operator is $k(s,\alpha) = (-s)^{-\alpha}$ defined for $-1 < s < 0$ and $0 < \alpha < 1$. We will work with both $L = 0$, in which case we have an associated Volterra equation of the first kind and $L \neq 0$, in which case we have an equation of the second kind. That is, we will be considering the following IVP:

$$\frac{dDx_t}{dt} = Lx_t + g(t), \quad 0 < t < T. \tag{3.0.1}$$

with initial data

$$x_0(s) = \varphi(s), \quad s \in [-1, 0] \tag{3.0.2}$$

where $x_t(s) = x(t+s)$ for $s \in [-1, 0]$, $t \geq 0$, g is a known function and the linear operator D has the following representation for $\varphi \in C[-1, 0]$

$$D\varphi = \int_{-1}^{0} \varphi(s)(-s)^{-\alpha} ds, \quad 0 < \alpha < 1. \tag{3.0.3}$$

The same ideas described here will be applied to IVP's with more general kernels. In particular, we will solve IVP (2.3.46)-(2.3.47) for different values of the functions $c(s)$ and

$p(s)$ in the kernel $k(s, \alpha) = c(s)(-s)^{-\alpha} + p(s)$. To make this work self contained we will first give a brief introduction on projection methods and particularly on collocation techniques. Non-polynomial spline collocation is described in detail since this is the technique we choose to employ.

3.1 Projection Methods and the Collocation Method

In this section we briefly discuss projection methods and their application to Volterra integral equations. Let X be a Banach space and let $X^N \subset X$, $N \geq 1$ be finite dimensional subspaces of X. These subspaces will be the ranges of the projections, from which the method derives its name. Two of the most well known projection methods are the collocation method and the Galerkin method. In the collocation method, the projections play a primary role, while the Galerkin method is completly characterised by the subspaces. These two methods are described extensively in many mathematical textbooks, see [7],[23]. In this work, we will only study the collocation method since this is the method we choose for solving our IVP.

Let the linear mapping $P^N : X \rightarrow X$ be a projection, i.e. $(P^N)^2 = P^N$. Each projection defines a subspace as the image space, i.e. $X^N = range(P^N)$. In this case we say that P^N is a projection onto X^N. Consider the space $X = C[0,1]$, with $\|x\| = \max_{t \in [0,1]} |x(t)|$, and consider the finite dimensional subspaces $X^N = \text{span}\{e_1, \ldots, e_N\}$, with projections $P^N : X \rightarrow X^N$,

$$P^N x(t) = \sum_{j=0}^{N} x(\tau_j) e_j^N(t), \quad x \in C[0,1]. \tag{3.1.4}$$

This choice of the projection defines the collocation method. The values $0 \leq \tau_1 \leq \ldots \leq \tau_N \leq 1$, are called collocation points.

Given the equation $Dx(t) = f(t)$ where $f \in X$ and the operator D is given by

$$Dx(t) = \int_0^t x(s)(-s)^{-\alpha} \, ds. \tag{3.1.5}$$

We apply collocation in the following way:

Define the family of operators $D^N : X \to X^N$ by $D^N = P^N D$ and the family of functions f^N, $N \geq 1$ as $f^N = P^N f$. The sequence of approximating functions

$$x^N(t) = \sum_{j=1}^{N} \gamma_j e_j(t)$$

are required to be solutions of the projected equation

$$P^N D x^N = P^N f \tag{3.1.6}$$

at each collocation point. Replacing x^N in (3.1.5) we have

$$Dx^N = \sum_{j=1}^{N} \gamma_j D e_j(t).$$

Since f in a known function the projection f^N of f is given by

$$P^N f = f^N = \sum_{j=1}^{N} f(t_j) e_j(t),$$

where t_j are the collocation points. Then the projected equation (3.1.6) at each collocation point become

$$Dx^N(t_j) = f(t_j)$$

or equivalently

$$\sum_{i=1}^{N} \gamma_i e_i(t_j) = f(t_j) \qquad j = 1, \cdots, N. \tag{3.1.7}$$

This is an algebraic system in the unknowns $\gamma_1, \gamma_2, \ldots, \gamma_N$. In matrix notation we have:

$$Q^N A^N = F^N$$

where

$$\begin{aligned} Q^N_{ik} &= De^N_i(\tau_k), \\ A^N &= (\gamma_1, \gamma_2, \ldots, \gamma_N)^T \\ F^N &= (f(\tau_1), \ldots, f(\tau_N))^T. \end{aligned} \tag{3.1.8}$$

In each case we want to choose the basis functions such that the collocation equation is given by equation (3.1.7) and solve the above algebraic system for the coefficients γ_j, $j = 1, \cdots, N$.

Remark 3.1.1 It is important to note that we will be solving a projected equation. That is, our "approximation" x^N is not really an approximation of the solution of the original equation but a solution to the "approximated" (projected) problem.

3.2 Non Polynomial Spline Collocation

Some high order convergence numerical techniques for solving Volterra integral equations, especially those of the second kind, are avaiable in the literature, see [17], [29], [7]. In our case we choose nonpolynomial spline collocation. This technique will allow us to use all the information we have on the solution of our problem to choose the projection spaces. That is, we consider collocation as a projection method into a space of non-polynomial splines suggested by the form of the solution of our IVP. We will make use of the particular structure of Volterra integral equations to apply the collocation method in a recursive way. We will use this technique for solving IVP (3.0.1)-(3.0.2) rewritten as a Volterra equation. That is, we will be solving Volterra equations of both the first and second kind with wealky singular kernels when considering IVP (3.0.1)-(3.0.2) with the right hand side operators $L = 0$ and $L \neq 0$, respectively.

3.3 Solving for $L = 0$

As a first step in our work we solved IVP (3.0.1)-(3.0.2) for $L = 0$. This choice of the right hand side operator, which at first appears to be the simpler of the two cases, is actually the more complicated one. Note that the D operator in the left hand side of (3.0.1) is non-atomic. This, together with Theorem 2.3.13, give us that IVP (3.0.1)-(3.0.2) can be view as a Volterra integral equation of the first kind. For this kind of equations numerical approximations are harder and less studied. For this reason we first focus our attention on this particular case. That is, we will consider the following IVP

$$\frac{dDx_t}{dt} = g(t), \quad 0 < t < T, \tag{3.3.9}$$

with initial data

$$x_0(s) = \varphi(s), \quad s \in [-1, 0] \tag{3.3.10}$$

where $x_t(s) = x(t+s)$ for $s \in [-1, 0]$, $t \geq 0$, $g \in C^m$ is a known function and the linear operator D has the following representation for $\varphi \in C[-1, 0]$

$$D\varphi = \int_{-1}^{0} \varphi(s)(-s)^{-\alpha}\, ds. \tag{3.3.11}$$

Integrating (3.3.9) and splitting the integral D we get

$$\int_{-1}^{-t} x(t+s)(-s)^{-\alpha} ds + \int_{-t}^{0} x(t+s)(-s)^{-\alpha} ds = D\varphi + \int_0^t f(s)\, ds$$

or equivalently,

$$\int_{-t}^{0} x(t+s)(-s)^{-\alpha} ds = D\varphi + \int_0^t f(s)\, ds - \int_{-1}^{-t} \varphi(t+s)(-s)^{-\alpha} ds.$$

Define

$$\overline{D}x(t) = \int_0^t x(t-s)s^{-\alpha} ds \tag{3.3.12}$$

and

$$\overline{f}(t) = D\varphi + \int_0^t f(s)\, ds - \int_{-1}^{-t} \varphi(t+s)(-s)^{-\alpha} ds. \tag{3.3.13}$$

Changing variables $\eta = t - s$ in equation (3.3.12) yields

$$\overline{D}x(t) = \int_0^t x(\eta)(t-\eta)^{-\alpha}\, d\eta$$

which allows the integrated form of IVP (3.3.9)-(3.3.10) to be written as

$$\overline{D}x(t) = \overline{f}(t). \tag{3.3.14}$$

Note that the left hand side of this last equation contains the unknown function x while the right hand side contains known "past" history of x. Accordingly, collocation is used in a recursive way as follows:

Let N be an integer, $h = 1/N$ and partition the interval $[0, T]$, $T > 0$ by $t_n = nh$. Denote by σ_0 the interval $[0, t_1]$, and by σ_n the interval $(t_n, t_{n+1}]$, $n = 1, ..., N-1$. The recursion is motivated by the fact that for $t \in \sigma_n$, we have $t = t_n + sh$, $s \in [0, 1]$. Letting $s = \tau$ equation (3.3.14) can be written as:

$$\int_0^{t_n+sh} x(\tau)(t_n + sh - \tau)^{-\alpha}\, d\tau = \overline{f}(t_n + sh). \tag{3.3.15}$$

Splitting the integral at t_n we have

$$\int_0^{t_n} x(\tau)(t_n + sh - \tau)^{-\alpha}\, d\tau + \int_{t_n}^{t_n+sh} x(\tau)(t_n + sh - \tau)^{-\alpha}\, d\tau = \overline{f}(t_n + sh),$$

which we can rewrite as

$$\int_{t_n}^{t_n+sh} x(\tau)(t_n + sh - \tau)^{-\alpha}\, d\tau = \overline{f}(t_n + sh) - \int_0^{t_n} x(\tau)(t_n + sh - \tau)^{-\alpha}\, d\tau.$$

Note that in the integral on the left hand side the argument of the function x lies in the interval σ_n, while in the right hand side its argument belongs to the interval of "past values" $[t_0, t_n)$. This will set the recursion. Our scheme consists on using collocation in each interval σ_n on equation (3.3.15). The basis functions used to build the projection spaces are suggested by the form of the solution given by the following results.

Theorem 3.3.2 (see [13], theorem 4.2) *Let* $\varphi \in C[-1, 0]$. *Then IVP* (3.0.1)-(3.0.2) *with* $f = 0$, $L = 0$ *and* $\alpha \in (0, 1)$ *has the unique integrable solution*

$$\begin{aligned} x(t) &= \frac{\sin \alpha\pi}{\pi} \int_{-1}^{0} \frac{1}{t - s} \left|\frac{t}{s}\right|^{\alpha} \varphi(s)\, ds \\ &\quad + \frac{\sin \alpha\pi}{\pi} \int_0^t \frac{(t - s)^{\alpha-1}}{(t - s) + 1} \varphi(s - 1)\, ds + [\eta - D\varphi] t^{\alpha-1}, \quad 0 < t \leq s. \end{aligned} \tag{3.3.16}$$

In what follows we will only consider the case of consistent initial data, i.e. $\eta = D\varphi$. In this case the representation given by (3.3.16) will only have two terms and the solution x will be continuous on $[0,1]$ provided $\varphi \in C[-1,0]$, (see [13], Remark 4.1). In our next Lemma we rewrite this result for the case where the initial data is a smooth function.

Lemma 3.3.3 *For a given $\varphi \in C^m[-1,0]$ the solution $x(t)$ of problem* (3.3.9)-(3.3.10) *with $f = 0$ has the form $x(t) = v_0(t) + t^\alpha v_1(t)$ where $v_1 \in C^\infty(0,1]$ and $v_0 \in C^m(0,1]$.*

Proof:

Clearly, using Theorem 3.3.2 we can write

$$x(t) = v_0(t) + t^\alpha v_1(t) \tag{3.3.17}$$

where

$$v_0(t) = \frac{\sin \alpha\pi}{\pi} \int_0^t \frac{(t-s)^{\alpha-1}}{(t-s)+1} \varphi(s-1)\, ds \tag{3.3.18}$$

and

$$v_1(t) = \frac{\sin \alpha\pi}{\pi} \int_{-1}^0 \frac{1}{t-s} \, |s|^\alpha \, \varphi(s)\, ds. \tag{3.3.19}$$

Using the change of variables $s = \eta t$ we can rewrite $v_0(t)$ as

$$\frac{\sin \alpha\pi}{\pi} \int_0^t \frac{(t-s)^{\alpha-1}}{(t-s)+1} \varphi(s-1)\, ds = \frac{\sin \alpha\pi}{\pi} t^\alpha \int_0^1 \frac{(1-\eta)^{\alpha-1}}{(t(1-\eta)+1)} \varphi(t\eta - 1)\, d\eta. \tag{3.3.20}$$

For simplicity of notation, we will call the integral term of v_0 in (3.3.20) V_0. Differentiating this term with respect to t we have

$$\begin{aligned}
\frac{dV_0}{dt} &= \frac{d}{dt}\int_0^1 \frac{(1-\eta)^{\alpha-1}}{t(1-\eta)+1}\varphi(t\eta-1)\,d\eta \\
&= \int_0^1 [\frac{-(1-\eta)^{\alpha}}{(t(1-\eta)+1)^2}\varphi(t\eta-1) \\
&+ \dot{\varphi}(t\eta-1)\frac{\eta(1-\eta)^{\alpha-1}}{t(1-\eta)+1}]\,d\eta.
\end{aligned}$$

Clearly $\frac{dV_0}{dt} \in C[0,1]$ as long as $\dot{\varphi} \in C[-1,0]$. Now we differentiate v_0 with respect to t to obtain

$$\begin{aligned}
\frac{d}{dt}v_0(t) &= \frac{sin(\alpha\pi)}{\pi}[\alpha t^{\alpha-1}\int_0^1 \frac{(1-\eta)^{\alpha-1}}{t(1-\eta)+1}\varphi(t\eta-1)\,d\eta \\
&+ t^{\alpha}\frac{d}{dt}V0(t)].
\end{aligned}$$

Then $\frac{d}{dt}v_0 \in C(0,1]$ as long as $\dot{\varphi} \in C[-1,0]$. Since $\varphi \in C^m[-1,0]$ we can repeat this process m times to get $v_0 \in C^m(0,1]$.

For $v_1(t)$ we have

$$\frac{dv_1}{dt} = \frac{\sin\alpha\pi}{\pi}\int_{-1}^0 \frac{d}{dt}(\frac{1}{t-s})\,|s|^{\alpha}\,\varphi(s)\,ds \tag{3.3.21}$$

and note that since $\frac{1}{t-s} \in C^{\infty}(0,1]$ we can repeat this process as often as we want, thus $v_1 \in C^{\infty}(0,1]$. This proves the lemma.

We will use the above result to choose the projection space in each subinterval of our recursive prosess. For each $n = 0,\cdots,N$, denote by X_n the space $C[t_n,t_{n+1}]$ with $\|x\| = \max_{t\in\sigma_n}|x(t)|$. Let m be as in the lemma above. We will define the projection spaces by considering the form of the solution given by Lemma 3.3.3. Let us assume, for a moment, that v_0 and $v_1 \in C^m[0,1]$, then we can use the Taylor expansion around 0 for both v_0 and v_1 and write the solution $x(t)$ as

$$x(t) = \sum_{k=0}^{m-1} \frac{v_0^k(0)}{k!} t^k + t^\alpha \sum_{k=0}^{m-1} \frac{v_1^k(0)}{k!} t^k + R_0(t) + t^\alpha R_1(t) \tag{3.3.22}$$

for $t \in [0, t_1]$, where $t_1 = 1/N$ and R_1, R_2 given by

$$\begin{aligned} R_1(t) &= \frac{v_1(\psi)^{k+1} t^{k+1}}{(k+1)!} \\ R_2(t) &= \frac{v_2(\psi)^{k+1} t^{k+1}}{(k+1)!}, \qquad \psi \in (0, t) \end{aligned}$$

are the remainders of the Taylor expansion. Recall that for $t \in [0, t_1]$ we have $t = sh$, $h = 1/N, s \in [0, 1]$ so we can rewrite equation (3.3.22) as

$$x(t) = \sum_{k=0}^{m-1} v_0^k(0) h^k s^k + h^\alpha \sum_{k=0}^{m-1} v_1^k(0) h^k s^{\alpha+k} + R_0(sh) + (sh)^\alpha R_1(sh). \tag{3.3.23}$$

It is convinient for us to take the projection space X_0^N, $n = 0, 1, \ldots, N-1$ to be

$$X_0^N = \text{span}\{1, s, ... s^{m-1}, s^\alpha, s^{\alpha+1}, ..., s^{\alpha+m-1}\}, \tag{3.3.24}$$

where $N = 2m$.

