# Optimization Based Domain Decomposition Methods for Linear and Nonlinear Problems

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## (ABSTRACT)

Optimization based domain decomposition methods for the solution of partial differential equations are considered. The crux of the method is a constrained minimization problem for which the objective functional measures the jump in the dependent variables across the common boundaries between subdomains; the constraints are the partial differential equations.

First, we consider a linear constraint. The existence of optimal solutions for the optimization problem is shown as is its convergence to the exact solution of the given problem. We then derive an optimality system of partial differential equations from which solutions of the domain decomposition problem may be determined. Finite element approximations to solutions of the optimality system are defined and analyzed as is an eminently parallelizable gradient method for solving the optimality system. The linear constraint minimization problem is also recast as a linear least squares problem and is solved by a conjugate gradient method.

The domain decomposition method can be extended to nonlinear problems such as the Navier-Stokes equations. This results from the fact that the objective functional for the minimization problem involves the jump in dependent variables across the interfaces between subdomains. Thus, the method does not require that the partial differential equations themselves be derivable through an extremal problem. An optimality system is derived by applying a Lagrange multiplier rule to a constrained optimization problem. Error estimates for finite element approximations are presented as is a gradient method to solve the optimality system. We also use a Gauss-Newton method to solve the minimization problem with the nonlinear constraint.

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# Chapter 1

# Introduction

Numerical methods based on domain decomposition are used in a wide variety of applications as a powerful technique to compute solutions of partial differential equations. The basic idea of a domain decomposition method is to split the whole domain into smaller ones so that the overall solution of a large problem can be obtained by solving smaller problems in subdomains. The method is especially good when a computer's memory is not large enough for the complete problem or when a domain has an irregular shape, which often happens in practical applications. Due to the obvious implication for parallel processing, domain decomposition methods have been the subject of great attention. There is a large-scale international conference held on the subject every year, and the processing of these and other conferences provide both a summary and a history of developments in the field; see e.g., [11], [12], [13], [29], [30], and [46].

Domain decomposition methods can be classified as either a non-overlapping or an overlapping subdomains problem. The classic domain decomposition method for overlapping subdomains is the Schwarz alternating method, in which subdomain problems are solved successively with boundary conditions obtained in the previous iteration. Schwarz's algorithm has been studied and developed from various points of view; e.g., [9], [19],[20], [21], [23], [24], [47], [48] and [49]. The classical Schwarz method was reexamined by Lions [47], [48] and extended to nonoverlapping subdomains [49]. As shown in Lions [47], [48], the classical Schwarz method can be interpretated as successive projections onto subspaces. The relevant error propagation operator of a particular Schwarz method is a polynomial of these projections and, in that case, the method is not ideal for parallel computing. Based on Lions' work, Dryja and Widlund developed and analyzed the additive Schwarz methods for more effective parallel computing in [20], [21], [23] and [56].

Another well-known family of domain decomposition method is the so-called substructuring iteration methods or Schur methods which are for non-overlapping subdomain problems. The main technique of these methods is to reduce an elliptic problem to an operator equation for lower-dimensional interfaces between subdomains. Such an operator is called a Poincaré-Steklov's operator. After the given differential equation is discretized by a finite element or finite different method, linear systems correponding to subdomains are derived. Then, a discretized Poincaré-Steklov's operator, which is called a Schur complement, is obtained from the coefficient matrix by a block elimination procedure, and solved by iterative methods of conjugate gradient type. In most cases, preconditioners are used for the conjugate gradient method to reduce the number of iterations. The main focus in the study of these methods has been to find and analyze these preconditioners; [14], [55]

In some cases, Schwarz iteration can be defined as a map on interfaces, thus regarded as substructuring methods. It was shown by Dryja and Widlund [22] that substructuring methods and Schwarz methods can be unified in certain cases. Both Schwarz methods and substructuring methods are based on the continuity of a solution or derivatives of the solution on interfaces. The continuity plays a role of intermediation between subdomains so that Dirichlet or Neumann boundary conditions for the local problem can be passed from one to the other. Our method also starts from imposing continuous Neumann boundary conditions for decomposed local problems.

In this thesis, we propose new domain decomposition methods based on optimization which we analyze, develop and implement numerical algorithms. We emphasize that the method easily generalizes to dimensions higher than two, to more than two subdomains, and to even nonlinear equations. In Chapter 2, we first consider the method in the context of the Poisson equation with a homogeneous Dirichlet boundary condition and for only two subdomains to simplify the exposition. In Chaper 3, the method is then extended to the multi-subdomain case for the same model problem. In Chapter 4, we apply the method to the incompressible stationary Navier-Stokes equations as an application for nonlinear problems.

## 1.1 Domain decomposition for a linear problem

Let $\Omega$ be a bounded open set in $\mathrm{R}^2$ with boundary $\Gamma$. We consider the model problem

$$-\Delta u = f \quad \text{in } \Omega \qquad \text{and} \qquad u = 0 \quad \text{on } \Gamma. \tag{1.1}$$

We consider a decomposition of the domain $\Omega$ into two disjoint open subdomains $\Omega_1$, $\Omega_2$ such that $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$. The *interface* between the two domains is denoted by $\Gamma_0$ so that

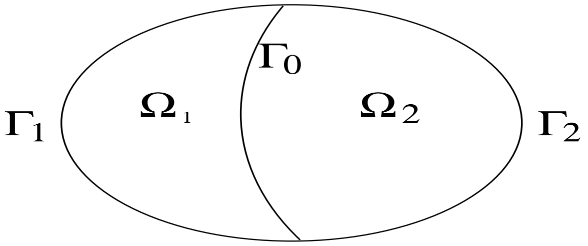

Figure 1.1: Decomposition of $\Omega$

$\Gamma_0 = \overline{\Omega}_1 \cap \overline{\Omega}_2$. Let $\Gamma_1 = \overline{\Omega}_1 \cap \Gamma$ and $\Gamma_2 = \overline{\Omega}_2 \cap \Gamma$. See Figure 1.1. We wish to determine the solution of (1.1) by solving the following pair of Poisson equations with mixed Dirichlet and Neumann boundary conditions:

$$-\Delta u_1 = f \quad \text{in } \Omega_1, \quad u_1 = 0 \quad \text{on } \Gamma_1 \quad \text{and} \quad \frac{\partial u_1}{\partial n_1} = g \quad \text{on } \Gamma_0 \tag{1.2}$$

and

$$-\Delta u_2 = f \quad \text{in } \Omega_2, \quad u_2 = 0 \quad \text{on } \Gamma_2 \quad \text{and} \quad \frac{\partial u_2}{\partial n_2} = -g \quad \text{on } \Gamma_0, \tag{1.3}$$

where, for $i = 1, 2$, $\partial/\partial n_i$ denotes the derivative in the direction of the unit outer normal to $\Omega_i$. We are particularly interested in the data $g$ in the Neumann boundary condition; in this thesis we will refer to $g$ as the *control*.

For an arbitrary choice for the control $g$, the solutions $u_1$ and $u_2$ of (1.2) and (1.3), respectively, do not agree with the solution $u$ of (1.1) in the respective subdomains, i.e., $u_1 \neq u|_{\Omega_1}$ and $u_2 \neq u|_{\Omega_2}$. The discrepancy is due to the fact that for an arbitrary choice of $g$, we have that $u_1 \neq u_2$ along $\Gamma_0$, even in a weak sense. On the other hand, there clearly exists a choice of $g$, namely $g = (\partial u/\partial n_1)|_{\Gamma_0} = -(\partial u/\partial n_2)|_{\Gamma_0}$, such that the solutions of (1.2) and (1.3) coincide with the solution of (1.1) on the corresponding subdomains.

Thus, our goal is to determine a function $g$ such that $u_1$ is as close as possible to $u_2$ along the interface $\Gamma_0$. One way of trying to achieve this goal is to minimize, over an appropriate class of candidate functions $g$, the functional

$$\mathcal{K}(u_1, u_2) = \frac{1}{2} \int_{\Gamma_0} (u_1 - u_2)^2 \, d\Gamma_0, \tag{1.4}$$

where $u_1$ and $u_2$ are determined from $g$ through (1.2) and (1.3), respectively. It is clear that there exists a minimizer for $\mathcal{K}(\cdot, \cdot)$; indeed, for the choice $g = (\partial u/\partial n_1)|_{\Gamma_0} = -(\partial u/\partial n_2)|_{\Gamma_0}$, where $u$ is the solution of (1.1), we have that $u_1 = u|_{\Omega_1}$ and $u_2 = u|_{\Omega_2}$ so that $\mathcal{K} = 0$.

In order to regularize the problem we penalize the functional $\mathcal{K}(\cdot,\cdot)$ by a measure of the size of the control. Thus, instead of minimizing (1.4), we minimize the functional

$$\begin{aligned} \mathcal{J}_\delta(u_1,u_2,g) &= \mathcal{K}(u_1,u_2) + \frac{\delta}{2}\int_{\Gamma_0} g^2 \, d\Gamma_0 \\ &= \frac{1}{2}\int_{\Gamma_0} (u_1-u_2)^2 \, d\Gamma_0 + \frac{\delta}{2}\int_{\Gamma_0} g^2 \, d\Gamma_0, \end{aligned} \tag{1.5}$$

where $\delta$ is a positive constant that can be chosen to change the relative importance of the two terms appearing in (1.5). Our domain decomposition method is then based on the following optimization problem:

*minimize* $\mathcal{J}_\delta(u_1,u_2,g)$ *over suitable functions* $g$ *subject to (1.2) and (1.3).*

The introduction of the penalty term in (1.5) also results in a simplification of the numerical algorithm. On the other hand, for any $\delta > 0$, minimizers of $\mathcal{J}_\delta(\cdot,\cdot,\cdot)$ do not satisfy $u_1|_{\Gamma_0} = u_2|_{\Gamma_0}$ so that a solution of the minimization problem does not exactly correspond to a solution of (1.1). We will study this issue in §2.3.

Domain decomposition methods based on optimization strategies have been previously proposed in [26], [27], [33] and [34]. For example, in [33], the minimization of an appropriate functional forces the optimal solution to satisfy the partial differential equations and the constraint imposed forces the solutions on the two subdomains to agree on the interface. Thus, in that paper, the role of the functional and constraints are reversed from that in our method. It is shown in [33] that the method proposed there is equivalent to a non-overlapping Schwarz method.

In Chapter 2, we give a precise restatement of the optimization problem, we prove that optimal solutions exist, and study the convergence of optimal solutions as the penalty parameter $\delta \to 0$. We use two different approaches to solve the optimization problem. First, we derive an optimality system of equations from which optimal solutions may be determined, we examine finite element approximations of the solutions of the optimality system, and study a parallelizable gradient method for the solution of the optimality system. As the second approach, the optimization problem is recast as a linear least squares problem by defining a linear operator, and a conjugate gradient algorithm for the linear least squares problem is examined. We also present the results of some numerical experiments. In Chapter 3, the method is examined for an arbitrary number of subdomains. We also discuss nonuniqueness of the solution of the Poisson equation and a solvability condition for interior subdomain problems.

## 1.2 Domain decomposition for a nonlinear problem

The appearance of the partial differential equations as a constraint in our methods renders them easy to generalize to other, more complex, even nonlinear problems. Also, different finite element discretizations based on different grid sizes and different polynomials may be used in each subdomain. For the same reasons, problems with discontinuous media that result in discontinuous coefficients in the partial differential equations can also be easily treated by the methods under discussion. Perhaps the best feature is the ease in which the methods can be extended to nonlinear problems. This results from the fact that the objective functional for the minimization problem involves the jump in dependent variables across the interfaces between subdomains and thus the method does not require that the partial differential equations themselves be derivable through an extremal problem.

Domain decomposition methods for solutions of the Stokes equations or the Navier-Stokes equations have been studied in many papers, e.g., [4], [8], [25], [32], [50] and [54]. For example, Fortin and Aboulaich [25] used the well-known Schwarz's method for the solutions of the Navier-Stokes equations, and Cahouet [8] used a substructuring method for the incompressible Stokes equations by defining a Poincaré-Steklov operator as a jump of a stress vector across the interface. Azaiez and Quarteroni [4], and Carlenzoli, Quateroni and Valli [10] used spectral domain decomposition methods for the Stokes equations and the Navier-Stokes equations. Various other methods such as a Fourier-method and a pseudospectral method can be found [50] and [54]. From the point of view of finite element discretizations, domain decomposition methods using mixed finite element methods were studied in, e.g., [24], [31], [32] and [34].

Suppose $\Omega$ denotes a bounded open set in $\mathrm{R}^2$ with boundary $\Gamma$ and is divided into two disjoint open subdomains $\Omega_1$ and $\Omega_2$. See Figure 1.1. If $\mathbf{u}$ denotes the velocity vector, $p$ the pressure, $\mathbf{f}$ a given body force, and $\nu$ the constant kinetic viscosity, then the problem to be solved is

$$-\nu\nabla\cdot(\nabla\mathbf{u}+(\nabla\mathbf{u})^T)+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p=\mathbf{f}\quad\text{in }\Omega,\tag{1.6}$$

$$\mathbf{u}=0\quad\text{on }\Gamma\tag{1.7}$$

and

$$\nabla\cdot\ \mathbf{u}=0\quad\text{in }\Omega.\tag{1.8}$$

We again decompose (1.6)-(1.8) into two subproblems for each subdomain by imposing a stress condition $\mathbf{g}$ on the interface $\Gamma_0$, which will be referred to as the *control*;

$$-\nu\nabla\cdot(\nabla\mathbf{u}_1+(\nabla\mathbf{u}_1)^T)+\mathbf{u}_1\cdot\nabla\mathbf{u}_1+\nabla p_1=\mathbf{f}\quad\text{in }\Omega_1,\tag{1.9}$$

$$\nabla\cdot\mathbf{u}_1=0\quad\text{in }\Omega_1,\tag{1.10}$$

$$\mathbf{u}_1 = 0 \quad \text{on } \Gamma_1, \tag{1.11}$$

$$-p_1\mathbf{n}_1 + \nu(\nabla\mathbf{u}_1 + (\nabla\mathbf{u}_1)^T)\cdot\mathbf{n}_1 - \frac{1}{2}(\mathbf{u}_1\cdot\mathbf{n}_1)\mathbf{u}_1 = \mathbf{g} \quad \text{on } \Gamma_0, \tag{1.12}$$

and

$$-\nu\nabla\cdot(\nabla\mathbf{u}_2 + (\nabla\mathbf{u}_2)^T) + \mathbf{u}_2\cdot\nabla\mathbf{u}_2 + \nabla p_2 = \mathbf{f} \quad \text{in } \Omega_2, \tag{1.13}$$

$$\nabla\cdot\mathbf{u}_2 = 0 \quad \text{in } \Omega_2, \tag{1.14}$$

$$\mathbf{u}_2 = 0 \quad \text{on } \Gamma_2, \tag{1.15}$$

$$-p_2\mathbf{n}_2 + \nu(\nabla\mathbf{u}_2 + (\nabla\mathbf{u}_2)^T)\cdot\mathbf{n}_2 - \frac{1}{2}(\mathbf{u}_2\cdot\mathbf{n}_2)\mathbf{u}_2 = -\mathbf{g} \quad \text{on } \Gamma_0, \tag{1.16}$$

where $\mathbf{n}_1$ and $\mathbf{n}_2$ denote the outward normal vectors to $\Omega_1$ and $\Omega_2$, respectively. Again, for an arbitrary choice for $\mathbf{g}$, solutions of (1.9)-(1.12) and (1.13)-(1.16) are not solutions of (1.6)-(1.8), e.g., $\mathbf{u}_1 \neq \mathbf{u}|_{\Omega_1}$ and $\mathbf{u}_2 \neq \mathbf{u}|_{\Omega_2}$. Here, we restrict our interest to the velocity vector $\mathbf{u}$ only for our domain decomposition method, thus we want to find $\mathbf{g}$ which minimizes $\mathbf{u}_1 - \mathbf{u}_2$ on $\Gamma_0$; i.e., the functional we want to minimize is

$$\mathcal{K}(\mathbf{u}_1, \mathbf{u}_2) = \frac{1}{2}\int_{\Gamma_0} (\mathbf{u}_1 - \mathbf{u}_2)^2 \, d\Gamma_0, \tag{1.17}$$

or

$$\mathcal{J}_\delta(\mathbf{u}_1, \mathbf{u}_2, \mathbf{g}) = \frac{1}{2}\int_{\Gamma_0} (\mathbf{u}_1 - \mathbf{u}_2)^2 + \frac{\delta}{2}\int_{\Gamma_0} \mathbf{g}^2 \, d\Gamma_0, \tag{1.18}$$

subject to the constraints (1.9)-(1.12) and (1.13)-(1.16). If $(\mathbf{u}, p)$ denotes a solution of (1.6)-(1.8), then, by choosing $\mathbf{g} = [-p\mathbf{n}_1 + \nu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)\cdot\mathbf{n}_1 - \frac{1}{2}(\mathbf{u}\cdot\mathbf{n}_1)\mathbf{u}]|_{\Gamma_0}$, we can have $\mathbf{u}_1 = \mathbf{u}_2$ on $\Gamma_0$ so that $\mathcal{K}(\mathbf{u}_1, \mathbf{u}_2) = 0$. But, unlike the Poisson equation, such $\mathbf{g}$ is not unique, since $p$ is determined only up to an arbitrary additive constant. Note that we did not impose any condition on $p$ in (1.6)-(1.8). On the other hand, $p_1$ and $p_2$ are determined uniquely by solving subproblems (1.9)-(1.12) and (1.13)-(1.16). The nonuniqueness of the optimal solution might yield some difficulties in actual computations, and we will discuss some problems encountered. The development of algorithms proceeds in a manner analogous to that for the Poisson equation.

# Chapter 2

# Domain decomposition for a linear problem

## 2.1 Statement of the problem, notation

Let $\Omega$ be a bounded open set in $\mathbf{R}^2$ with boundary $\Gamma$. We consider the Poisson equation

$$-\Delta u = f \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \Gamma \tag{2.1}$$

as our model problem for the linear case. Let $\Omega$ be divided into two disjoint open subdomains $\Omega_1$, $\Omega_2$ such that $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$. The *interface* between the two subdomains is denoted by $\Gamma_0$ so that $\Gamma_0 = \overline{\Omega}_1 \cap \overline{\Omega}_2$. Let $\Gamma_1 = \overline{\Omega}_1 \cap \Gamma$ and $\Gamma_2 = \overline{\Omega}_2 \cap \Gamma$. See Figure 1.1.

Now consider the following pair of Poisson equations with mixed Dirichlet and Neumann boundary conditions:

$$-\Delta u_1 = f \quad \text{in } \Omega_1, \quad u_1 = 0 \quad \text{on } \Gamma_1, \quad \text{and} \quad \frac{\partial u_1}{\partial n_1} = g \quad \text{on } \Gamma_0 \tag{2.2}$$

and

$$-\Delta u_2 = f \quad \text{in } \Omega_2, \quad u_2 = 0 \quad \text{on } \Gamma_2, \quad \text{and} \quad \frac{\partial u_2}{\partial n_2} = -g \quad \text{on } \Gamma_0. \tag{2.3}$$

Let the functional $\mathcal{J}_\delta(\cdot,\cdot,\cdot)$ be defined by

$$\mathcal{J}_\delta(u_1, u_2, g) = \frac{1}{2}\int_{\Gamma_0} (u_1 - u_2)^2 \, d\Gamma_0 + \frac{\delta}{2}\int_{\Gamma_0} g^2 \, d\Gamma_0. \tag{2.4}$$

Then we consider the following optimization problem:

minimize $\mathcal{J}_\delta(\cdot,\cdot,\cdot)$ over suitable functions $g$ subject to (2.2) and (2.3).

We now introduce some notation that will be used throughout this chapter. Let $L^2(D)$, $H^1(D)$ denote the standard Sobolev spaces with respect to a domain $D$ with the standard norm $\|\cdot\|_{1,D}$, $\|\cdot\|_{0,D}$. We then define the subspaces

$$\begin{aligned} H^1_0(\Omega) &= \{v \in H^1(\Omega) : \ v = 0 \quad \text{on } \Gamma\}, \\ H^1_{\Gamma_1}(\Omega_1) &= \{v \in H^1(\Omega_1) : \ v = 0 \quad \text{on } \Gamma_1\} \end{aligned}$$

and

$$H^1_{\Gamma_2}(\Omega_2) = \{v \in H^1(\Omega_2) : \ v = 0 \quad \text{on } \Gamma_2\}.$$

Inner products on $L^2(\Omega_i)$, i=1,2, and $L^2(\Gamma_0)$ are defined by

$$(v,w)_{\Omega_i} = \int_{\Omega_i} vw \, d\Omega_i$$

and

$$(v,w)_{\Gamma_0} = \int_{\Gamma_0} vw \, d\Gamma_0,$$

respectively. Dual spaces will be denoted by $(\cdot)^*$.

## 2.2 The existence of an optimal solution

Let $g$ denote the boundary control and assume $u_1$, $u_2$ satisfy (2.2) and (2.3), respectively. Then, a weak formulation corresponding to (2.2) and (2.3) is given by:
find $u_1 \in H^1_{\Gamma_1}(\Omega_1)$ and $u_2 \in H^1_{\Gamma_2}(\Omega_2)$ such that

$$a_1(u_1,v) = (f,v)_{\Omega_1} + (g,v)_{\Gamma_0} \quad \forall v \in H^1_{\Gamma_1}(\Omega_1), \tag{2.5}$$

$$a_2(u_2,v) = (f,v)_{\Omega_2} - (g,v)_{\Gamma_0} \quad \forall v \in H^1_{\Gamma_2}(\Omega_2) \tag{2.6}$$

where, for $i = 1, 2$,

$$a_i(u,v) = \int_{\Omega_i} \nabla u \cdot \nabla v \, d\Omega_i.$$

It is well known that the bilinear forms $a_i(\cdot,\cdot)$, $i=1,2$, are coercive and continuous, i.e., there exist constants $C_a>0$ and $C_b>0$ such that

$$a_i(u,u) \geq C_a \, \|u\|^2_{1,\Omega_i} \quad \forall u \in H^1(\Omega_i),\ i=1,2,$$

and

$$a_i(u,v) \leq C_b \, \|u\|_{1,\Omega_i} \|v\|_{1,\Omega_i} \quad \forall u,\, v \in H^1(\Omega_i),\ i=1,2;$$

see, e.g., [28] and [52]. The existence of a unique solution of (2.5) and (2.6) follows from the Lax-Milgram Theorem. Also, that theorem yields the continuous dependence on the data, i.e., for $i=1,2$, there exists a constant $C>0$ such that

$$\|u_i\|_{1,\Omega_i} \leq C(\|f\|_{0,\Omega_i} + \|g\|_{0,\Gamma_0}). \tag{2.7}$$

Of course, we also have the standard weak formulation corresponding to (2.1). If we denote the solution of (2.1) by $u^{ex}$, then

$$\int_\Omega \nabla u^{ex} \cdot \nabla v \; d\Omega = \int_\Omega f v \; d\Omega \quad \forall v \in H^1_0(\Omega). \tag{2.8}$$

Note that if $g^{ex} = (\partial u^{ex}/\partial n_1)|_{\Gamma_0} = -(\partial u^{ex}/\partial n_2)|_{\Gamma_0}$, then

$$a_1(u^{ex},v) = (f,v)_{\Omega_1} + (g^{ex},v)_{\Gamma_0}, \quad \forall v \in H^1_{\Gamma_1}(\Omega_1) \tag{2.9}$$

and

$$a_2(u^{ex},v) = (f,v)_{\Omega_2} - (g^{ex},v)_{\Gamma_0}, \quad \forall v \in H^1_{\Gamma_2}(\Omega_2). \tag{2.10}$$

Next, we examine the existence of an optimal solution that minimizes $\mathcal{J}_\delta(\cdot,\cdot,\cdot)$. Let the *admissibility set* be defined by

$$\begin{aligned} \mathcal{U}_{ad} \;=\; & \{(u_1,u_2,g) \in H^1_{\Gamma_1}(\Omega_1) \times H^1_{\Gamma_2}(\Omega_2) \times L^2(\Gamma_0) : \\ & (2.5),\ (2.6) \text{ are satisfied and } \mathcal{J}_\delta(u_1,u_2,g) < \infty\}. \end{aligned} \tag{2.11}$$

Then, $(\hat{u}_1,\hat{u}_2,\hat{g}) \in \mathcal{U}_{ad}$ is called an *optimal solution*, if there exists $\epsilon>0$ such that

$$\mathcal{J}_\delta(\hat{u}_1,\hat{u}_2,\hat{g}) \leq \mathcal{J}_\delta(u_1,u_2,g) \tag{2.12}$$

for all $(u_1,u_2,g) \in \mathcal{U}_{ad}$ satisfying

$$\|u_1-\hat{u}_1\|_{1,\Omega_1} + \|u_2-\hat{u}_2\|_{1,\Omega_2} + \|g-\hat{g}\|_{0,\Gamma_0} \leq \epsilon. \tag{2.13}$$

**Theorem 2.1** *There is a unique optimal solution* $(\hat{u}_1,\hat{u}_2,\hat{g}) \in \mathcal{U}_{ad}$.

**Proof:** Clearly, $\mathcal{U}_{ad}$ is not empty. Let $\{(u_1^{(n)}, u_2^{(n)}, g^{(n)})\}$ be a minimizing sequence in $\mathcal{U}_{ad}$; i.e.,

$$\lim_{n\to\infty} \mathcal{J}_\delta(u_1^{(n)}, u_2^{(n)}, g^{(n)}) = \inf_{(u_1,u_2,g)\in\mathcal{U}_{ad}} \mathcal{J}_\delta(u_1, u_2, g).$$

Then, from (2.11), we have that the sequence $\{g^{(n)}\}$ is uniformly bounded in $L^2(\Gamma_0)$. Moreover, by (2.7), $\{u_1^{(n)}\}$ and $\{u_2^{(n)}\}$ are uniformly bounded. Consequently, there exists a subsequence $\{(u_1^{(n_i)}, u_2^{(n_i)}, g^{(n_i)})\}$ such that

$$\begin{aligned} u_1^{(n_i)} &\rightharpoonup \hat{u}_1 \quad \text{in } H^1_{\Gamma_1}(\Omega_1), \\ u_2^{(n_i)} &\rightharpoonup \hat{u}_2 \quad \text{in } H^1_{\Gamma_2}(\Omega_2), \\ g^{(n_i)} &\rightharpoonup \hat{g} \quad \text{in } L^2(\Gamma_0) \end{aligned}$$

for some $(\hat{u}_1, \hat{u}_2, \hat{g}) \in \mathcal{U}_{ad}$. Now, by the process of passing to the limit, we have that $(\hat{u}_1, \hat{u}_2, \hat{g})$ satisfies (2.5) and (2.6). Also, the fact that the functional $\mathcal{J}_\delta(\cdot,\cdot,\cdot)$ is lower semi-continuous implies that

$$\begin{aligned} \inf_{(u_1,u_2,g)\in\mathcal{U}_{ad}} \mathcal{J}_\delta(u_1, u_2, g) &= \lim_{i\to\infty} \inf \mathcal{J}_\delta(u_1^{(n_i)}, u_2^{(n_i)}, g^{(n_i)}) \\ &\geq \mathcal{J}_\delta(\hat{u}_1, \hat{u}_2, \hat{g}). \end{aligned}$$

Hence, we must have $\mathcal{J}_\delta(\hat{u}_1, \hat{u}_2, \hat{g}) = \inf_{(u_1,u_2,g)\in\mathcal{U}_{ad}} \mathcal{J}_\delta(u_1, u_2, g)$ so that we conclude $(\hat{u}_1, \hat{u}_2, \hat{g})$ is an optimal solution. Uniqueness follows from the convexity of the functional, the admissibility set and the linearity of the constraints. □

## 2.3 Convergence with vanishing penalty parameter

The functional (2.4) contains the penalty parameter $\delta$ that controls the relative importance of the two terms. Clearly, for finite values of $\delta$, solutions of the optimization problem (2.11)-(2.12) will not satisfy (2.8), the latter being a weak formulation of (2.1). In the next theorem we show that optimal solutions converge to the solution of (2.1) as $\delta \to 0$. In this section, for simplicity of notation, we denote an optimal solution satisfying (2.11)-(2.12) and which corresponds to a given value of $\delta$ by $(u_1^\delta, u_2^\delta, g^\delta)$.