For $t \in (t_n, t_{n+1}]$ with $n \geq 1$, we can write $t = t_n + sh$ with $s \in [0, 1]$, $h = 1/N$. Using the same argument as above, we will expand the solution around t_n to get

$$x(t) = \sum_{k=0}^{m-1} \frac{v_0^k(t_n)}{k!} (t - t_n)^k + t^\alpha \sum_{k=0}^{m-1} \frac{v_1^k(t_n)}{k!} (t - t_n)^k + R_0(t) + t^\alpha R_1(t). \tag{3.3.25}$$

In this case we have

$$
\begin{aligned}
R_0(t) &= \frac{v_0(\psi)^{k+1}(t-t_n)^{k+1}}{(k+1)!} \\
R_1(t) &= \frac{v_1(\psi)^{k+1}(t-t_n)^{k+1}}{(k+1)!}, \qquad \psi \in (t_n, t).
\end{aligned}
$$

Recall that $t_n = 1/N$, thus for any $t \in (t_n, t_{n+1}]$ we can write $t = t_n + sh = t_n(1 + s/n)$ and equation (3.3.25) becomes

$$
\begin{aligned}
x(t) &= \sum_{k=0}^{m-1} v_0^k(t_n)h^k s^k + t_n^\alpha \sum_{k=0}^{m-1} v_1^k(t_n)h^k(1+s/n)^\alpha s^k \\
&\quad + R_0(t) + (t_n(1+s/n))^\alpha R_1(t).
\end{aligned} \tag{3.3.26}
$$

As suggested by equation (3.3.26) we will take the projection spaces $X_n, n \geq 1$, to be

$$
X_n^N = \text{span}\{1, s, ..., s^{m-1}, (1+s/n)^\alpha, (1+s/n)^\alpha s, ..., (1+s/n)^\alpha s^{m-1}\}, \tag{3.3.27}
$$

for $n = 1, \ldots, N-1$.

Let the projections P_n^N for $n \geq 0$ be given by equation (3.1.4) where the basis functions $\{e_1^N, \cdots, e_N^N\}$ are given by equations (3.3.24) and (3.3.27). Then the sequence of approximating functions $\{x_n^N\}_{N\geq 1}$, have the form

$$
x_n^N(t) = P_n^N x(t) = \sum_{j=1}^{N} \gamma_j e_j^n(t) \tag{3.3.28}
$$

with each x_n^N, defined on the interval σ_n, is a solution of the system

$$
(P_n^N D x_n^N)(t) = (P_n^N f)(t), \quad t \in \sigma_n \tag{3.3.29}
$$

and t a collocation point. Or equivalently, the funtion x_n^N will be require to satisfy equation (3.3.15) at each collocation point. That is, letting $t_i = t_n + \tau_{n_i}^N$, $i = 0, 1, \ldots, m-1$, the collocation points on the interval σ_n. The approximation x_n^N is required to satisfy the set of N equations with N unknowns $\gamma_0, \cdots, \gamma_N$ given by

$$\begin{aligned}&\int_0^{t_n+\tau_{n_i}^N} x_n^N(s)(t_n+\tau_{n_i}^N-s)^{-\alpha}ds \qquad (3.3.30)\\ &= f(t_n+\tau_{n_i}^N) - \int_0^{t_n} x_n^N(s)(t_n+\tau_{n_i}^N-s)^{-\alpha}ds, \qquad i=1,\ldots,N.\end{aligned}$$

Replacing x^N by equation (3.3.28) we have

$$\begin{aligned}&\int_{t_n}^{t_n+\tau_{n_i}^N} \sum_{j=1}^{N} \gamma_j e_j^n(s)(t_n+\tau_{n_i}^N-s)^{-\alpha}\,ds \qquad (3.3.31)\\ &= \sum_{j=1}^{N}\gamma_j \int_{t_n}^{t_n+\tau_{n_i}^N} e_j^n(s)(t_n+\tau_{n_i}^N-s)^{-\alpha}\,ds\\ &= f(t_n+\tau_{n_i}^N) - \int_0^{t_n}\sum_{j=1}^{N}\gamma_j e_j^n(s)(t_n+\tau_{n_i}^N-s)^{-\alpha}\,ds\\ &= f(t_n+\tau_{n_i}^N) - \sum_{j=0}^{N}\gamma_j \int_0^{t_n} e_j^n(s)(t_n+\tau_{n_i}^N-s)^{-\alpha}\,ds, \qquad i=1,\ldots,N.\end{aligned}$$

The entries of the matrix $Q_{i,k}^N$ in equation (3.1.8) will be of the form:

$$\begin{aligned}Q_{i,k}^n &= \overline{D}e_i^n(\tau_k) = \int_{t_n}^{t_n+\tau_k^n} e_i^n(s)(t_n+\tau_{n_i}^N-s)^{-\alpha}\,ds, \qquad i=1,\cdots,N, \qquad (3.3.32)\\ F^n &= (\overline{f}(t_1^n),\cdots,\overline{f}(t_N^n))^T. \qquad (3.3.33)\end{aligned}$$

for each collocation point $t_k^n \in \sigma_n$, $k=1,\cdots,N$, $n=1,\cdots,N$. The functions $\overline{f}(t_k^n)$ are given by

$$\overline{f}(t_k^n) = f(t_n+\tau_k^n) - \sum_{j=1}^{N}\gamma_j \int_0^{t_n+\tau_k^n} e_j^n(s)(t_n+\tau_k^n-s)^{-\alpha}\,ds,$$

with γ_j, $j \geq 0$ given by the solutions of the previous $n-1$ systems. That is, we are solving a system of N equations with N unknowns for each subinterval σ_n, $n=1,\cdots,N$.

For convergence results see section 6.4.2 of [8].

3.4 Solving for $L \neq 0$

In this section we study the approximation problem for IVP (2.3.46)-(2.3.47). That is, we will consider the problem

$$\frac{dDx_t}{dt} = Lx_t + g(t), \quad 0 \leq t \leq T, \tag{3.4.34}$$

with initial data

$$x_0(s) = \varphi(s), \quad s \in [-1, 0] \tag{3.4.35}$$

where $x_t(s) = x(t+s)$ for $s \in [-1, 0]$, $t \geq 0$, $g \in C^m$ is a known function and the linear operators D and L have the following representation for $\varphi \in C[-1, 0]$

$$Dx_t = \int_{-1}^{0} [c(s)(-s)^{-\alpha} + p(s)]x(t+s)\, ds, \tag{3.4.36}$$

where $c, p \in C^m[-1, 0]$.

The right hand side L operator has the following form

$$L\varphi = a\varphi(0) + b\varphi(-1) + \int_{-1}^{0} h(s)\varphi(s)\, ds, \tag{3.4.37}$$

with $h \in C^m$.

Using Theorem 2.3.13 we know that IVP (3.4.34)-(3.4.35) can be viewed as the following Volterra integral equation of the second kind

$$x(t) - \int_{0}^{t} [(t-\tau)^{\alpha-1} K(\tau - t) + M(\tau - t)]x(\tau)\, d\tau = f(t) \tag{3.4.38}$$

where the functions K and M are given by (2.3.51) and (2.3.52) respectively and the right hand side function f is given by (2.3.53). Moreover, Theorem 2.4.21 gives us a representation of the solution of IVP (3.4.34)-(3.4.35). That is, we have the following result

Theorem 3.4.4 *Let x be the solution of IVP (2.3.46)-(2.3.47). Then x can be represented as*

$$\begin{aligned} x(t) &= \sum_{j=1}^{m} t^{j-1} q_j(t) + t^{\alpha} q(t) + \sum_{n=1}^{\infty}\sum_{j=1}^{m} t^{n\alpha-1+j} p_{j,n}(t) \\ &+ \sum_{n=1}^{\infty} t^{(n+1)\alpha} p_n(t). \end{aligned}$$

where $p_{j,n}(t)$, $j = 1, \cdots, m$, $n \geq 0$ are given by

$$\begin{aligned} p_{j,n}(t) &= \int_0^1 (1-\tau)^{n\alpha-1} \tau^{j-1} \varphi_n(t, \tau t, \alpha) q_j(\tau t)\, d\tau \qquad j = 1, \cdots, m, \quad n \geq 1 \\ p_n(t) &= \int_0^1 (1-\tau)^{n\alpha-1} \varphi_n(t, \tau t, \alpha) \tau^{\alpha-1} \tau q(t\tau)\, d\tau \qquad n \geq 1. \end{aligned}$$

As suggested by the form of the solution given by the above theorem we consider the following set of basis functions:

$$\{1, t, t^2, \cdots, t^m, t^{\alpha+j}, t^{2\alpha+j}, \cdots, t^{m\alpha+j}\}, \qquad j = 1, \cdots, m. \tag{3.4.39}$$

Note that equation (3.4.38) can be rewritten as

$$x(t) - \int_0^t (t-s)^{\alpha-1} KM(t, s, \alpha) x(s)\, ds = f(t) \tag{3.4.40}$$

where $KM(t, s, \alpha) = K(s-t) + (t-s)^{1-\alpha} M(s-t)$ and K and M are given by (2.3.51) and (2.3.52) respectively.

We use recursive collocation in the following way:

Let N be an integer, $h = 1/N$ and partition the interval $[0,1]$, by $t_n = nh$. Denote by σ_0 the interval $[0, t_1]$, and by σ_n the interval $(t_n, t_{n+1}]$, $n = 1, ..., N-1$. The recursion is motivated by the fact that for $t \in \sigma_n$, $t = t_n + sh$, $s \in [0,1)$ and equation (3.4.40) can be written as:

$$x(t_n + sh) - \int_0^{t_n+sh} (t_n + sh - \tau)^{\alpha-1} KM(t_n + sh, \tau, \alpha) x(\tau)\, d\tau = f(t_n + sh). \tag{3.4.41}$$

Splitting the integral at t_n we have

$$\begin{aligned} x(t_n + sh) \;&-\; \int_0^{t_n} (t_n + sh - \tau)^{\alpha-1} KM(t_n + sh, \tau, \alpha) x(\tau)\, d\tau \\ &-\; \int_{t_n}^{t_n+sh} (t_n + sh - \tau)^{\alpha-1} KM(t_n + sh, \tau, \alpha) x(\tau)\, d\tau = f(t_n + sh). \end{aligned}$$

Note that we already solved the problem for $t \in [0, t_n)$, so $x^N(\tau)$, $\tau \in [0, t_n)$ is known data. Then we can rewrite the problem as

$$\begin{aligned} x(t_n + sh) \;&-\; \int_{t_n}^{t_n+sh} (t_n + sh - \tau)^{\alpha-1} KM(t_n + sh, \tau, \alpha) x(\tau)\, d\tau \\ &=\; f(t_n + sh) + \int_0^{t_n} (t_n + sh - \tau)^{\alpha-1} KM(t_n + sh - \tau) x(\tau)\, d\tau. \end{aligned}$$

Using the change of variables $\eta = \tau - t_n$ the above equation becomes

$$\begin{aligned} x(t_n + sh) \;&-\; \int_0^{sh} (sh - \eta)^{\alpha-1} KM(sh - \eta, \eta + t_n, \alpha) x(\eta + t_n)\, d\eta \\ &=\; f(t_n + sh) - \int_0^{t_n} (t_n + sh - \tau)^{\alpha-1} KM(t_n + sh, \tau, \alpha) x(\tau)\, d\tau. \end{aligned} \tag{3.4.42}$$

For each interval σ_n, $n \geq 0$ we consider the projection P_n as

$$P_n^N x(t) = \sum_{j=0}^{N} \gamma_j e_j^n(t) \tag{3.4.43}$$

where $e_j^n(t)$ are given by (3.4.39).

We will require x^N to be a solution of equation (3.4.42) at each collocation point. That is, we will expect x^N to satisfy

$$\begin{aligned} \sum_{j=1}^{N} \gamma_j e_j(t_n + sh) \quad &- \quad \sum_{j=1}^{N} \gamma_j \int_0^{sh} (sh-\eta)^{\alpha-1} KM(sh-\eta, \eta + t_n, \alpha) e_j^n(\eta + t_n)\, d\eta \\ &= \quad \bar{f}(t_n + sh). \end{aligned} \tag{3.4.44}$$

for $t_n + sh \in \sigma_n = [t_n, t_{n+1})$ a collocation point for each $s \in [0,1]$ and $\bar{f}(t_n + sh)$ is given by

$$\bar{f}(t_n + sh) = f(t_n + sh) - \int_0^{t_n} (t_n + sh - \tau)^{\alpha-1} KM(t_n + sh, \tau, \alpha) x(\tau)\, d\tau.$$

The convergence of our numerical scheme is given by Theorem 6.2.6 in [8] that states:

Theorem 3.4.5 *Let the function f and the kernel K in equation* (3.4.38) *belong to $C^m([0,T])$ and $C^m([0,T] \times [0,T])$ respectively, with $m \geq 1$, and assume that neither function vanishes identically. Then there exists an $\bar{h} > 0$ such that the collocation equation* (3.4.44) *defines for each $h \in (0, \bar{h})$ a unique approximation x^N. The error induced by this approximation satisfies, for every choice of the collocation parameters $\{t_j\}$ with $0 \leq t_1 < \cdots < t_m \leq 1$, and for all quasi-uniform mesh sequences,*

$$\|e\|_\infty = O(N^{-(1-\alpha)}). \tag{3.4.45}$$

Moreover, given the representation of the solution obtained in chapter 2 and choosing the projection spaces as in equations (3.3.27) and (3.4.39), we satisfy the conditions of section 6.2.5 of [8]. That is, we can get an error of the order $O(N^{-m})$ where m is the smoothness of the initial data of IVP (2.3.46)-(2.3.47).

3.5 Conclusions and Open Problems

We presented a numerical scheme for both the first kind and the second kind Volterra equations. For the second kind Volterra equation the non-polynomial spline collocation technique is discussed in [8]. This technique can be formally extended to the first kind equation as was presented in this chapter and local convergence can be guarantee by Theorem 6.4.1 page 397 of [8].

We are particularly interested on applying this technique to approximate the solution of a general aerodynamic model developed to control the vibrations that contributes to flutter in the wings of an aircraft, [12]. In 1983, Burns, Herdman and Cliff [12], formulated a complete dynamical model in terms of functional differential equations of neutral type for the elastic motions of a three-degree-of-freedom typical airfoil section, with flap, in a two dimensional, incompressible flow. This model contains a singular integral equation component that can be view as

$$\frac{d}{dt} D x_t = L x_t + g(t),$$

where the D operator is given by

$$D x_t = \int_{-1}^{0} \sqrt{\frac{Us - 2}{s}}\, x_t(s)\, ds,$$

with U a positive constant.

Note that the kernel of the D operator can be written as

$$k(s) = \sqrt{\frac{Us - 2}{s}} = \sqrt{-Us + 2}\sqrt{\frac{1}{-s}}, \quad s \in [-1, 0].$$

That is, $k(s)$ is the product of $\sqrt{\frac{1}{-s}}$ with a smooth function. Thus, the numerical scheme descriebed in this chapter can be apply to approximate the solution of the general aerodynamic problem. We belive that we will have numerical results for this problem in the near future.

In the next section we present numerical results. It is interesting to note that these results were satisfactory even for the case of inconcistent initial data for IVP (2.3.46)-(2.3.47). However, the theory concerning this case is still an open problem.

3.6 Numerical Examples

3.6.1 Non-perturbed Case

Example 3.6.1 $L = 0$

We consider IVP (2.3.46)-(2.3.47) for $p = 0$, $c = 1$, $L = 0$, $g = 0$ and initial data $\varphi(s) = -s$ for both $\alpha = 0.5$ and $\alpha = 0.75$. Figure 3.1 shows the numerical and true solution for both cases.

Example 3.6.2 $L \neq 0$

Let the initial condition of IVP (2.3.46)-(2.3.47) be $\varphi(t) = t$ and let the functions $c(s) = 1$, $p(s) = 0$ and $g(t) = 0$ and the right hand side operator $L\varphi = \varphi(0) + \varphi(-1)$. Let $\beta(\cdot, \cdot)$ be the usual Beta function. Using Theorem 2.3.13, our IVP is equivalent to a Volterra equation of the second kind with right hand side function $f(t)$ given by

$$f(t) = \frac{2\beta(3-\alpha,\alpha)t}{(1-\alpha)(2-\alpha)} + \frac{(2+\alpha)\beta(3,\alpha)t^{1+\alpha}}{2} - \frac{(1+\alpha)(2-\alpha)}{(1-\alpha)}\beta(2,\alpha)t^{\alpha}$$

Figure 3.2 shows numerical solutions of the Volterra equation of the second kind with right hand side $f(t)$ and $\alpha = 0.5$ for different values of the stepsize $h = 1/N^2$.

Example 3.6.3 $L \neq 0$

Let the initial condition of IVP (2.3.46)-(2.3.47) be $\varphi(t) = t^2$ and let the functions $p(s) = 0$, $c(s) = 0$ and $g(t) = 0$ and the right hand side operator $L\varphi = \varphi(0) + \varphi(-1)$, then our IVP is equivalent to a Volterra equation of the second kind with right hand side $f(t)$ given by:

$$\begin{aligned} f(t) &= 3C(\alpha)\beta(4-\alpha,\alpha)t^2 + \frac{(3+\alpha)}{3}\beta(4,\alpha)t^{\alpha+2} - \frac{(4-\alpha^2)}{(1-\alpha)}\beta(3,\alpha)t^{1+\alpha} \\ &+ \frac{(4-\alpha)(1+\alpha)}{(2-\alpha)}\beta(2,\alpha)t^{\alpha} \end{aligned}$$

where the constant $C(\alpha) = \frac{1}{3-\alpha} + \frac{1}{1-\alpha} - \frac{2}{2-\alpha}$, and $\beta(\cdot, \cdot)$ is the beta function.

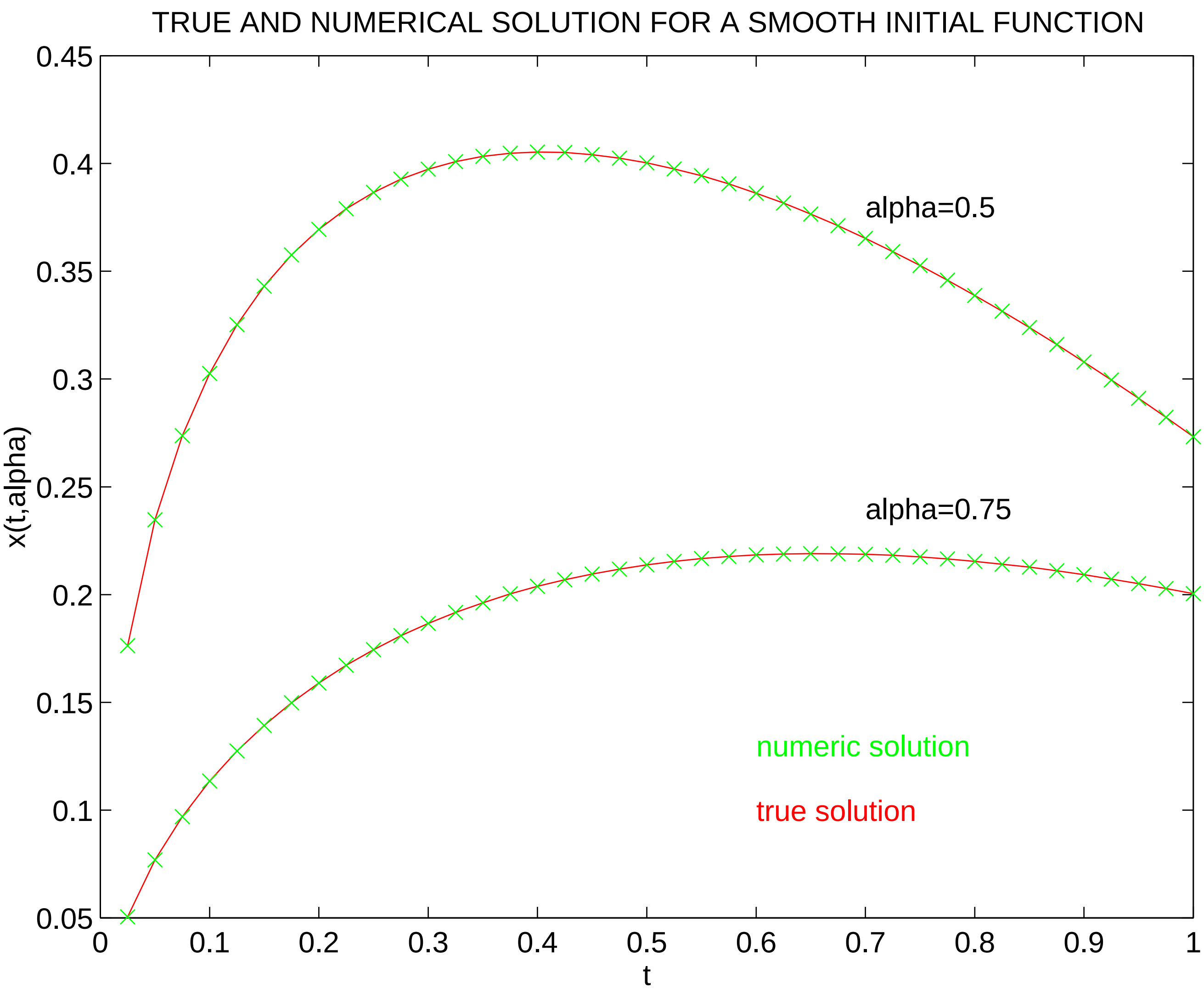

Figure 3.1: Non-perturbed case, $L = 0$

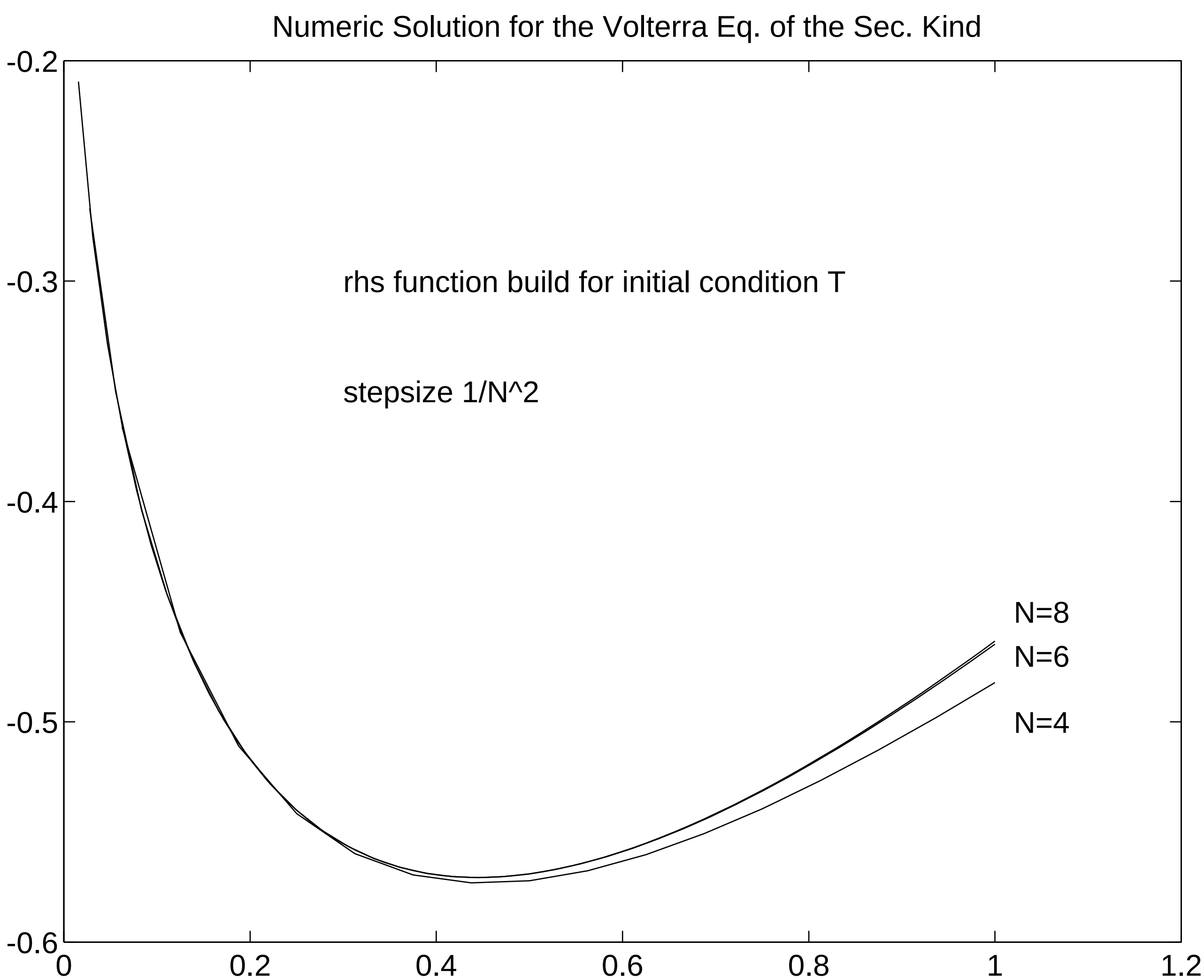

Figure 3.2: Non-perturbed case, $L \neq 0$

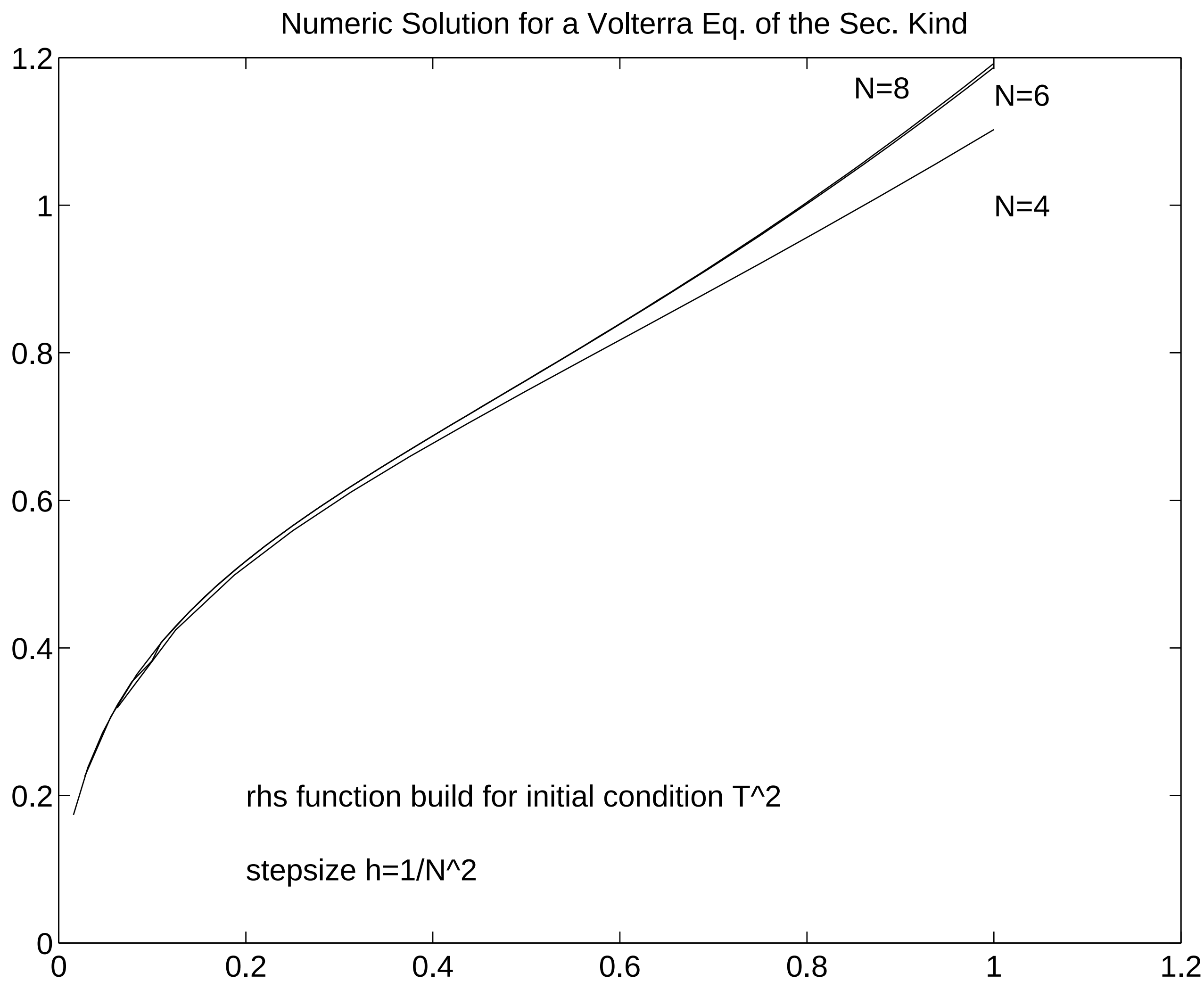

Figure 3.3: Non-perturbed case, $L \neq 0$

Numerical solutions for the Volterra equation of the second kind with right hand side function $f(t)$ and $\alpha = 0.5$ are shown in Figure 3.3 for different values of the stepsize $h = 1/N^2$.