**Theorem 2.2** *For each $\delta$, let $(u_1^\delta, u_2^\delta, g^\delta)$ denote an optimal solution satisfying (2.11)-(2.12). Let $u^{ex}$ denote the solution of (2.1) and let $u_1^{ex} = u^{ex}|_{\Omega_1\cup\Gamma_0}$ and $u_2^{ex} = u^{ex}|_{\Omega_2\cup\Gamma_0}$. Then $\|u_1^\delta - u_1^{ex}\|_{1,\Omega_1} \to 0$ and $\|u_2^\delta - u_2^{ex}\|_{1,\Omega_2} \to 0$ as $\delta \to 0$.*

**Proof:** Let $g^{ex} = \partial u_1^{ex}/\partial n_1$ on $\Gamma_0$. Suppose $\{(u_1^\delta, u_2^\delta, g^\delta)\}$ is a sequence of optimal solutions and $\delta$ goes to 0. Then, we have that

$$\mathcal{J}_\delta(u_1^\delta, u_2^\delta, g^\delta) \leq \mathcal{J}_\delta(u_1^{ex}, u_2^{ex}, g^{ex}) \quad \forall \delta, \tag{2.14}$$

i.e.,

$$\frac{1}{2}\int_{\Gamma_0} (u_1^\delta - u_2^\delta)^2 \, d\Gamma_0 + \frac{\delta}{2}\int_{\Gamma_0} (g^\delta)^2 \, d\Gamma_0 \leq \frac{\delta}{2}\int_{\Gamma_0} (g^{ex})^2 \, d\Gamma_0 \quad \forall \delta. \tag{2.15}$$

Then, $\|g^\delta\|_{0,\Gamma_0}$ is uniformly bounded in $L^2(\Gamma_0)$ and $\|u_1^\delta - u_2^\delta\|_{0,\Gamma_0} \to 0$ if $\delta \to 0$. We also obtain, by (2.7), that $\|u_1^\delta\|_{1,\Omega_1}$ and $\|u_2^\delta\|_{1,\Omega_2}$ are uniformly bounded. Hence, as $\delta \to 0$, there is a subsequence which converges to some $(u_1^*, u_2^*, g^*) \in H^1_{\Gamma_1}(\Omega_1) \times H^1_{\Gamma_2}(\Omega_2) \times L^2(\Gamma_0)$ and the fact that $\|u_1^\delta - u_2^\delta\|_{\Gamma_0} \to 0$ yields $u_1^* = u_2^*$. By passing to the limit, we have

$$\begin{aligned} a_1(u_1^*, v) &= (f, v)_{\Omega_1} + (g^*, v)_{\Gamma_0} \quad \forall v \in H^1_{\Gamma_1}(\Omega_1), \\ a_2(u_2^*, v) &= (f, v)_{\Omega_2} - (g^*, v)_{\Gamma_0} \quad \forall v \in H^1_{\Gamma_2}(\Omega_2). \end{aligned}$$

If we define $u^* \in H^1_0(\Omega)$ by

$$u^* = \begin{cases} u_1^* & \text{in } \Omega_1 \cup \Gamma_0 \\ u_2^* & \text{in } \Omega_2 \cup \Gamma_0, \end{cases} \tag{2.16}$$

then, $u^*$ satisfies

$$\int_\Omega \nabla u^* \cdot \nabla v \, d\Omega = (f, v)_\Omega \quad \forall v \in H^1_0(\Omega)$$

and thus, by the uniqueness of the solution of (2.8), we conclude that $u^* = u^{ex}$. □

We can actually obtain a rate of convergence for $(u_1^\delta - u_1^{ex})$ and $(u_2^\delta - u_2^{ex})$ as $\delta \to 0$.

**Theorem 2.3** *Assume the hypothesis of Theorem 2.2. Let* $g^{ex} = \partial u_1^{ex}/\partial n_1$ *on* $\Gamma_0$. *Then,*

$$\begin{aligned} \|u_1^\delta - u_1^{ex}\|_{1,\Omega_1} + \|u_2^\delta - u_2^{ex}\|_{1,\Omega_2} &\leq C\delta^{1/4}\|g^{ex}\|_{0,\Gamma_0}^{1/4}\|g^\delta - g^{ex}\|_{0,\Gamma_0}^{3/4} \\ &\leq C\delta^{1/4}\|g^{ex}\|_{0,\Gamma_0}, \end{aligned} \tag{2.17}$$

*where* $C$ *does not depend on* $\delta$.

**Proof:** From (2.5), (2.6), (2.9) and (2.10), we have that

$$\begin{aligned} a_1(u_1^\delta - u_1^{ex}, v_1) &= (g^\delta - g^{ex}, v_1)_{\Gamma_0} \quad \forall v_1 \in H^1_{\Gamma_1}(\Omega_1), \\ a_2(u_2^\delta - u_2^{ex}, v_2) &= -(g^\delta - g^{ex}, v_2)_{\Gamma_0} \quad \forall v_2 \in H^1_{\Gamma_2}(\Omega_2). \end{aligned}$$

Setting $v_1 = u_1^\delta - u_1^{ex}$ and $v_2 = u_2^\delta - u_2^{ex}$ and noting that $u_1^{ex} = u_2^{ex}$ on $\Gamma_0$, we obtain

$$a_1(u_1^\delta - u_1^{ex}, u_1^\delta - u_1^{ex}) + a_2(u_2^\delta - u_2^{ex}, u_2^\delta - u_2^{ex}) = (g^\delta - g^{ex}, u_1^\delta - u_2^\delta)_{\Gamma_0}.$$

Then, by the coercivity properties of the forms $a_i(\cdot,\cdot)$, $i = 1, 2$, we obtain

$$C_a(\|u_1^\delta - u_1^{ex}\|_{1,\Omega_1}^2 + \|u_2^\delta - u_2^{ex}\|_{1,\Omega_2}^2) \le \|g^\delta - g^{ex}\|_{0,\Gamma_0}\|u_1^\delta - u_2^\delta\|_{0,\Gamma_0}. \tag{2.18}$$

Next, as in the proof of Theorem 2.2, we obtain

$$\int_{\Gamma_0} |u_1^\delta - u_2^\delta|^2 \, d\Gamma_0 \le \delta \int_{\Gamma_0} (|g^\delta|^2 - |g^{ex}|^2) \, d\Gamma_0 \quad \forall \delta$$

so that, for all $\delta$,

$$\begin{aligned} \|u_1^\delta - u_2^\delta\|_{0,\Gamma_0}^2 &\le \delta \int_{\Gamma_0} (g^\delta + g^{ex})(g^\delta - g^{ex}) \, d\Gamma_0 \\ &\le \delta \|g^\delta + g^{ex}\|_{0,\Gamma_0}\|g^\delta - g^{ex}\|_{0,\Gamma_0} \le 2\delta \|g^{ex}\|_{0,\Gamma_0}\|g^\delta - g^{ex}\|_{0,\Gamma_0}, \end{aligned} \tag{2.19}$$

where we have used $\|g^\delta\|_{0,\Gamma_0} \le \|g^{ex}\|_{0,\Gamma_0}$, a result that follows directly from the functional $\mathcal{J}_\delta(\cdot,\cdot,\cdot)$ and the definitions of $g^{ex}$ and $g^\delta$. The combination of (2.18) and (2.19) yields (2.17). □

## 2.4 The optimality system

In this section we use the Lagrange multiplier rule to derive an optimality system of equations from which solutions of the optimization problem (2.11)-(2.12) may be determined. We first show the existence of suitable Lagrange multipliers. In the present context it suffices to show that the Fréchet derivatives of the linear constraint functions with respect to $u_1$ and $u_2$, respectively, have closed ranges. For more details, see [53]. For $i = 1, 2$, let the linear operators $M_i : H^1_{\Gamma_i}(\Omega_i) \times L^2(\Gamma_0) \to H^1_{\Gamma_i}(\Omega_i)^*$ be defined as follows: $M_i(u_i, g) = f_i$ for $(u_i, g) \in H^1_{\Gamma_i}(\Omega_i) \times L^2(\Gamma_0)$ and $f_i \in H^1_{\Gamma_i}(\Omega_i)^*$ if and only if

$$a_1(u_1, v)_{\Omega_1} = (f_1, v)_{\Omega_1} + (g, v)_{\Gamma_0} \quad \forall v \in H^1_{\Gamma_1}(\Omega_1)$$

and

$$a_2(u_2, v)_{\Omega_2} = (f_2, v)_{\Omega_2} - (g, v)_{\Gamma_0} \quad \forall v \in H^1_{\Gamma_2}(\Omega_2).$$

Note that the Fréchet derivative of $M_i$ is the operator itself; i.e., if $(\hat{u}_1, \hat{u}_2, \hat{g})$ is an optimal solution, $M_i'(\hat{u}_i, \hat{g}) \in \mathcal{L}(H^1_{\Gamma_i}(\Omega_i) \times L^2(\Gamma_0), H^1_{\Gamma_i}(\Omega_i)^*)$ is defined as

$$M_i'(\hat{u}_i, \hat{g}) \cdot (w_i, s) = \bar{f}_i$$

if and only if

$$a_1(w_1, v) = (\bar{f}_1, v)_{\Omega_1} + (s, v)_{\Gamma_0} \quad \forall v \in H^1_{\Gamma_1}(\Omega_1)$$

and

$$a_2(w_2, v) = (\bar{f}_2, v)_{\Omega_2} - (s, v)_{\Gamma_0} \quad \forall v \in H^1_{\Gamma_2}(\Omega_2).$$

It is clear that the operators $M_1$, $M_2$ are onto for any $s \in L^2(\Gamma_0)$, and therefore they have closed ranges.

Now, for $(u_1, u_2, g, \lambda_1, \lambda_2) \in H^1_{\Gamma_1}(\Omega_1) \times H^1_{\Gamma_2}(\Omega_2) \times L^2(\Gamma_0) \times H^1_{\Gamma_1}(\Omega_1) \times H^1_{\Gamma_2}(\Omega_2)$, define the Lagrangian

$$\begin{aligned} \mathcal{L}(u_1, u_2, g, \lambda_1, \lambda_2) &= \mathcal{J}_\delta(u_1, u_2, g) - a_1(u_1, \lambda_1) + (f, \lambda_1)_{\Omega_1} + (g, \lambda_1)_{\Gamma_0} \\ &- a_2(u_2, \lambda_2) + (f, \lambda_2)_{\Omega_2} - (g, \lambda_2)_{\Gamma_0}, \end{aligned} \tag{2.20}$$

where $\mathcal{J}_\delta(\cdot, \cdot, \cdot)$ is defined by (2.4). The constrained problem (2.11)-(2.12) can now be recast as the unconstrained problem of finding stationary points of $\mathcal{L}(\cdot, \cdot, \cdot, \cdot, \cdot)$. We now apply the necessary conditions for the latter problem. Clearly, setting to zero the first variations with respect to the Lagrange multipliers $\lambda_1$ and $\lambda_2$ yields the constaints (2.5) and (2.6), respectively. Setting to zero the first variations with respect to $u_1$ and $u_2$ yields the adjoint, or co-state, equations

$$a_1(\xi, \lambda_1) = (u_1 - u_2, \xi)_{\Gamma_0} \quad \forall \xi \in H^1_{\Gamma_0}(\Omega_1) \tag{2.21}$$

and

$$a_2(\xi, \lambda_2) = -(u_1 - u_2, \xi)_{\Gamma_0} \quad \forall \xi \in H^1_{\Gamma_0}(\Omega_2), \tag{2.22}$$

respectively. Finally, setting to zero the first variation with respect to $g$ yields the optimality condition

$$(g, r)_{\Gamma_0} = -\frac{1}{\delta}(\lambda_1 - \lambda_2, r)_{\Gamma_0} \quad \forall r \in L^2(\Gamma_0). \tag{2.23}$$

To summarize, solutions of the optimization problem (2.11)-(2.12) may be determined by solving the optimality system (2.5)-(2.6) and (2.21)-(2.23). This optimality system may be viewed as a weak formulation of the problem (2.2) and (2.3),

$$\Delta\lambda_1 = 0 \quad \text{in } \Omega_1, \qquad \lambda_1 = 0 \quad \text{on } \Gamma_1 \quad \text{and} \quad \frac{\partial \lambda_1}{\partial n_1} = u_1 - u_2 \quad \text{on } \Gamma_0, \tag{2.24}$$

$$\Delta\lambda_2 = 0 \quad \text{in } \Omega_2 \qquad \lambda_2 = 0 \quad \text{on } \Gamma_2 \quad \text{and} \quad \frac{\partial \lambda_2}{\partial n_2} = -(u_1 - u_2) \quad \text{on } \Gamma_0 \tag{2.25}$$

and

$$g = -\frac{1}{\delta}(\lambda_1 - \lambda_2) \quad \text{on } \Gamma_0. \tag{2.26}$$

Note that we may use (2.26) to eliminate $g$ from (2.2) and (2.3).

The optimality system (2.5)-(2.6) and (2.21)-(2.23) may also be derived in a direct manner using "sensitivity" derivatives instead of the Lagrange multiplier rule; see [17] or [38]. Let

$$\mathcal{M}_\delta(g) = \mathcal{J}_\delta(u_1(g), u_2(g), g), \tag{2.27}$$

where, for given $g$,

$$u_i(g) \ : \ g \in L^2(\Gamma_0) \ \ \rightarrow \ \ H^1_{\Gamma_i}(\Omega_i) \quad \text{for } i = 1, 2,$$

are defined as the solutions of (2.5) and (2.6), respectively. Then, the minimization problem (2.11)-(2.12) is equivalent to the problem of determining $g \in L^2(\Gamma_0)$ such that $\mathcal{M}(g)$ is minimized.

Now, the first derivative of $\mathcal{M}_\delta$, $d\mathcal{M}_\delta(g)/dg$, is defined through its action on a variation $\tilde{g}$ by

$$< \frac{d\mathcal{M}_\delta(g)}{dg}, \tilde{g} >= \delta(g, \tilde{g})_{\Gamma_0} + (u_1 - u_2, \tilde{u}_1 - \tilde{u}_2)_{\Gamma_0} \ \ \forall \tilde{g} \in L^2(\Gamma_0), \tag{2.28}$$

where $\tilde{u}_1 \in H^1_{\Gamma_1}(\Omega_1)$ and $\tilde{u}_2 \in H^1_{\Gamma_2}(\Omega_2)$ are the solutions of

$$\begin{aligned} a_1(\tilde{u}_1, v) &= (\tilde{g}, v)_{\Gamma_0} \ \ \forall v \in H^1_{\Gamma_1}(\Omega_1), && (2.29)\\ a_2(\tilde{u}_2, v) &= -(\tilde{g}, v)_{\Gamma_0} \ \ \forall v \in H^1_{\Gamma_2}(\Omega_2), && (2.30)\end{aligned}$$

respectively. Thus, $\tilde{u}_1$ and $\tilde{u}_2$ give the changes in $u_1$ and $u_2$ that result from the change $\tilde{g}$ in $g$. Let $u_1$, $u_2$ be the solutions of (2.5), (2.6) and define $\lambda_1$ and $\lambda_2$ be the solutions of the adjoint problems (2.21) and (2.22), respectively. Set $v = \lambda_1$ in (2.29), $v = \lambda_2$ in (2.30), $\xi = \tilde{u}_1$ in (2.21) and $\xi = \tilde{u}_2$ in (2.22). Combining the results yields that

$$(\tilde{g}, \lambda_1 - \lambda_2)_{\Gamma_0} = (\tilde{u}_1 - \tilde{u}_2, u_1 - u_2)_{\Gamma_0}$$

so that, from (2.28),

$$< \frac{d\mathcal{M}_\delta(g)}{dg}, \tilde{g} >= \delta(g, \tilde{g})_{\Gamma_0} + (\tilde{g}, \lambda_1 - \lambda_2)_{\Gamma_0} \ \ \forall \tilde{g} \in L^2(\Gamma_0). \tag{2.31}$$

Thus, the first order necessary condition $d\mathcal{M}_\delta/dg = 0$ yields that

$$\delta(g, \tilde{g})_{\Gamma_0} = -(\tilde{g}, \lambda_1 - \lambda_2)_{\Gamma_0} \ \ \forall \tilde{g} \in L^2(\Gamma_0),$$

which is exactly (2.23).

**Remark 2.1** *Note that (2.31) yields an explicit formula for the gradient of* $\mathcal{M}_\delta$*, i.e.,*

$$\frac{d\mathcal{M}_\delta(g)}{dg} = \delta g + (\lambda_1 - \lambda_2)|_{\Gamma_0}. \tag{2.32}$$

*Identity (2.32) gives information needed if one were to use a gradient method, e.g., a method that requires* $\mathcal{M}_\delta$ *and* $d\mathcal{M}_\delta(g)/dg$ *for a given approximation of* $g$*, to solve our optimization problem.*

## 2.5 Finite element approximations

Let ${W_1}^h$, ${W_2}^h$ denote finite element spaces such that ${W_1}^h \subset H^1_{\Gamma_1}(\Omega_1)$ and ${W_2}^h \subset H^1_{\Gamma_2}(\Omega_2)$. We assume that the usual approximation properties hold, i.e., for $i = 1, 2$, there exist an integer $k$ and a constant $C$ such that

$$\inf_{v^h \in W_i^h} \|v - v^h\|_{1,\Omega_i} \le Ch^m \|v\|_{m+1,\Omega_i} \tag{2.33}$$

$\forall v \in H^1_{\Gamma_i}(\Omega_i) \cap H^{m+1}(\Omega_i)$ and $0 \le m \le k$; see [15] and [28]. The finite element approximations of solutions of the optimality system (2.5)-(2.6) and (2.21)-(2.23) are defined as follows. Seek ${u_1}^h \in {W_1}^h$, ${u_2}^h \in {W_2}^h$, ${\lambda_1}^h \in {W_1}^h$ and ${\lambda_2}^h \in {W_2}^h$ such that

$$a_1({u_1}^h, v^h) = (f, v^h)_{\Omega_1} + \frac{1}{\delta}(-{\lambda_1}^h + {\lambda_2}^h, v^h)_{\Gamma_0} \quad \forall v^h \in {W_1}^h, \tag{2.34}$$

$$a_2({u_2}^h, v^h) = (f, v^h)_{\Omega_2} - \frac{1}{\delta}(-{\lambda_1}^h + {\lambda_2}^h, v^h)_{\Gamma_0} \quad \forall v^h \in {W_2}^h, \tag{2.35}$$

$$a_1({\lambda_1}^h \xi^h) = ({u_1}^h - {u_2}^h, \xi^h)_{\Gamma_0} \quad \forall \xi^h \in {W_1}^h, \tag{2.36}$$

and

$$a_2({\lambda_2}^h, R^h) = -({u_1}^h - {u_2}^h, \xi^h)_{\Gamma_0} \quad \forall \xi^h \in {W_2}^h. \tag{2.37}$$

Note that we have used (2.26) to eliminate the control $g$ from the optimality system.

**Remark 2.2** *The finite element spaces* $W_1^h$ *and* $W_2^h$ *are defined separately in each subdomain* $\Omega_1$ *and* $\Omega_2$*, respectively. Thus, different grids and/or different degree polynomial may be used in each subdomain. In particular, there is no requirement that the grids in the two subdomains match along the interface* $\Gamma_0$ *nor that the finite element functions be continuous across that interface.*

We obtain error estimates for the finite element approximations through the use of the Brezzi-Rappaz-Raviart theory; see [7]. Let

$$X = H^1_{\Gamma_1}(\Omega_1) \times H^1_{\Gamma_1}(\Omega_1) \times H^1_{\Gamma_2}(\Omega_2) \times H^1_{\Gamma_2}(\Omega_2),$$

$$Y = H^1_{\Gamma_1}(\Omega_1)^* \times H^1_{\Gamma_1}(\Omega_1)^* \times H^1_{\Gamma_2}(\Omega_2)^* \times H^1_{\Gamma_2}(\Omega_2)^*,$$

$$Z = L^2(\Gamma_0) \times L^2(\Gamma_0) \times L^2(\Gamma_0) \times L^2(\Gamma_0),$$

and

$$X^h = {W_1}^h \times {W_1}^h \times {W_2}^h \times {W_2}^h,$$

where $H^1_{\Gamma_i}(\Omega_i)^*$ is the dual space of $H^1_{\Gamma_i}(\Omega_i)$ for $i = 1, 2$. Note that the imbedding $Z \subset Y$ is compact. Let $\Lambda$ be a positive compact interval containing $1/\delta$ and define the operator $B \in \mathcal{L}(Y, X)$ as follows: For $(\tilde{Q}_1, \tilde{P}_1, \tilde{Q}_2, \tilde{P}_2) \in Y$ and $(\tilde{u}_1, \tilde{\lambda}_1, \tilde{u}_2, \tilde{\lambda}_2) \in X$

$$B(\tilde{Q}_1, \tilde{P}_1, \tilde{Q}_2, \tilde{P}_2) = (\tilde{u}_1, \tilde{\lambda}_1, \tilde{u}_2, \tilde{\lambda}_2)$$

if and only if

$$a_i(\tilde{u}_i, v) = (\tilde{Q}_i, v)_{\Omega_i} \quad \forall v \in H^1_{\Gamma_i}(\Omega_i),$$

$$a_i(\tilde{\lambda}_i, r) = (\tilde{P}_i, r)_{\Omega_i} \quad \forall r \in H^1_{\Gamma_i}(\Omega_i)$$

for $i = 1, 2$. Also the discretized operator $B_h \in \mathcal{L}(Y, X^h)$ is defined as follows: for $(\tilde{Q}_1, \tilde{P}_1, \tilde{Q}_2, \tilde{P}_2) \in Y$ and $(\tilde{u}_1^h, \tilde{\lambda}_1^h, \tilde{u}_2^h, \tilde{\lambda}_2^h) \in X^h$

$$B_h(\tilde{Q}_1, \tilde{P}_1, \tilde{Q}_2, \tilde{P}_2) = (\tilde{u}_1^h, \tilde{\lambda}_1^h, \tilde{u}_2^h, \tilde{\lambda}_2^h)$$

if and only if

$$a_i(\tilde{u}_i^h, s^h) = (\tilde{Q}_i, s^h)_{\Omega_i} \quad \forall s^h \in {W_i}^h, \tag{2.38}$$

$$a_i(\tilde{\lambda}_i^h, r^h) = (\tilde{P}_i^h, r^h)_{\Omega_i} \quad \forall r^h \in {W_i}^h \tag{2.39}$$

for $i = 1, 2$. Then, by the approximation properties (2.33), we have that

$$\lim_{h \to 0} \|(\tilde{u}_1, \tilde{\lambda}_1, \tilde{u}_2, \tilde{\lambda}_2) - (\tilde{u}_1^h, \tilde{\lambda}_1^h, \tilde{u}_2^h, \tilde{\lambda}_2^h)\|_X = 0 \tag{2.40}$$

and

$$\begin{aligned} &\|(\tilde{u}_1, \tilde{\lambda}_1, \tilde{u}_2, \tilde{\lambda}_2) - (\tilde{u}_1^h, \tilde{\lambda}_1^h, \tilde{u}_2^h, \tilde{\lambda}_2^h)\|_X \\ &\leq Ch^m(\|\tilde{u}_1\|_{m+1,\Omega_1} + \|\tilde{\lambda}_1\|_{m+1,\Omega_1} + \|\tilde{u}_2\|_{m+1,\Omega_2} + \|\tilde{\lambda}_2\|_{m+1,\Omega_2}), \end{aligned} \tag{2.41}$$

if $(\tilde{u}_1, \tilde{\lambda}_1, \tilde{u}_2, \tilde{\lambda}_2) \in H^{m+1}(\Omega_1) \times H^{m+1}(\Omega_1) \times H^{m+1}(\Omega_2) \times H^{m+1}(\Omega_2)$. We now define the operator $G$ from $\Lambda \times X$ to $Y$ as follows: For $(\tilde{Q}_1, \tilde{P}_1, \tilde{Q}_2, \tilde{P}_2) \in Y$ and $(1/\delta, (\tilde{u}_1, \tilde{\lambda}_1, \tilde{u}_2, \tilde{\lambda}_2)) \in \Lambda \times X$

$$G(1/\delta, (u_1, \lambda_1, u_2, \lambda_2)) = (\tilde{Q}_1, \tilde{P}_1, \tilde{Q}_2, \tilde{P}_2)$$

if and only if

$$(\tilde{Q}_1, s)_{\Omega_1} = -(f, s)_{\Omega_1} - \frac{1}{\delta}(-\lambda_1 + \lambda_2, s)_{\Gamma_0} \quad \forall s \in H^1_{\Gamma_1}(\Omega_1),$$

$$(\tilde{P}_1, r)_{\Omega_1} = -(u_1 - u_2, r)_{\Gamma_0} \quad \forall r \in H^1_{\Gamma_1}(\Omega_1),$$

$$(\tilde{Q}_2, s)_{\Omega_2} = -(f, s)_{\Omega_2} + \frac{1}{\delta}(-\lambda_1 + \lambda_2, s)_{\Gamma_0} \quad \forall s \in H^1_{\Gamma_2}(\Omega_2),$$

and

$$(\tilde{P}_2, r)_{\Omega_2} = (u_1 - u_2, r)_{\Gamma_0} \quad \forall r \in H^1_{\Gamma_2}(\Omega_2).$$

Now, by the definition of the operators $B$, $B_h$ and $G$, $BG(1/\delta, (u_1, \lambda_1, u_2, \lambda_2))$ is equivalent to

$$a_1(u_1, s) = -(f, s)_{\Omega_1} - \frac{1}{\delta}(-\lambda_1 + \lambda_2, s)_{\Gamma_0} \quad \forall s \in H^1_{\Gamma_1}(\Omega_1),$$

$$a_1(\lambda_1, r) = -(u_1 - u_2, r)_{\Gamma_0} \quad \forall r \in H^1_{\Gamma_1}(\Omega_1),$$

$$a_2(u_2, s) = -(f, s)_{\Omega_2} + \frac{1}{\delta}(-\lambda_1 + \lambda_2, s)_{\Gamma_0} \quad \forall s \in H^1_{\Gamma_2}(\Omega_2),$$

and

$$a_2(\lambda_2, r) = (u_1 - u_2, r)_{\Gamma_0} \quad \forall r \in H^1_{\Gamma_2}(\Omega_2).$$

Therefore, we can see that (2.5)-(2.6), with $g$ given by (2.26), and (2.21)-(2.22) and the discrete optimality system (2.34)-(2.37) can be expressed as the operator equations

$$(u_1, \lambda_1, u_2, \lambda_2) + BG(1/\delta, (u_1, \lambda_1, u_2, \lambda_2)) = 0$$

and

$$({u_1}^h, {\lambda_1}^h, {u_2}^h, {\lambda_2}^h) + B_h G(1/\delta, ({u_1}^h, {\lambda_1}^h, {u_2}^h, {\lambda_2}^h)) = 0,$$

respectively.

The derivative of $G$ with respect to $(u_1, \lambda_1, u_2, \lambda_2)$ can be defined by

$$G_X(1/\delta, (u_1, \lambda_1, u_2, \lambda_2)) = \begin{pmatrix} 0 & 1/\delta & 0 & -1/\delta \\ -1 & 0 & 1 & 0 \\ 0 & -1/\delta & 0 & 1/\delta \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

for all $(1/\delta, (u_1, \lambda_1, u_2, \lambda_2)) \in \Lambda \times X$.

Note that $G_X(1/\delta, (u_1, \lambda_1, u_2, \lambda_2)) \in \mathcal{L}(X, Z)$. Since $\Lambda$ is a compact interval, $G$ and its first and second Frechet derivatives are locally bounded operators.

We have shown that the hypotheses of the Brezzi-Rappaz-Raviart theory are satisfied, hence we have the estimate

$$\begin{aligned}\|(u_1,\lambda_1,u_2,\lambda_2) &- ({u_1}^h,{\lambda_1}^h,{u_2}^h,{\lambda_2}^h)\|_X \\ &\leq C\|(B-B_h)G(1/\delta,(u_1,\lambda_1,u_2,\lambda_2))\|_X \\ &\leq Ch^m(\|u_1\|_{m+1,\Omega_1}+\|\lambda_1\|_{m+1,\Omega_1}+\|u_2\|_{m+1,\Omega_2}+\|\lambda_2\|_{m+1,\Omega_2}).\end{aligned} \tag{2.42}$$

If we assume that the regularity property of partial differential equations

$$\|u_1\|_{m+1,\Omega_1}+\|\lambda_1\|_{m+1,\Omega_1}+\|u_2\|_{m+1,\Omega_2}+\|\lambda_2\|_{m+1,\Omega_2}\leq C\|f\|_{m-1,\Omega} \tag{2.43}$$

holds, then combining this with (2.42) yields the following result.

**Theorem 2.4** *Let* $(u_1,u_2,\lambda_1,\lambda_2)$ *denote the solution of (2.5)-(2.6) and (2.21)-(2.23) and let* $({u_1}^h,{u_2}^h,{\lambda_1}^h,{\lambda_2}^h)$ *denote the solutions of (2.34)-(2.37). Assume the approximation property (2.33) holds and* $u_1,\ \lambda_1\in H^{m+1}(\Omega_1)\cap H^1_{\Gamma_1}(\Omega_1)$ *and* $u_2,\ \lambda_2\in H^{m+1}(\Omega_2)\cap H^1_{\Gamma_2}(\Omega_2)$ *for some* $1\leq m<k$*; also assume that the regularity property (2.43) holds. Then,*

$$\begin{aligned}\|u_1-{u_1}^h\|_{1,\Omega_1}+\|\lambda_1-{\lambda_1}^h\|_{1,\Omega_1}+\|u_2-{u_2}^h\|_{1,\Omega_2}+\|\lambda_2-{\lambda_2}^h\|_{1,\Omega_2}& \\ \leq Ch^m\|f\|_{m-1,\Omega},&\end{aligned} \tag{2.44}$$

*where* $C$ *is independent of* $h$.

## 2.6 A gradient method

The optimality system (2.5)-(2.6) and (2.21)-(2.23) is a coupled system whose solutions yield solutions of the optimization problem (2.11)-(2.12). In order to obtain a parallelizable algorithm, we must uncouple the calculations in the two subdomains. One way of accomplishing this is through a gradient method.

The simple gradient method we consider is defined as follows. Given a starting guess $g^{(0)}$, let

$$g^{(n+1)}=g^{(n)}-\frac{\alpha}{\delta}\frac{d\mathcal{M}(g^{(n)})}{dg}\quad \text{for } n=1,2,\cdots, \tag{2.45}$$

where $\alpha/\delta$ is a step size. Combining with (2.32) yields, for $n=1,2,\cdots,$

$$g^{(n+1)}=(1-\alpha)g^{(n)}-\frac{\alpha}{\delta}({\lambda_1}^{(n)}-{\lambda_2}^{(n)}) \tag{2.46}$$

where ${\lambda_1}^{(n)}$, ${\lambda_2}^{(n)}$ are determined from (2.21) and (2.22), respectively, with $g$ replaced by $g^{(n)}$. In summary, the algorithm is given as follows.

**Algorithm 2.1**

1. *Choose* $g^{(0)}$.
2. *For* $n = 0, 1, 2, \cdots$,
   a. *determine* ${u_1}^{(n)}$, ${u_2}^{(n)}$ *from*

$$\begin{aligned} a_1({u_1}^{(n)}, v) &= (f, v)_{\Omega_1} + (g^{(n)}, v)_{\Gamma_0} \quad \forall v \in H^1_{\Gamma_1}(\Omega_1), \\ a_2({u_2}^{(n)}, v) &= (f, v)_{\Omega_2} - (g^{(n)}, v)_{\Gamma_0} \quad \forall v \in H^1_{\Gamma_2}(\Omega_1); \end{aligned}$$

   b. *determine* ${\lambda_1}^{(n)}$, ${\lambda_2}^{(n)}$ *from*

$$\begin{aligned} a_1({\lambda_1}^{(n)}, R) &= (u_1^{(n)} - u_2^{(n)}, R)_{\Gamma_0} \quad \forall R \in H^1_{\Gamma_1}(\Omega_1), \\ a_2({\lambda_2}^{(n)}, R) &= -(u_1^{(n)} - u_2^{(n)}, R)_{\Gamma_0} \quad \forall R \in H^1_{\Gamma_2}(\Omega_2); \end{aligned}$$

   c. *determine* $g^{(n+1)}$ *from*

$$g^{(n+1)} = (1-\alpha) g^{(n)} - \frac{\alpha}{\delta}({\lambda_1}^{(n)} - {\lambda_2}^{(n)}).$$

Of course, in practice, a stopping criteria should be added in order to terminate the iteration. In the above algorithm, $\delta$ is the penalty parameter inherited from the functional $\mathcal{J}_\delta(\cdot,\cdot,\cdot)$ and $\alpha$ is an additional parameter that can be chosen for improving the convergence properties of the algorithm. We can choose a suitable step size by controlling $\alpha$ when $\delta$ is fixed. Note that, if $\alpha = 1$, the gradient method is equivalent to the simple iteration scheme of the optimality system (2.5)-(2.6) and (2.21)-(2.23). However, the iteration may not converge in the case $\alpha = 1$ and $\delta << 1$ because the resulting step size $1/\delta$ becomes so large.

The following result is useful in determining a sufficient condition for the convergence of Algorithm 2.1.

**Lemma 2.1** *Let* $X$ *be a Hilbert space equipped with the inner product* $(\cdot,\cdot)_X$ *and norm* $\|\cdot\|_X$. *Suppose* $\mathcal{M}$ *is a functional on* $X$ *such that* *(i)* $\mathcal{M}$ *has a local minimum at* $\hat{x}$ *and is twice differentiable in an open ball* $B$ *centered at* $\hat{x}$*;*

*(ii)* $| < x, \mathcal{M}''(u)y > | \le m_a \|x\|_X \|y\|_X \quad \forall u \in B,\ x \in X,\ y \in X;$

*(iii)* $| < x, \mathcal{M}''(u)x > | \ge m_b \|x\|_X^2 \quad \forall u \in B,\ x \in X,$

*where* $m_a$ *and* $m_b$ *are positive real costants. Let* $R$ *denote the Riesz map, i.e.,* $< f, x >= (Rf, x)_X$ *for all* $x \in X$ *and all* $f \in X^*$*. Choose* $x^{(0)}$ *sufficiently close to* $\hat{x}$ *and choose a sequence* $\{\rho_n\}$ *such that* $0 < \rho_* \le \rho_n \le \rho^* < 2m_b/m_a^2$*. Then, the sequence* $\{x^{(n)}\}$ *defined by*

$$x^{(n)} = x^{(n-1)} - \rho_n R\mathcal{M}'(x^{(n-1)}) \quad \text{for } n = 1, 2, \cdots, \tag{2.47}$$

*converges to* $\hat{x}$*. Furthermore, if* $B = X$ *and* $\hat{x}$ *is a global minimum, then the sequence generated by (2.47) converges to* $\hat{x}$ *for any initial guess* $x^{(0)}$*.*

**Proof:** See, e.g., [16]. □

Applying Lemma 2.1 to Algorithm 2.1 yields the following convergence result.

**Theorem 2.5** *Let* $({u_1}^{(n)}, {u_2}^{(n)}, {\lambda_1}^{(n)}, {\lambda_2}^{(n)})$ *denote a sequence obtained by Algorithm 2.1 and* $(u_1, u_2, \lambda_1, \lambda_2)$ *denote the solution of the optimality system (2.5)-(2.6) and (2.21)-(2.23). If* $\alpha < 1/2$ *and* $\delta^2/2\alpha$ *is sufficiently large, then* ${u_1}^{(n)} \to u_1$*,* ${u_2}^{(n)} \to u_2$*,* ${\lambda_1}^{(n)} \to \lambda_1$ *and* ${\lambda_2}^{(n)} \to \lambda_2$ *as* $n \to \infty$*.*

**Proof:** For each $g \in L^2(\Gamma_0)$, the second Fréchet derivative of $\mathcal{M}_\delta(g)$, $\mathcal{M}''_\delta(g)$ is defined by

$$< \mathcal{M}''_\delta(g), (z, w) > \;=\; \delta(w, z)_{\Gamma_0} + (U_1 - U_2, V_1 - V_2)_{\Gamma_0} \quad \forall z, w \in L^2(\Gamma_0)$$

where $U_i, V_i \in H^1_{\Gamma_i}(\Omega_i)$ are the solutions of

$$\begin{aligned} a_1(U_1, S) &= (w, S)_{\Gamma_0} \quad \forall S \in H^1_{\Gamma_1}(\Omega_1), \\ a_2(U_2, S) &= -(w, S)_{\Gamma_0} \quad \forall S \in H^1_{\Gamma_2}(\Omega_2), \\ a_1(V_1, S) &= (z, S)_{\Gamma_0} \quad \forall S \in H^1_{\Gamma_1}(\Omega_1), \\ a_2(V_2, S) &= -(z, S)_{\Gamma_0} \quad \forall S \in H^1_{\Gamma_2}(\Omega_2). \end{aligned}$$

It is easily seen that there exist constants $C$ satisfying

$$\|U_i\|_{1,\Omega_i} \le C\|w\|_{0,\Gamma_0}, \quad \|V_i\|_{1,\Omega_i} \le C\|z\|_{0,\Gamma_0}.$$

Thus, we have

$$\begin{aligned} | < \mathcal{M}''_\delta(g), (z, w) > | &\leq \delta \|w\|_{0,\Gamma_0} \|z\|_{0,\Gamma_0} + 4C^2 \|w\|_{0,\Gamma_0} \|z\|_{0,\Gamma_0} \\ &\leq M \|w\|_{0,\Gamma_0} \|z\|_{0,\Gamma_0}, \end{aligned}$$

where $M = 2max\{\delta, 4C^2\}$. Also,

$$\begin{aligned} | < \mathcal{M}''_\delta(g), (z, z) > | &= \delta \|z\|^2_{0,\Gamma_0} + \|V_1 - V_2\|^2_{0,\Gamma_0} \\ &\geq \delta \|z\|^2_{0,\Gamma_0}. \end{aligned}$$

Note that the step size in (2.45) is fixed as $\alpha/\delta$. Thus, if $\alpha < 1/2$ and $\delta^2/\alpha$ is sufficiently large, $2m_b/m_a^2 > \alpha/\delta$. Hence, by Lemma 2.1, $g^{(n)} \to g$ in $L^2(\Gamma_0)$ as $n \to \infty$. Now, using the estimate (2.7), we can obtain the desired convergence results. □

## 2.7 Numerical results

Let the domain $\Omega$ be the rectangle $\{(x, y) : 0 < x < 2, 0 < y < 1\}$; $\Omega$ is divided into two parts $\Omega_1$, $\Omega_2$ such that $\Omega_1 = \{(x, y) : 0 < x < 1, 0 < y < 1\}$ and $\Omega_2 = \{(x, y) : 1 < x < 2, 0 < y < 1\}$ with the interface $\Gamma_0 = \{(x, y) : x = 1, 0 < y < 1\}$. The finite element spaces ${W_1}^h$, ${W_2}^h$ were chosen to consist of the standard continuous, piecewise quadratic polynomials based on triangle meshes.

The computation was carried out for the problem which has the exact solution $(x - 2)y \sin x \cos(\pi y/2)$. First, we need to find a good step size $\alpha/\delta$. It is necessary to use

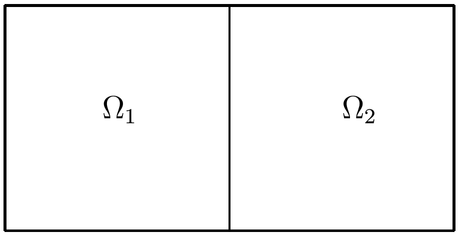

Figure 2.1: Subdomains of $\Omega = (0, 2) \times (0, 1)$

both parameters $\alpha$, $\delta$ in the algorithm so that we can obtain satisfactory convergence results for the gradient method while still obtaining good agreement with the exact solution. We choose $\delta$ to be small so that we do not overpenalize the functional. Then, $\alpha$ can be chosen so that the gradient method converges.

Calculations were performed with a fixed small $\delta$ and various values for $\alpha$. We used the stopping criterion defined by

$$\|\text{change in successive value of } u\| < 10^{-5}$$

and the number of iterations are presented in Table 2.1. The gradient method did not converge if step size $\alpha/\delta$ was larger than 9; evidently, in this case, the sufficient condition in Theorem 2.5 is not satisfied. The $L^2$ distance between the exact solution and the finite element solution was computed for mesh sizes $h = 1/4$, $h = 1/8$, $h = 1/16$ and are presented in Table 2.1. We also calculated the rate of convergence of the finite element solution. These rates were obtained by comparing the errors on pairs of grids $\{h = 1/4, h = 1/8\}$ and $\{h = 1/8, h = 1/16\}$. $H^1$-errors and $H^1$-rates were calculated in the same way and the results are presented in Table 2.2. Since quadratic elements were used, we expected the $L^2$-rates and $H^1$-rates to be 3 and 2, respectively; see [15] or [28]. Table 2.1 and Table 2.2 show that the step size $\alpha/\delta = 9$ gives the smallest error and the best rate, especially for the finer mesh pair. However, some of the smaller step sizes; e.g., $\alpha/\delta = 5$, require fewer iterations in order to meet the tolerance criterion.

In the second experiment, we wished to find a relationship between $\delta$ and the mesh size $h$; i.e., how small should $\delta$ be for a mesh size $h$ so that the rate of convergence with respect to $h$ is not compromised. To this purpose we varied $\delta$ for each mesh size and set $\alpha = 9\delta$. Table 2.3 provides the $L^2$-errors for various mesh sizes. The highlighted (in bold face) errors in Table 2.3 were chosen as the largest error which has the same first two significant digits as the smallest error so that the highlighted ones can be regarded to be sufficiently small. Choosing the $\delta$ which generates the highlighted error for each mesh size, we found that $\delta$ should be chosen proportional to $h$; i.e., roughly speaking, $\delta = Ch$, where $C \approx 10^{-3}$.

Table 2.1: $L^2$-errors and $L^2$-rates for $\delta = 10^{-5}$

| $\alpha$ | No.of iter. | $L^2 - error$ $h = 1/4$ | $L^2$ rate | $L^2 - error$ $h = 1/8$ | $L^2$ rate | $L^2 - error$ $h = 1/16$ |
|---|---|---|---|---|---|---|
| $0.9 \cdot 10^{-5}$ | 31 | $0.6913 \cdot 10^{-3}$ | 2.33 | $0.1377 \cdot 10^{-3}$ | 0.35 | $0.1082 \cdot 10^{-3}$ |
| $1.0 \cdot 10^{-5}$ | 28 | $0.6911 \cdot 10^{-3}$ | 2.34 | $0.1369 \cdot 10^{-3}$ | 0.35 | $0.1072 \cdot 10^{-3}$ |
| $3.0 \cdot 10^{-5}$ | 12 | $0.6837 \cdot 10^{-3}$ | 2.71 | $0.1048 \cdot 10^{-3}$ | 0.77 | $0.6152 \cdot 10^{-4}$ |
| $5.0 \cdot 10^{-5}$ | 10 | $0.6832 \cdot 10^{-3}$ | 2.88 | $0.9268 \cdot 10^{-4}$ | 1.31 | $0.3747 \cdot 10^{-4}$ |
| $7.0 \cdot 10^{-5}$ | 11 | $0.6828 \cdot 10^{-3}$ | 2.98 | $0.8653 \cdot 10^{-4}$ | 2.30 | $0.1756 \cdot 10^{-4}$ |
| $8.0 \cdot 10^{-5}$ | 18 | $0.6826 \cdot 10^{-3}$ | 3.00 | $0.8545 \cdot 10^{-4}$ | 2.88 | $0.1163 \cdot 10^{-4}$ |
| $9.0 \cdot 10^{-5}$ | 39 | $0.6826 \cdot 10^{-3}$ | 3.00 | $0.8550 \cdot 10^{-4}$ | 2.95 | $0.1110 \cdot 10^{-4}$ |
| $10.0 \cdot 10^{-5}$ | diverges | | | | | |

Table 2.2: $H^1$-errors and $H^1$-rates for $\delta = 10^{-5}$

| $\alpha$ | $H^1 - error$ $h = 1/4$ | $H^1$ rate | $H^1 - error$ $h = 1/8$ | $H^1$ rate | $H^1 - error$ $h = 1/16$ |
|---|---|---|---|---|---|
| $0.9 \cdot 10^{-5}$ | $0.1457 \cdot 10^{-1}$ | 1.98 | $0.3702 \cdot 10^{-2}$ | 1.69 | $0.1148 \cdot 10^{-2}$ |
| $1.0 \cdot 10^{-5}$ | $0.1457 \cdot 10^{-1}$ | 1.98 | $0.3701 \cdot 10^{-2}$ | 1.69 | $0.1146 \cdot 10^{-2}$ |
| $3.0 \cdot 10^{-5}$ | $0.1456 \cdot 10^{-1}$ | 1.99 | $0.3663 \cdot 10^{-2}$ | 1.86 | $0.1010 \cdot 10^{-2}$ |
| $5.0 \cdot 10^{-5}$ | $0.1457 \cdot 10^{-1}$ | 2.00 | $0.3649 \cdot 10^{-2}$ | 1.93 | $0.9547 \cdot 10^{-3}$ |
| $7.0 \cdot 10^{-5}$ | $0.1457 \cdot 10^{-1}$ | 2.00 | $0.3642 \cdot 10^{-2}$ | 1.98 | $0.9212 \cdot 10^{-3}$ |
| $8.0 \cdot 10^{-5}$ | $0.1457 \cdot 10^{-1}$ | 2.00 | $0.3640 \cdot 10^{-2}$ | 2.00 | $0.9116 \cdot 10^{-3}$ |
| $9.0 \cdot 10^{-5}$ | $0.1458 \cdot 10^{-1}$ | 2.00 | $0.3640 \cdot 10^{-2}$ | 2.00 | $0.9101 \cdot 10^{-3}$ |
| $10.0 \cdot 10^{-5}$ | diverges | | diverges | | diverges |

Table 2.3: $L^2$-errors for $\alpha = 9 \times \delta$

| $\delta$ | $h = 1/4$ | $h = 1/8$ | $h = 1/10$ | $h = 1/16$ | $h = 1/20$ |
|---|---|---|---|---|---|
| $3.0 \cdot 10^{-3}$ | $0.7121 \cdot 10^{-3}$ | · | · | · | · |
| $1.5 \cdot 10^{-3}$ | $0.6902 \cdot 10^{-3}$ | · | · | · | · |
| $1.0 \cdot 10^{-3}$ | $\mathbf{0.6869 \cdot 10^{-3}}$ | $0.1120 \cdot 10^{-3}$ | · | · | · |
| $9.0 \cdot 10^{-4}$ | $0.6862 \cdot 10^{-3}$ | · | · | · | · |
| $5.0 \cdot 10^{-4}$ | $0.6839 \cdot 10^{-3}$ | · | $0.5774 \cdot 10^{-4}$ | $0.3916 \cdot 10^{-4}$ | · |
| $3.0 \cdot 10^{-4}$ | $0.6829 \cdot 10^{-3}$ | $0.8694 \cdot 10^{-4}$ | · | · | · |
| $2.0 \cdot 10^{-4}$ | · | $\mathbf{0.8591 \cdot 10^{-4}}$ | $0.4477 \cdot 10^{-4}$ | $0.1439 \cdot 10^{-4}$ | · |
| $1.0 \cdot 10^{-4}$ | $0.6826 \cdot 10^{-3}$ | $0.8539 \cdot 10^{-4}$ | $\mathbf{0.4380 \cdot 10^{-4}}$ | $0.1111 \cdot 10^{-4}$ | $0.6276 \cdot 10^{-5}$ |
| $7.0 \cdot 10^{-5}$ | · | · | $0.4370 \cdot 10^{-4}$ | $\mathbf{0.1074 \cdot 10^{-4}}$ | $0.5608 \cdot 10^{-5}$ |
| $5.0 \cdot 10^{-5}$ | · | · | $0.4368 \cdot 10^{-4}$ | $0.1070 \cdot 10^{-4}$ | $\mathbf{0.5529 \cdot 10^{-5}}$ |
| $3.0 \cdot 10^{-5}$ | · | · | · | · | $0.5763 \cdot 10^{-5}$ |
| $1.0 \cdot 10^{-5}$ | · | $0.8550 \cdot 10^{-4}$ | $0.4377 \cdot 10^{-4}$ | $0.1110 \cdot 10^{-4}$ | $0.6276 \cdot 10^{-5}$ |
| $1.0 \cdot 10^{-7}$ | $0.6826 \cdot 10^{-3}$ | $0.8546 \cdot 10^{-4}$ | $0.4382 \cdot 10^{-4}$ | $0.1129 \cdot 10^{-4}$ | $0.6615 \cdot 10^{-5}$ |

## 2.8 Linear least squares approach

The optimization problem can be recast as a linear least squares problem so that one can use a more effective numerical implementation such as the conjugate gradient method. Even though it was shown in the previous section that Algorithm 2.1 converges with suitably chosen step size, choosing the best step size is not always guaranteed, especially, for the multi-subdomain case. Thus, in this section, we consider a least squares problem from which the solution of (2.11)-(2.12) is obtained.

### 2.8.1 Linear operator

We formulate the optimization problem (2.11)-(2.12) as a linear least squares problem by defining a linear operator $B$ and a vector $d \in L^2(\Gamma_0)$ as follows:
Let $\bar{u}_1, \bar{u}_2$ be the solutions of

$$-\Delta \bar{u}_1 = f \text{ in } \Omega_1, \quad \bar{u}_1 = 0 \text{ on } \Gamma_1 \quad \text{and} \quad \frac{\partial \bar{u}_1}{\partial n_1} = 0 \text{ on } \Gamma_0 \tag{2.48}$$

and

$$-\Delta \bar{u}_2 = f \text{ in } \Omega_2, \quad \bar{u}_2 = 0 \text{ on } \Gamma_2 \quad \text{and} \quad \frac{\partial \bar{u}_2}{\partial n_2} = 0 \text{ on } \Gamma_0. \tag{2.49}$$

Define $d \in L^2(\Gamma_0)$ by

$$d = (\bar{u}_1 - \bar{u}_2)|_{\Gamma_0} \tag{2.50}$$

and consider the following homogeneous partial differential equations

$$-\Delta \tilde{u}_1 = 0 \text{ in } \Omega_1, \quad \tilde{u}_1 = 0 \text{ on } \Gamma_1 \quad \text{and} \quad \frac{\partial \tilde{u}_1}{\partial n_1} = g \text{ on } \Gamma_0 \tag{2.51}$$

and

$$-\Delta \tilde{u}_2 = 0 \text{ in } \Omega_2, \quad \tilde{u}_2 = 0 \text{ on } \Gamma_2 \quad \text{and} \quad \frac{\partial \tilde{u}_2}{\partial n_2} = -g \text{ on } \Gamma_0. \tag{2.52}$$

Note that if $\tilde{u}_1, \tilde{u}_2$ are the solutions of (2.51) and (2.52), then $\bar{u}_1 + \tilde{u}_1$, $\bar{u}_2 + \tilde{u}_2$ are the solutions of (2.2) and (2.3). We define a linear operator $B : L^2(\Gamma_0) \to L^2(\Gamma_0)$ by

$$Bg = (\tilde{u}_1 - \tilde{u}_2)|_{\Gamma_0}, \tag{2.53}$$

where $\tilde{u}_1, \tilde{u}_2$ are the solutions of (2.51) and (2.52). Then, by the definitions of $d$ and $B$, the functional (2.4) can be written as

$$\mathcal{J}_\delta(g) = \frac{1}{2}\|Bg + d\|^2_{0,\Gamma_0} + \frac{\delta}{2}\|g\|^2_{0,\Gamma_0} \tag{2.54}$$

and the constrained minimization problem is turned into the unconstrained linear least squares problem

$$\min_g \frac{1}{2}\|Bg+d\|^2_{0,\Gamma_0}+\frac{\delta}{2}\|g\|^2_{0,\Gamma_0}. \tag{2.55}$$

Therefore, if $(\hat{u}_1, \hat{u}_2, \hat{g})$ is the minimizer of (2.4), then $\hat{g}$ is the solution of (2.55).

We now show that $B$ is a self-adjoint operator. For a given $h \in L^2(\Gamma_0)$ consider the adjoint equations

$$-\Delta\lambda_1 = 0 \text{ in } \Omega_1, \quad \lambda_1 = 0 \text{ on } \Gamma_1 \quad \text{and} \quad \frac{\partial\lambda_1}{\partial n_1} = h \text{ on } \Gamma_0 \tag{2.56}$$

and

$$-\Delta\lambda_2 = 0 \text{ in } \Omega_2 \quad \lambda_2 = 0 \text{ on } \Gamma_2 \quad \text{and} \quad \frac{\partial\lambda_2}{\partial n_2} = -h \text{ on } \Gamma_0. \tag{2.57}$$

Then by the definition of $B$,

$$Bh = (\lambda_1 - \lambda_2)|_{\Gamma_0}, \tag{2.58}$$

where $\lambda_1, \lambda_2$ are the solutions of

$$a_1(\lambda_1, \xi) = (h, \xi)_{\Gamma_0} \quad \forall \xi \in H^1_{\Gamma_1}(\Omega_1), \tag{2.59}$$

$$a_2(\lambda_2, \xi) = -(h, \xi)_{\Gamma_0} \quad \forall \xi \in H^1_{\Gamma_2}(\Omega_2), \tag{2.60}$$

which are weak formulations corresponding to (2.56)-(2.57). On the other hand, a weak formulation corresponding to (2.51)-(2.52) is

$$a_1(w_1, v) = (g, v)_{\Gamma_0} \quad \forall v \in H^1_{\Gamma_1}(\Omega_1), \tag{2.61}$$

$$a_2(w_2, v) = -(g, v)_{\Gamma_0} \quad \forall v \in H^1_{\Gamma_2}(\Omega_2). \tag{2.62}$$

Thus,

$$\begin{aligned}
(Bg, h)_{\Gamma_0} &= (w_1 - w_2, h)_{\Gamma_0} \\
&= (w_1, h)_{\Gamma_0} - (w_2, h)_{\Gamma_0} \\
&= a_1(\lambda_1, w_1) + a_2(\lambda_2, w_2) \qquad \text{by } (2.59), (2.60) \\
&= (g, \lambda_1)_{\Gamma_0} - (g, \lambda_1)_{\Gamma_0} \qquad \text{by } (2.61), (2.62) \\
&= (g, \lambda_1 - \lambda_2)_{\Gamma_0} \\
&= (g, Bh)_{\Gamma_0}.
\end{aligned}$$

By the definition of an adjoint operator, we conclude that

$$B^* = B, \tag{2.63}$$

therefore, $B$ is a self-adjoint operator.

Now define a linear operator $A \in \mathcal{L}(L^2(\Gamma_0), L^2(\Gamma_0) \times L^2(\Gamma_0))$ by

$$A = \begin{pmatrix} B \\ \sqrt{\delta} I \end{pmatrix}, \tag{2.64}$$

where $I$ denotes the identity operator and $b \in L^2(\Gamma_0) \times L^2(\Gamma_0)$ by

$$b = \begin{pmatrix} d \\ 0 \end{pmatrix}. \tag{2.65}$$

Then we can write the linear least squares problem (2.55) in the form of

$$\min_g \frac{1}{2} \|Ag + b\|^2_{L^2(\Gamma_0) \times L^2(\Gamma_0)}. \tag{2.66}$$

Note that $A^*A$ is self-adjoint and positive definite.

**Lemma 2.2** *If $\delta > 0$, then the linear least square problem (2.66) admits a unique solution. The solution of (2.55) is the solution of the normal equation*

$$A^*Ag = -A^*b. \tag{2.67}$$

**Proof:** It can be easily seen that $A$ has a closed range. See §3.3 for the more general case. Then the result follows from [36, Thm. 2.1.1]. □

**Remark 2.3** *$A^*Ag + A^*b$ is equivalent to $\lambda_1 - \lambda_2 + \delta g$, i.e., if $g$ solves $A^*Ag + A^*b = 0$, then $g$ also satisfies (2.26).*

### 2.8.2 Conjugate gradient method

The fact that $A^*A$ is self-adjoint and positive definite enables us to use effective numerical implementations to solve (2.66). Here we adopt the following basic conjugate gradient algorithm for the linear least squares problem (2.66), which can be found in many references. For example, see [35], [36].

**Algorithm 2.2 (Conjugate Gradient Method for Least Squares)**

*Given $A$, $b$, $g^{(0)}$*

```
1. Set $r^{(0)} = -b - Ag^{(0)}$,
       $p^{(0)} = A^* r^{(0)}$.

2. For $n = 0, 1, 2, \cdots$,

   a. if $\|A^* r^{(n)}\|_{0,\Gamma_0} < \epsilon$ stop,
   b. $\sigma^{(n)} = \|A^* r^{(n)}\|^2_{0,\Gamma_0} / \|Ap^{(n)}\|^2_{L^2(\Gamma_0)\times L^2(\Gamma_0)}$ ,
   c. $g^{(n+1)} = g^{(n)} + \sigma^{(n)} p^{(n)}$ ,
   d. $r^{(n+1)} = r^{(n)} - \sigma^{(n)} Ap^{(n)}$,
   e. $\tau^{(n)} = \|A^* r^{(n+1)}\|^2_{0,\Gamma_0} / \|A^* r^{(n)}\|^2_{0,\Gamma_0}$ ,
   f. $p^{(n+1)} = A^* r^{(n+1)} + \tau^{(n)} p^{(n)}$.
```

Applying Algorithm 2.2 to (2.66) and combining with (2.50), (2.53), (2.63) and (2.64), we obtain the following algorithm.

**Algorithm 2.3**

```
1. a. Compute $\bar{u}_1$, $\bar{u}_2$ by

      $a_1(\bar{u}_1, v) = (f, v)_{\Omega_1} \quad \forall v \in H^1_{\Gamma_1}(\Omega_1)$,
      $a_2(\bar{u}_2, v) = (f, v)_{\Omega_2} \quad \forall v \in H^1_{\Gamma_2}(\Omega_2)$,

   b. set $b = \bar{u}_1 - \bar{u}_2$.

2. Choose $g^{(0)}$.

3. a. Compute $w_1$, $w_2$ by

      $a_1(w_1, v) = (g^{(0)}, v)_{\Gamma_0} \quad \forall v \in H^1_{\Gamma_1}(\Omega_1)$,
      $a_2(w_2, v) = -(g^{(0)}, v)_{\Gamma_0} \quad \forall v \in H^1_{\Gamma_2}(\Omega_2)$,

   b. set
      $r_1^{(0)} = -b - (w_1 - w_2)|_{\Gamma_0}$,
      $r_2^{(0)} = -\sqrt{\delta} g^{(0)}$.

4. a. Compute $\tilde{\lambda}_1^{(0)}$, $\tilde{\lambda}_2^{(0)}$ by

      $a_1(\tilde{\lambda}_1^{(0)}, v) = (r_1^{(0)}, v)_{\Gamma_0} \quad \forall v \in H^1_{\Gamma_1}(\Omega_1)$,
      $a_2(\tilde{\lambda}_2^{(0)}, v) = -(r_1^{(0)}, v)_{\Gamma_0} \quad \forall v \in H^1_{\Gamma_2}(\Omega_2)$,
```

   b. *set* $p^{(0)} = (\tilde\lambda_1^{(0)} - \tilde\lambda_2^{(0)})|_{\Gamma_0} + \sqrt{\delta}\, r_2^{(0)}$,
   c. *set* $y^{(0)} = p^{(0)}$.