3.6.2 Perturbed Case

Example 3.6.4 Constant Perturbation

Consider the perturbed IVP (2.3.46)-(2.3.47) with perturbation $p(s) = 1$, $c(s) = 1$ and right hand side operator $L\varphi = \varphi(0) + \varphi(-1)$ and $g(t) = 0$. Let the initial condition be $\varphi(t) = t$. In this particular case we can solve the IVP explicitly. The true solution is given by:

$$x(t) = \frac{t}{(1-\alpha)(2-\alpha)} \frac{1}{\beta(2, 1-\alpha)} + \frac{(3-2\alpha)t^{\alpha}}{(\alpha-1)\beta(1+\alpha, 1-\alpha)} + \frac{t^{\alpha+1}}{\beta(2+\alpha, 1-\alpha)}.$$

The true and numeric solution for the perturbed case with $\alpha = 0.5$ are shown in Figure 3.4.

Remark 3.6.5 Note that in this example the perturbation $p(s) = 1$ is canceled out with the L operator. That is, the perturbed IVP with $p(s) = 1$ and with the right hand side operator L being point evaluation is actually a first kind integral equation problem.

Example 3.6.6 Linear Perturbation

Consider the perturbed IVP (2.3.46)-(2.3.47) with perturbation $p(s) = 6s$, $c(s) = 1$ and right hand side operator $L\varphi = \varphi(0) + \varphi(-1)$. Let the initial condition be $\varphi(t) = t$ and let the function $g(t) = -2t$. In this case the IVP can be solved explicitly. The true solution is given by $x(t) = t$. Figure 3.5 shows both the numerical and true solutions for this problem.

Example 3.6.7 Linear Perturbation

Consider the perturbed IVP (2.3.46)-(2.3.47) with perturbation $p(s) = -s$ and $c(s) = 1$, and let the right hand side operator $L\varphi = \varphi(0) + \varphi(-1)$. Let the initial condition be $\varphi(t) = t$ and let the function $g(t) = -t$. Numerical solutions for this IVP with different values of the stepsize are shown in Figure 3.6.

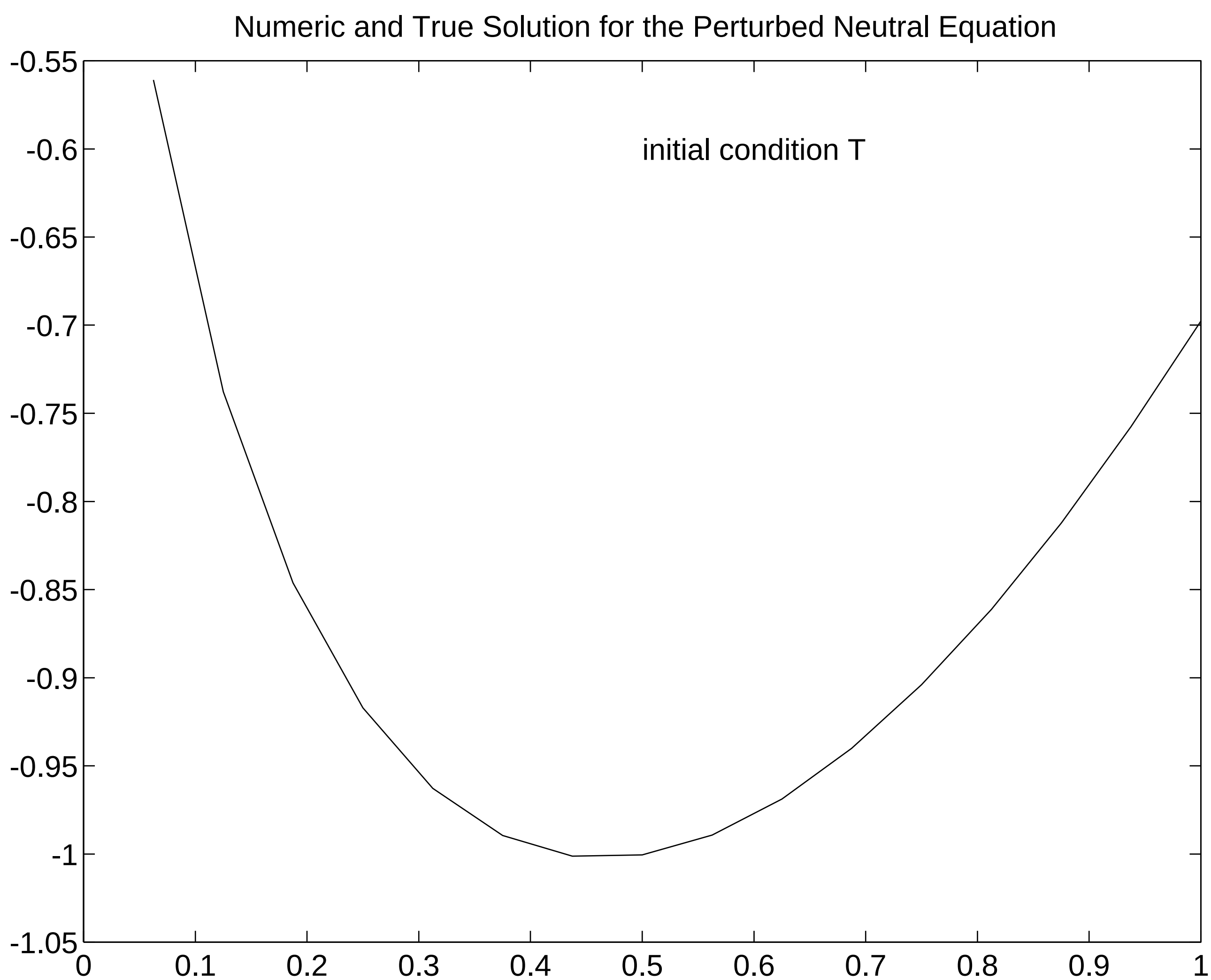

Figure 3.4: Perturbed case, constant perturbation

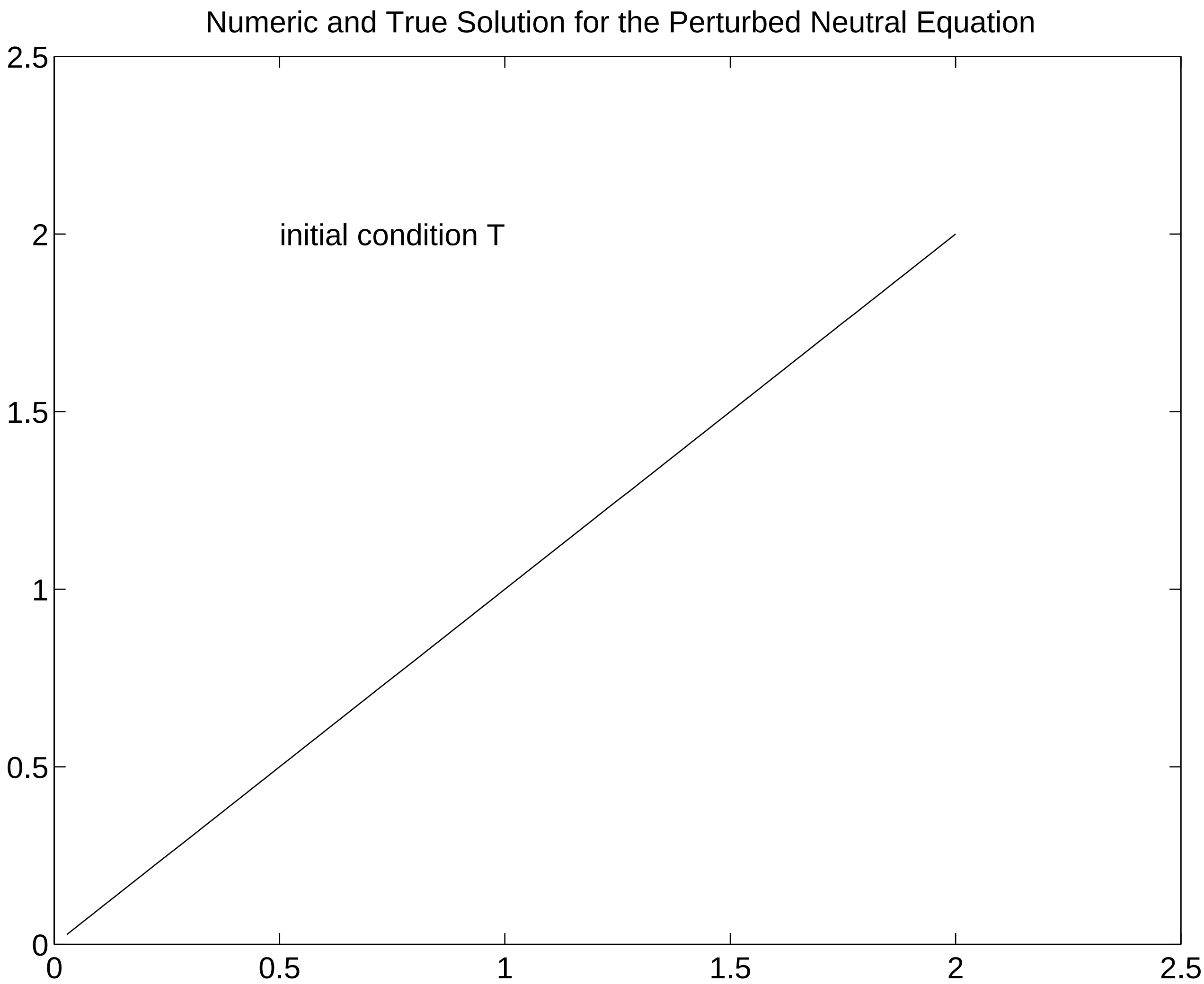

Figure 3.5: Perturbed case, linear perturbation

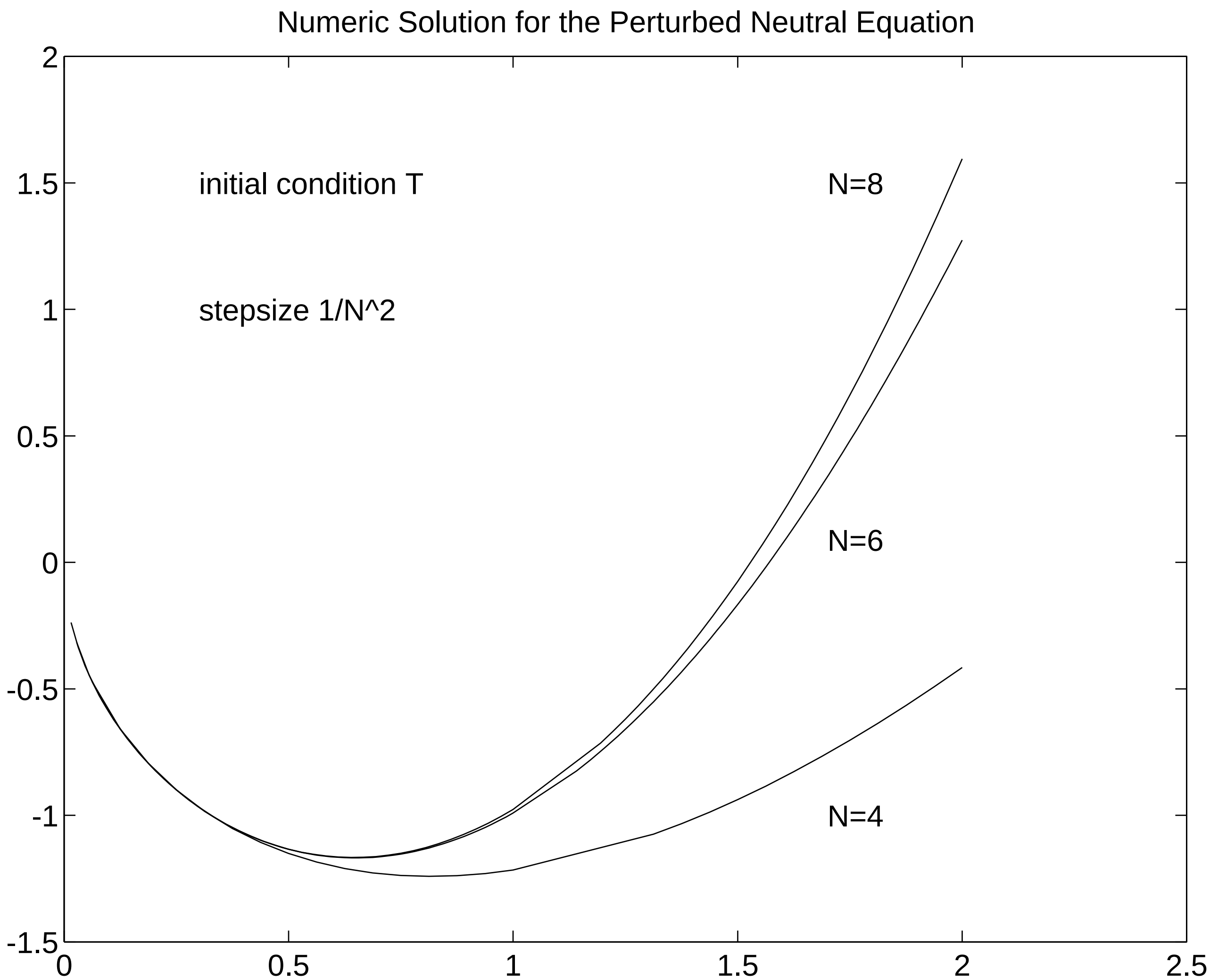

Figure 3.6: Perturbed case, linear perturbation

3.6.3 A Special Case

Example 3.6.8 Discontinuous initial condition

We consider the scalar equation

$$\int_{-1}^{0} (-s)^{-\alpha} x(t+s)\, ds = 1, \qquad t > 0 \tag{3.6.46}$$

with initial data

$$\varphi(s) = \begin{cases} 0 & \text{if } s \in (-1,0) \\ 1/2 & \text{if } s = 0. \end{cases} \tag{3.6.47}$$

This example is a generalization of Example 4.1 in [27].

The solution of equation (3.6.46)-(3.6.47) is given by

$$x(t,\alpha) = \frac{1}{\pi} t^{\alpha-1} \sin \alpha\pi, \quad t \in [0,1). \tag{3.6.48}$$

Note that in this case the initial data φ is not a continuous function therefore our result concerning the form of the solution does not apply. Thus the scheme described in [8] can not be applied directly. To approximate the solution we approximate, in a suitable way, the initial function φ by a sequence of differentiable functions. In particular, for $\varepsilon \in (-1,0)$ consider the family of functions φ_ε given by

$$\varphi_\varepsilon(s) = \begin{cases} 0 & \text{if } s < \varepsilon \\ a(\varepsilon)s^2 + b(\varepsilon)s + 1/2 & \text{if } s \in (\varepsilon, 0) \\ 1/2 & \text{otherwise.} \end{cases}$$

Let $\{\varepsilon_n\}$ be a sequence with $\varepsilon_n \to 0$. The sequence φ_{ε_n} is not a Cauchy sequence in $(C^1[-1,0], \|\cdot\|_\infty)$, but converges to φ in L^2, therefore, $\lim_{n\to\infty}(x_n)_t = x_t$ in L^2.