5. *For* $n = 0, 1, 2, \cdots$,

   a. *if* $\|y^{(n)}\|_{0,\Gamma_0} < \epsilon$ *go to 6,*

   b. *compute* $\tilde u_1^{(n)}$, $\tilde u_1^{(n)}$ *by*

$$\begin{array}{rcl} a_1(\tilde u_1^{(n)}, v) & = & (p^{(n)}, v)_{\Gamma_0} \quad \forall v \in H^1_{\Gamma_1}(\Omega_1), \\ a_2(\tilde u_2^{(n)}, v) & = & -(p^{(n)}, v)_{\Gamma_0} \quad \forall v \in H^1_{\Gamma_2}(\Omega_2), \end{array}$$

   c. *set*

$$\begin{array}{l} z_1^{(n)} = (\tilde u_1^{(n)} - \tilde u_2^{(n)})|_{\Gamma_0}, \\ z_2^{(n)} = \sqrt{\delta}\, p^{(n)}, \end{array}$$

   d. *set*

$$\sigma^{(n)} = \frac{\|y^{(n)}\|^2_{0,\Gamma_0}}{\|(z_1^{(n)}, z_2^{(n)})^T\|_{L^2(\Gamma_0)\times L^2(\Gamma_0)}},$$

   e. *compute* $g^{(n)}$ *by*

$$g^{(n+1)} = g^{(n)} + \sigma^{(n)} p^{(n)},$$

   f. *compute* $r_1^{(n+1)}$, $r_2^{(n+1)}$ *by*

$$\begin{array}{rcl} r_1^{(n+1)} & = & r_1^{(n)} - \sigma^{(n)} z_1^{(n)}, \\ r_2^{(n+1)} & = & r_2^{(n)} - \sigma^{(n)} z_2^{(n)}, \end{array}$$

   g. *compute* $\tilde\lambda_1^{(n+1)}$, $\tilde\lambda_2^{(n+1)}$ *by*

$$\begin{array}{rcl} a_1(\tilde\lambda_1^{(n+1)}, v) & = & (r_1^{(n+1)}, v)_{\Gamma_0} \quad \forall v \in H^1_{\Gamma_1}(\Omega_1), \\ a_2(\tilde\lambda_2^{(n+1)}, v) & = & -(r_1^{(n+1)}, v)_{\Gamma_0} \quad \forall v \in H^1_{\Gamma_2}(\Omega_2), \end{array}$$

   h. *set* $y^{(n+1)} = (\tilde\lambda_1^{(n+1)} - \tilde\lambda_2^{(n+1)})|_{\Gamma_0} + \sqrt{\delta}\, r_2^{(n+1)}$,
   i. *set*

$$\tau^{(n)} = \frac{\|y^{(n+1)}\|^2_{0,\Gamma_0}}{\|y^{(n)}\|^2_{0,\Gamma_0}},$$

   j. *set* $p^{(n+1)} = y^{(n+1)} + \tau^{(n)} p^{(n)}$.

*6. Compute $u_1$, $u_2$ by*

$$\begin{aligned} a_1(u_1, v) &= (f, v)_{\Omega_1} + (g^{(n)}, v)_{\Gamma_0} \quad \forall v \in H^1_{\Gamma_1}(\Omega_1), \\ a_2(u_2, v) &= (f, v)_{\Omega_2} - (g^{(n)}, v)_{\Gamma_0} \quad \forall v \in H^1_{\Gamma_2}(\Omega_2). \end{aligned}$$

**Theorem 2.6** *If $\delta > 0$, iterates generated by the conjugate gradient method for the solution of the least squares problem (2.66) converge to a unique solution $g^*$ of the least squares problem. The iterates satisfy*

$$\|g^{(n+1)} - g^*\|_{A^*A} \le \frac{\|A\|^2 - \delta}{\|A\|^2 + \delta} \|g^{(n)} - g^*\|_{A^*A}, \tag{2.68}$$

*where $\|g\|^2_{A^*A} = (A^*Ag, g)_{\Gamma_0}$.*

**Proof:** Since

$$\delta \|g\|_{0,\Gamma_0} \le ((B^*B + \delta I)g, g)_{\Gamma_0} = (A^*Ag, g)_{\Gamma_0} \le \|A\|^2 \|g\|_{0,\Gamma_0},$$

this result follows immediately from the standard convergence results for the conjugate gradient method. See e.g., [3, Sec. 13], [36, Sec. II.5]. □

The following lemma shows that the operator $A^*A$ in the normal equation (2.67) is a compact perturbation of the identity operator so that the linear convergence result (2.68) is improved.

**Lemma 2.3** *The operator $B^*B \in \mathcal{L}(L^2(\Gamma_0), L^2(\Gamma_0))$ with $B$ and $B^*$ defined by (2.53) (2.63), respectively, is compact.*

**Proof:** The operator $B^*B \in \mathcal{L}(L^2(\Gamma_0), L^2(\Gamma_0))$ can be viewed as a composition of the following bounded linear operators. Define

$$\begin{aligned} T_1 &: L^2(\Gamma_0) \to H^1(\Omega_1) \times H^1(\Omega_2) \quad \text{by} \quad T_1 g = (w_1, w_2), \\ T_2 &: H^1(\Omega_1) \times H^1(\Omega_2) \to H^{1/2}(\Gamma_0) \quad \text{by} \quad T_2(w_1, w_2) = (w_1 - w_2)|_{\Gamma_0}, \\ T_3 &: H^{1/2}(\Gamma_0) \to H^2(\Omega_1) \times H^2(\Omega_2) \quad \text{by} \quad T_3 z = (\lambda_1, \lambda_2), \\ T_4 &: H^2(\Omega_1) \times H^2(\Omega_2) \to H^1(\Omega_1) \times H^1(\Omega_2) \quad \text{by} \quad T_4(\lambda_1, \lambda_2) = (\lambda_1, \lambda_2) \end{aligned}$$

and

$$T_5 : H^1(\Omega_1) \times H^1(\Omega_2) \to L^2(\Gamma_0) \quad \text{by} \quad T_5(\lambda_1, \lambda_2) = (\lambda_1 - \lambda_2)|_{\Gamma_0}.$$

Here, $w_1, w_2$ are the solutions of (2.51), (2.52), respectively. Since $g \in L^2(\Gamma_0)$, $w_i \in H^1(\Omega_i)$, $i = 1, 2$. This implies $T_1 \in \mathcal{L}(L^2(\Gamma_0), H^1(\Omega_1) \times H^1(\Omega_2))$. By the trace theorem, $z = (w_1 - w_2)|_{\Gamma_0} \in H^{1/2}(\Gamma_0)$. Hence $T_2 \in \mathcal{L}(H^1(\Omega_1) \times H^1(\Omega_2), H^{1/2}(\Gamma_0))$. The functions $\lambda_1, \lambda_2$ are the solutions of (2.51), (2.52), respectively, with $g$ replaced by $z$. Since $z \in H^{1/2}(\Gamma_0)$, $\lambda_i \in H^2(\Omega_i)$, $i = 1, 2$. This shows that $T_3 \in \mathcal{L}(H^{1/2}(\Gamma_0), H^2(\Omega_1) \times H^2(\Omega_2))$. Since the imbedding $H^2(\Omega_i) \hookrightarrow H^1(\Omega_i)$, $i = 1, 2$, is compact, the linear operator $T_4$ is compact. Finally, by the trace theorem, $T_5 \in \mathcal{L}(H^1(\Omega_1) \times H^1(\Omega_2), L^2(\Gamma_0))$ is bounded. Thus, $B^*B$ is the composition of linear bounded operators $T_1, \cdots, T_5$, of which one is compact. This yields the compactness of $B^*B$. □

**Theorem 2.7** *If $\delta > 0$, the iterates generated by Algorithm 2.3 for the solution of the linear least squares problem (2.66) converge r-superlinearly; i.e.,*

$$\|g^{(n)} - g^*\|_{A^*A} \leq (c_n)^n \|g^{(0)} - g^*\|_{A^*A},$$

*where* $\lim_{n\to\infty} c_n = 0$.

**Proof:** Since $A^*A = B^*B + \delta I$ and $B^*B$ is compact, the result follows from [57]. □

### 2.8.3 Numerical results

Let the domain $\Omega$ be the rectangle $\{(x, y) : 0 < x < 2, 0 < y < 1\}$. $\Omega$ is divided into two parts $\Omega_1$, $\Omega_2$ such that $\Omega_1 = \{(x, y) : 0 < x < 1, 0 < y < 1\}$ and $\Omega_2 = \{(x, y) : 1 < x < 2, 0 < y < 1\}$ with the interface $\Gamma_0 = \{(x, y) : x = 1, 0 < y < 1\}$. See Figure 2.1. The finite element spaces ${W_1}^h$, ${W_2}^h$ are chosen to consist of the standard continuous, piecewise quadratic elements on triangular meshes. The data $f$ in (2.1) is adjusted so that (2.1) has the exact solution $(x-2)y \sin x \cos(\pi y/2)$. $\delta$ is fixed as $10^{-5}$ and the tolerance $10^{-7}$ for the stopping criterion is chosen for all grid sizes. We choose the same domain, the same exact solution and all other conditions the same as the example in §2.7 for comparison. Table 2.4 presents errors and rates for the mesh sizes $h = 1/4$, $h = 1/8$, $h = 1/16$, $h = 1/32$ in each direction and the number of iterations. The rates were obtained by comparing the errors on pairs of grids $\{h = 1/4, h = 1/8\}$, $\{h = 1/8, h = 1/16\}$ and $\{h = 1/16, h = 1/32\}$. Since quadratic elements were used, we expected that $L^2$-rates and $H^1$-rates are close to 3 and 2, respectively; see [28]. Comparing with the results obtained by using the best step size in Table 2.1 and Table 2.2, Table 2.4 shows larger errors for mesh grids $h = 1/4$, $h = 1/8$. But, for finer grids, it shows smaller errors and better convergence rates. Also the number of iterations to meet the tolerance criterion is much less. For example, the number of iterations performed for $h = 1/16$ was 39 with step size 9 in §2.7.

Table 2.4: errors and rates for $\delta = 10^{-5}$

| mesh size | no. of iter. | $L^2 - error$ | $H^1 - error$ |
|---|---|---|---|
| $\frac{1}{4} \times \frac{1}{4}$ | 8 | $7.125 \cdot 10^{-4}$ | $2.488 \cdot 10^{-2}$ |
| rate | | 3.01 | 2.00 |
| $\frac{1}{8} \times \frac{1}{8}$ | 9 | $8.866 \cdot 10^{-5}$ | $6.214 \cdot 10^{-3}$ |
| rate | | 3.00 | 2.00 |
| $\frac{1}{16} \times \frac{1}{16}$ | 7 | $1.108 \cdot 10^{-5}$ | $1.553 \cdot 10^{-3}$ |
| rate | | 2.96 | 2.00 |
| $\frac{1}{32} \times \frac{1}{32}$ | 7 | $1.428 \cdot 10^{-6}$ | $3.884 \cdot 10^{-4}$ |

# Chapter 3

# Extension to multi-subdomain problems

In this chapter we consider the domain decomposition methods discussed in Chapter 2 for multi-subdomain problems. We introduced two different approaches to solve the domain decomposition problem in the two subdomain setting. Both methods can be easily extended to multi-subdomain problems, if the decomposed subdomains have some boundaries common with the domain $\Omega$. Hence, we first consider the case in which there are no interior subdomains. The first method is extended to that case; we define an objective functional and derive an optimality system using a Lagrange multiplier rule. Relative results such as the existence of the optimal solution, convergence of optimal solutions to an exact solution, error estimates for the finite element solution and convergence of iterations by numerical algorithms can be obtained easily by the same technique shown in Chapter 2. In the case that there are interior subdomains which do not have common boundaries with $\Omega$, some of the local problems we need to solve are pure Neumann problems. Hence, the solutions are not unique and, in addition, the Neumann conditions on the boundaries of the interior subdomains should satisfy the compatibility condition for the existence of the solution. We consider the linear least squares approach for this more general case in the second part of this chapter.

## 3.1 Decomposed problem and the cost functional

Suppose we decompose the domain $\Omega$ with boundary $\Gamma$ into n disjoint open subdomains $\Omega_1, \ldots, \Omega_n$ such that $\overline{\Omega} = \overline{\Omega}_1 \cup \cdots \cup \overline{\Omega}_n$. Define the index set

$$N(i) = \{j \mid int(\overline{\Omega}_i \cap \overline{\Omega}_j) \neq \emptyset\},$$

where *int* denotes the relative interior, i.e., $N(i)$ consists of indices of subdomains which have interfaces common with $\Omega_i$. To make sure that each interface contributes to exactly one term in the objective functional, we define another index set $N_<(i) \subset N(i)$ by

$$N_<(i) = \{j \mid j \in N(i),\ j < i\}, \quad i = 2, \ldots, n.$$

Note that $N_<(1) = \emptyset$. Let $\Gamma_{ij}$ $(= \Gamma_{ji})$ denote the interface between $\Omega_i$ and $\Omega_j$, i.e., $\Gamma_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j$ for $i = 1, \ldots, n$ and $j \in N(i)$. Let $\Gamma_i = \overline{\Omega}_i \cap \Gamma$ for $i = 1, \ldots, n$. Now we consider the following family of elliptic partial differential equations: for $i = 1, \ldots, n$,

$$-\Delta u_i = f \text{ in } \Omega_i, \quad u_i = 0 \text{ on } \Gamma_i \quad \text{and} \quad \frac{\partial u_i}{\partial n_{ij}} = g_{ij} \text{ on } \Gamma_{ij}, \ \ j \in N(i), \tag{3.1}$$

where $n_{ij}$ $(= -n_{ji})$ is the unit outward normal vector to $\Omega_i$ on $\Gamma_{ij}$. Note that $g_{ij} = -g_{ji}$.

We wish to find functions $g_{ij}$ on each interface $\Gamma_{ij}$ so that the differences between $u_i$ and $u_j$ on $\Gamma_{ij}$ are as small as possible. So we define a functional $\mathcal{J}_\delta(\cdot, \ldots, \cdot)$ by

$$\mathcal{J}_\delta \ = \ \frac{1}{2}\sum_{i=2}^{n}\sum_{j\in N_<(i)} \int_{\Gamma_{ij}} (u_i - u_j)^2 \, d\Gamma_{ij} + \frac{1}{2}\sum_{i=2}^{n}\sum_{j\in N_<(i)} \delta_{ij} \int_{\Gamma_{ij}} g_{ij}^2 \, d\Gamma_{ij}, \tag{3.2}$$

where $u_i$ are the solutions of (3.1) and $\delta_{ij}$ are positive constants which can change the relative importance of the terms in (3.2).

We now define the spaces which will be used throughout this chapter. Let $L^2(D)$, $H^1(D)$ denote the usual Sobolev spaces with respect to a domain $D$ with the standard norms $\|\cdot\|_{0,D}$ and $\|\cdot\|_{1,D}$. The subspaces $H_0^1(\Omega)$, $H_{\Gamma_i}^1(\Omega_i)$, for $i = 1, \ldots, n$, are defined by

$$\begin{aligned} H_0^1(\Omega) &= \{v \in H^1(\Omega) : \ v = 0 \text{ on } \Gamma\}, \\ H_{\Gamma_i}^1(\Omega_i) &= \{v \in H^1(\Omega_i) : \ v = 0 \text{ on } \Gamma_i\}. \end{aligned}$$

Let $(\cdot,\cdot)_{\Omega_i}$, $(\cdot,\cdot)_{\Gamma_{ij}}$ denote the usual inner product in $L^2(\Omega_i)$ and $L^2(\Gamma_{ij})$. We also define the space

$$G = \Pi_{i=2}^{n}\Pi_{j\in N_<(i)} \ L^2(\Gamma_{ij}) \tag{3.3}$$

equipped with the scalar product

$$(\mathbf{g}^1, \mathbf{g}^2)_G = \sum_{i=2}^{n} \sum_{j \in N_<(i)} (g_{ij}^1, g_{ij}^2)_{\Gamma_{ij}}$$

and the induced norm $\| \cdot \|_{0,G}^2 = (\cdot, \cdot)_G$.

## 3.2 Optimality system

In this section we assume that every subdomain has common boundaries with the domain $\Omega$, i.e., $int(\overline{\Omega}_i \cap \Gamma) \neq \emptyset$ so that the decomposed problems (3.1) have mixed Dirichlet and Neumann boundary conditions for $i = 1, 2, \ldots, n$.

### 3.2.1 The existence of an optimal solution

A weak formulation corresponding to (3.1) is as follows: for $i = 1, \ldots, n$, find $u_i \in H_{\Gamma_i}^1(\Omega_i)$ such that

$$a_i(u_i, v) = (f, v)_{\Omega_i} + \sum_{j \in N(i)} (g_{ij}, v)_{\Gamma_{ij}} \quad \forall v \in H_{\Gamma_i}^1(\Omega_i), \tag{3.4}$$

where

$$a_i(u, v) = \int_{\Omega_i} \nabla u \cdot \nabla v \, d\Omega_i.$$

The bilinear forms $a_i(\cdot, \cdot)$ are coercive and continuous, i.e., there exist constants $C_a > 0$ and $C_b > 0$ such that

$$a_i(u, u) \geq C_a \|u\|_{1,\Omega_i}^2 \quad \forall u \in H^1(\Omega_i), \ i = 1, \ldots, n,$$

and

$$a_i(u, v) \leq C_b \|u\|_{1,\Omega_i} \|v\|_{1,\Omega_i} \quad \forall u, v \in H^1(\Omega_i), \ i = 1, \ldots, n.$$

The existence of a unique solution of (3.4) follows from the Lax-Milgram Theorem. Also, that theorem yields the continuous dependence on the data, i.e., there exists a constant $C > 0$ such that

$$\|u_i\|_{1,\Omega_i} \leq C(\|f\|_{0,\Omega_i} + \sum_{j \in N(i)} \|g_{ij}\|_{0,\Gamma_{ij}}),$$

for $i = 1, \ldots, n$. See, e.g., [28].

Now we define the admissibility set by

$$\begin{aligned} \mathcal{U}_{ad} &= \{(u_1,\ldots,u_n,\mathbf{g}) \in H^1_{\Gamma_1}(\Omega_1) \times \ldots \times H^1_{\Gamma_n}(\Omega_n) \times G \\ &\quad : (3.4) \text{ is satisfied for } i=1,\ldots,n \text{ and } \mathcal{J}_\delta(u_1,\ldots,u_n,\mathbf{g}) < \infty\}. \end{aligned} \tag{3.5}$$

Proof for existence of optimal solutions follows analogously to the proof for the 2-subdomain problem.

**Theorem 3.1** *There is a unique optimal solution* $(\hat{u}_1,\ldots,\hat{u}_n,\hat{\mathbf{g}}) \in \mathcal{U}_{ad}$ *such that*

$$\mathcal{J}_\delta(\hat{u}_1,\ldots,\hat{u}_n,\hat{\mathbf{g}}) \le \mathcal{J}_\delta(u_1,\ldots,u_n,\mathbf{g}) \tag{3.6}$$

*for all* $(u_1,\ldots,u_n,\mathbf{g}) \in \mathcal{U}_{ad}$.

### 3.2.2 Optimality system

It was shown in the previous chapter that there exist suitable Lagrange multipliers for the 2-subdomain case. Thus, here we use the Lagrange multiplier rule without a proof of the existence to derive an optimality system from which solutions of the optimality problem may be determined. Due to the constraint (3.4), the Lagrangian is given by

$$\begin{aligned} \mathcal{L}(u_1,\ldots,u_n,\mathbf{g}) &= \mathcal{J}_\delta(u_1,\ldots,u_n,\mathbf{g}) \\ &\quad + \sum_{i=1}^{n}(-a_i(u_i,\lambda_i) + (f,\lambda_i)_{\Omega_i} + \sum_{j\in N(i)}^{n}(g_{ij},\lambda_i)_{\Gamma_{ij}}), \end{aligned} \tag{3.7}$$

where $\mathcal{J}_\delta(u_1,\ldots,u_n,\mathbf{g})$ is defined by (3.2). We then find stationary points of $\mathcal{L}(\cdot,\ldots,\cdot)$. Setting to zero the variations with respect to the Lagrange multipliers $\lambda_i$, for $i=1,\ldots,n$, recovers the constraints (3.4). $\partial\mathcal{L}/\partial u_i = 0$, for $i=1,\ldots,n$, yield the adjoint equations

$$a_i(\xi,\lambda_i) = \sum_{j\in N(i)}^{n}(u_i-u_j,\xi)_{\Gamma_{ij}} \quad \forall \xi \in H^1_{\Gamma_i}(\Omega_i). \tag{3.8}$$

Also $\partial\mathcal{L}/\partial g_{ij} = 0$ for $i=1,\ldots,n$, $j \in N(i)$, yield the optimality condition

$$(g_{ij},r)_{\Gamma_{ij}} = -\frac{1}{\delta_{ij}}(\lambda_i-\lambda_j,r)_{\Gamma_{ij}} \quad \forall r \in L^2(\Gamma_{ij}). \tag{3.9}$$

Combining the above results, we have the optimality system:

$$a_i(u_i,v) = (f,v)_{\Omega_i} - \sum_{j\in N(i)}\frac{1}{\delta_{ij}}(\lambda_i-\lambda_j,v)_{\Gamma_{ij}} \quad \forall v \in H^1_{\Gamma_i}(\Omega_i) \tag{3.10}$$

and

$$a_i(\lambda_i, \xi_i) = \sum_{j \in N(i)}^{n} (u_i - u_j, \xi)_{\Gamma_{ij}} \quad \forall \xi \in H^1_{\Gamma_i}(\Omega_i), \tag{3.11}$$

which are weak formulations of

$$-\Delta u_i = f \quad \text{in } \Omega_i, \qquad u_i = 0 \quad \text{on } \Gamma_i, \qquad \frac{\partial u_i}{\partial n_{ij}} = -\frac{1}{\delta_{ij}}(\lambda_i - \lambda_j) \quad \text{on } \Gamma_{ij}, \quad j \in N(i) \tag{3.12}$$

and the adjoint equations

$$-\Delta \lambda_i = 0 \quad \text{in } \Omega_i, \qquad \lambda_i = 0 \quad \text{on } \Gamma_i, \qquad \frac{\partial \lambda_i}{\partial n_{ij}} = u_i - u_j \quad \text{on } \Gamma_{ij}, \quad j \in N(i). \tag{3.13}$$

### 3.2.3 Numerical algorithm using the gradient method

We consider a gradient algorithm to solve the coupled system (3.10)-(3.11). Our purpose is to find $\mathbf{g} \in G$ which minimizes

$$\mathcal{K}_\delta(\mathbf{g}) = \mathcal{J}_\delta(u_1(\mathbf{g}), \ldots, u_n(\mathbf{g}), \mathbf{g}),$$

where $u_i$ for $i = 1, \ldots, n$ are the solutions of (3.12). The gradient method is then described as follows:
For $k = 1, 2, 3, \cdots$,

$$g_{ij}^{(k+1)} = g_{ij}^{(k)} - \frac{\alpha_{ij}}{\delta_{ij}} \frac{d\mathcal{K}_\delta(\mathbf{g}^{(k)})}{dg_{ij}} \quad \text{for } i = 1, \ldots, n, \quad j \in N(i), \tag{3.14}$$

where $\alpha_{ij}/\delta_{ij}$ is a step size. Using the "sensitivity" argument described in §2.4.1, the first derivatives of $\mathcal{K}_\delta$ with respect to $g_{ij}$ can be obtained as

$$\frac{\partial \mathcal{K}_\delta(\mathbf{g})}{\partial g_{ij}} = \delta_{ij} g_{ij} + (\lambda_i - \lambda_j)|_{\Gamma_{ij}}, \tag{3.15}$$

where $\lambda_i$ are determined from (3.13). Combining (3.15) with (3.14), we have the following algorithm.

**Algorithm 3.4**

1. *Choose* $g_{ij}^{(0)}$ *for* $i = 1, \ldots, n$, $j \in N_<(i)$.
2. *For* $k = 0, 1, 2, \cdots$,

   a. *for* $i = 1, \ldots, n$, *compute* ${u_i}^{(k)}$ *by*

   $$a_i({u_i}^{(k)}, v) = (f, v)_{\Omega_i} + \sum_{j \in N(i)} (g_{ij}^{(k)}, v)_{\Gamma_{ij}} \quad \forall v \in H^1_{\Gamma_i}(\Omega_i),$$

   b. *for* $i = 1, \ldots, n$, *compute* ${\lambda_i}^{(k)}$ *by*

   $$a_i({\lambda_i}^{(k)}, \xi) = \sum_{j \in N(i)} (u_i^{(k)} - u_j^{(k)}, \xi)_{\Gamma_{ij}} \quad \forall \xi \in H^1_{\Gamma_i}(\Omega_i),$$

   d. *for* $i = 1, \ldots, n$ *and* $j \in N_<(i)$, *compute* ${g_{ij}}^{(k+1)}$ *by*

   $$g_{ij}^{(k+1)} = (1 - \alpha_{ij}){g_{ij}}^{(k)} - \frac{\alpha_{ij}}{\delta_{ij}}({\lambda_i}^{(k)} - {\lambda_j}^{(k)}).$$

Convergence of the above algorithm with suitable step sizes was proved in Theorem 2.5 for the 2-subdomain problem and the proof in this multi-subdomain case follows analogously.

### 3.2.4 Numerical results

We present numerical results obtained by using Algorithm 3.4. The domain $\Omega$ was chosen as $\{(x, y) : 0 < x < 1, 0 < y < 1\}$ and divided into 2, 4 and 6 subdomains. See (a)-(c) of Figure 3.1 for sample decompositions of $\Omega$. The finite element spaces in each subdomain are chosen to consist of piecewise quadratic elements on triangular meshes and the data $f$ in (2.1) was adjusted so that $u = (x-1)y \sin x \cos(\pi y/2)$ is the exact solution and the boundary values are zero. We used a stopping criterion for Algorithm 3.4 defined by

$$\sum_{j=2}^{n} \sum_{j \in N_<(i)} \|\text{relative change in successive values of } g_{ij}\|_{0,\Gamma_{ij}} < tol.$$

We chose a relatively large *tol* for coarse mesh grids to avoid unnecessary iterations. The values of *tol* chosen, for example, for the 2-subdomain case are $10^{-2}$, $10^{-3}$, $10^{-4}$ and $10^{-5}$ for mesh grids $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$ and $\frac{1}{32}$, respectively. The penalty parameters $\delta_{ij}$ were fixed as $10^{-7}$ and step sizes for the gradient method were 16, 1.9 and 0.65 for 2 subdomains, 4 subdomains and 6 subdomains, respectively, which were found to be the optimal by various precalculations; we computed the optimal solutions with various step sizes and chose the one which yields the smallest error. It was again observed that the optimal step size does not depend on a grid size as in the 2-subdomain case presented in §2.7. We chose all the same step sizes for each interface for 4 and 6 subdomains cases, even though choosing step sizes in that way may result in slow convergence of iterations. In fact, it was found that, as the number of interfaces is increased, convergence of iterates becomes significantly slow for fine mesh grids. Results for the 2-, 4- and 6-subdomain problems are given in Table 3.1, 3.2 and 3.3, respectively. Table 3.3 shows that 500 iterations, which was chosen as the maximum number of iterations, are not enough to yield a good rate for the grid $1/48 \times 1/32$. Comparing the errors for each grid size in Table 3.1 with the ones in Table 3.2, we do not see much difference between them. We have almost the same numerical results for 4 subdomains as those for 2 subdomains. However, the 4 subdomains case requires approximately three times as many iterations. In the 6-subdomain case, we obtained the rates of convergence which almost coincide with the theoretical ones for relatively coarse grids, but the finest grid that we chose requires more than 500 iterations due to the slow convergence.

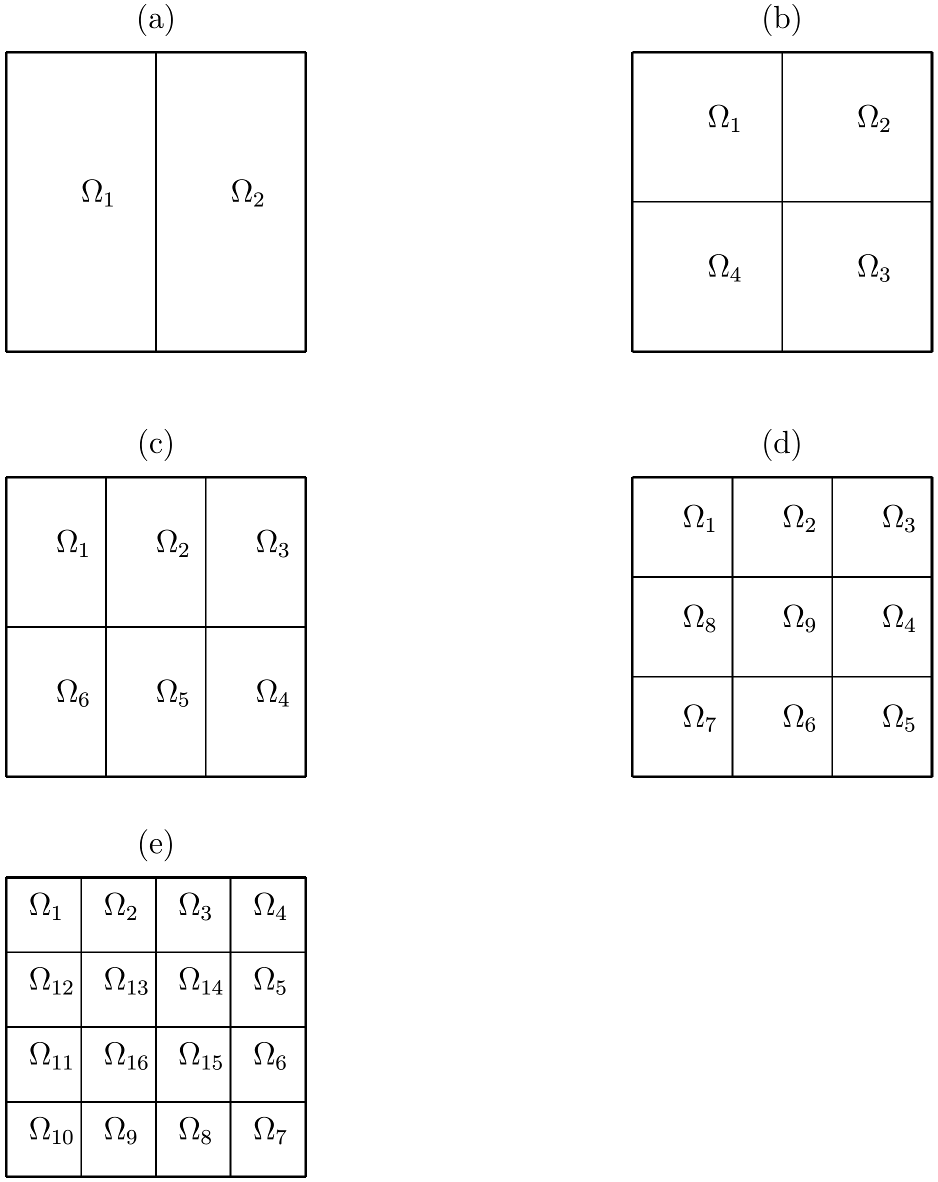

Figure 3.1: Subdomains of $\Omega = (0,1)^2$

Table 3.1: errors and rates for 2-subdomain problem

| mesh size | no. of iter. | $L^2 - error$ | $H^1 - error$ |
|---|---|---|---|
| $\frac{1}{4} \times \frac{1}{4}$ | 17 | $3.117 \cdot 10^{-4}$ | $1.149 \cdot 10^{-2}$ |
| rate | | 3.00 | 1.99 |
| $\frac{1}{8} \times \frac{1}{8}$ | 24 | $3.908 \cdot 10^{-5}$ | $2.891 \cdot 10^{-3}$ |
| rate | | 3.00 | 2.00 |
| $\frac{1}{16} \times \frac{1}{16}$ | 31 | $4.893 \cdot 10^{-6}$ | $7.238 \cdot 10^{-4}$ |
| rate | | 3.00 | 2.00 |
| $\frac{1}{32} \times \frac{1}{32}$ | 66 | $6.119 \cdot 10^{-7}$ | $1.810 \cdot 10^{-4}$ |

Table 3.2: errors and rates for 4-subdomain problem

| mesh size | no. of iter. | $L^2 - error$ | $H^1 - error$ |
|---|---|---|---|
| $\frac{1}{4} \times \frac{1}{4}$ | 56 | $3.121 \cdot 10^{-4}$ | $1.150 \cdot 10^{-2}$ |
| rate | | 3.00 | 1.99 |
| $\frac{1}{8} \times \frac{1}{8}$ | 73 | $3.914 \cdot 10^{-5}$ | $2.892 \cdot 10^{-3}$ |
| rate | | 3.00 | 2.00 |
| $\frac{1}{16} \times \frac{1}{16}$ | 107 | $4.899 \cdot 10^{-6}$ | $7.240 \cdot 10^{-4}$ |
| rate | | 2.99 | 2.00 |
| $\frac{1}{32} \times \frac{1}{32}$ | 182 | $6.158 \cdot 10^{-7}$ | $1.811 \cdot 10^{-4}$ |

Table 3.3: errors and rates for 6-subdomain problem

| mesh size | no. of iter. | $L^2 - error$ | $H^1 - error$ |
|---|---|---|---|
| $\frac{1}{6} \times \frac{1}{4}$ | 96 | $1.853 \cdot 10^{-4}$ | $8.067 \cdot 10^{-3}$ |
| rate | | 3.00 | 1.99 |
| $\frac{1}{12} \times \frac{1}{8}$ | 258 | $2.322 \cdot 10^{-5}$ | $2.037 \cdot 10^{-3}$ |
| rate | | 2.99 | 1.99 |
| $\frac{1}{24} \times \frac{1}{16}$ | 500 | $2.927 \cdot 10^{-6}$ | $5.115 \cdot 10^{-4}$ |
| rate | | 2.60 | 1.99 |
| $\frac{1}{48} \times \frac{1}{32}$ | 500 | $4.825 \cdot 10^{-7}$ | $1.287 \cdot 10^{-4}$ |

## 3.3 Linear least squares approach

In this section we generalize the linear least squares approach discussed in Chapter 2 for multi-subdomain problems, which include the case that some of the decomposed subdomains are interior subdomains which don't have a common boundary with $\Omega$. As mentioned before, there are two reasons that we can't simply apply the methods discussed to this case. The first difficulty encountered is the nonuniqueness of the solutions on interior subdomains because the local problems on those subdomains are pure Neumann problems, on the other hand, the original problem for $\Omega$ has a unique solution. Also, due to the same reason, the interface functions $g_{ij}$ for those subdomains need to satisfy the compatibility condition for the existence of solutions. To overcome these problems, we use an extra condition to guarantee a unique solution on the interior subdomains and look for the difference between the solution obtained by using the extra condition and the solution wanted by an optimization method. In addition, a projection operator defined on the interfaces is introduced to cover the compatibility condition.

Define the index sets

$$\begin{aligned} E &= \{i \mid int(\Gamma \cap \overline{\Omega}_i) \neq \emptyset\}, && (3.16)\\ I &= \{1, \ldots, n\} \setminus E. && (3.17) \end{aligned}$$

That is, the set $E$ consists of indices of exterior subdomains, which have common boundaries with $\Omega$. The subdomains with index in $I$ are the interior domains. We introduce the function spaces

$$V_i = \begin{cases} H^1_{\Gamma_i}(\Omega_i) & \text{for } i \in E, \\ H^1(\Omega_i) & \text{for } i \in I, \end{cases} \tag{3.18}$$

for $i = 1, 2, \ldots, n$.

Consider the problems

$$-\Delta u_i = f \text{ in } \Omega_i, \qquad u_i = 0 \text{ on } \Gamma_i, \qquad \frac{\partial u_i}{\partial n_{ij}} = g_{ij} \text{ on } \Gamma_{ij} \;\text{ for } j \in N(i), \tag{3.19}$$

for $i = 1, \ldots, n$ and their corresponding weak formulations

$$a_i(u_i, v_i) = (f, v_i)_{\Omega_i} + \sum_{j \in N(i)} (g_{ij}, v_i)_{\Gamma_{ij}} \quad \forall v_i \in V_i. \tag{3.20}$$

For $i \in E$, (3.20) is uniquely solvable. For $i \in I$, the problem has a solution $u_i \in V_i$ if and only if the compatibility condition

$$\sum_{j \in N(i)} \int_{\Gamma_{ij}} g_{ij} \, d\Gamma_{ij} = -\int_{\Omega_i} f \, d\Omega_i \tag{3.21}$$

is satisfied. In addition, since the solution is not unique, the additional condition

$$\int_{\Omega_i} u_i \, d\Omega_i = 0 \tag{3.22}$$

will be used to guarantee the uniqueness of the solution.

### 3.3.1 Formulation as a linear least squares problem

First, we consider particular solutions $\bar{u}_i$ of (3.19) by fixing the interface functions $\bar{g}_{ij}$ in a suitable way. The differences of the traces $\bar{u}_i - \bar{u}_j$ of solutions on neighboring domains are assembled into a vector $\mathbf{d}$. Then, for suitable interface functions $\tilde{g}_{ij}$, we consider the solutions $\tilde{u}_i$ of (3.19) with the right hand side $f = 0$. This defines a bounded linear operator $B$ that maps these interface functions $\tilde{g}_{ij}$ into the differences of the traces $\tilde{u}_i - \tilde{u}_j$ of solutions on neighboring domains. Since the solution of (3.19) with interface functions $g_{ij} = \bar{g}_{ij} + \tilde{g}_{ij}$ and the right hand side $f$ is given by $u_i = \bar{u}_i + \tilde{u}_i$, we need to find functions $\tilde{g}_{ij}$ to minimize the $L^2$ norm of the traces of $u_i - u_j = \bar{u}_i - \bar{u}_j + \tilde{u}_i - \tilde{u}_j$. If there are no interior subdomains, this can be formulated as $\min_{\tilde{\mathbf{g}} \in G} \|B\tilde{\mathbf{g}} + \mathbf{d}\|^2$ in some appropriate norm.

#### Particular solutions of the subproblems

Let $\bar{\mathbf{g}} \in G$ satisfy

$$\sum_{j \in N(i)} \int_{\Gamma_{ij}} \bar{g}_{ij} \, d\Gamma_{ij} = - \int_{\Omega_i} f \, d\Omega_i \quad \forall i \in I, \tag{3.23}$$

where $\bar{g}_{ij} = -\bar{g}_{ji}$. On the interior domains $\Omega_i$, $i \in I$, we consider the problem:

$$\begin{aligned} a_i(\bar{u}_i, v_i) &= (f, v_i)_{\Omega_i} + \sum_{j \in N(i)} (\bar{g}_{ij}, v_i)_{\Gamma_{ij}} \quad \forall v_i \in V_i, && (3.24) \\ \int_{\Omega_i} \bar{u}_i \, d\Omega_i &= 0. && (3.25) \end{aligned}$$

Equation (3.24) is a weak formulation corresponding to

$$-\Delta \bar{u}_i = f \quad \text{in } \Omega_i, \quad \frac{\partial \bar{u}_j}{\partial n_{ij}} = \bar{g}_{ij} \quad \text{on } \Gamma_{ij} \tag{3.26}$$

and the condition (3.25) is introduced to enforce uniqueness of the solution of (3.26).

On the exterior domains $\Omega_i$, $i \in E$, we let $\bar{u}_i$ denote the solution of

$$a_i(\bar{u}_i, v_i) = (f, v_i)_{\Omega_i} + \sum_{j \in N(i)} (\bar{g}_{ij}, v_i)_{\Gamma_{ij}} \quad \forall v_i \in V_i, \tag{3.27}$$

which is a weak formulation corresponding to

$$-\Delta \bar{u}_i = f \quad \text{in } \Omega_i, \quad \frac{\partial \bar{u}_i}{\partial n_{ij}} = \bar{g}_{ij} \quad \text{on } \Gamma_{ij} \quad \text{and} \quad \bar{u}_i = 0 \quad \text{on } \Gamma \cap \bar{\Omega}_i. \tag{3.28}$$

If $I = \emptyset$, i.e., if there are no interior subdomains, then condition (3.23) is void and the interface functions $\bar{\mathbf{g}}$ can be chosen arbitrarily, e.g., $\bar{\mathbf{g}} = 0$. If there are interior subdomains, a function $\bar{g}_{ij}$ satisfying (3.23) can be easily constructed. To describe the construction process, we assume that the subdomains are numbered in layers starting with the exterior subdomains. This assumption is made to simplify the presentation of the construction process, but is not necessary in an actual implementation. Let the exterior subdomains have indices $1, \ldots, m$.

We compute constant interface functions $\bar{g}_{ij}$ satisfying (3.23) starting with the inner most interior subdomain and working outwards. For $i = n$, which is in $I$, we set

$$\bar{g}_{nk} = -\frac{\int_{\Omega_n} f \, d\Omega_n}{\sum_{j \in N(n)} |\Gamma_{nj}|}, \quad k \in N(n). \tag{3.29}$$

For the other interior domains $i = n-1, \ldots, m+1$, we successively set

$$\bar{g}_{ik} = -\frac{\int_{\Omega_i} f \, d\Omega_i - \sum_{j \in N(i) \cap \{i+1, \ldots, n\}} \int_{\Gamma_{ij}} \bar{g}_{ij} \, d\Gamma_{ij}}{\sum_{j \in N(i) \setminus \{i+1, \ldots, n\}} |\Gamma_{ij}|}, \quad k \in N(i) \setminus \{i+1, \ldots, n\}. \tag{3.30}$$

Notice that $N(i) \setminus \{i+1, \ldots, n\} \neq \emptyset$. Hence, for all $i = n-1, \ldots, m+1$ at least one function $\bar{g}_{ik}$ is determined by (3.30), i.e., there is enough freedom in the choice of interface functions to satisfy (3.23) on all interior domains.

It is important to start the construction process at the inner most interior subdomain and work outwards. Otherwise, one might end up with an interior subdomain, say $\Omega_l$, for which all boundary functions $\bar{g}_{lj}$ have been determined previously. In this case it cannot be guaranteed that (3.23) holds for $i = n$.

**Solutions of the homogeneous problem**

We define the subspace $G_o$ of $G$ by

$$G_0 = \left\{ \mathbf{g} \in G \;\Big|\; \sum_{j \in N(i)} \int_{\Gamma_{ij}} g_{ij} \, d\Gamma_{ij} = 0, \; i \in I \right\},$$

where $g_{ij} = -g_{ji}$. If $I = \emptyset$, i.e., if there are no interior subdomains, then let $G_0 = G$. For given $\tilde{\mathbf{g}} \in G_0$ we consider

$$\begin{aligned} a_i(\tilde{u}_i, v_i) &= (f, v_i)_{\Omega_i} + \sum_{j \in N(i)} (\tilde{g}_{ij}, v_i)_{\Gamma_{ij}} \quad \forall v_i \in V_i, && (3.31) \\ \int_{\Omega_i} \tilde{u}_i \, d\Omega &= 0, && (3.32) \end{aligned}$$

for $i \in I$, and

$$a_i(\tilde{u}_i, v_i) = (f, v_i)_{\Omega_i} + \sum_{j \in N(i)} (\tilde{g}_{ij}, v_i)_{\Gamma_{ij}} \quad \forall v_i \in V_i \tag{3.33}$$

for $i \in E$. The equations (3.31), (3.33) are weak formulations corresponding to

$$-\Delta \tilde{u}_i = 0 \quad \text{in } \Omega_i, \quad \text{and} \quad \frac{\partial \tilde{u}_i}{\partial n_{ij}} = \tilde{g}_{ij} \quad \text{on } \Gamma_{ij}, \; j \in N(i), \tag{3.34}$$

and

$$-\Delta \tilde{u}_i = 0 \quad \text{in } \Omega_i, \quad \tilde{u}_i = 0 \quad \text{on } \Gamma_i \quad \text{and} \quad \frac{\partial \tilde{u}_i}{\partial n_{ij}} = \tilde{g}_{ij} \quad \text{on } \Gamma_{ij}, \; j \in N(i), \tag{3.35}$$

respectively.

We define the linear operators $B_i : G_0 \to V_i$, $i = 1, \ldots, n$, by

$$B_i \tilde{\mathbf{g}} = \tilde{u}_i, \tag{3.36}$$

where for $i \in I$, $\tilde{u}_i$ is the unique solution of (3.31)-(3.32) and for $i \in E$, $\tilde{u}_i$ is the unique solution of (3.33).

In the following optimization formulation of the domain decomposition problem it is crucial that the interface functions $\tilde{g}_{ij}$ satisfy the compatibility condition $\tilde{\mathbf{g}} \in G_0$. Of course, if there are no interior domains, then $G = G_0$ and no interface conditions need to be imposed. Otherwise, we enforce this condition by introducing the projection $P$ onto the subspace

$G_0$. Since $G_0$ is a closed linear subspace of $G$, the projection $P : G \rightarrow G_0$ is well defined. It satisfies

$$P^* = P, \quad \|P\|_{0,G} = 1.$$

If $I = \emptyset$, then $P = I_G$, where $I_G$ denotes the identity on $G$. The projection $\tilde{\mathbf{g}} = P\mathbf{g}$ is the solution of

$$\min \frac{1}{2}\|\tilde{\mathbf{g}} - \mathbf{g}\|_G^2 \tag{3.37}$$

$$\text{such that} \sum_{j\in N(i)} \int_{\Gamma_{ij}} \tilde{g}_{ij}\, d\Gamma_{ij} = 0, \quad i \in I. \tag{3.38}$$

**Lemma 3.1** *Let* $\mathbf{g} \in G$. *Then the projection* $\tilde{\mathbf{g}} = P\mathbf{g}$ *is given by*

$$\tilde{g}_{ij} = g_{ij} \quad \textit{for } i,j \in E \tag{3.39}$$

*and*

$$\tilde{g}_{ij} = \begin{cases} g_{ij} - (\lambda_i - \lambda_j) & \textit{if } i \in I,\ j \in N(i) \cap I, \\ g_{ij} - \lambda_i & \textit{if } i \in I,\ j \in N(i) \setminus I, \end{cases} \tag{3.40}$$

*where the scalars* $\lambda_i$ *are the solution of*

$$\Pi\lambda = r, \tag{3.41}$$

*with*

$$\Pi_{ik} = \begin{cases} \sum_{j\in N(i)} |\Gamma_{ij}| & \textit{if } i = k, \\ -|\Gamma_{ik}| & \textit{if } i \neq k, k \in N(i), \\ 0 & \textit{else}, \end{cases} \tag{3.42}$$

$$r_i = \sum_{j\in N(i)} (g_{ij}, 1)_{\Gamma_{ij}} \tag{3.43}$$

*for* $i, k \in I$.

**Proof:** The necessary and sufficient optimality conditions for (3.37), (3.38) are

$$\sum_{i=2}^{n} \sum_{j\in N_<(i)} (\tilde{g}_{ij} - g_{ij}, h_{ij})_{\Gamma_{ij}} + \sum_{i\in I} \lambda_i \sum_{j\in N(i)} (1, h_{ij})_{\Gamma_{ij}} = 0 \tag{3.44}$$

for all $h_{ij} \in L^2(\Gamma_{ij})$ with $h_{ij} = -h_{ji}$. Setting $h_{ij} = 0$ for $i \in I$ and $j \in N(i)$, in (3.44) yields

$$\tilde{g}_{ij} = g_{ij} \quad \text{for } i,j \in E. \tag{3.45}$$

The remaining functions $\tilde{g}_{ij}$ are computed from

$$\sum_{i\in I}\sum_{j\in N_<(i)}(\tilde{g}_{ij},h_{ij})_{\Gamma_{ij}}+\sum_{i\in I}\lambda_i\sum_{j\in N(i)}(1,h_{ij})_{\Gamma_{ij}}=\sum_{i\in I}\sum_{j\in N_<(i)}(g_{ij},h_{ij})_{\Gamma_{ij}}. \tag{3.46}$$

For $i_0 \in I$, setting $h_{i_0,j}=1$ for all $j\in N(i_0)$ and $h_{ij}=0$ otherwise in (3.46), yields

$$\lambda_{i_0}\sum_{j\in N(i_0)}|\Gamma_{i_0,j}|-\sum_{j\in N(i_0)\cap I}\lambda_j|\Gamma_{i_0,j}|=\sum_{j\in N(i_0)}\int_{\Gamma_{i_0,j}}g_{i_0,j}\,d\Gamma \tag{3.47}$$

by (3.38) and the fact that $h_{ij}=-h_{ji}$. Similarly, for other $i\in I$, we have a similar equation and obtain the system (3.41) for the $\lambda_i$'s.

Also, for each $i_0\in I$, $j_0\in N(i_0)$, setting $h_{ij}=0$ if $(i,j)\neq(i_0,j_0)$, we have

$$\begin{aligned}(\tilde{g}_{i_0,j_0},h_{i_0,j_0})_{\Gamma_{i_0,j_0}}&+\lambda_{i_0}(1,h_{i_0,j_0})_{\Gamma_{i_0,j_0}}-\lambda_{j_0}(1,h_{i_0,j_0})_{\Gamma_{i_0,j_0}}\\&=(g_{i_0,j_0},h_{i_0,j_0})_{\Gamma_{i_0,j_0}}\quad\forall h_{i_0,j_0}\in L^2(\Gamma_{i_0,j_0}),\quad\text{if } j\in I\end{aligned}$$

and

$$(\tilde{g}_{i_0,j_0},h_{i_0,j_0})_{\Gamma_{i_0,j_0}}+\lambda_i(1,h_{i_0,j_0})_{\Gamma_{i_0,j_0}}=(g_{i_0,j_0},h_{i_0,j_0})_{\Gamma_{i_0,j_0}}\quad\forall h_{i_0,j_0}\in L^2(\Gamma_{i_0,j_0}),\quad\text{if } j\notin I,$$

which yield (3.40). □

**Remark 3.1** *The system (3.41) is always solvable, since the matrix* $\Pi$ *is an invertible matrix. See [37] for more details.*

### The linear least squares problem

It will be helpful to index the sets $N_<(i)$ for $i=2,\ldots,n$, for defining operators and a vector, which will be used to formulate a linear least squares problem. Let

$$N_<(i)=\{j(i,1),\ldots,j(i,l_i)\},\quad i=2,\ldots,n. \tag{3.48}$$

We use the convention that $N_<(i)=\emptyset$ if and only if $l_i=0$.

Let $\bar{u}_i$ denote the particular solutions introduced in the beginning of this section and let $B_i$ be the operators defined by (3.36). Then solutions of (3.20) are given by

$$u_i=\begin{cases}\bar{u}_i+B_i\tilde{\mathbf{g}}, & \text{if } i\in E,\\ \bar{u}_i+B_i\tilde{\mathbf{g}}+c_i, & \text{if } i\in I,\end{cases}$$

where $\tilde{\mathbf{g}} \in G_0$ and $c_i$ is an arbitrary constant.

Define the vector

$$\mathbf{d} = \begin{pmatrix} \gamma_{2,j(2,1)}(\bar{u}_2 - \bar{u}_{j(2,1)}) \\ \vdots \\ \gamma_{2,j(2,l_2)}(\bar{u}_2 - \bar{u}_{j(2,l_2)}) \\ \vdots \\ \vdots \\ \gamma_{n,j(n,1)}(\bar{u}_n - \bar{u}_{j(n,1)}) \\ \vdots \\ \gamma_{n,j(n,l_n)}(\bar{u}_n - \bar{u}_{j(n,l_n)}) \end{pmatrix} \in G, \tag{3.49}$$

the operator $M \in \mathrm{R^n} \to \mathrm{G}$ by

$$M \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} c_2 - c_{j(2,1)} \\ \vdots \\ c_2 - c_{j(2,l_2)} \\ \vdots \\ \vdots \\ c_n - c_{j(n,1)} \\ \vdots \\ c_n - c_{j(n,l_n)} \end{pmatrix}, \tag{3.50}$$

and the operator $B \in \mathcal{L}(G_0, G)$ by

$$B = \begin{pmatrix} \gamma_{2,j(2,1)}(B_2 - B_{j(2,1)}) \\ \vdots \\ \gamma_{2,j(2,l_2)}(B_2 - B_{j(2,l_2)}) \\ \vdots \\ \vdots \\ \gamma_{n,j(n,1)}(B_n - B_{j(n,1)}) \\ \vdots \\ \gamma_{n,j(n,l_n)}(B_n - B_{j(n,l_n)}) \end{pmatrix}. \tag{3.51}$$

Then, the domain decomposition problem is now formulated as a constrained linear least squares problem:

$$\begin{aligned} &\min\ \mathcal{J}_\delta(\mathbf{g},\mathbf{c}) = \tfrac{1}{2}\|B\mathbf{g}+M\mathbf{c}+\mathbf{d}\|_G^2 + \tfrac{\delta}{2}\|\mathbf{g}\|_G^2 \\ &\text{such that } \sum_{j\in N(i)} \int_{\Gamma_{ij}} g_{ij}\, d\Gamma = 0 \quad \forall i \in I. \end{aligned} \tag{3.52}$$

Notice that the constraint must be strictly enforced to ensure well–posedness of $B\mathbf{g}$. The constraint can be eliminated (implicitly enforced) using the projection $P$. Problem (3.52) is equivalent to the unconstrained linear least squares problem:

$$\min \mathcal{J}_\delta(\mathbf{g},\mathbf{c}) = \frac{1}{2}\|BP\mathbf{g}+M\mathbf{c}+\mathbf{d}\|_G^2 + \frac{\delta}{2}\|P\mathbf{g}\|_G^2. \tag{3.53}$$

In fact, if $\mathbf{g}$ solves (3.53), then $P\mathbf{g}$ solves (3.52) and if $\mathbf{g}$ solves (3.52), then $\mathbf{g}$ solves (3.53). In (3.52), (3.53) we fix $c_i = 0$ for $i \in E$. Only the scalars $c_i$, $i \in I$ and the control $\mathbf{g}$ on the interfaces are variables. Rather than introducing additional notation, we simply write $\mathbf{c}$. If there are no interior subdomains, $I = \emptyset$, then the constraint in (3.52) is void and $P = I_G$. Moreover, all scalars $c_i$ are zero (by our convention) and the least squares problem (3.53) is given by

$$\min \mathcal{J}_\delta(\mathbf{g}) = \frac{1}{2}\|B\mathbf{g}+\mathbf{d}\|_G^2 + \frac{\delta}{2}\|\mathbf{g}\|_G^2. \tag{3.54}$$

### 3.3.2 Solution of the linear least squares problem

#### Adjoint computations

We start with the adjoint computation of $B \in \mathcal{L}(G_0, G)$. Since $G$ and $G_0$ are Hilbert spaces, the adjoint $B^*$ maps $G$ into $G_0$. Due to the image space $G_0$, the proper computation of $B^*$ is rather complicated, so it will be advantageous to introduce another operator $\widetilde{B} \in \mathcal{L}(G,G)$. For $i \in E$ we set $\widetilde{B}_i = B_i$. For $i \in I$ we define $\widetilde{B}_i \in \mathcal{L}(G, V_i)$ through

$$\widetilde{B}_i\mathbf{g} = u_i,$$

where $u_i$ is the unique solution of

$$a_i(u_i, v_i) = \widetilde{k}_i(1, v_i)_{\Omega_i} + \sum_{j\in N(i)} (g_{ij}, v_i)_{\Gamma_{ij}} \quad \forall v_i \in V_i \tag{3.55}$$

and

$$\int_{\Omega_i} u_i\, d\Omega_i = 0, \tag{3.56}$$

where

$$\widetilde{k}_i = -\frac{1}{|\Omega_i|} \sum_{j \in N(i)} \int_{\Gamma_{ij}} g_{ij} \, d\Gamma_{ij}. \tag{3.57}$$

Equation (3.55) is a weak formulation corresponding to

$$-\Delta u_i = \widetilde{k}_i \quad \text{in } \Omega_i, \qquad \frac{\partial u_i}{\partial n_{ij}} = g_{ij} \quad \text{on } \Gamma_{ij}, \ j \in N(i). \tag{3.58}$$

Due to the definition of $\widetilde{k}_i$ and (3.56), (3.55) admits a unique solution. The operator $\widetilde{B} \in \mathcal{L}(G, G)$ is now defined using (3.51) with $B_i$ replaced by $\widetilde{B}_i$. Since $\widetilde{k}_i = 0$ for $\mathbf{g} \in G_0$, the restriction of $\widetilde{B}$ onto $G_0$ is equal to $B$, i.e.,

$$\widetilde{B}|_{G_0} = B.$$

In particular, (3.53) is equivalent to

$$\min \mathcal{J}(\mathbf{g}, \mathbf{c}) = \frac{1}{2} \| \widetilde{B} P \mathbf{g} + M \mathbf{c} + \mathbf{d} \|_G^2 + \frac{\delta}{2} \| P \mathbf{g} \|_G^2. \tag{3.59}$$

The adjoint of $\widetilde{B}$ is introduced in the following lemma.