Figure 3.7 shows both the numerical and the true solution for this case. Figure 3.8 show how the approximation of the solution converges to the true solution as $\varphi_\varepsilon \to \varphi$, $\varepsilon \to 0$. Figure 3.9 show how the approximation of the solution converges to the real solution for a given initial data φ_ε, $\varepsilon = 10^{-4}$, and different values of the stepsize.

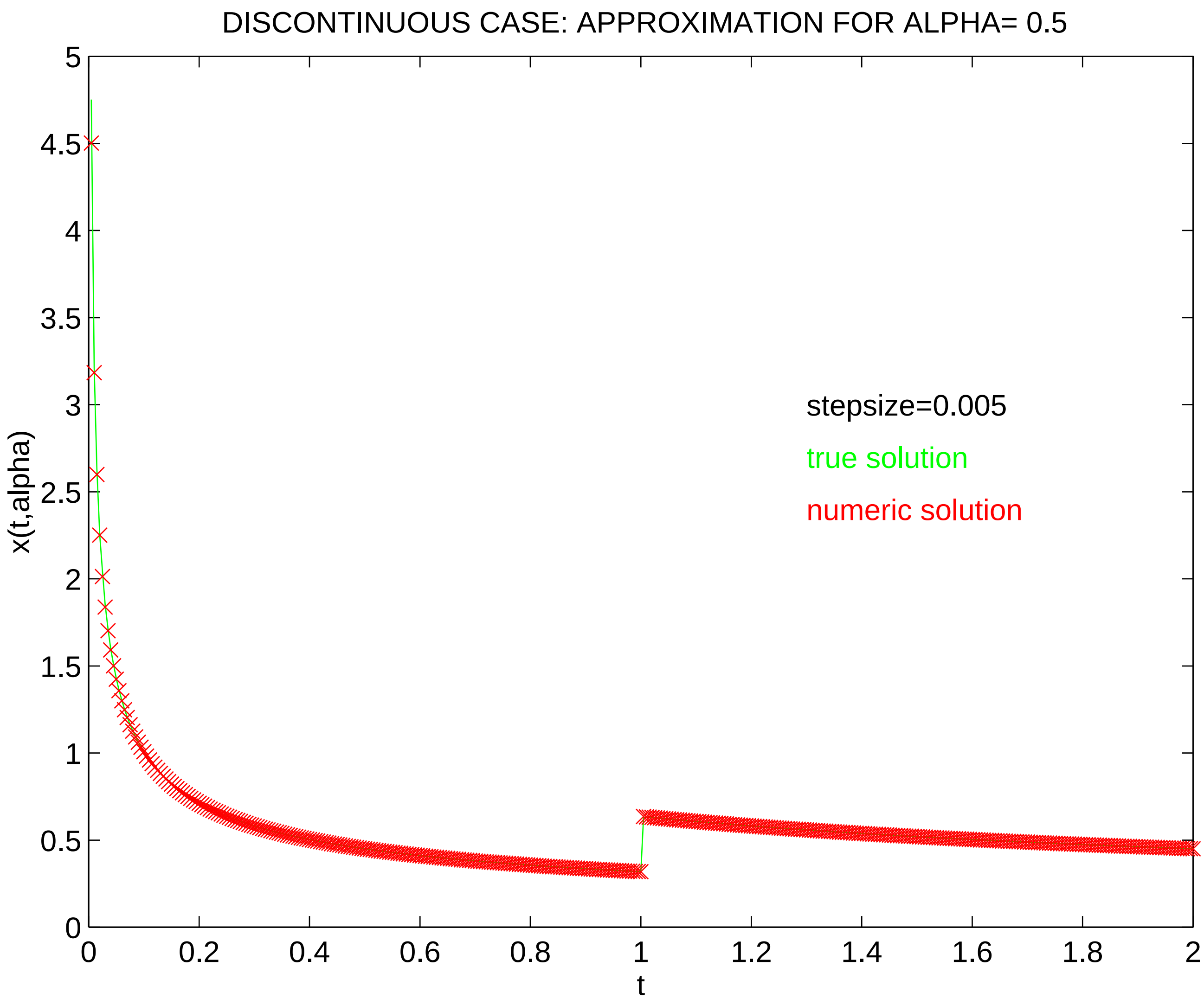

Figure 3.7: Discontinuous case

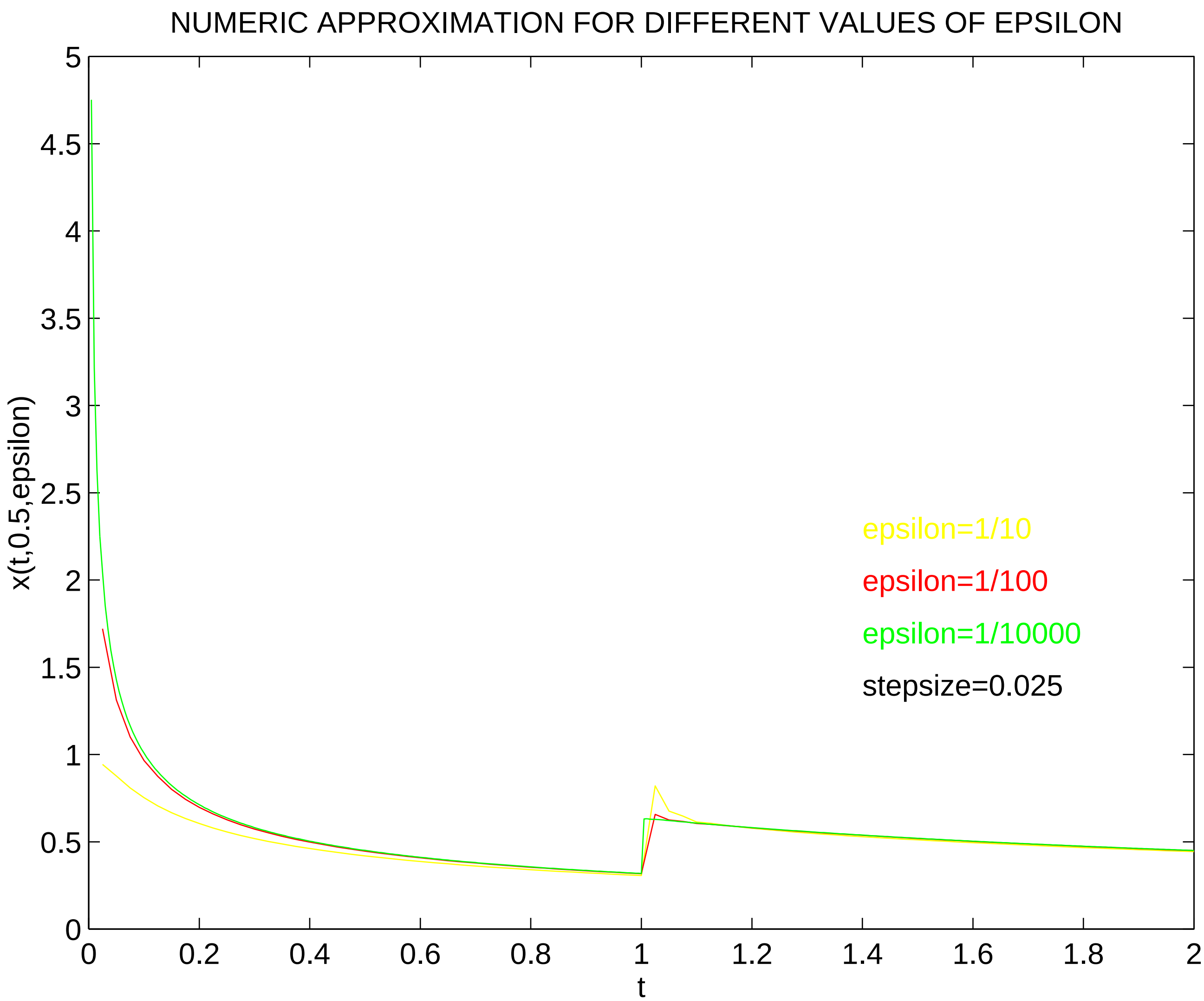

Figure 3.8: Discontinuous case

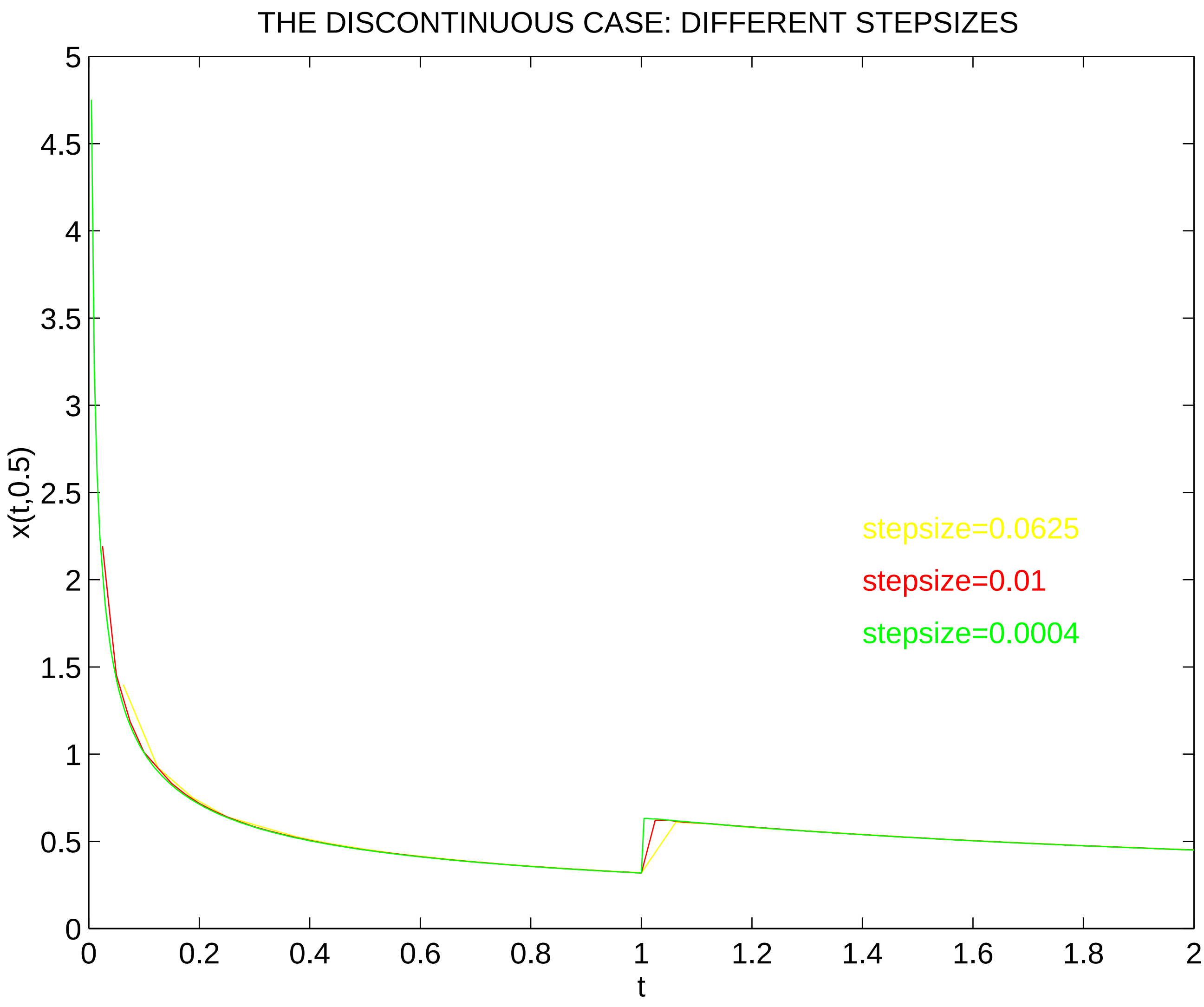

Figure 3.9: Discontinuous case

Chapter 4

The Identification Problem

4.1 Introduction

In this chapter we study the parameter identification problem for IVP (4.1.1)-(4.1.2). We will focus our attention on the identification of the parameter α appearing in the kernel of the D operator given by equation (4.1.3). This identification problem is particularly challenging due to the weak singularity of the D operator at zero. We will solve the identification problem for both consistent and inconsistent initial data. Note that the case of inconsistent initial data is not covered by the theory presented in the previous chapters, yet numerical results where successful.

The IVP we studied has the following structure:

$$\frac{dDx_t}{dt} = Lx_t + g(t), \quad 0 < t < T, \tag{4.1.1}$$

with initial data

$$x_0(s) = \varphi(s), \quad s \in [-1, 0] \tag{4.1.2}$$

where $x_t(s) = x(t+s)$ for $s \in [-1, 0]$, $t \geq 0$, $g(t) \in C$ is a known function and the linear operator D has the following representation for $\varphi \in C[-1, 0]$

$$D\varphi = \int_{-1}^{0} \varphi(s) k(s, \alpha)\, ds \tag{4.1.3}$$

where $k(\cdot, \alpha) \in L^1(-1, 0)$ is a non-negative, nondecreasing function in $[-1, 0]$ which is weakly singular at $s = 0$.

Since our main goal in this work is to identify the parameter appearing in the singular kernel of the D operator given by equation (4.1.3) we will consider the right hand side operator $L = 0$.

4.2 The Identification Problem

Given the measurements $y_i, i = 1, \cdots, n$ of $x(t)$, solution of (4.1.1)-(4.1.2), at discrete points $t_i, i = 1, \cdots, n$ we want to identify the parameter α in equation (4.1.3). That is, we are looking for that α which minimizes the least square fit to data criterion

$$J(\alpha) = \sum_{i=1}^{n} \|x(t_i, \alpha) - y_i\|^2 \tag{4.2.1}$$

We will denote this problem by P. Identification problems are widely studied, see [2], [4], [5], and the references therein. Assuming that the solution to IVP (4.1.1)-(4.1.2) is differentiable with respect to the parameter α, we can follow the same ideas used by these authors to solve problem P. That is, we will follow the following steps:

- STEP 1 Consider a sequence of approximated initial value problems IVP_N corresponding to a discretization of IVP (4.1.1)-(4.1.2) for some fixed parameter $\alpha^N \in Q$ with solution $x_t^N(\cdot, \varphi, \alpha^N)$ satisfying that $x_t^N(\cdot, \varphi, \alpha^N) \to x_t(\cdot, \varphi, \alpha)$ as $N \to \infty$, uniformly in compact time sets.
- STEP 2 For each IVP_N we define an associated minimization problem P^N by:

$$\min_{\alpha \in Q} J^N(\alpha) \tag{4.2.2}$$

where

$$J^N(\alpha) = \sum_{i=1}^{n} \|x^N(t_i, \alpha) - y_i\|^2 \tag{4.2.3}$$

Assume $J^N(\alpha^N)$ is the solution to the minimization problem P^N and $x^N(t, \alpha^N)$ is the solution of IVP_N given by the approximation scheme developed in chapter 3.

- STEP 3 By solving the approximated minimization problems P^N we define a sequence of approximated parameters α^N. Since the admissible parameter set Q is compact, the sequence $\{\alpha^N\} \in Q$ has a convergent subsequence, i.e. there is an $\alpha^* \in Q$ such that $\alpha^N \to \alpha^*$, $N \to \infty$.

- STEP 4 We will prove that α^* is a solution for the minimization problem P.