**Lemma 3.2** *For $\mathbf{h} \in G$ and $h_{ij} = -h_{ji}$*

$$\widetilde{B}^* \mathbf{h} = \begin{pmatrix} \gamma_{2,j(2,1)}(z_2 - z_{j(2,1)}) \\ \vdots \\ \gamma_{2,j(2,l_2)}(z_2 - z_{j(2,l_2)}) \\ \vdots \\ \vdots \\ \gamma_{n,j(n,1)}(z_n - z_{j(n,1)}) \\ \vdots \\ \gamma_{n,j(n,l_n)}(z_n - z_{j(n,l_n)}) \end{pmatrix}, \tag{3.60}$$

*where, for $i \in I$, $z_i$ is the unique solution of*

$$a_i(v_i, z_i) = \widetilde{k}_i^* (1, v_i)_{\Omega_i} + \sum_{j \in N(i)} (h_{ij}, v_i)_{\Gamma_{ij}} \quad \forall v_i \in V_i, \tag{3.61}$$

$$\int_{\Omega_i} z_i \, d\Omega_i = 0, \tag{3.62}$$

*where*

$$\widetilde{k}_i^* = -\frac{1}{|\Omega_i|} \sum_{j \in N(i)} \int_{\Gamma_{ij}} h_{ij}\, d\Gamma_{ij}, \tag{3.63}$$

*and for $i \in E$, $z_i$ is the unique solution of*

$$a_i(v_i, z_i) = \sum_{j \in N(i)} (h_{ij}, v_i)_{\Gamma_{ij}} \quad \forall v_i \in V_i. \tag{3.64}$$

**Proof:** Define the operator $\widehat{B} \in \mathcal{L}(G, G)$ by

$$\widehat{B}\mathbf{h} = \begin{pmatrix} \gamma_{2,j(2,1)}(z_2 - z_{j(2,1)}) \\ \vdots \\ \gamma_{2,j(2,l_2)}(z_2 - z_{j(2,l_2)}) \\ \vdots \\ \vdots \\ \gamma_{n,j(n,1)}(z_n - z_{j(n,1)}) \\ \vdots \\ \gamma_{n,j(n,l_n)}(z_n - z_{j(n,l_n)}) \end{pmatrix},$$

where $z_i$ are the solutions of (3.61)- (3.64). Then, by the definitions of $\widetilde{B}$ and $\widehat{B}$,

$$\begin{aligned}
(\widetilde{B}\mathbf{g}, \mathbf{h})_G &= \sum_{i=2}^{n} \sum_{j \in N_<(i)} (\widetilde{u}_i - \widetilde{u}_j, h_{ij})_{\Gamma_{ij}} \\
&= \sum_{i=2}^{n} \sum_{j \in N_<(i)} (\widetilde{u}_i, h_{ij})_{\Gamma_{ij}} - \sum_{i=2}^{n} \sum_{j \in N_<(i)} (\widetilde{u}_j, h_{ij})_{\Gamma_{ij}} \\
&= \sum_{i \in I} a_i(\widetilde{u}_i, z_i) - \sum_{i \in I} \widetilde{k}_i^* \int_{\Omega_i} \widetilde{u}_i\, d\Omega_i + \sum_{i \in E} a_i(\widetilde{u}_i, z_i) \\
&= \sum_{i=2}^{n} \sum_{j \in N_<(i)} (g_{ij}, z_i)_{\Gamma_{ij}} - \sum_{i=2}^{n} \sum_{j \in N_<(i)} (g_{ij}, z_j)_{\Gamma_{ij}} \\
&\quad + \sum_{i \in I} \widetilde{k}_i \int_{\Omega_i} z_i\, d\Omega_i - \sum_{i \in I} \widetilde{k}_i^* \int_{\Omega_i} \widetilde{u}_i\, d\Omega_i \\
&= \sum_{i=2}^{n} \sum_{j \in N_<(i)} (g_{ij}, z_i - z_j)_{\Gamma_{ij}} + \sum_{i \in I} \widetilde{k}_i \int_{\Omega_i} z_i\, d\Omega_i - \sum_{i \in I} \widetilde{k}_i^* \int_{\Omega_i} \widetilde{u}_i\, d\Omega_i \\
&= (\mathbf{g}, \widehat{B}\mathbf{h})_G + \sum_{i \in I} \widetilde{k}_i \int_{\Omega_i} z_i\, d\Omega_i - \sum_{i \in I} \widetilde{k}_i^* \int_{\Omega_i} \widetilde{u}_i\, d\Omega_i.
\end{aligned}$$

And, we have that

$$(\widetilde{B}\mathbf{g}, \mathbf{h})_G = (\mathbf{g}, \widehat{B}\mathbf{h})_G, \tag{3.65}$$

due to (3.56) and (3.62). Thus, by the definition of an adjoint, we have that

$$\widetilde{B}^* = \widehat{B}.$$

□

**Remark 3.2** *Equation (3.61) is a weak formulation corresponding to*

$$-\Delta z_i = \widetilde{k}_i^* \quad in\ \Omega_i, \quad \frac{\partial z_i}{\partial n_{ij}} = h_{ij} \quad on\ \Gamma_{ij},\ j \in N(i). \tag{3.66}$$

*Due to the definition of $\widetilde{k}_i^*$ and (3.62), this differential equation admits a unique solution. Equation (3.64) is a weak formulation corresponding to*

$$-\Delta z_i = 0 \quad in\ \Omega_i, \quad z_i = 0 \quad on\ \Gamma_i \quad and \quad \frac{\partial z_i}{\partial n_{ij}} = h_{ij} \quad on\ \Gamma_{ij},\ j \in N(i). \tag{3.67}$$

The adjoint $M^*$ of the operator $M \in \mathcal{L}(\mathrm{R^n}, \mathrm{G})$ is computed as follows:

$$(\mathbf{h}, M\mathbf{c})_G = \sum_{i=2}^{n} \sum_{j\in N_<(i)} \int_{\Gamma_{ij}} h_{ij}(M\mathbf{c})_{ij}\, d\Gamma_{ij} = \sum_{k=1}^{n} c_k \sum_{i=2}^{n} \sum_{j\in N_<(i)} \int_{\Gamma_{ij}} h_{ij}(Me_k)_{ij}\, d\Gamma_{ij}.$$

Thus, one can see that

$$M^*\mathbf{h} = \begin{pmatrix} \sum_{i=2}^{n} \sum_{j\in N_<(i)} \int_{\Gamma_{ij}} h_{ij}(Me_1)_{ij}\, d\Gamma_{ij} \\ \vdots \\ \sum_{i=2}^{n} \sum_{j\in N_<(i)} \int_{\Gamma_{ij}} h_{ij}(Me_n)_{ij}\, d\Gamma_{ij} \end{pmatrix}, \tag{3.68}$$

where $e_k$ denotes the $k$th unit vector in $\mathrm{R^n}$.

### Properties of the linear least squares problem

We introduce the notation

$$\mathbf{A} = \begin{pmatrix} \widetilde{B}P & M \\ \sqrt{\delta}\, P & 0 \end{pmatrix} \in \mathcal{L}(G \times \mathrm{R^n}, \mathrm{G}\times\mathrm{G}), \qquad \mathbf{b} = \begin{pmatrix} \mathbf{d} \\ 0 \end{pmatrix} \in \mathrm{G}\times\mathrm{G}, \quad \mathbf{x} = \begin{pmatrix} \mathbf{g} \\ \mathbf{c} \end{pmatrix} \in \mathrm{G}\times\mathrm{R^n} \tag{3.69}$$

if $I \neq \emptyset$ and

$$\mathbf{A} = \begin{pmatrix} B \\ \sqrt{\delta}\, I \end{pmatrix} \in \mathcal{L}(G, G \times G), \qquad \mathbf{b} = \begin{pmatrix} \mathbf{d} \\ 0 \end{pmatrix} \in G \times G, \quad \mathbf{x} = \begin{pmatrix} \mathbf{g} \end{pmatrix} \in G \tag{3.70}$$

otherwise. Here, we used $\mathbf{c} \in \mathrm{R}^{\mathrm{n}}$ to simplify notation and only the scalars $c_i, i \in I$, are variable. Using the notation, the minimization problem can be written in the form

$$\min_{\mathbf{x}} \frac{1}{2} \|\mathbf{A}\mathbf{x} + \mathbf{b}\|^2_{G \times G}, \tag{3.71}$$

where $G$ is defined by (3.3).

**Lemma 3.3** *If $I = \emptyset$ and $\delta > 0$, then the operator $\mathbf{A}$ has a closed range and trivial nullspace $N(\mathbf{A}) = \{\mathbf{0}\}$. If $I \neq \emptyset$ and $\delta > 0$, then the operator $\mathbf{A}$ has closed range and its nullspace is given by*

$$N(\mathbf{A}) = \left\{ \begin{pmatrix} \mathbf{g} \\ \mathbf{0} \end{pmatrix} \mid \mathbf{g} \in N(P) \right\},$$

*where $P$ is the projection operator. Moreover, there exists $\sigma > 0$ such that*

$$\|\mathbf{A}(\mathbf{g}, \mathbf{c})\| \geq \sigma \|(\mathbf{g}, \mathbf{c})\| \tag{3.72}$$

*for all $(\mathbf{g}, \mathbf{c}) \in N(P)^{\perp} \times \mathrm{R}^{\mathrm{n}}$. In particular, $\mathbf{A}^*\mathbf{A}$ is positive definite on $N(P)^{\perp} \times \mathrm{R}^{\mathrm{n}}$.*

**Proof:** The proof of the first statement follows from the second one with proper adjustment of notation. Let $(\mathbf{u}^k, \mathbf{h}^k)^T \in R(\mathbf{A})$ with $(\mathbf{u}^k, \mathbf{h}^k)^T \to (\hat{\mathbf{u}}, \hat{\mathbf{h}})$. There exist $\mathbf{g}^k, \mathbf{c}^k$ with $\widetilde{B}P\mathbf{g}^k + M\mathbf{c}^k = \mathbf{u}^k$ and $\sqrt{\delta}\ \mathbf{g}^k = \mathbf{h}^k$. We can assume that $\mathbf{g}^k \in N(P)^{\perp} = R(P)$ and $\mathbf{c}^k \perp N(M)$. The solution $\mathbf{c}^k$ of $M\mathbf{c}^k = \mathbf{u}^k - \widetilde{B}P\mathbf{g}^k$ is given by $\mathbf{c}^k = M^{\dagger}(\mathbf{u}^k - \widetilde{B}P\mathbf{g}^k)$, where $M^{\dagger}$ denotes the pseudoinverse. Since $\delta > 0$ and $\mathbf{g}^k = \mathbf{h}^k/\sqrt{\delta}$, $\mathbf{g}^k \to \hat{\mathbf{g}} = \hat{\mathbf{h}}/\sqrt{\delta}$ and $\mathbf{c}^k = M^{\dagger}(\mathbf{u}^k - \widetilde{B}P\mathbf{g}^k) \to \hat{\mathbf{c}} = M^{\dagger}(\hat{\mathbf{u}} - \widetilde{B}P\hat{\mathbf{g}})$. This yields that $\widetilde{B}P\hat{\mathbf{g}} + M\hat{\mathbf{c}} = \hat{\mathbf{u}}$ and $\sqrt{\delta}\ \hat{\mathbf{g}} = \hat{\mathbf{h}}$ and thus, $R(\mathbf{A})$ is closed.

Let $\mathbf{g} \in N(P)^{\perp}$, $\mathbf{c} \in \mathrm{R}^{\mathrm{n}}$ satisfy $\widetilde{B}P\mathbf{g} + M\mathbf{c} = \mathbf{0}$ and $\sqrt{\delta}\, P\mathbf{g} = \sqrt{\delta}\, \mathbf{g} = \mathbf{0}$. Then $\mathbf{g} = \mathbf{0}$ and $M\mathbf{c} = \mathbf{0}$. Since $c_i = 0$ for $i \in E$, the definition (3.50) of $M$ implies that $c_j = 0$ for all $j \in N_<(i)$, $i \in E$, and for all $j$ such that $i \in N_<(j)$ for some $i \in E$. Working our way successively from the exterior domains to all interior domains, we find that $\mathbf{c} = \mathbf{0}$.

Suppose there exists $(\mathbf{g}^k, \mathbf{c}^k)$ with $\mathbf{g}^k \in N(P)^{\perp}$ and $\|(\mathbf{g}^k, \mathbf{c}^k)\| = 1$ such that $\|\mathbf{A}(\mathbf{g}^k, \mathbf{c}^k)\| \to 0$. Then, by the definition of $A$, $\sqrt{\delta}\|\mathbf{g}^k\| \to 0$, thus $M\mathbf{c}^k \to \mathbf{0}$. Since $N(M) = \{\mathbf{0}\}$ and $M$ is finite dimensional, we obtain $\mathbf{c}^k \to \mathbf{0}$, which contradicts $\|(\mathbf{g}^k, \mathbf{c}^k)\| = 1$. Hence, there exists $\sigma > 0$ such that $\|\mathbf{A}(\mathbf{g}, \mathbf{c})\| \geq \sigma \|(\mathbf{g}, \mathbf{c})\|$. $\square$

Using Lemma 3.3, we obtain the following result:

**Theorem 3.2** *Let $\delta > 0$. The linear least squares problem (3.71) admits a solution. Moreover, if $I = \emptyset$, then the solution of (3.71) is unique and is characterized as the solution of the normal equation*

$$B^*B\mathbf{g} + \delta\mathbf{g} = -B^*\mathbf{d}.$$

*If $I \neq \emptyset$, then solutions of (3.71) are solutions of the normal equation*

$$\begin{pmatrix} P^*\widetilde{B}^*\widetilde{B}P + \delta\, P^*P & P^*\widetilde{B}^*M \\ M^*\widetilde{B}P & M^*M \end{pmatrix} \begin{pmatrix} \mathbf{g} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} -P^*\widetilde{B}^*\mathbf{d} \\ -M^*\mathbf{d} \end{pmatrix} \tag{3.73}$$

*and vice versa. The solution is unique if $P = I$, i.e., if no interior subdomains exist.*

**Proof:** See, e.g., [36, Thm. 2.1.1]. □

**Lemma 3.4** *The operators $\widetilde{B}^*\widetilde{B} \in \mathcal{L}(G, G)$ and $P^*\widetilde{B}^*\widetilde{B}P \in \mathcal{L}(G, G)$ are compact.*

**Proof:** The operator $\widetilde{B}^*\widetilde{B} \in \mathcal{L}(G, G)$ can be viewed as a composition of five bounded linear operators. The first operator $T_1 \in \mathcal{L}(G, \prod_{i=1}^n H^1(\Omega_i))$ maps $\mathbf{g} \in G$ into the solutions $u_i = \widetilde{B}_i\mathbf{g} \in H^1(\Omega_i)$. The second operator $T_2 \in \mathcal{L}(\prod_{i=1}^n H^1(\Omega_i), \prod_{i,j} H^{1/2}(\Gamma_{ij}))$ maps these functions $u_i \in H^1(\Omega_i)$ into

$$T_2(u_1, \dots, u_n) = \begin{pmatrix} \gamma_{2,j(2,1)}(u_2 - u_{j(2,1)}) \\ \vdots \\ \gamma_{2,j(2,l_2)}(u_2 - u_{j(2,l_2)}) \\ \vdots \\ \vdots \\ \gamma_{d,j(d,1)}(u_n - u_{j(n,1)}) \\ \vdots \\ \gamma_{d,j(d,l_d)}(u_n - u_{j(n,l_n)}) \end{pmatrix}.$$

Note that $\widetilde{B}$ is the imbedding of $T_2 \circ T_1$ into $G$. The third operator $T_3 \in \mathcal{L}(\prod_{i,j} H^{1/2}(\Gamma_{ij}), \prod_{i=1}^n H^2(\Omega_i))$ maps the traces $\gamma_{ij}(u_i - u_j)$ into the solutions $z_i$ of (3.61)-(3.62) with $h_{ij}$ replaced by $\gamma_{ij}(u_i - u_j)$. The fourth operator $T_4 \in \mathcal{L}(\prod_{i=1}^n H^2(\Omega_i), \prod_{i=1}^n H^1(\Omega_i))$ is the simple imbedding. Finally, for given $z_i \in H^1(\Omega_i)$, $T_5 \in \mathcal{L}(\prod_{i=1}^n H^1(\Omega_i), G)$ is defined by the right hand side of (3.60). Clearly, $\widetilde{B}^*\widetilde{B} = T_5 \circ \ldots \circ = T_1$. All operators $T_1, \dots, T_5$, are bounded and, since the the imbedding $H^2(\Omega_i) \hookrightarrow H^1(\Omega_i)$ is compact, the linear operator $T_4$ is compact. Thus, $\widetilde{B}^*\widetilde{B}$ is the composition of linear bounded operators $T_1, \dots, T_5$, of which one is compact. This yields the compactness of $\widetilde{B}^*\widetilde{B}$. Since $P$ is bounded, compactness of $\widetilde{B}^*\widetilde{B}$ implies compactness of $P^*\widetilde{B}^*\widetilde{B}P$. □

### Convergence of the conjugate gradient algorithm

Again, we use the conjugate gradient algorithm 2.2 presented in §2.8.2 to solve the linear least squares problem (3.71). Convergence results for the conjugate gradient algorithm are well established. For the finite dimensional case these can be found in textbooks on iterative methods such as [3] and [45]. These results extend to the infinite dimensional case when the Hilbert space $\mathbf{X}$ is separable and $\mathbf{A}$ has closed range. See, e.g., [36]. In our application $\mathbf{X} = G$ or $\mathbf{X} = G \times \mathrm{R}^{\mathrm{n}}$ is separable and $\mathbf{A}$ has closed range, provided $\delta > 0$ (Lemma 3.3).

**Theorem 3.3** *If $\delta > 0$, the conjugate gradient method for the solution of the least squares problem (3.71) converges to a solution $\hat{\mathbf{x}}$ of the least squares problem. If $I = \emptyset$, then $\hat{\mathbf{x}}$ is unique. If $I \neq \emptyset$ the conjugate gradient method converges to the solution $\hat{\mathbf{x}} = (\hat{\mathbf{g}}, \hat{\mathbf{c}})$ which is given by $\hat{\mathbf{g}} = \mathbf{g}_+ + P\mathbf{g}^{(0)}$. Here $\mathbf{g}_+ \in N(P)^\perp$ denotes the minimum norm solution and $\mathbf{x}^{(0)} = (\mathbf{g}^{(0)}, \mathbf{c}^{(0)})$ is the initial iterate. The conjugate gradient iterates obey*

$$\|\mathbf{x}^{(k+1)} - \hat{\mathbf{x}}\|_{\mathbf{A}^*\mathbf{A}} \leq \frac{\|\mathbf{A}\|^2 - \sigma^2}{\|\mathbf{A}\|^2 + \sigma^2} \|\mathbf{x}^{(k)} - \hat{\mathbf{x}}\|_{\mathbf{A}^*\mathbf{A}},$$

*where $\sigma > 0$ is the largest constant satisfying (3.72) and $\|\mathbf{x}\|^2_{\mathbf{A}^*\mathbf{A}} = (\mathbf{A}^*\mathbf{A}\mathbf{x}, \mathbf{x})_{\mathbf{X}}$.*

**Proof:** Since

$$\sigma^2 \|\mathbf{x}\|_{\mathbf{X}} \leq (\mathbf{A}^*\mathbf{A}\mathbf{x}, \mathbf{x})_{\mathbf{X}} \leq \|\mathbf{A}\|^2 \|\mathbf{x}\|_{\mathbf{X}},$$

this result follows immediately from the standard convergence results for the conjugate gradient method. See e.g. [3, Sec. 13], [36, Sec. II.5]. □

The following observation is relevant for the case $I \neq \emptyset$. If started with $\mathbf{g}^{(0)} = \mathbf{0} \in G_0$, the conjugate gradient algorithm will generate iterates $(\mathbf{g}^{(k)}, \mathbf{c}^{(k)})$ satisfying

$$\begin{pmatrix} \mathbf{g}^{(k)} \\ \mathbf{c}^{(k)} \end{pmatrix} \in \mathcal{K}_k \left( \begin{pmatrix} P^*\widetilde{B}^*\widetilde{B}P + \delta\, P^*P & P^*\widetilde{B}^*M \\ M^*\widetilde{B}P & M^*M \end{pmatrix}, \begin{pmatrix} -P^*\widetilde{B}^*\mathbf{d} \\ -M^*\mathbf{d} \end{pmatrix} \right),$$

where for $C, b$, $\mathcal{K}_k(C, b)$ is the Krylov subspace

$$\mathcal{K}_k(C, b) = \mathrm{span}\{C^{k-1}b, \ldots, Cb, b\}.$$

In particular, all iterates satisfy $\mathbf{g}^{(k)} \in R(P^*) = G_0$.

Now consider the least squares problem

$$\min \mathcal{J}_\delta(\mathbf{g},\mathbf{c}) = \min \frac{1}{2}\|\widetilde{B}P\mathbf{g} + M\mathbf{c} + \mathbf{d}\|_G^2 + \frac{\delta}{2}\|\mathbf{g}\|_G^2. \tag{3.74}$$

The problem (3.74) differs from (3.59) in the penalty term. Since $P$ is missing in the penalty term of (3.74), this least squares problem can be shown to have a unique solution. The solution of (3.74) is the solution of the normal equation

$$\begin{pmatrix} P^*\widetilde{B}^*\widetilde{B}P + \delta\, I & P^*\widetilde{B}^*M \\ M^*\widetilde{B}P & M^*M \end{pmatrix} \begin{pmatrix} \mathbf{g} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} -P^*\widetilde{B}^*\mathbf{d} \\ -M^*\mathbf{d} \end{pmatrix}. \tag{3.75}$$

If the initial iterate satisfies $\mathbf{g}^{(0)} \in G_0$, then a simple induction proof shows that all iterates of the conjugate gradient method applied to (3.74) satisfy $\mathbf{g}^{(k)} \in R(P^*) = G_0$. Hence, if $\mathbf{g}^{(0)} \in G_0$, then the conjugate gradient method applied to (3.59) and to (3.74) produce the same iterates.

When the operator in the normal equation is a compact perturbation of the identity, such as in our case for $\delta > 0$, then the linear convergence result Theorem 3.3 can be improved.

**Theorem 3.4** *If $\delta > 0$, the iterates of the conjugate gradient method for the solution of the least squares problem (3.74) converge r–superlinearly.*

**Proof:** We write

$$\mathbf{A}^*\mathbf{A} = \underbrace{\begin{pmatrix} P^*\widetilde{B}^*\widetilde{B}P & P^*\widetilde{B}^*M \\ M^*\widetilde{B}P & M^*M - \delta I \end{pmatrix}}_{=\mathbf{C}} + \begin{pmatrix} \delta\, I & 0 \\ 0 & \delta I \end{pmatrix}.$$

By Lemma 3.3, $\mathbf{A}^*\mathbf{A}$ is positive definite on the Hilbert space $G \times \mathrm{R}^\mathrm{n}$. Moreover, by Lemma 3.4, $\mathbf{C}$ is compact. Hence, on $G \times \mathrm{R}^\mathrm{n}$, $\mathbf{A}^*\mathbf{A} = \mathbf{C} + \delta I$. Therefore, the result follows from [57]. □

### 3.3.3 Numerical results

We chose the domain $\Omega = \{(x,y) : 0 < x < 1, 0 < y < 1\}$ and adjusted the data $f$ so that $u = (x-1)y \sin x \cos(\pi y/2)$ is the exact solution. Numerical tests were performed with 2, 4, 6, 9 and 16 subdomains. In 9- and 16-subdomain problems, 1 and 4 subdomains, respectively, are interior subdomains which do not have common boundaries with $\Omega$. See Figure 3.1 for the decomposition of $\Omega$. We used the finite element space which consists of piecewise quadratic elements on triangular meshes on each subdomain. The values of stopping tolerance for the conjugate gradient iterations were chosen as $C \times (mesh\ size)^3$, where $C = 10^{-3}$ for 2-, 4-, 6- and 9-subdomain problems and $C = 5\times10^{-4}$ for 16-subdomain problem. We fixed the penalty parameter $\delta$ as $10^{-7}$ and the maximum number of iterations as 500.

We computed the $L^2$ distance and the $H^1$ distance between the exact solution and the finite element solutions for each mesh size chosen, and also calculated the rate of convergence of the finite element solution by comparing the errors on pairs of grids. Comparing the errors in Table 3.8 with the ones in Table 3.4 and Table 3.5, we see that the errors for $h = 1/16$ and $h = 1/32$ coincide up to the first two significant digits. The coarse grad $h = 1/8$ generates a little smaller error on 16 subdomains. But We noticed the rate of the convergence of the conjugate gradient iterations gets smaller as the number of subdomains increases. In particular, when the number of interior subdomains is increased from 1 to 4, Table 3.7 and Table 3.8 show that many more iterations are needed.

Table 3.4: errors and rates for 2-subdomain problem

| mesh size | no. of iter. | $L^2 - error$ | $H^1 - error$ |
|---|---|---|---|
| $\frac{1}{4} \times \frac{1}{4}$ | 3 | $3.116 \cdot 10^{-4}$ | $1.149 \cdot 10^{-2}$ |
| rate | | 3.00 | 1.99 |
| $\frac{1}{8} \times \frac{1}{8}$ | 4 | $3.907 \cdot 10^{-5}$ | $2.891 \cdot 10^{-3}$ |
| rate | | 3.00 | 2.00 |
| $\frac{1}{16} \times \frac{1}{16}$ | 5 | $4.889 \cdot 10^{-6}$ | $7.238 \cdot 10^{-4}$ |
| rate | | 3.00 | 2.00 |
| $\frac{1}{32} \times \frac{1}{32}$ | 7 | $6.111 \cdot 10^{-7}$ | $1.810 \cdot 10^{-4}$ |

Table 3.5: errors and rates for 4-subdomain problem

| mesh size | no. of iter. | $L^2 - error$ | $H^1 - error$ |
|---|---|---|---|
| $\frac{1}{4} \times \frac{1}{4}$ | 4 | $3.122 \cdot 10^{-4}$ | $1.150 \cdot 10^{-2}$ |
| rate | | 3.00 | 1.99 |
| $\frac{1}{8} \times \frac{1}{8}$ | 5 | $3.911 \cdot 10^{-5}$ | $2.892 \cdot 10^{-3}$ |
| rate | | 3.00 | 2.00 |
| $\frac{1}{16} \times \frac{1}{16}$ | 6 | $4.896 \cdot 10^{-6}$ | $7.240 \cdot 10^{-4}$ |
| rate | | 3.00 | 2.00 |
| $\frac{1}{32} \times \frac{1}{32}$ | 9 | $6.119 \cdot 10^{-7}$ | $1.811 \cdot 10^{-4}$ |

Table 3.6: errors and rates for 6-subdomain problem

| mesh size | no. of iter. | $L^2 - error$ | $H^1 - error$ |
|---|---|---|---|
| $\frac{1}{6} \times \frac{1}{4}$ | 9 | $1.839 \cdot 10^{-4}$ | $8.076 \cdot 10^{-3}$ |
| rate | | 2.99 | 1.99 |
| $\frac{1}{12} \times \frac{1}{8}$ | 12 | $2.317 \cdot 10^{-5}$ | $2.037 \cdot 10^{-3}$ |
| rate | | 3.00 | 1.99 |
| $\frac{1}{24} \times \frac{1}{16}$ | 18 | $2.903 \cdot 10^{-6}$ | $5.114 \cdot 10^{-4}$ |
| rate | | 3.00 | 2.00 |
| $\frac{1}{48} \times \frac{1}{32}$ | 27 | $3.640 \cdot 10^{-7}$ | $1.281 \cdot 10^{-4}$ |

Table 3.7: errors and rates for 9-subdomain problem

| mesh size | no. of iter. | $L^2 - error$ | $H^1 - error$ |
|---|---|---|---|
| $\frac{1}{6} \times \frac{1}{6}$ | 10 | $9.080 \cdot 10^{-5}$ | $5.063 \cdot 10^{-3}$ |
| rate | | 2.98 | 1.99 |
| $\frac{1}{12} \times \frac{1}{12}$ | 13 | $1.147 \cdot 10^{-5}$ | $1.278 \cdot 10^{-3}$ |
| rate | | 2.99 | 1.99 |
| $\frac{1}{24} \times \frac{1}{24}$ | 22 | $1.442 \cdot 10^{-6}$ | $3.207 \cdot 10^{-4}$ |
| rate | | 2.99 | 2.00 |
| $\frac{1}{48} \times \frac{1}{48}$ | 39 | $1.811 \cdot 10^{-7}$ | $8.033 \cdot 10^{-5}$ |

Table 3.8: errors and rates for 16-subdomain problem

| mesh size | no. of iter. | $L^2 - error$ | $H^1 - error$ |
|---|---|---|---|
| $\frac{1}{8} \times \frac{1}{8}$ | 32 | $3.796 \cdot 10^{-5}$ | $2.834 \cdot 10^{-3}$ |
| rate | | 2.97 | 1.98 |
| $\frac{1}{16} \times \frac{1}{16}$ | 78 | $4.857 \cdot 10^{-6}$ | $7.204 \cdot 10^{-4}$ |
| rate | | 2.96 | 1.98 |
| $\frac{1}{32} \times \frac{1}{32}$ | 218 | $6.226 \cdot 10^{-7}$ | $1.830 \cdot 10^{-4}$ |
| rate | | 2.88 | 1.93 |
| $\frac{1}{64} \times \frac{1}{64}$ | 500 | $8.435 \cdot 10^{-8}$ | $4.799 \cdot 10^{-5}$ |

# Chapter 4

# Domain decomposition for a nonlinear problem

We consider the Navier-Stokes equations of incompressible viscous flow in the simple two subdomain setting as a nonlinear model problem for our domain decomposition method. Let $\Omega$ denote a bounded open set in $\mathrm{R}^2$ with boundary $\Gamma$. If $\mathbf{u}$ denotes the velocity vector, $p$ the pressure, $\mathbf{f}$ a given body force, and $\nu$ the constant kinetic viscosity, then the problem to be solved is

$$-\nu\nabla\cdot(\nabla\mathbf{u}+(\nabla\mathbf{u})^T)+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p=\mathbf{f}\quad\text{in }\Omega, \tag{4.1}$$

$$\mathbf{u}=0\quad\text{on }\Gamma \tag{4.2}$$

and

$$\nabla\cdot\ \mathbf{u}=0\quad\text{in }\Omega. \tag{4.3}$$

Suppose $\Omega$ is divided into two disjoint open subdomains $\Omega_1$, $\Omega_2$ such that $\overline{\Omega}=\overline{\Omega}_1\cup\overline{\Omega}_2$. The *interface* between the two domains is denoted by $\Gamma_0$ so that $\Gamma_0=\overline{\Omega}_1\cap\overline{\Omega}_2$. Let $\Gamma_1=\overline{\Omega}_1\cap\Gamma$ and $\Gamma_2=\overline{\Omega}_2\cap\Gamma$. See Figure 1.1. Then we consider the following pair of Navier-Stokes equations with mixed boundary conditions:

$$-\nu\nabla\cdot(\nabla\mathbf{u}_1+(\nabla\mathbf{u}_1)^T)+\mathbf{u}_1\cdot\nabla\mathbf{u}_1+\nabla p_1=\mathbf{f}\quad\text{in }\Omega_1, \tag{4.4}$$

$$\nabla\cdot\mathbf{u}_1=0\quad\text{in }\Omega_1, \tag{4.5}$$

$$\mathbf{u}_1=0\quad\text{on }\Gamma_1, \tag{4.6}$$

$$-p_1\mathbf{n}_1+\nu(\nabla\mathbf{u}_1+(\nabla\mathbf{u}_1)^T)\cdot\mathbf{n}_1-\frac{1}{2}(\mathbf{u}_1\cdot\mathbf{n}_1)\mathbf{u}_1=\mathbf{g}\quad\text{on }\Gamma_0, \tag{4.7}$$

and

$$-\nu\nabla\cdot(\nabla\mathbf{u}_2+(\nabla\mathbf{u}_2)^T)+\mathbf{u}_2\cdot\nabla\mathbf{u}_2+\nabla p_2=\mathbf{f}\quad\text{in }\Omega_2,\tag{4.8}$$

$$\nabla\cdot\mathbf{u}_2=0\quad\text{in }\Omega_2,\tag{4.9}$$

$$\mathbf{u}_2=0\quad\text{on }\Gamma_2,\tag{4.10}$$

$$-p_2\mathbf{n}_2+\nu(\nabla\mathbf{u}_2+(\nabla\mathbf{u}_2)^T)\cdot\mathbf{n}_2-\frac{1}{2}(\mathbf{u}_2\cdot\mathbf{n}_2)\mathbf{u}_2=-\mathbf{g}\quad\text{on }\Gamma_0,\tag{4.11}$$

where $\mathbf{n}_1$ and $\mathbf{n}_2$ denote the outward nomal vectors to $\Omega_1$ and $\Omega_2$, respectively. Note that the left hand sides of (4.7) and (4.11) appear naturally in a weak formulation if a skew-symmetric form is used for the nonlinear terms. For an arbitrary choice for $\mathbf{g}$, solutions of (4.4)-(4.7) and (4.8)-(4.11) are not solutions of (4.1)-(4.3), e.g., $\mathbf{u}_1\neq\mathbf{u}|_{\Omega_1}$ and $\mathbf{u}_2\neq\mathbf{u}|_{\Omega_2}$. Here, we restrict our interest to the velocity vector $\mathbf{u}$ for our domain decomposition method and want to find $\mathbf{g}$ that minimizes the $L^2$-norm of $\mathbf{u}_1-\mathbf{u}_2$ on $\Gamma_0$. Thus, the functional to be minimized is

$$\mathcal{K}(\mathbf{u}_1,\mathbf{u}_2)=\frac{1}{2}\int_{\Gamma_0}(\mathbf{u}_1-\mathbf{u}_2)^2\,d\Gamma_0,\tag{4.12}$$

or

$$\mathcal{J}_\delta(\mathbf{u}_1,\mathbf{u}_2,\mathbf{g})=\frac{1}{2}\int_{\Gamma_0}(\mathbf{u}_1-\mathbf{u}_2)^2\,d\Gamma_0+\frac{\delta}{2}\int_{\Gamma_0}\mathbf{g}^2\,d\Gamma_0.\tag{4.13}$$

Again, we want to minimize the penalized functional (4.13) instead of (4.12) because we need to control the size of $\mathbf{g}$. Therefore, the optimization problem we want to solve is to find $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{g}$ that minimize (4.13) subject to (4.4)-(4.7) and (4.8)-(4.11).