Note that for steps 3 and 4 we only need to use compactness arguments and step 2, but no particular approximation method will be nessesary, see [30].

4.2.1 Statement of the Minimization Problem

Let α_0 be an initial guess for our identification problem and let ε be fixed we will consider the compact set $Q = [\alpha_0 - \varepsilon, \alpha_0 + \varepsilon]$ to be our admissible parameter set. Then the identification problem P for the parameter α can be written as:

$$\min_{\alpha \in Q} J(\alpha) \tag{4.2.4}$$

where

$$J(\alpha) = \sum_{i=1}^{n} \|x(t_i, \alpha) - y_i\|^2. \tag{4.2.5}$$

To prove that the solution to the minimization problem (4.2.4) exists we will first prove that the solution of IVP (4.1.1)-(4.1.2) is a continuous function of the parameter α.

Lemma 4.2.1 *Let the kernel function $k(\cdot,\alpha)$ in (4.1.3) be continuous with respect to α. Then the solution $x_t(s,\varphi,\alpha) = x(t+s,\varphi,\alpha)$ of IVP (4.1.1)-(4.1.2) is a continuous function of the parameter α.*

Proof: Let $\alpha^* \in Q$ be fixed and consider a sequence $\alpha^N \in Q$ such that $\alpha^N \to \alpha^*$. Consider the integrated form of IVP (4.1.1)-(4.1.2) with $\alpha = \alpha^*$ and denote by $x_t(\cdot,\varphi,\alpha^*)$ its unique continuous solution. Let $\{IVP^N\}_{N\geq 1}$ be a family of IVP's given by

$$
\begin{aligned}
D_{\alpha^N} x_t &= D_\alpha \varphi + \int_0^t g(s)\,ds \quad t \in [0,T] &\quad (4.2.6)\\
x_0(s) &= \varphi(s) \quad s \in [-1,0], &\quad (4.2.7)
\end{aligned}
$$

where

$$
D_{\alpha^N}\varphi = \int_{-1}^0 \varphi(s)k(s,\alpha^N)\,ds.
$$

Using well-posedness of the above IVP's we have that for each α^N the corresponding IVP^N will have a unique solution $x_t(\cdot,\varphi,\alpha^N)$ and the family $T_{\alpha^N}(t)\varphi = x_t(s,\varphi,\alpha^N) = x(t + s,\varphi,\alpha^N)$ $s \in [-1,0]$ defines a C_0 semigroup in C. We now consider a new family of IVP's given by substracting each IVP^N from the integrated form of IVP (4.1.1)-(4.1.2) with $\alpha = \alpha^*$. That is, we will consider the equation

$$
\begin{aligned}
\int_{-1}^0 x_t(s,\varphi,\alpha^*)k(s,\alpha^*)\,ds &- \int_{-1}^0 x_t(s,\varphi,\alpha^N)k(s,\alpha^N)\,ds\\
&= \int_{-1}^0 \varphi(s)k(s,\alpha^*)\,ds - \int_{-1}^0 \varphi(s)k(s,\alpha^N)\,ds. \quad (4.2.8)
\end{aligned}
$$

We can write (4.2.8) as

$$
\int_{-1}^0 [x_t(s,\varphi,\alpha^*) - x_t(s,\varphi,\alpha^N)]k(s,\alpha^N)\,ds
$$

$$
\begin{aligned}
&= \int_{-1}^{0} \varphi(s)k(s,\alpha^*)\,ds - \int_{-1}^{0} \varphi(s)k(s,\alpha^N)\,ds \\
&- \int_{-1}^{0} x_t(s,\varphi,\alpha^*)[k(s,\alpha^*) - k(s,\alpha^N)]\,ds
\end{aligned}
$$

That is, we consider the family of IVP's $\{\overline{P}^N\}_{N\geq 1}$ given by

$$
\begin{aligned}
\int_{-1}^{0} y_t(s,\alpha^N,\alpha^*)k(s,\alpha^N)\,ds &= F^N(t), \quad t \in [0,T] \\
y_0(s) &= 0, \quad s \in [-1,0],
\end{aligned}
$$

where $y_t(\cdot,0,\alpha^N,\alpha^*) = x_t(\cdot,\varphi,\alpha^*) - x_t(\cdot,\varphi,\alpha^N)$ and

$$
\begin{aligned}
F^N(t) &= F(x_t(\cdot,\varphi,\alpha^*),\alpha^N,\alpha^*) \\
&= \int_{-1}^{0} \varphi(s)[k(s,\alpha^*) - k(s,\alpha^N)]\,ds - \int_{-1}^{0} x_t(s,\varphi,\alpha^*)[k(s,\alpha^*) - k(s,\alpha^N)]\,ds.
\end{aligned}
$$

We recall that the function $x_t(\cdot,\varphi,\alpha^*)$ is a known continuous function, since it is the solution of IVP (4.1.1)-(4.1.2) with $\alpha = \alpha^*$. Thus the function F^N is a continuous function of t and α and we have that $F^N \to 0$ as $\alpha^N \to \alpha^*$.

Using well posedness of IVP^N for each N we have that the solution $y_t(\cdot,0,\alpha^N,\alpha^*)$ is a continuous function of the initial data $y_0(s) = 0$, $s \in [-1,0]$ and the right hand side function F^N. Thus,

$$
\lim_{N\to\infty} y_t(\cdot,0,F^N) = y_t(\cdot,0,0) = 0 \tag{4.2.9}
$$

or equivalently,

$$
\lim_{N\to\infty} x_t(\cdot,\varphi,\alpha^N) = x_t(\cdot,\varphi,\alpha^*) \tag{4.2.10}
$$

then, $x_t(\cdot,\varphi,\alpha)$ is continuous at $\alpha = \alpha^*$. Since α^* is arbitrary this proves the lemma.

Remark 4.2.2 Since we are working in a compact set the above lemma establishes the existence of a minimum for problem P. Thus, we can state the following theorem.

Theorem 4.2.3 *The minimization problem P has at least one solution.*

4.2.2 Differentiability of the Solution wrt the Parameter

Lemma 4.2.4 *Let the kernel function $k(s,\cdot)$ in (4.1.3) be differentiable with respect to α and assume $\frac{d}{d\alpha}k(\cdot,\alpha) \in L^1[-1,0]$. Then the solution $x_t(s,\varphi,\cdot)$ of IVP (4.1.1)-(4.1.2) is differentiable with respect to the parameter α. Moreover, the function $\frac{d}{d\alpha}x_t(s,\varphi,\cdot)$ satisfies the following sensitivity equation*

$$\int_{-1}^{0} z_t(s,\alpha)k(s,\alpha)\,ds = \int_{-1}^{0}[\varphi(s)-x_t(s)]\frac{d}{d\alpha}k(s,\alpha)\,ds, \tag{4.2.11}$$

$$z_0(s) = 0, \quad s \in [-1,0]. \tag{4.2.12}$$

Proof: We first note that since $x_t(s,\varphi,\alpha)$ is a continuous function of t and α and $\frac{d}{d\alpha}k(s,\alpha) \in L^1[-1,0]$, $x_t(s,\varphi,\alpha)\frac{d}{d\alpha}k(s,\alpha) \in L^1[-1,0]$. Moreover, $G(t) = \int_{-1}^{0}[\varphi(s)-x_t(s,\varphi,\alpha)]\frac{d}{d\alpha}k(s,\alpha)\,ds$ is a continuous function of t and α and $G(0)=0$. Thus, IVP (4.2.11)-(4.2.12) is well-posed and its solution defines a C_0-semigroup on C.

Let $\{\alpha^N\}_{N\geq 1} \in Q$ and let $\alpha^* \in Q$ fixed and such that $\alpha^N \to \alpha^*$ as $N \to \infty$. Let $x_t(\cdot,\varphi,\alpha^*)$ be the solution of IVP (4.1.1)-(4.1.2) with $\alpha = \alpha^*$. Consider the family of IVP's IVP_N given by

$$\int_{-1}^{0} x_t(s,\varphi,\alpha^N)k(s,\alpha^N)\,ds = \int_{-1}^{0}\varphi(s)k(s,\alpha^N)\,ds + \int_{0}^{t} g(s)\,ds \tag{4.2.13}$$

$$x_0(s,\alpha^N) = \varphi \quad s \in [-1,0], \tag{4.2.14}$$

where g is the right hand side function of IVP (4.1.1)-(4.1.2).

Substracting the integrated form of IVP (4.1.1)-(4.1.2) with $\alpha = \alpha^*$ from IVP's (4.2.13)-(4.2.14) we have

$$\begin{aligned} &\int_{-1}^{0} x_t(s,\varphi,\alpha^N)k(s,\alpha^N)\,ds - \int_{-1}^{0} x_t(s,\varphi,\alpha^*)k(s,\alpha^*)\,ds \\ = &\int_{-1}^{0}\varphi(s)k(s,\alpha^N)\,ds - \int_{-1}^{0}\varphi(s)k(s,\alpha^*)\,ds, \end{aligned}$$

or equivalently

$$
\begin{aligned}
& \int_{-1}^{0} [x_t(s,\varphi,\alpha^N) - x_t(s,\varphi,\alpha^*)] k(s,\alpha^*)\, ds \\
= & \int_{-1}^{0} \varphi(s) k(s,\alpha^N)\, ds - \int_{-1}^{0} \varphi(s) k(s,\alpha^*)\, ds \\
- & \int_{-1}^{0} x_t(s,\varphi,\alpha^N)[k(s,\alpha^N) - k(s,\alpha^*)]\, ds. \qquad (4.2.15)
\end{aligned}
$$

Dividing by $\alpha^N - \alpha^*$ and taking the limit as $\alpha^N \to \alpha^*$ at the right hand side of (4.2.15) we have that

$$
\int_{-1}^{0} \varphi(s) \frac{k(s,\alpha^N) - k(s,\alpha^*)}{\alpha^N - \alpha^*}\, ds \to \int_{-1}^{0} \varphi(s) \frac{d}{d\alpha} k(s,\alpha^*)\, ds, \qquad \alpha^N \to \alpha^*, \qquad (4.2.16)
$$

and

$$
\int_{-1}^{0} x_t(s,\varphi,\alpha^N) \frac{k(s,\alpha^N) - k(s,\alpha^*)}{\alpha^N - \alpha^*}\, ds \to \int_{-1}^{0} x_t(s,\varphi,\alpha^*) \frac{d}{d\alpha} k(s,\alpha^*)\, ds, \qquad \alpha^N \to \alpha^*. \qquad (4.2.17)
$$

Define the function

$$
y_t(s,\alpha^N,\alpha^*) = \frac{x_t(s,\varphi,\alpha^N) - x_t(s,\varphi,\alpha^*)}{\alpha^N - \alpha^*} - z_t(s,0,\alpha^*), \qquad (4.2.18)
$$

where $z_t(\cdot,0,\alpha^*)$ is the solution of IVP (4.2.11)-(4.2.12) for $\alpha = \alpha^*$.

Now we consider a new family of IVP's given by

$$
\begin{aligned}
\int_{-1}^{0} y_t(s,\alpha^N,\alpha^*) k(s,\alpha^*)\, ds &= F(t,\alpha^N,\alpha^*) \qquad (4.2.19) \\
y_0(s) &= 0. \qquad (4.2.20)
\end{aligned}
$$

where

$$
\begin{aligned}
F(t,\alpha^N,\alpha^*) &= \int_{-1}^{0}[\varphi(s)+x_t(s,\varphi,\alpha^N)]\frac{k(s,\alpha^N)-k(s,\alpha^*)}{\alpha^N-\alpha^*}\,ds \\
&- \int_{-1}^{0}[\varphi(s)+x_t(s,\alpha^*)]\frac{d}{d\alpha}k(s,\alpha^*)\,ds. \qquad (4.2.21)
\end{aligned}
$$

From equations (4.2.16) and (4.2.17) we have that $F(t,\alpha^N,\alpha^*)\to 0$ as $\alpha^N\to\alpha^*$. That, together with well-posedness of IVP (4.2.19)-(4.2.20) give us

$$
\lim_{N\to\infty} y_t(\cdot,0,F(t,\alpha,\alpha^N)) = y_t(\cdot,0,0) = 0. \qquad (4.2.22)
$$

Thus,

$$
\begin{aligned}
z_t(\cdot,\alpha^*) &= \lim_{N\to\infty}\frac{x_t(\cdot,\alpha^N)-x_t\cdot,\alpha^*)}{\alpha^N-\alpha^*} \\
&= \frac{d}{d\alpha}x_t(\cdot,\varphi,\alpha^*).
\end{aligned}
$$

Then $z_t(\cdot,\alpha^*)$ is the derivative of x_t with respect to α at $\alpha=\alpha^*$. Since α^* is arbitrary this proves the lemma.

4.3 The Approximated Minimization Problems

We define a sequence of approximated problems P^N by:

$$
\min_{\alpha\in Q} J^N(\alpha) \qquad (4.3.23)
$$

where

$$
J^N(\alpha) = \sum_{i=1}^{n}\|x^N(t_i,\alpha)-y_i\|^2 \qquad (4.3.24)
$$

and $x^N(t,\alpha)$ is the solution of IVP_N given by the approximation problem developed in chapter 3.

Using the same arguments of compactness and continuity of the approximate solutions with respect to the parameter α we can state the existence of solutions for the approximate problems.

Lemma 4.3.5 *Problem P^N has at least one solution.*

4.3.1 Convergence of the Identification Scheme

To prove that the identification scheme is convergent we first need to prove the following lemma.

Lemma 4.3.6 *Let $\{\alpha^N\} \in Q$ with Q a compact set. Let $x_t^N(\cdot,\varphi,\alpha^N)$ be a solution of IVP_N and let $x_t(\cdot,\varphi,\alpha)$ be a solution of IVP* (4.1.1)-(4.1.2) *for $\alpha \in Q$ such that*

$$x_t^N(\cdot,\varphi,\alpha) \to x_t(\cdot,\varphi,\alpha) \tag{4.3.25}$$

as $N \to \infty$. Then there is a subsequence $\{\alpha^{N_j}\} \in Q$ such that $\alpha^{N_j} \to \alpha$, $j \to \infty$ and

$$x_t^N(\cdot,\varphi,\alpha^{N_j}) \to x_t(\cdot,\varphi,\alpha) \tag{4.3.26}$$

as $N, j \to \infty$.

Proof:

The proof clearly follows from the fact that Q is a compact set and the functions x_t^N and x_t are continuous with respect to α.

Now we are at a point where we can prove the convergence of the identification scheme.

Lemma 4.3.7 *Let $\varphi \in C^m[-1,0]$ be the initial function for problem* (4.1.1)-(4.1.2)*. Let α^N be the solution of* (4.3.23) *for each N. Then there is a subsequence $\{\alpha^{N_j}\}$ of $\{\alpha^N\} \in Q$ and an $\alpha^* \in Q$ such that $\alpha^{N_j} \to \alpha^*$, as $N_j \to \infty$ and α^* is the solution of the identification problem* (4.2.5)*.*

Proof:

Using equation (4.3.26) we have

$$\|x_t^N(\cdot,\varphi,\alpha^{N_j}) - x_t(\cdot,\varphi,\alpha^*)\| \to 0 \quad N, j \to \infty \tag{4.3.27}$$

let J^{N_j} be a subsequence of the sequence of approximate minimization problems, then we have

$$J(\alpha^*) = \lim_{N_j \to \infty} J^{N_j}(\alpha^{N_j}) \leq \lim_{N_j \to \infty} J^{N_j}(\alpha), \quad \alpha \in Q \tag{4.3.28}$$

since by definition α^{N_j} is the solution of the approximated minimization problem (4.3.23) for $N = N_j$.