## 4.1 The existence of an optimal solution

First, we define some spaces that will be used throughout Chapter 4. Let $H^r(D)$ denote the standard Sobolev space of order $r$ with respect to a domain $D$ with the standard norm $\|\cdot\|_{r,D}$ and let $\mathbf{H}^r(D)$ denote the corresponding Sobolev space of vector-valued functions. We then define the spaces

$$\mathbf{H}^1_0(\Omega)=\{\mathbf{v}\in\mathbf{H}^1(\Omega):\ \mathbf{v}=\mathbf{0}\quad\text{on }\Gamma\},$$

$$\mathbf{H}^1_{\Gamma_1}(\Omega_1)=\{\mathbf{v}\in\mathbf{H}^1(\Omega_1):\ \mathbf{v}=\mathbf{0}\quad\text{on }\Gamma_1\}$$

and

$$\mathbf{H}^1_{\Gamma_2}(\Omega_2)=\{\mathbf{v}\in\mathbf{H}^1(\Omega_2):\ \mathbf{v}=\mathbf{0}\quad\text{on }\Gamma_2\}.$$

We also define, for $i = 1, 2,$

$$(\mathbf{u}, \mathbf{v})_{\Omega_i} = \int_{\Omega_i} \mathbf{u} \cdot \mathbf{v} \, d\Omega_i,$$

$$(p, q)_{\Omega_i} = \int_{\Omega_i} pq \, d\Omega_i,$$

if $\mathbf{u} \cdot \mathbf{v}$, $pq \in L^1(\Omega_i)$, and

$$(\mathbf{u}, \mathbf{v})_{\Gamma_0} = \int_{\Gamma_0} \mathbf{u} \cdot \mathbf{v} \, d\Gamma_0,$$

$$(p, q)_{\Gamma_0} = \int_{\Gamma_0} pq \, d\Gamma_0,$$

if $\mathbf{u} \cdot \mathbf{v}$, $pq \in L^1(\Gamma_0)$.

Let $\mathbf{g}$ be the boundary control and assume $(\mathbf{u}_1, p_1)$ and $(\mathbf{u}_2, p_2)$ satisfy (4.4)-(4.7) and (4.8)-(4.11), respectively. Then, a weak formulation corresponding to (4.4)-(4.7) and (4.8)-(4.11) is

$$\nu a_1(\mathbf{u}_1, \mathbf{v}) + c_1(\mathbf{u}_1, \mathbf{u}_1, \mathbf{v}) + b_1(\mathbf{v}, p_1) = (f, \mathbf{v})_{\Omega_1} + (\mathbf{g}, \mathbf{v})_{\Gamma_0} \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_1}(\Omega_1), \tag{4.14}$$

$$b_1(\mathbf{u}_1, q) = 0 \quad \forall q \in L^2(\Omega_1), \tag{4.15}$$

$$\nu a_2(\mathbf{u}_2, \mathbf{v}) + c_2(\mathbf{u}_2, \mathbf{u}_2, \mathbf{v}) + b_1(\mathbf{v}, p_2) = (f, \mathbf{v})_{\Omega_2} - (\mathbf{g}, \mathbf{v})_{\Gamma_0} \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_2}(\Omega_2), \tag{4.16}$$

and

$$b_2(\mathbf{u}_2, q) = 0 \quad \forall q \in L^2(\Omega_2), \tag{4.17}$$

where, for $i = 1, 2,$

$$a_i(\mathbf{u}, \mathbf{v}) = \frac{1}{2} \int_{\Omega_i} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) : (\nabla \mathbf{v} + (\nabla \mathbf{v})^T) \, d\Omega_i \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}^1_{\Gamma_i}(\Omega_i),$$

$$b_i(\mathbf{v}, q) = -\int_{\Omega_i} q \operatorname{div} \mathbf{v} \, d\Omega_i \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_i}(\Omega_i), \ \forall q \in L^2(\Omega_i)$$

and

$$c_i(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \frac{1}{2} \int_{\Omega_i} (\mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{w} - \mathbf{u} \cdot \nabla \mathbf{w} \cdot \mathbf{v}) \, d\Omega_i \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}^1_{\Gamma_i}(\Omega_i).$$

Note that

$$c_i(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0 \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}^1_{\Gamma_i}(\Omega_i). \tag{4.18}$$

It is well known that the forms $a_i(\cdot, \cdot)$, $b_i(\cdot, \cdot)$ and $c_i(\cdot, \cdot, \cdot)$, for $i = 1, 2$, are continuous, i.e., there exist positive constants $K_a$, $K_b$ and $K_c$ such that

$$|a_i(\mathbf{u}, \mathbf{v})| \leq K_a \|\mathbf{u}\|_{1,\Omega_i} \|\mathbf{v}\|_{1,\Omega_i} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}^1_{\Gamma_i}(\Omega_i), \tag{4.19}$$

$$|b_i(\mathbf{v},q)| \le K_b \|\mathbf{v}\|_{1,\Omega_i}\|q\|_{0,\Omega_i} \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_i}(\Omega_i),\ \forall q \in L^2(\Omega_i), \tag{4.20}$$

and

$$|c_i(\mathbf{u},\mathbf{v},\mathbf{w})| \le K_c \|\mathbf{u}\|_{1,\Omega_i}\|\mathbf{v}\|_{1,\Omega_i}\|\mathbf{w}\|_{1,\Omega_i} \quad \forall \mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{H}^1_{\Gamma_i}(\Omega_i). \tag{4.21}$$

Also, $a_i(\cdot,\cdot)$ satisfies a coercivity property: there exists a positive constant $K_d$ such that

$$a_i(\mathbf{u},\mathbf{u}) \ge K_d \|\mathbf{u}\|^2_{1,\Omega_i} \quad \forall \mathbf{u} \in \mathbf{H}^1_{\Gamma_i}(\Omega_i) \tag{4.22}$$

and $b_i(\cdot,\cdot)$ satisfies the *inf-sup* condition

$$\sup_{0\ne\mathbf{v}\in\mathbf{H}^1_{\Gamma_i}(\Omega_i)} \frac{b_i(\mathbf{v},q)}{\|\mathbf{v}\|_{1,\Omega_i}} \ge K_e \|q\|_{0,\Omega_i} \quad \forall q \in L^2(\Omega_i), \tag{4.23}$$

where $K_e$ is a positive constant.

Using the above conditions, we can obtain an a priori estimate for the solution of the weak problem (4.14)-(4.17), i.e.,

$$\|\mathbf{u}_i\|_{1,\Omega_i} \le \frac{\|\mathbf{f}\|_{0,\Omega_i} + \|\mathbf{g}\|_{0,\Gamma_0}}{K_d\,\nu}. \tag{4.24}$$

To show this, let $\mathbf{u}_i$, $p_i$ for $i=1,2$, satisfy the weak form (4.14)-(4.17). Then,

$$\nu a_i(\mathbf{u}_i,\mathbf{v}) + b_i(\mathbf{v},p_i) = (\mathbf{f},\mathbf{v})_{\Omega_i} + (-1)^{i+1}(\mathbf{g},\mathbf{v})_{\Gamma_0} - c_i(\mathbf{u}_i,\mathbf{u}_i,\mathbf{v}) \tag{4.25}$$

for all $\mathbf{v} \in \mathbf{H}^1_{\Gamma_i}(\Omega_i)$ and

$$b_i(\mathbf{u}_i,q) = 0 \quad \forall q \in L^2(\Omega_i). \tag{4.26}$$

Setting $\mathbf{v} = \mathbf{u}_i$ in (4.25) and by (4.18), (4.26), we have

$$\nu a_i(\mathbf{u}_i,\mathbf{u}_i) = (f,\mathbf{u}_i)_{\Omega_i} + (-1)^{i+1}(\mathbf{g},\mathbf{u}_i)_{\Gamma_0}. \tag{4.27}$$

Then (4.24) follows from (4.22) and (4.27).

If we assume that the control $\mathbf{g}$ satisfies

$$\frac{K_c(\|\mathbf{f}\|_{0,\Omega_i} + \|\mathbf{g}\|_{0,\Gamma_0})}{K_d^2\,\nu^2} < 1, \tag{4.28}$$

then, in addition to (4.24), the solutions of the linearized Navier-Stokes equations are bounded. Consider the linearized Navier-Stokes equation

$$\nu a_i(\mathbf{w}_i,\mathbf{v}) + c_i(\mathbf{w}_i,\mathbf{u}_i,\mathbf{v}) + c_i(\mathbf{u}_i,\mathbf{w}_i,\mathbf{v}) + b_i(\mathbf{v},\overline{p}_i) = (\overline{\mathbf{g}},v)_{\Gamma_0} \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_i}(\Omega_i), \tag{4.29}$$

$$b_i(\mathbf{w}_i,q) = 0 \quad \forall q \in L^2(\Omega_i), \tag{4.30}$$

where $\mathbf{u}_i$ for $i = 1, 2$, are the solutions of (4.14)-(4.17). Setting $\mathbf{v} = \mathbf{w}_i$ in (4.29), we have

$$\nu a_i(\mathbf{w}_i, \mathbf{w}_i) + c_i(\mathbf{w}_i, \mathbf{u}_i, \mathbf{w}_i) = (\overline{\mathbf{g}}, \mathbf{w}_i)_{\Gamma_0}$$

by (4.18), (4.30). Then, from (4.22) and (4.21),

$$(\nu K_d - K_c\|\mathbf{u}_i\|_{1,\Omega_i})\|\mathbf{w}_i\|_{1,\Omega_i} \leq \|\overline{\mathbf{g}}\|_{0,\Gamma_0}.$$

We can easily show that $\nu K_d - K_c\|u_i\|_{1,\Omega_i} > 0$ using (4.28), (4.24) and thus, there exists a positive constant C such that

$$\|\mathbf{w}_i\|_{1,\Omega_i} \leq C\|\overline{\mathbf{g}}\|_{0,\Gamma_0}. \tag{4.31}$$

Let $(\mathbf{u}^{ex}, p^{ex})$ be a solution of (4.1)-(4.3) and $\mathbf{g}^{ex} = [-p^{ex}\mathbf{n}_1 + \nu(\nabla\mathbf{u}^{ex} + (\nabla\mathbf{u}^{ex})^T) \cdot \mathbf{n}_1 - \frac{1}{2}(\mathbf{u}^{ex} \cdot \mathbf{n}_1)\mathbf{u}^{ex}]|_{\Gamma_0}$. We assume that $\|\mathbf{g}^{ex}\|_{0,\Gamma_0}$ is small enough so that it satisfies (4.28). Then it is also true that an optimal control $\mathbf{g}$ satisfies (4.28), since the $L^2$-norm of the optimal $\mathbf{g}$ is always less than $\|\mathbf{g}^{ex}\|_{0,\Gamma_0}$ (see (4.37)), thus there exists a neighborhood $B$ of the optimal $\mathbf{g}$ in which (4.28) is satisfied.

Now, we define the *admissibility set* $\mathcal{U}_{ad}$ by

$$\begin{aligned}\mathcal{U}_{ad} \;=\; &\{(\mathbf{u}_1, \mathbf{u}_2, \mathbf{g}) \in \mathbf{H}^1_{\Gamma_1}(\Omega_1) \times \mathbf{H}^1_{\Gamma_2}(\Omega_2) \times \mathbf{L}^2(\Gamma_0) : \text{ there exist } p_i \in L^2(\Omega_i) \text{ for } i = 1, 2,\\ &\text{so that } (4.14) - (4.17) \text{ are satisfied and } \mathcal{J}_\delta(\mathbf{u}_1, \mathbf{u}_2, \mathbf{g}) < \infty\}.\end{aligned} \tag{4.32}$$

Then, $(\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{g}}) \in \mathcal{U}_{ad}$ is called an *optimal solution* if there exists $\epsilon > 0$ such that

$$\mathcal{J}_\delta(\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{g}}) \leq \mathcal{J}_\delta(\mathbf{u}_1, \mathbf{u}_2, \mathbf{g}) \tag{4.33}$$

for all $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{g}) \in \mathcal{U}_{ad}$ satisfying

$$\|\mathbf{u}_1 - \hat{\mathbf{u}}_1\|_{1,\Omega_1} + \|\mathbf{u}_2 - \hat{\mathbf{u}}_2\|_{1,\Omega_2} + \|\mathbf{g} - \hat{\mathbf{g}}\|_{0,\Gamma_0} \leq \epsilon. \tag{4.34}$$

We show the existence of optimal solutions in the next theorem.

**Theorem 4.1** *There exists a* $(\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{g}}) \in \mathcal{U}_{ad}$ *such that (4.13) is minimized.*

**Proof:** It is clear that $\mathcal{U}_{ad}$ is not empty. Let $\{(\mathbf{u}_1^{(n)}, \mathbf{u}_2^{(n)}, \mathbf{g}^{(n)})\}$ be a sequence in $\mathcal{U}_{ad}$ such that

$$\lim_{n\to\infty} \mathcal{J}_\delta(\mathbf{u}_1^{(n)}, \mathbf{u}_2^{(n)}, \mathbf{g}^{(n)}) = \inf_{(\mathbf{u}_1^{(n)}, \mathbf{u}_2^{(n)}, \mathbf{g}^{(n)}) \in \mathcal{U}_{ad}} \mathcal{J}_\delta(\mathbf{u}_1, \mathbf{u}_2, \mathbf{g}).$$

Then, by (4.24),

$$\|\mathbf{u}_i^{(n)}\|_{1,\Omega_i} \le \frac{\|\mathbf{f}\|_{0,\Omega_i} + \|\mathbf{g}^{(n)}\|_{0,\Gamma_0}}{K_d\,\nu}, \tag{4.35}$$

i.e., $(\mathbf{u}_1^{(n)}, \mathbf{u}_2^{(n)}, \mathbf{g}^{(n)})$ is uniformly bounded by (4.32) and (4.35). On the other hand, there exist $p_1^{(n)} \in L^2(\Omega_1)$ and $p_2^{(n)} \in L^2(\Omega_2)$ such that $(\mathbf{u}_i^{(n)}, p_i^{(n)})$ for $i = 1, 2$, are the solutions of (4.14)-(4.17) and it can be shown that $p_1^{(n)}$ and $p_2^{(n)}$ are uniformly bounded by (4.14), (4.16), (4.23) and (4.24). Thus, there exist subsequences such that

$$\begin{aligned}
\mathbf{u}_i^{(n_j)} &\rightharpoonup \hat{\mathbf{u}}_i \quad \text{in } \mathbf{H}^1_{\Gamma_i}(\Omega_i),\\
p_i^{(n_j)} &\rightharpoonup \hat{p}_i \quad \text{in } L^2(\Omega_i),\\
\mathbf{g}^{(n_j)} &\to \hat{\mathbf{g}} \quad \text{in } \mathbf{L}^2(\Gamma_0),\\
\mathbf{u}_i^{(n_j)} &\to \hat{\mathbf{u}}_i \quad \text{in } \mathbf{L}^2(\Omega_i),\\
\mathbf{u}_i^{(n_j)}|_{\Gamma_0} &\to \hat{\mathbf{u}}_i|_{\Gamma_0} \quad \text{in } \mathbf{L}^2(\Gamma_0)
\end{aligned}$$

for some $(\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{g}}) \in \mathcal{U}_{ad}$, $\hat{p}_1 \in L^2(\Omega_1)$ and $\hat{p}_2 \in L^2(\Omega_2)$. The last two convergence results follow from the compact imbeddings $\mathbf{H}^1(\Omega_i) \subset \mathbf{L}^2(\Omega_i)$ and $\mathbf{H}^{1/2}(\Gamma_0) \subset \mathbf{L}^2(\Gamma_0)$. First, note that

$$\begin{aligned}
\lim_{n\to\infty} c_i(\mathbf{u}_i^{(n)}, \mathbf{u}_i^{(n)}, \mathbf{v}) &= \lim_{n\to\infty} \frac{1}{2}\left(\int_{\Gamma_0} (\mathbf{u}_i^{(n)} \cdot \mathbf{n}_i)\mathbf{u}_i^{(n)} \cdot \mathbf{v}\, d\Gamma_0 - 2\int_{\Omega_i} \mathbf{u}_i^{(n)} \cdot \nabla \mathbf{v} \cdot \mathbf{u}_i^{(n)}\, d\Omega_i\right)\\
&= \frac{1}{2}\left(\int_{\Gamma_0} (\hat{\mathbf{u}}_i \cdot \mathbf{n}_i)\hat{\mathbf{u}}_i \cdot \mathbf{v}\, d\Gamma_0 - 2\int_{\Omega_i} \hat{\mathbf{u}}_i \cdot \nabla \mathbf{v} \cdot \hat{\mathbf{u}}_i\, d\Omega_i\right)\\
&= c_i(\hat{\mathbf{u}}_i, \hat{\mathbf{u}}_i, \mathbf{v}) \quad \mathbf{v} \in C^\infty(\overline{\Omega}_i),
\end{aligned}$$

since $\mathbf{u}_i^{(n_j)} \to \hat{\mathbf{u}}_i$ in $\mathbf{L}^2(\Omega_i)$ and $\mathbf{u}_i^{(n_j)}|_{\Gamma_0} \to \hat{\mathbf{u}}_i|_{\Gamma_0}$ in $\mathbf{L}^2(\Gamma_0)$. And, since $C^\infty(\overline{\Omega}_i)$ is dense in $\mathbf{H}^1(\Omega_i)$, we have

$$\lim_{n\to\infty} c_i(\mathbf{u}_i^{(n)}, \mathbf{u}_i^{(n)}, \mathbf{v}) = c_i(\hat{\mathbf{u}}_i, \hat{\mathbf{u}}_i, \mathbf{v}) \quad \mathbf{v} \in \mathbf{H}^1(\Omega_i).$$

Hence, by passing to the limit, it can be easily shown that $(\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{g}})$ satisfies (4.14)-(4.17) and, by the fact that the functional $\mathcal{J}(\cdot,\cdot,\cdot)$ is lower semi-continuous, we conclude that

$$\mathcal{J}_\delta(\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{g}}) = \inf_{(\mathbf{u}_1,\mathbf{u}_2,\mathbf{g}) \in \mathcal{U}_{ad}} \mathcal{J}_\delta(\mathbf{u}_1, \mathbf{u}_2, \mathbf{g})$$

i.e., $(\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{g}})$ is an optimal solution. □

## 4.2 Convergence with vanishing penalty parameter

In the next theorem we show that optimal solutions converge to the solution of (4.1)-(4.3) as $\delta \to 0$. For simplicity of notation, we denote an optimal solution satisfying (4.32)-(4.34) and which corresponds to a given value of $\delta$ by $(\mathbf{u}_1^\delta, \mathbf{u}_2^\delta, \mathbf{g}^\delta)$.

**Theorem 4.2** *For each $\delta$, let $(\mathbf{u}_1^\delta, \mathbf{u}_2^\delta, \mathbf{g}^\delta)$ denote an optimal solution satisfying (4.32)-(4.34). Let $\mathbf{u}^{ex}$ denote the solution of (4.1)-(4.3) and let $\mathbf{u}_1^{ex} = \mathbf{u}^{ex}|_{\Omega_1\cup\Gamma_0}$ and $\mathbf{u}_2^{ex} = \mathbf{u}^{ex}|_{\Omega_2\cup\Gamma_0}$. Then $\|\mathbf{u}_1^\delta - \mathbf{u}_1^{ex}\|_{1,\Omega_1} \to 0$ and $\|\mathbf{u}_2^\delta - \mathbf{u}_2^{ex}\|_{1,\Omega_2} \to 0$ as $\delta \to 0$.*

**Proof:** Assume $(\mathbf{u}^{ex}, p^{ex})$ denotes an exact solution of (4.1)-(4.3), where $p^{ex}$ has zero mean over $\Omega$. Let $\mathbf{u}_i^{ex} = \mathbf{u}^{ex}|_{\Omega_i\cup\Gamma_0}$ and $p_i^{ex} = p^{ex}|_{\Omega_i\cup\Gamma_0}$ for $i = 1, 2$. Let $\mathbf{g}^{ex} = -p_1^{ex}\mathbf{n}_1 + \nu(\nabla\mathbf{u}_1^{ex} + (\nabla\mathbf{u}_1^{ex})^T)\cdot\mathbf{n}_1 - \frac{1}{2}(\mathbf{u}_1^{ex}\cdot\mathbf{n}_1)\mathbf{u}_1^{ex}$ on $\Gamma_0$. Suppose $\{(\mathbf{u}_1^\delta, \mathbf{u}_2^\delta, \mathbf{g}^\delta)\}$ is a sequence of optimal solutions and $\delta$ goes to 0. Then there exist $p_i^\delta \in L^2(\Omega_i)$ for $i = 1, 2$, such that $(\mathbf{u}_i^\delta, p_i^\delta)$ satisfies (4.14)-(4.17). We have that

$$\mathcal{J}(\mathbf{u}_1^\delta, \mathbf{u}_2^\delta, \mathbf{g}^\delta) \le \mathcal{J}(\mathbf{u}_1^{ex}, \mathbf{u}_2^{ex}, \mathbf{g}^{ex}) \quad \forall\delta, \tag{4.36}$$

i.e.,

$$\frac{1}{2}\int_{\Gamma_0} (\mathbf{u}_1^\delta - \mathbf{u}_2^\delta)^2 \, d\Gamma_0 + \frac{\delta}{2}\int_{\Gamma_0} (\mathbf{g}^\delta)^2 \, d\Gamma_0 \le \frac{\delta}{2}\int_{\Gamma_0} (\mathbf{g}^{ex})^2 \, d\Gamma_0 \quad \forall\delta. \tag{4.37}$$

Hence, $\|\mathbf{g}^\delta\|_{0,\Gamma_0}$ is uniformly bounded in $\mathbf{L}^2(\Gamma_0)$ and $\|\mathbf{u}_1^\delta - \mathbf{u}_2^\delta\|_{0,\Gamma_0} \to 0$ if $\delta \to 0$. We also obtain, by (4.24) and (4.32), that $\|\mathbf{u}_1^\delta\|_{1,\Omega_1}$, $\|\mathbf{u}_2^\delta\|_{1,\Omega_2}$ are uniformly bounded and that $\|p_1^\delta\|_{0,\Omega_1}$, $\|p_2^\delta\|_{0,\Omega_2}$ are uniformly bounded by (4.14), (4.16), (4.23) and (4.24). Hence, as $\delta \to 0$, there exist subsequences of $\{(\mathbf{u}_1^\delta, \mathbf{u}_2^\delta, \mathbf{g}^\delta)\}$ and $\{p_i^\delta\}$ which converge to some $(\mathbf{u}_1^*, \mathbf{u}_2^*, \mathbf{g}^*) \in \mathbf{H}_{\Gamma_1}^1(\Omega_1) \times \mathbf{H}_{\Gamma_2}^1(\Omega_2) \times \mathbf{L}^2(\Gamma_0)$ and to some $p_i^* \in L^2(\Omega_i)$ for $i = 1, 2$. The fact that $\|\mathbf{u}_1^\delta - \mathbf{u}_2^\delta\|_{0,\Gamma_0} \to 0$ yields $\mathbf{u}_1^* = \mathbf{u}_2^*$ on $\Gamma_0$ and by passing to the limit, we have that $(\mathbf{u}_i^*, p_i^*)$ for $i = 1, 2$, satisfy (4.14)-(4.17). If we define $\mathbf{u}^* \in \mathbf{H}_0^1(\Omega)$ by

$$\mathbf{u}^* = \begin{cases} \mathbf{u}_1^* & \text{in } \Omega_1 \cup \Gamma_0 \\ \mathbf{u}_2^* & \text{in } \Omega_2 \cup \Gamma_0, \end{cases} \tag{4.38}$$

and $p^* \in L^2(\Omega)$ by

$$p^* = \begin{cases} p_1^* & \text{in } \Omega_1 \cup \Gamma_0 \\ p_2^* & \text{in } \Omega_2, \end{cases} \tag{4.39}$$

then $(\mathbf{u}^*, p^*)$ satisfies (4.1)-(4.3). Now we can conclude that $\mathbf{u}^* = \mathbf{u}^{ex}$, since $\mathbf{u}$ satisfying (4.1)-(4.3) is unique by the assumption that $\mathbf{g}^{ex}$ satisfies (4.28). □

## 4.3 The optimality system

### 4.3.1 The existence of Lagrange multipliers

We use a Lagrange multiplier rule to reduce the constrained minimization problem (4.32)-(4.34) to an unconstrained problem. First, we show that suitable Lagrange multipliers exist.

Let $A = \mathbf{H}^1_{\Gamma_1}(\Omega_1) \times L^2(\Omega_1) \times \mathbf{H}^1_{\Gamma_2}(\Omega_2) \times L^2(\Omega_2) \times \mathbf{L}^2(\Gamma_0)$ and $B = (\mathbf{H}^1_{\Gamma_1}(\Omega_1))^* \times L^2(\Omega_1) \times (\mathbf{H}^1_{\Gamma_2}(\Omega_2))^* \times L^2(\Omega_2)$. Suppose the nonlinear operator $M : A \to B$ denotes the constraint equations, i.e., $M : A \to B$ is defined by

$$M(\mathbf{u}_1, p_1, \mathbf{u}_2, p_2, \mathbf{g}) = (\mathbf{f}_1, \varphi_1, \mathbf{f}_2, \varphi_2)$$

if and only if

$$\nu a_i(\mathbf{u}_i, \mathbf{v}) + c_i(\mathbf{u}_i, \mathbf{u}_i, \mathbf{v}) + b_i(\mathbf{v}, p_i) + (-1)^i(\mathbf{g}, \mathbf{v})_{\Gamma_0} = (\mathbf{f}_i, \mathbf{v})_{\Omega_i} \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_i}(\Omega_i)$$

and

$$b_i(\mathbf{u}_i, q) = (\varphi_i, q)_{\Omega_i} \quad \forall q \in L^2(\Omega_i)$$

for $i = 1, 2$. Note that $M$ is linear about $\mathbf{g}$ and thus convex.