Using equation (4.3.25) and the form of the functionals J^N we have that

$$\lim_{N_j \to \infty} J^{N_j}(\alpha) = J(\alpha). \tag{4.3.29}$$

The above result, together with equation (4.3.28) gives us

$$J(\alpha^*) \leq J(\alpha) \tag{4.3.30}$$

for all $\alpha \in Q$, and this proves the theorem.

4.4 Conclusions

In this chapter we studied identification techniques for IVP (4.5.1)-(4.5.2), however our main goal is to identify a parameter appearing in the singular kernel of the left hand side operator D. We are particularly interested in this problem since IVP (4.5.1)-(4.5.2) can be thought as a simplification of the general aerodynamic model described in chapter 3, subsection 3.5.1. The model contains a weakly singular component. This lead to difficulties when applying numerical schemes. In the next section we present numerical results for this problem. That is, we identify the parameter $\alpha \in (0,1)$ where the kernel of the D operator is given by $k(s,\alpha) = (-s)^{-\alpha}$. Is interesting to note that we solved the identification problem for IVP (4.5.1)-(4.5.2) for both consistent and inconsistent initial data. The identification problem for a parameter appearing at the right hand side operator L was studied by Herdman and Brewer in [6].

4.5 Numerical Examples

For our numerical examples we consider IVP (4.1.1)-(4.1.2) with $g = 0$, $L = 0$ and the kernel function $k(s,\alpha) = (-s)^{-\alpha}$ at equation (4.1.3).

Example 4.5.1 Smooth Initial Condition

For the problem (4.1.1)-(4.1.2) with initial function $\varphi(s) = -s$ we identify the parameter α for both $\alpha = 0.5$ and $\alpha = 0.75$. The numerical approximations use the scheme described in chapter 3. The initial guess were $\alpha_{inic} = 0.4$ and $\alpha_{inic} = 0.85$ respectively. The true solutions $x(t,\alpha)$ are $x(t,0.5) = -t + \frac{4}{\pi}t^{1/2}$ and $x(t,0.75) = -t + 4./\beta(0.25,1.75)t^{3/4}$, respectively. Figures 4.1 and 4.2 show some of the curves for the α's furnished by the minimization subroutines as they converge to $\alpha = 0.75$ and $\alpha = 0.5$, respectively. The optimal values obtained were $\alpha_{opt} = 0.7499$ and $\alpha_{opt} = 0.4999$, respectively. The stepzise used for the numerical approximations was $1/h^2$ with $h = 10$. The measurements $y_i, i = 1,\cdots,10$ of the real solution $x(t,\alpha)$ for both $\alpha = 0.5$ and $\alpha = 0.75$ were taken at the fixed points $t_i = 1/40 + i/h, i = 1,\cdots,10$, that is we considered one measurement $y_i = y(t_i)$ for each subinterval σ_i defined for the numerical approximation.

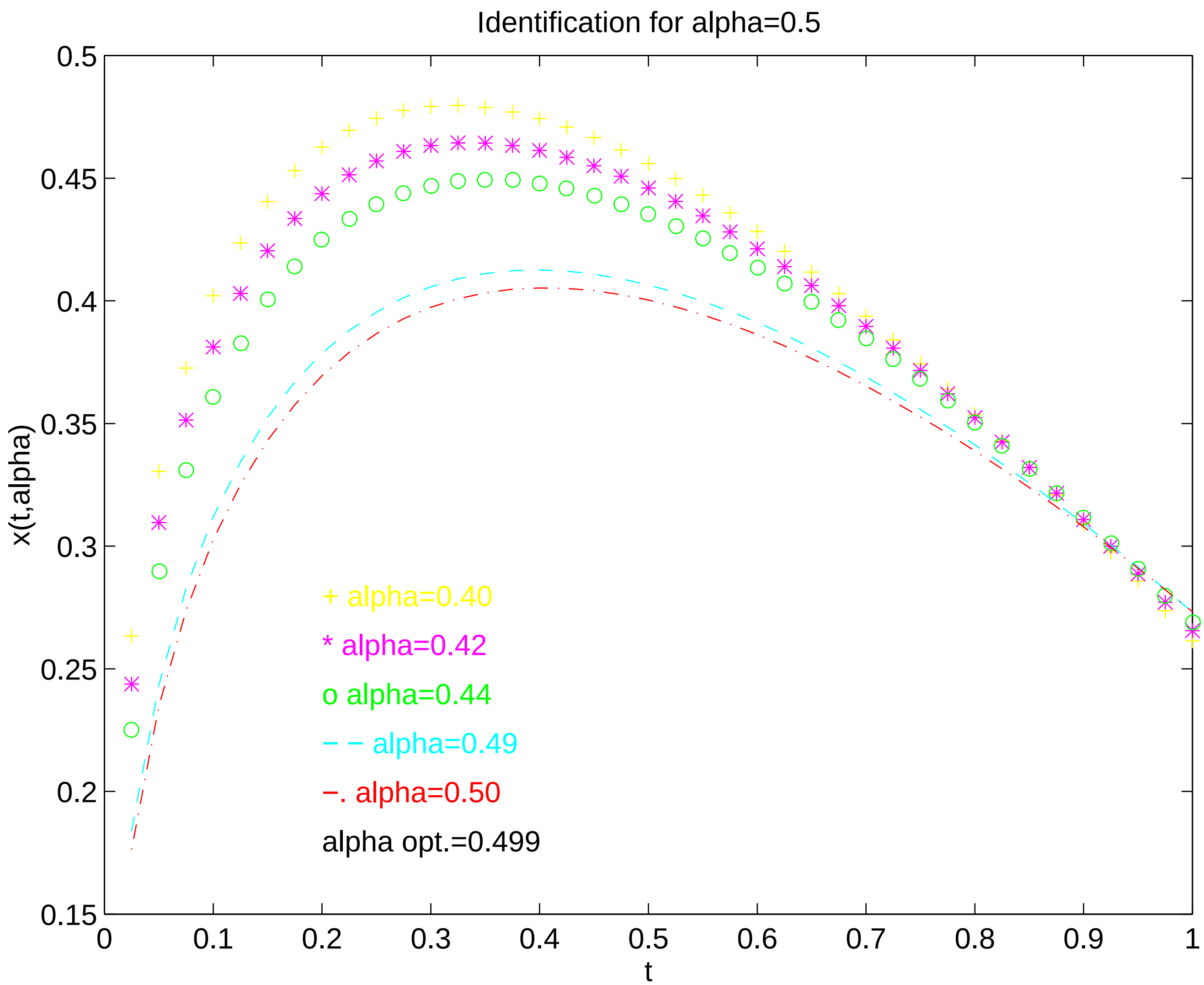

Figure 4.1: Smooth initial condition

Example 4.5.2 Discontinuous Initial Condition

We consider the scalar equation

$$\int_{-1}^{0} (-s)^{-\alpha} x(t+s)\, ds = 1, \qquad t > 0 \tag{4.5.1}$$

with initial data

$$\varphi(s) = \begin{cases} 0 & \text{if } s \in (-1,0) \\ 1/2 & \text{if } s = 0. \end{cases} \tag{4.5.2}$$

For more details on the numerical approximation to the solution of this problem see chapter 3 section 3.6.3.

Figures 4.3 and 4.4 show the numerical results of identifying the parameter α on the IVP for (4.5.1)-(4.5.2). In Figure 4.3 we considered measurements on $[0,1]$. The initial guess was $\alpha_{inic} = 0.4$. The optimun value obtained was $\alpha_{opt} = 0.518$. In Figure 4.4 we identify $\alpha = 0.5$ but the measurements x_i were taken for t_i in the interval $[1,2]$. The initial guess was $\alpha_{inic} = 0.4$. The optimum value obtained was $\alpha_{opt} = 0.512$. Figure 4.5 shows the same curves for $t \in (0,2)$. The stepzise use for the numerical approximation in all of these cases was $1/h^2$ with $h = 50$. The measurements $y_i, i = 1, \cdots, 50$ of the real solution $x(t, 0.5)$ were taken at the fixed points $t_i = 1/40 + i/h, i = 1, \cdots, 50$, that is we considered one measurement $y_i = y(t_i)$ for each subinterval σ_i defined for the numerical approximation.

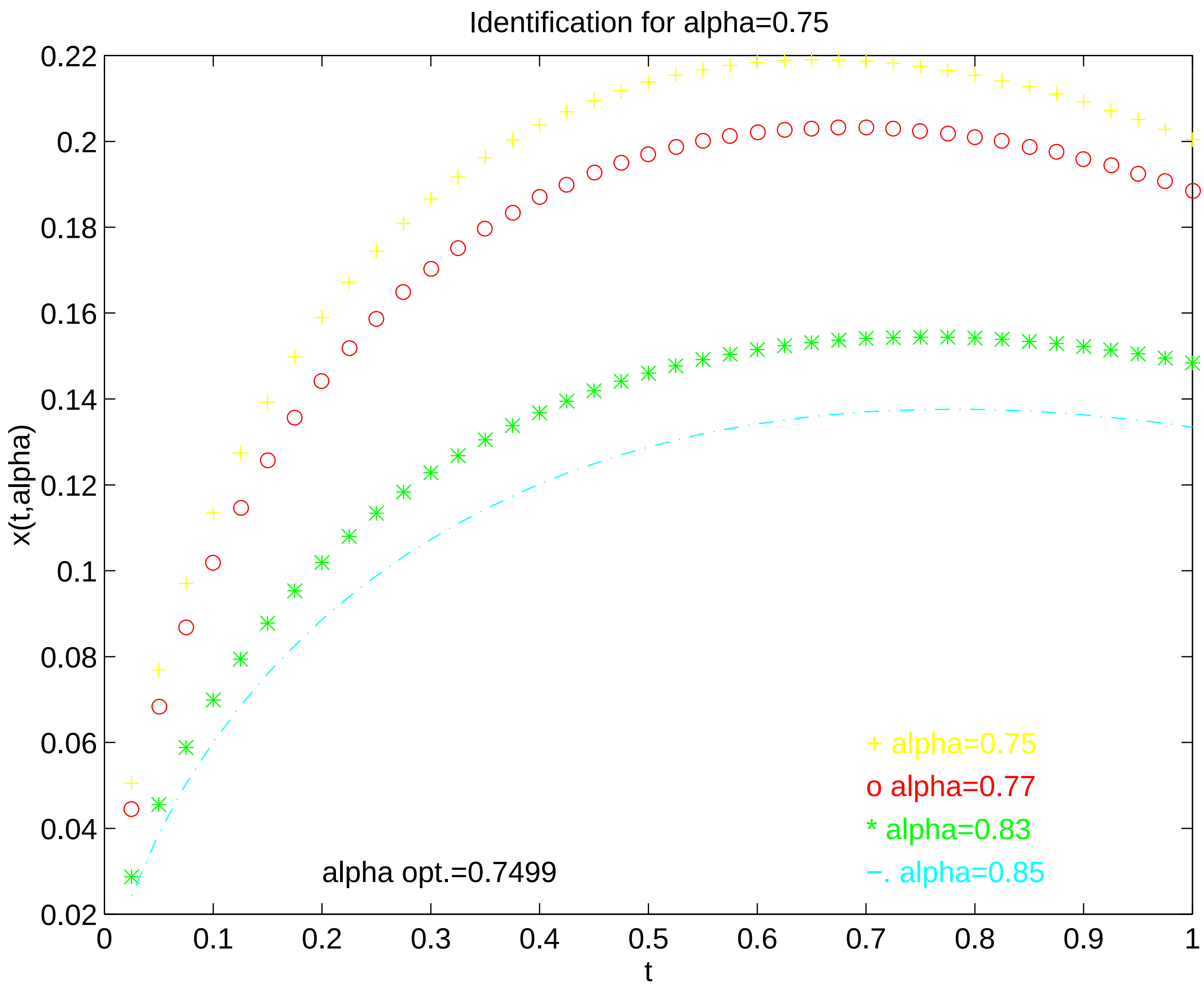

Figure 4.2: Smooth initial condition

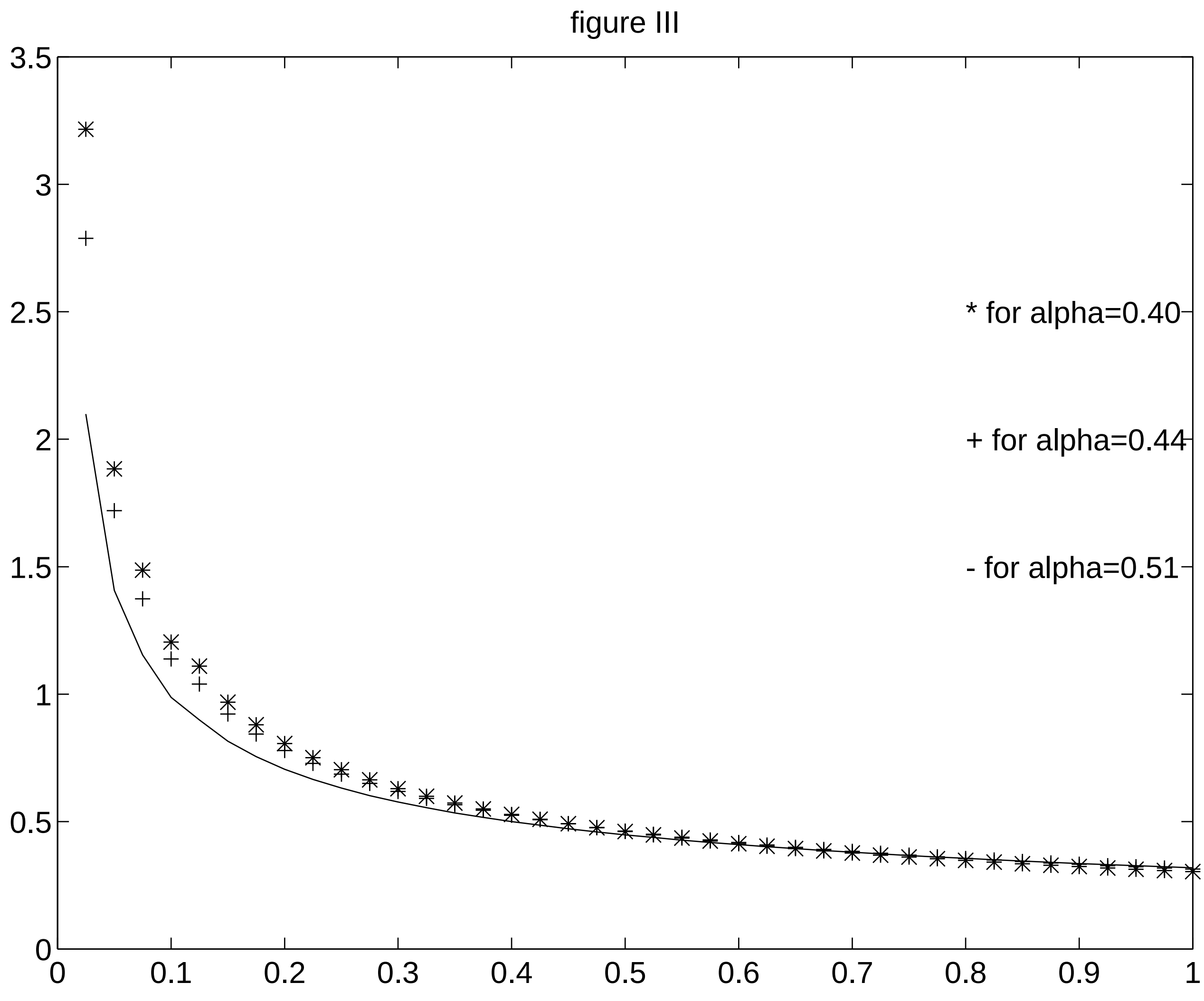

Figure 4.3: Discontinuous initial condition

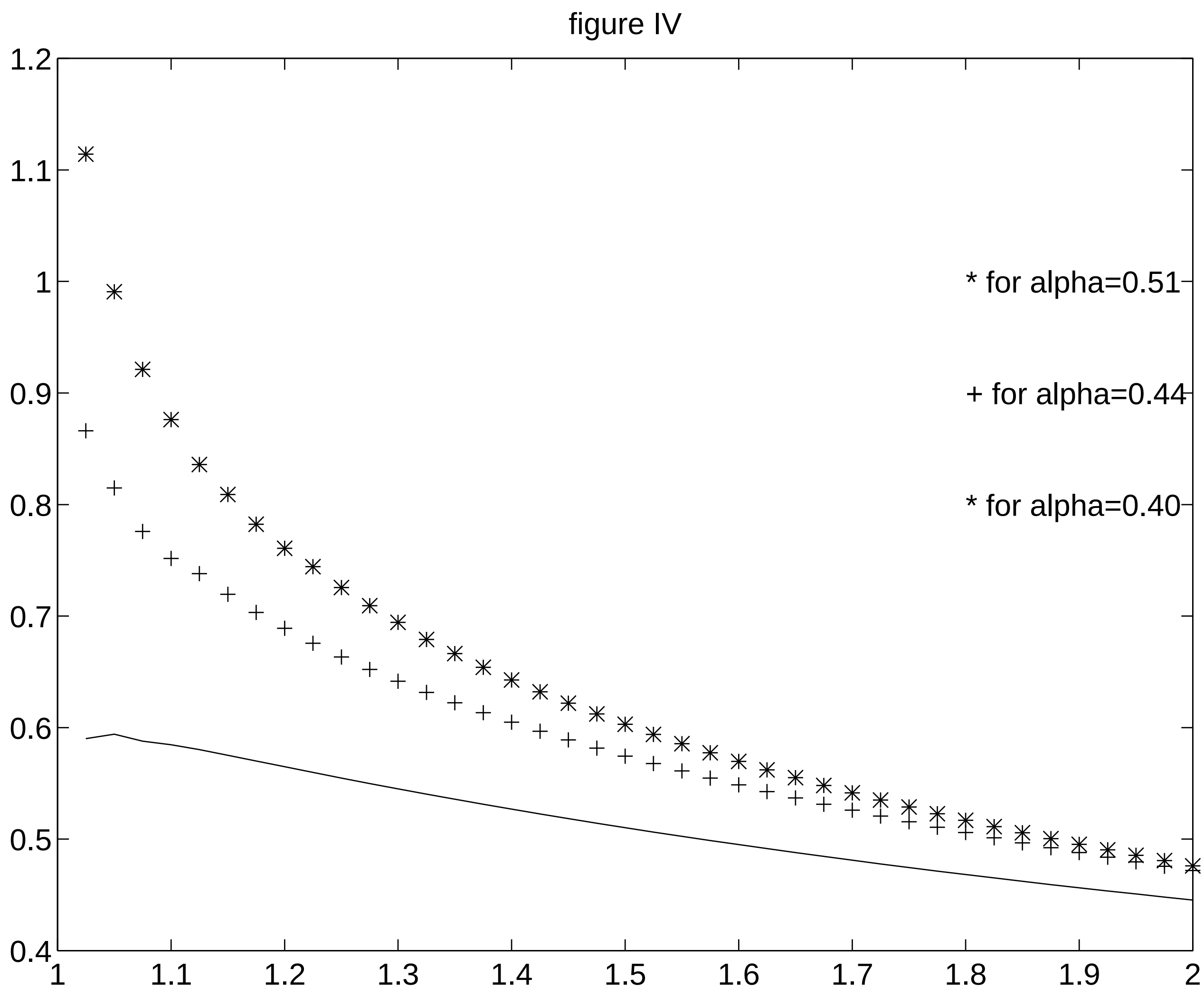

Figure 4.4: Discontinuous initial condition

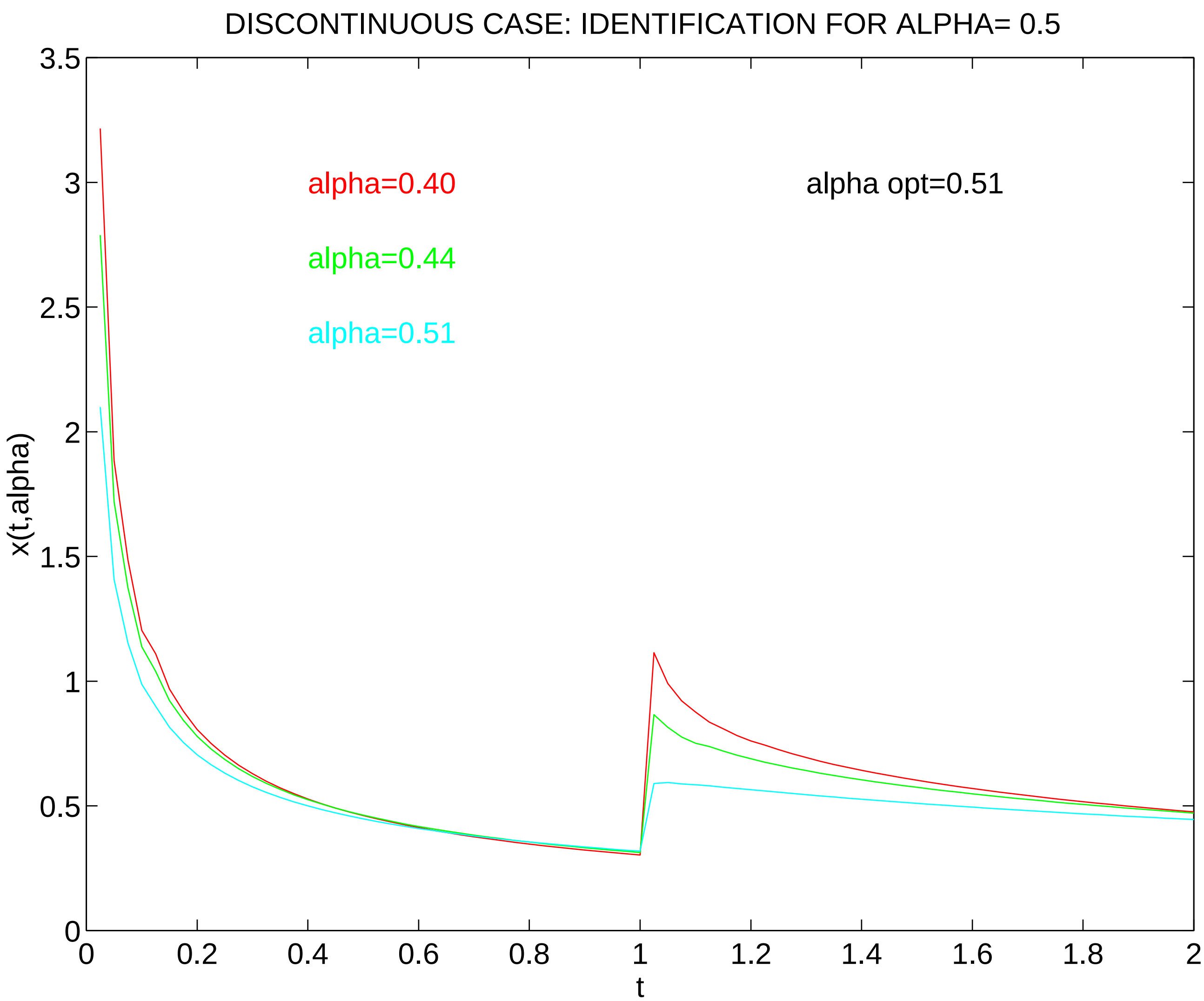

Figure 4.5: Discontinuous initial condition

Bibliography

[1] H. T. Banks, J. A. Burns, *Hereditary control problems: numerical methods based on averaging approximations*, SIAM J. Control and Optimization, Vol 16, 2, March 1978, pp. 169-208.

[2] H. T. Banks, J.A. Burns and E. M. Cliff, Parameter Estimation and *Identification for Systems with Delays, SIAM J. Control and Optimization, Vol 19, 6, pp 791-829, 1981.*

[3] H. T. Banks, J. A. Burns, E. M. Cliff, *Spline-based approximation methods for control and identification of hereditary systems*, International Symposium on Systems Optimization and Analysis, A. Bensoussan and J. L. Lions, eds.,Lecture Notes in Control and Information Science, 14, Springer, Heidelberg, 1979, pp. 314-320.

[4] H. T. Banks and K. Kunich, *Estimation Techniques for Distributed Parameter Systems, Birkhauser, 1989.*

[5] H. T. Banks and P.K. Daniel Lamm, *Estimation of Delays and Other Parameters in Nonlinear Functional Differential Equations, SIAM J. Control and Optimization, Vol 21, 6, pp 895-915, 1983.*

[6] D. W. Brewer and T. L. Herdman, *A parameter dependence problem in neutral functional differential equations, Pitman Resaerch Notes in Mathematics Series, Vol 273*

[7] H. Brunner *Non polynomial spline collocation for Volterra equations with weakly singular kernels*, SIAM J.Numer. Anal., 6 (1983) pp. 1106-1119.

[8] H. Brunner. *The numerical solution of Volterra Equations North-Holland. 1986*

[9] H. Brunner. *On the discretization of Volterra integral equations, Nieuw Archief Voor Wisskunde (4), Vol. 2, 1984.*

[10] H. Brunner. *Superconvergence of Collocation Methods for Volterra Integral Equations of the First Kind, Computing 21, pp. 151-157, 1979.*

[11] H. Brunner. *A Note on Collocation Methods for Volterra Integral Equations of the First Kind. Computing 23, pp. 179-187, 1979.*

[12] J. Burns, E.Cliff and T. Herdman,*A State-Space Model for an aeroelastic system*, Proc. 22nd IEEE Conf. on Decision and Control, San Antonio, Tx, Dec 14-16,1983, pp. 1074-1077.

[13] J. Burns, T. Herdman and H. Stech, *Linear Functional Differential Equations as Semigroups on Product Spaces*, SIAM J. Math. Anal., 14 (1983), pp. 98-116.

[14] J. A. Burns, T. L. Herdman, J. Turi *Neutral Functional Integro Differential Equations with Weakly Singular Kernels* , J. Math. Anal. Appl., 145 (1990), pp. 371-401.

[15] J. Burns, K. Ito, *On well-posedness of integro- differential equations in weighted L^2 -spaces*, Differential and Integral Equations, Vol 8, Number 3 (March 1995), pp. 627-646.

[16] Y. Cao, G. M. Cerezo, T. L. Herdman, J. Turi, *Singular expantion of solutions for neutral equations*, preprint.

[17] Y. Cao, Y. Xu *Singulatrity Preserving Galerking Methods for Weakly Singular Fredholm Integral Equations*, preprint.

[18] G. M. Cerezo, *Approximation and Identification Problems for Neutral Functional Differential Equations*, Master's Thesis, Virginia Polytechnic Institute and State University, Blacksburg, Virginia, 1994.

[19] G. M. Cerezo, T. L. Herdman, E. Fernandez, J. Turi *Collocation tecchniques for singular equations*, Proc. 33nd IEEE Conf. on Decision and Control, Orlando, Fl, Dec 14-16, 1994.

[20] G. M. Cerezo, T. L. Herdman, E. Fernandez, J. Turi *Parameter identification techniques for singular neutral equations*, preprint.

[21] Eggermont P.P. *Collocation for Volterra Integral Equations of the first kind with iterated kernel. Siam J. Numer Amal. Vol 20, No5, Oct. 1983.*

[22] M. A. Golberg. *A Survey of Numerical Methods for Integral Equations, Solution Methods for Integral Equations. Theory and Applications. Plenum Press N.Y.*

[23] W. Hackbusch . *Integral Equations. Theory and numerical treatment. ISNM Vol. 120, 1995.*

[24] E. Hewitt, K.A. Ross *Abstract Harmonic Analysis I.* Springer-Verlag, Berlin, 1970.

[25] K. Ito, F. Kappel, J. Turi *On Well-Posedness of Singular Neutral Equations in The State Space C* , J. Diff. Eqts., Vol 125, No 1, 99 40-72, 1996.

[26] K. Ito, F. Kappel *On Integro Differential Equations with Weakly Singular Kernels*, "Differential Equations with Applications in Biology, Physics and Engineering" ,(J. A. Goldstein, F. Kappel and W. Schappacher, Eds.) , 1991, pp. 209-218, Marcel Dekker, New York.

[27] K. Ito and J. Turi, *Numerical Methods for a Class of Singular Integro-Differential Equations Based on Semigroup Approximation, SIAM J. Num. Anal. Vol 28, No 6,pp.1698-1722, Dec. 1991*

[28] F. Kappel, K. P. Zhang *On Neutral Functional Differential Equations with Nonatomis Difference Operator*, J. Math. Anal. Appl.,113 (1986) pp. 311-343.

[29] P. Linz *Analytical and Numerical Methods for Volterra Equations.*, SIAM, Philadelphia, 1985.

[30] Murphy,K. A. *Estimation of time and state dependent delays and other parameters in functional differential equations*, SIAM J. Appl. Math., 50, 4, pp 972-1000, 1990.

[31] D. Salamon *Control and observation of neutral systems.* Pitman Advanced Publishing Program, 1984.

[32] L. Schumaker. *Spline Functions: Basic theory, Wiley, N. York, 1981.*

[33] A. Pazzi, *Semigroups of linear operators and applications to partial differential equations.*, Springer-Verlag.

[34] J. L. Phillips, *The use of collocation as a projection method for solving linear operator equations*, SIAM J. Numer. Anal., 9 (1972), pp. 14-28.

VITA

The author was born in Buenos Aires, Argentina in 1962. She got her B.S./M.S. in Applied Mathematics form the University of Buenos Aires in Marh 1990. Since 1987 until 1990 she worked as a teaching assistant at the university of Buenos Aires. During that period she taught mathematics and biostatistics courses for undregraduate students. Form 1990 to 1992 she worked as a researcher at the National Commission of Atomic Energy in Buenos Aires. In 1992 she joined the graduate program in Applied Mathematics at the Virginia Polytechnic Institute and State University in Blacksburg, Virginia. From 1992 to 1996 she worked as a teaching assistant at Virginia Tech. She got her MS from that university in May 1994, and her Ph. D. in Mathematics in December 1996.