**Theorem 4.3** *Suppose* $(\hat{\mathbf{u}}_1, \hat{p}_1, \hat{\mathbf{u}}_2, \hat{p}_2, \hat{\mathbf{g}}) \in A$ *denotes an optimal solution. Then there exists a nonzero Lagrange multiplier* $(\boldsymbol{\lambda}_1, \beta_1, \boldsymbol{\lambda}_2, \beta_2) \in \mathbf{H}^1_{\Gamma_1}(\Omega_1) \times L^2(\Omega_1) \times \mathbf{H}^1_{\Gamma_2}(\Omega_2) \times L^2(\Omega_2)$ *such that* $((\hat{\mathbf{u}}_1, \hat{p}_1, \hat{\mathbf{u}}_2, \hat{p}_2, \hat{\mathbf{g}}), (\boldsymbol{\lambda}_1, \beta_1, \boldsymbol{\lambda}_2, \beta_2))$ *is a saddle point of the Lagrangian*

$$\mathcal{L}(\mathbf{u}_1, p_1, \mathbf{u}_2, p_2, \mathbf{g}, \boldsymbol{\lambda}_1, \beta_1, \boldsymbol{\lambda}_2, \beta_2) = \mathcal{J}_\delta(\mathbf{u}_1, \mathbf{u}_2, \mathbf{g}) + < M(\mathbf{u}_1, p_1, \mathbf{u}_2, p_2, \mathbf{g}), (\boldsymbol{\lambda}_1, \beta_1, \boldsymbol{\lambda}_2, \beta_2) > . \tag{4.40}$$

**Proof:** First, we show that the Fréchet derivative of $M$, $M'$, has a closed range. The operator
$M'(\hat{\mathbf{u}}_1, \hat{p}_1, \hat{\mathbf{u}}_2, \hat{p}_2, \hat{\mathbf{g}}) \in \mathcal{L}(A, B)$ is defined as follows:

$$M'(\hat{\mathbf{u}}_1, \hat{p}_1, \hat{\mathbf{u}}_2, \hat{p}_2, \hat{\mathbf{g}}) \cdot (\mathbf{w}_1, r_1, \mathbf{w}_2, r_2, \mathbf{s}) = (\overline{\mathbf{f}}_1, \overline{\varphi}_1, \overline{\mathbf{f}}_2, \overline{\varphi}_2)$$

for $(\mathbf{w}_1, r_1, \mathbf{w}_2, r_2, \mathbf{s}) \in A$ and $(\overline{\mathbf{f}}_1, \overline{\varphi}_1, \overline{\mathbf{f}}_2, \overline{\varphi}_2) \in B$ if and only if

$$\nu a_i(\mathbf{w}_i, \mathbf{v}) + c_i(\mathbf{w}_i, \hat{\mathbf{u}}_i, \mathbf{v}) + c_i(\hat{\mathbf{u}}_i, \mathbf{w}_i, \mathbf{v}) + b_i(\mathbf{v}, r_i) + (-1)^i(s, \mathbf{v})_{\Gamma_0} = (\overline{\mathbf{f}}_i, \mathbf{v})_{\Omega_i} \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_i}(\Omega_i) \tag{4.41}$$

and

$$b_i(\mathbf{w}_i, q) = (\overline{\varphi}_i, q)_{\Omega_i} \quad \forall q \in L^2(\Omega_i) \tag{4.42}$$

for $i = 1, 2$. It can be shown that $M'$ is a compact perturbation of a Fredholm operator as follows: define the linear operator $S : A \rightarrow B$ by

$$S(\hat{\mathbf{u}}_1, \hat{p}_1, \hat{\mathbf{u}}_2, \hat{p}_2, \hat{\mathbf{g}}) \cdot (\mathbf{w}_1, r_1, \mathbf{w}_2, r_2, \mathbf{s}) = (\widetilde{\mathbf{f}}_1, \widetilde{\varphi}_1, \widetilde{\mathbf{f}}_2, \widetilde{\varphi}_2)$$

for $(\mathbf{w}_1, r_1, \mathbf{w}_2, r_2, \mathbf{s}) \in A$ and $(\widetilde{\mathbf{f}}_1, \widetilde{\varphi}_1, \widetilde{\mathbf{f}}_2, \widetilde{\varphi}_2) \in B$ if and only if

$$\nu a_i(\mathbf{w}_i, \mathbf{v}) + b_i(\mathbf{v}, r_i) + (-1)^i (\mathbf{s}, \mathbf{v})_{\Gamma_0} = (\widetilde{\mathbf{f}}_i, \mathbf{v})_{\Omega_i} \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_i}(\Omega_i)$$

and

$$b_i(\mathbf{w}_i, q) = (\widetilde{\varphi}_i, q)_{\Omega_i} \quad \forall q \in L^2(\Omega_i)$$

for $i = 1, 2$. Then, since the Stokes operator $S : A \rightarrow B$ is an isomorphism (see [28], [52]), S is a Fredholm operator. Now, define the operators $C_i : \mathbf{H}^1_{\Gamma_i}(\Omega_i) \times \mathbf{H}^1_{\Gamma_i}(\Omega_i) \rightarrow (\mathbf{H}^1_{\Gamma_i}(\Omega_i))^*$, for $i = 1, 2$, by

$$< C_i(\mathbf{u}, \mathbf{v}), \mathbf{w} >= c_i(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \frac{1}{2} \int_{\Omega_i} (\mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{w} - \mathbf{u} \cdot \nabla \mathbf{w} \cdot \mathbf{v}) \, d\Omega_i \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}^1_{\Gamma_i}(\Omega_i).$$

Then, $C_i(\hat{\mathbf{u}}_i, \cdot)$ is a operator from $\mathbf{H}^1_{\Gamma_i}(\Omega_i)$ into $\mathbf{L}^{3/2}(\Omega_i)$, since $\nabla \hat{\mathbf{u}}, \nabla \mathbf{v} \in \mathbf{L}^2(\Omega_i)$ and $\hat{\mathbf{u}}, \mathbf{v} \in \mathbf{H}^1_{\Gamma_i}(\Omega_i) \subset \mathbf{L}^6(\Omega_i)$. Thus, $C_i(\hat{\mathbf{u}}_i, \cdot)$ is compact from $\mathbf{H}^1_{\Gamma_i}(\Omega_i)$ into $(\mathbf{H}^1_{\Gamma_i}(\Omega_i))^*$, since $\mathbf{L}^{3/2}(\Omega_i)$ is compactly imbedded in $(\mathbf{H}^1_{\Gamma_i}(\Omega_i))^*$. Similarly, it can be shown that $C_i(\cdot, \hat{\mathbf{u}}_i)$ is compact from $\mathbf{H}^1_{\Gamma_i}(\Omega_i)$ into $(\mathbf{H}^1_{\Gamma_i}(\Omega_i))^*$. Now it follows that $M'$ is a compact perturbation of the Fredholm operator $S$ and therefore $M'$ has a closed range.

We now prove that $M' : A \rightarrow B$ is onto. Suppose $M'(\hat{\mathbf{u}}_1, \hat{p}_1, \hat{\mathbf{u}}_2, \hat{p}_2, \hat{\mathbf{g}})$ is not onto. Then, since $M'$ has a closed range, there exists a nonzero $(\boldsymbol{\mu}_1, \phi_1, \boldsymbol{\mu}_2, \phi_2) \in B^* = \mathbf{H}^1_{\Gamma_1}(\Omega_1) \times L^2(\Omega_1) \times \mathbf{H}^1_{\Gamma_2}(\Omega_2) \times L^2(\Omega_2)$ such that

$$< (\overline{\mathbf{f}}_1, \overline{\varphi}_1, \overline{\mathbf{f}}_2, \overline{\varphi}_2), (\boldsymbol{\mu}_1, \phi_1, \boldsymbol{\mu}_2, \phi_2) >= 0$$

for all $(\overline{\mathbf{f}}_1, \overline{\varphi}_1, \overline{\mathbf{f}}_2, \overline{\varphi}_2) \in R(M')$, i.e.,

$$(\overline{\mathbf{f}}_1, \boldsymbol{\mu}_1) + (\overline{\varphi}_1, \phi_1) + (\overline{\mathbf{f}}_2, \boldsymbol{\mu}_2) + (\overline{\varphi}_2, \phi_2) = 0.$$

Then, by (4.41) and (4.42), there exists a nonzero $(\boldsymbol{\mu}_1, \phi_1, \boldsymbol{\mu}_2, \phi_2) \in \mathbf{H}^1_{\Gamma_1}(\Omega_1) \times L^2(\Omega_1) \times \mathbf{H}^1_{\Gamma_2}(\Omega_2) \times L^2(\Omega_2)$ such that, for $i = 1, 2$,

$$\nu a_i(\boldsymbol{\mu}_i, \mathbf{w}) + c_i(\hat{\mathbf{u}}_i, \mathbf{w}, \boldsymbol{\mu}_i) + c_i(\mathbf{w}, \hat{\mathbf{u}}_i, \boldsymbol{\mu}_i) + b_i(\mathbf{w}, \phi_i) = 0 \quad \forall \mathbf{w} \in \mathbf{H}^1_{\Gamma_i}(\Omega_i), \tag{4.43}$$

$$b_i(\boldsymbol{\mu}_i, r) = 0 \quad \forall r \in L^2(\Omega_i) \tag{4.44}$$

and

$$(\mathbf{s}, \boldsymbol{\mu}_i)_{\Gamma_0} = 0 \quad \forall \mathbf{s} \in \mathbf{L}^2(\Gamma_0). \tag{4.45}$$

The system (4.43)-(4.45) is a weak formulation corresponding to

$$-\nu \nabla \cdot (\nabla \boldsymbol{\mu}_i + (\nabla \boldsymbol{\mu}_i)^T) + \frac{1}{2}(\boldsymbol{\mu}_i \cdot (\nabla \hat{\mathbf{u}}_i)^T - \hat{\mathbf{u}}_i \cdot (\nabla \boldsymbol{\mu}_i)^T) - \hat{\mathbf{u}}_i \cdot \nabla \boldsymbol{\mu}_i + \nabla \phi_i = \mathbf{0} \quad \text{in } \Omega_i, \tag{4.46}$$

$$\text{div } \boldsymbol{\mu}_i = 0 \quad \text{in } \Omega_i, \tag{4.47}$$

$$\boldsymbol{\mu}_i = \mathbf{0} \quad \text{on } \Gamma_0 \cup \Gamma_i \tag{4.48}$$

and

$$-\phi_i \mathbf{n}_i + \nu(\nabla \boldsymbol{\mu}_i + (\nabla \boldsymbol{\mu}_i)^T) \cdot \mathbf{n}_i - \frac{1}{2}(\hat{\mathbf{u}}_i \cdot \mathbf{n}_i)\boldsymbol{\mu}_i = \mathbf{0} \quad \text{on } \Gamma_0. \tag{4.49}$$

Now, let the domain $\Omega_1'$ and $\Omega_1^e$ be constructed as indicated in Figure 4.1. $\Omega_2'$ and $\Omega_2^e$ can be constructed similarly. Let $\Omega_i^e = \Omega_i \cup \Omega_i' \cup (\Gamma_0 \cap \Gamma_i')$, where $\Gamma_i'$ denotes the boundary of $\Omega_i'$. Let $\mathbf{u}_i^e$ denote a fixed extension of $\hat{\mathbf{u}}_i$ such that $\mathbf{u}_i^e \in \mathbf{H}^1(\Omega_i^e)$. Then using the boundary condition (4.48) and (4.49), we can define extensions $\boldsymbol{\mu}_i^e$ and $\phi_i^e$ such that $\boldsymbol{\mu}_i^e = \boldsymbol{\mu}_i$ and $\phi_i^e = \phi_i$ on $\Omega_i$ and $\boldsymbol{\mu}_i^e = 0$ and $\phi_i^e = 0$ on $\Omega_i'$. Also $(\boldsymbol{\mu}_1^e, \phi_1^e, \boldsymbol{\mu}_2^e, \phi_2^e)$ satisfies the differential equations

$$-\nu \nabla \cdot (\nabla \boldsymbol{\mu}_i^e + (\nabla \boldsymbol{\mu}_i^e)^T) + \frac{1}{2}(\boldsymbol{\mu}_i^e \cdot (\nabla \mathbf{u}_i^e)^T - \mathbf{u}_i^e \cdot (\nabla \boldsymbol{\mu}_i^e)^T) - \mathbf{u}_i^e \cdot \nabla \boldsymbol{\mu}_i^e + \nabla \phi_i^e = \mathbf{0} \quad \text{in } \Omega_i^e, \tag{4.50}$$

$$\text{div } \boldsymbol{\mu}_i^e = 0 \quad \text{in } \Omega_i^e \tag{4.51}$$

and

$$\boldsymbol{\mu}_i^e = \mathbf{0} \quad \text{on } \Gamma_i^e \tag{4.52}$$

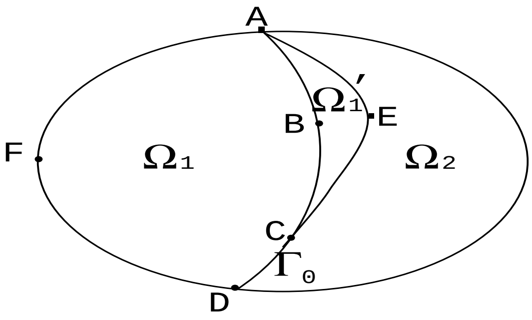

Figure 4.1: Extension of $\Omega_1$, $\Gamma_0 = ABCD$, $\Gamma_1' = ABCEA$, $\Gamma_1^e = AFDCEA$

for $i = 1, 2$. For fixed domains $\Omega_i^e$, it is possible for (4.50)-(4.52) to have a nontrivial solution $(\boldsymbol{\mu}_1^e, \phi_1^e, \boldsymbol{\mu}_2^e, \phi_2^e)$, i.e., for $1/\nu$ to be an eigenvalue of the problem (4.50)-(4.52). But the problem involves a compact perturbation of the Stokes operator, and thus its spectrum is discrete. Then, by appropriately choosing the extended domains $\Omega_i^e$, for $i = 1, 2$, we can guarantee that $\nu$ is not an eigenvalue of (4.50)-(4.52), and therefore the homogeneous linear equations have only the trivial solution $(\boldsymbol{\mu}_1^e, \phi_1^e, \boldsymbol{\mu}_1^e, \phi_1^e) = \mathbf{0}$. But this is a contradiction, thus $M'$ is onto. Now the existence of the Lagrange multiplier follows from [53, Sec. 1.1]. □

### 4.3.2 Lagrange multipliers rule

For $(\mathbf{u}_1, p_1, \mathbf{u}_2, p_2, \mathbf{g}, \boldsymbol{\lambda}_1, \beta_1, \boldsymbol{\lambda}_2, \beta_2) \in \mathbf{H}^1_{\Gamma_1}(\Omega_1) \times L^2(\Omega_1) \times \mathbf{H}^1_{\Gamma_2}(\Omega_2) \times L^2(\Omega_2) \times \mathbf{L}^2(\Gamma_0) \times \mathbf{H}^1_{\Gamma_1}(\Omega_1) \times L^2(\Omega_1) \times \mathbf{H}^1_{\Gamma_2}(\Omega_2) \times L^2(\Omega_2)$, define the Lagrangian

$$\begin{aligned}&\mathcal{L}(\mathbf{u}_1, p_1, \mathbf{u}_2, p_2, \mathbf{g}, \boldsymbol{\lambda}_1, \beta_1, \boldsymbol{\lambda}_2, \beta_2) = \mathcal{J}_\delta(\mathbf{u}_1, \mathbf{u}_2, \mathbf{g}) + \\ &\quad \sum_{i=1}^{2}(-a_i(\mathbf{u}_i, \boldsymbol{\lambda}_i) - c_i(\mathbf{u}_i, \mathbf{u}_i, \boldsymbol{\lambda}_i) - b_i(\boldsymbol{\lambda}_i, p_i) + (\mathbf{f}_i, \boldsymbol{\lambda}_i)_{\Omega_i} + (-1)^{i+1}(\mathbf{g}, \boldsymbol{\lambda}_i)_{\Gamma_0} + b_i(\mathbf{u}_i, \beta_i)).\end{aligned}$$

Setting to zero the first variations with respect to $\mathbf{u}_1$ and $\mathbf{u}_2$ yields the adjoint equations

$$\nu a_i(\boldsymbol{\lambda}_1, \boldsymbol{\xi}) + c_i(\mathbf{u}_i, \boldsymbol{\xi}, \boldsymbol{\lambda}_i) + c_i(\boldsymbol{\xi}, \mathbf{u}_i, \boldsymbol{\lambda}_i) + b_i(\boldsymbol{\xi}, \beta_i) = (-1)^{i+1}(\mathbf{u}_1 - \mathbf{u}_2, \boldsymbol{\xi})_{\Gamma_0} \quad \forall \boldsymbol{\xi} \in \mathbf{H}^1_{\Gamma_i}(\Omega_i) \tag{4.53}$$

for $i = 1, 2$. Setting to zero the first variations with respect to $p_1$ and $p_2$ yields the optimality conditions

$$b_i(\boldsymbol{\lambda}_i, \eta) = 0 \quad \forall \eta \in L^2(\Omega_i) \tag{4.54}$$

for $i = 1, 2$. Finally, setting to zero the first variations with respect to g yields

$$(\mathbf{g}, \mathbf{r})_{\Gamma_0} = -\frac{1}{\delta}(\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2, \mathbf{r})_{\Gamma_0} \quad \forall \mathbf{r} \in \mathbf{L}^2(\Gamma_0). \tag{4.55}$$

Then, solutions of the optimal problem are determined by solving the system (4.14)-(4.17) and (4.53)-(4.55). The system (4.14)-(4.17) and (4.53)-(4.55) is a weak formulation corresponding to

$$-\nu\nabla\cdot(\nabla\mathbf{u}_i + (\nabla\mathbf{u}_i)^T) + \mathbf{u}_i\cdot\nabla\mathbf{u}_i + \nabla p_i = \mathbf{f} \quad \text{in } \Omega_i, \tag{4.56}$$

$$\operatorname{div}\mathbf{u}_i = 0 \quad \text{in } \Omega_i, \tag{4.57}$$

$$-p_i\mathbf{n}_i + \nu(\nabla\mathbf{u}_i + (\nabla\mathbf{u}_i)^T)\cdot\mathbf{n}_i - \frac{1}{2}(\mathbf{u}_i\cdot\mathbf{n}_i)\mathbf{u}_i = -\frac{(-1)^{i+1}}{\delta}(\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2) \quad \text{on } \Gamma_0 \tag{4.58}$$

and

$$-\nu\nabla\cdot(\nabla\boldsymbol{\lambda}_i+(\nabla\boldsymbol{\lambda}_i)^T)+\frac{1}{2}(\boldsymbol{\lambda}_i\cdot(\nabla\mathbf{u}_i)^T-\mathbf{u}_i\cdot(\nabla\boldsymbol{\lambda}_i)^T)-\mathbf{u}_i\cdot\nabla\boldsymbol{\lambda}_i+\nabla\beta_i=0 \quad \text{in } \Omega_i, \tag{4.59}$$

$$\operatorname{div}\boldsymbol{\lambda}_i=0 \quad \text{in } \Omega_i, \tag{4.60}$$

$$-\beta_i\mathbf{n}_i+\nu(\nabla\boldsymbol{\lambda}_i+(\nabla\boldsymbol{\lambda}_i)^T)\cdot\mathbf{n}_i-\frac{1}{2}(\mathbf{u}_i\cdot\mathbf{n}_i)\boldsymbol{\lambda}_i=(-1)^{i+1}(\mathbf{u}_1-\mathbf{u}_2) \quad \text{on } \Gamma_0. \tag{4.61}$$

The optimality system (4.14)-(4.17) and (4.53)-(4.55) can be obtained directly using "sensitivity" derivatives as in the case of the Poisson equation problem. See §2.4. Let

$$\mathcal{M}_\delta(\mathbf{g})=\mathcal{J}_\delta(\mathbf{u}_1(\mathbf{g}),\mathbf{u}_2(\mathbf{g}),\mathbf{g}), \tag{4.62}$$

where, for given $\mathbf{g}$,

$$\mathbf{u}_i(\mathbf{g}):\mathbf{g}\in\mathbf{L}^2(\Gamma_0)\to\mathbf{H}^1_{\Gamma_i}(\Omega_i) \quad \text{for } i=1,2$$

are defined as the solutions of (4.14)-(4.17), respectively. The first derivative of $\mathcal{M}_\delta$, $d\mathcal{M}_\delta(\mathbf{g})/d\mathbf{g}$, is defined by

$$<\frac{d\mathcal{M}_\delta(\mathbf{g})}{d\mathbf{g}},\tilde{\mathbf{g}}>=\delta(\mathbf{g},\tilde{\mathbf{g}})_{\Gamma_0}+(\mathbf{u}_1-\mathbf{u}_2,\tilde{\mathbf{u}}_1-\tilde{\mathbf{u}}_2)_{\Gamma_0} \quad \forall\tilde{\mathbf{g}}\in\mathbf{L}^2(\Gamma_0), \tag{4.63}$$

where $\tilde{\mathbf{u}}_1\in\mathbf{H}^1_{\Gamma_1}(\Omega_1)$ and $\tilde{\mathbf{u}}_2\in\mathbf{H}^1_{\Gamma_2}(\Omega_2)$ are the solutions of

$$\nu a_i(\tilde{\mathbf{u}}_i,\mathbf{v})+c_i(\mathbf{u}_i,\tilde{\mathbf{u}}_i,\mathbf{v})+c_i(\tilde{\mathbf{u}}_i,\mathbf{u}_i,\mathbf{v})+b_i(\mathbf{v},\tilde{p}_i)=(-1)^{i+1}(\tilde{\mathbf{g}},\mathbf{v})_{\Gamma_0} \quad \forall\mathbf{v}\in\mathbf{H}^1_{\Gamma_i}(\Omega_i) \tag{4.64}$$

and

$$b_i(\tilde{\mathbf{u}}_i,q)=0 \quad \forall q\in L^2(\Omega_i) \tag{4.65}$$

for $i=1,2$, and $\mathbf{u}_1$, $\mathbf{u}_2$ are the solutions of (4.14)-(4.17).

Let $\boldsymbol{\lambda}_1$, $\beta_1$, $\boldsymbol{\lambda}_2$ and $\beta_2$ be the solutions of (4.53)-(4.55). Set $\mathbf{v}=\boldsymbol{\lambda}_i$ in (4.64), $q=\beta_i$ in (4.65), $\boldsymbol{\xi}=\tilde{\mathbf{u}}_i$ in (4.53) and $\eta=\tilde{p}_i$ in (4.54). Then we have that

$$(\tilde{\mathbf{g}},\boldsymbol{\lambda}_1-\boldsymbol{\lambda}_2)_{\Gamma_0}=(\mathbf{u}_1-\mathbf{u}_2,\tilde{\mathbf{u}}_1-\tilde{\mathbf{u}}_2)_{\Gamma_0}$$

so that, from (4.63),

$$<\frac{d\mathcal{M}_\delta(\mathbf{g})}{d\mathbf{g}},\tilde{\mathbf{g}}>=\delta(\mathbf{g},\tilde{\mathbf{g}})_{\Gamma_0}+(\tilde{\mathbf{g}},\boldsymbol{\lambda}_1-\boldsymbol{\lambda}_2)_{\Gamma_0} \quad \forall\tilde{\mathbf{g}}\in\mathbf{L}^2(\Gamma_0). \tag{4.66}$$

Then the first order necessary condition $d\mathcal{M}_\delta/d\mathbf{g}=0$ yields that

$$\delta(\mathbf{g},\tilde{\mathbf{g}})_{\Gamma_0}=-(\tilde{\mathbf{g}},\boldsymbol{\lambda}_1-\boldsymbol{\lambda}_2)_{\Gamma_0} \quad \forall\tilde{\mathbf{g}}\in\mathbf{L}^2(\Gamma_0),$$

which is equal to (4.55).

## 4.4 Finite element approximations

### 4.4.1 Finite element discretization

We choose finite element spaces $\mathbf{W}_i^h$ and $S_i^h$ for $i = 1, 2$, such that $\mathbf{W}_i^h \subset \mathbf{H}^1_{\Gamma_i}(\Omega_i)$ and $S_i^h \subset L^2(\Omega_i)$. Assume that the finite element spaces satisfy the standard approximation properties, i.e., for $i = 1, 2$, there exist an integer $k$ and a constant $C$ such that

$$\inf_{\mathbf{v}^h \in W_i^h} \|\mathbf{v} - \mathbf{v}^h\|_1 \le C h^m \|\mathbf{v}\|_{m+1} \tag{4.67}$$

$\forall \mathbf{v} \in \mathbf{H}^{m+1}(\Omega_i)$, $1 \le m \le k$ and

$$\inf_{q^h \in S_i^h} \|q - q^h\|_0 \le C h^m \|q\|_m \tag{4.68}$$

$\forall q \in H^m(\Omega_i)$, $1 \le m \le k$. See [28], [52]. We also assume the *inf-sup* (or *LBB*) condition,

$$\inf_{0 \ne q^h \in S_i^h} \sup_{0 \ne \mathbf{v}^h \in \mathbf{W}_i^h} \frac{b_i(\mathbf{v}^h, q^h)}{\|\mathbf{v}^h\|_1 \|q^h\|_0} \ge C, \tag{4.69}$$

where $C$ is a positive constant independent of $h$. See [28], [52]. Finite element approximations of solutions of the optimality system (4.14)-(4.17) and (4.53)-(4.55) are defined as follows: for $i = 1, 2$, seek $\mathbf{u}_i^h \in \mathbf{W}_i^h$, $p_i^h \in S_i^h$, $\boldsymbol{\lambda}_i^h \in \mathbf{W}_i^h$ and $\beta_i^h \in S_i^h$ such that

$$\begin{aligned} \nu a_i(\mathbf{u}_i^h, \mathbf{w}^h) + c_i(\mathbf{u}_i^h, \mathbf{u}_i^h, \mathbf{w}^h) + b_i(\mathbf{w}^h, p_i^h) \\ = (\mathbf{f}, \mathbf{w}^h)_{\Omega_i} + \frac{(-1)^{i+1}}{\delta}((-\boldsymbol{\lambda}_1^h + \boldsymbol{\lambda}_2^h), \mathbf{w}^h)_{\Gamma_0} \quad \forall \mathbf{w}^h \in \mathbf{H}^1_{\Gamma_i}(\Omega_i), \end{aligned} \tag{4.70}$$

$$b_i(\mathbf{u}_i^h, q^h) = 0 \quad \forall q^h \in L^2(\Omega_i), \tag{4.71}$$

$$\begin{aligned} \nu a_i(\boldsymbol{\lambda}_i^h, \mathbf{w}^h) + c_i(\mathbf{w}^h, \mathbf{u}_i^h, \boldsymbol{\lambda}_i^h) + c_i(\mathbf{u}_i^h, \mathbf{w}^h, \boldsymbol{\lambda}_i^h) + b_i(\mathbf{w}^h, \beta_i^h) \\ = (-1)^{i+1}(\mathbf{u}_1^h - \mathbf{u}_2^h, \mathbf{w}^h)_{\Gamma_0} \quad \forall \mathbf{w}^h \in \mathbf{H}^1_{\Gamma_i}(\Omega_i), \end{aligned} \tag{4.72}$$

and

$$b_i(\boldsymbol{\lambda}_i^h, q^h) = 0 \quad \forall q^h \in L^2(\Omega_i). \tag{4.73}$$

We will use a result from [7] to obtain error estimates for finite element approximations of the nonlinear optimality system. Let $X$ and $Y$ be Banach spaces and $\Lambda$ be a positive compact interval of $R$ containing a positive constant $\alpha$. We consider a nonlinear problem of the type

$$F(\alpha, \varphi) = \varphi + TG(\alpha, \varphi) = 0, \tag{4.74}$$

where $T \in \mathcal{L}(Y, X)$ and $G$ is a $C^2$ mapping from $\Lambda \times X$ to $Y$. If $F(\alpha, \varphi(\alpha)) = 0$ for $\alpha \in \Lambda$ and the map $\alpha \to \varphi(\alpha)$ is a continuous function from $\Lambda$ into $X$, then $\{(\alpha, \varphi(\alpha)) : \alpha \in \Lambda\}$ is called a *branch of solutions* of (4.74). If, in addition, the Fréchet derivative $D_\varphi F(\alpha, \varphi(\alpha))$ is an isomorphism from $X$ into $X$ for all $\alpha \in \Lambda$, then we call the branch $\{(\alpha, \varphi(\alpha)) : \alpha \in \Lambda\}$ nonsingular.

Let $X^h \subset X$ be a finite element space and $T^h \in \mathcal{L}(Y, X^h)$ be an discretized operator of $T$. Then the approximation problem for (4.74) is to seek $\varphi^h \in X^h$ such that

$$F^h(\alpha, \varphi^h) = \varphi^h + TG(\alpha, \varphi^h) = 0. \tag{4.75}$$

We assume that there exists a Banach space $Z$ continuously imbedded in $Y$ such that

$$D_\varphi G(\alpha, \varphi(\alpha)) \in \mathcal{L}(X, Z) \tag{4.76}$$

$\forall \alpha \in \Lambda$ and $\varphi \in X$. We also assume the following approximation properties for the operator $T^h$:

$$\lim_{h \to 0} \|(T^h - T)r\|_X = 0 \quad \forall r \in Y \tag{4.77}$$

and

$$\lim_{h \to 0} \|T^h - T\|_{\mathcal{L}(Z,X)} = 0. \tag{4.78}$$

Under the assumptions, we now quote the following theorem.

**Theorem 4.4** *Let $X$ and $Y$ be Banach spaces and $\Lambda$ a compact interval of the real line* R. *Assume that $G$ is a $C^2$ mapping from $\Lambda \times X$ into $Y$ and that $D^2G$ is bounded on all bounded sets of $\Lambda \times X$. Assume that (4.76) - (4.78) hold and that $\{(\alpha, \varphi(\alpha)) : \alpha \in \Lambda\}$ is a branch of nonsingular solutions of (4.74). Then there exists a neighborhood $\mathcal{O}$ of the origin in $X$ and, for $h \leq h_0$ small enough, a unique $C^2$ function $\alpha \to \varphi^h(\alpha) \in X^h$ such that $\{(\alpha, \varphi^h(\alpha)) : \alpha \in \Lambda\}$ is a branch of nonsingular solutions of (4.75) and $\varphi^h(\alpha) - \varphi(\alpha) \in \mathcal{O}$ for all $\alpha \in \Lambda$. Furthermore, there exists a positive constant $C$, independent of $h$ and $\alpha$, such that*

$$\|\varphi^h(\alpha) - \varphi(\alpha)\|_X \leq C\|(T^h - T)G(\alpha, \varphi(\alpha))\|_X \quad \forall \alpha \in \Lambda. \tag{4.79}$$

### 4.4.2 Error estimates

We use Theorem 4.4 to obtain error estimates for the optimality system (4.14)-(4.17), (4.53)-(4.55) and the discretized optimality system (4.70)-(4.73). Let

$$\begin{aligned}
X_i &= \mathbf{H}^1_{\Gamma_i}(\Omega_i) \times L^2(\Omega_i) \times \mathbf{H}^1_{\Gamma_i}(\Omega_i) \times L^2(\Omega_i) \quad \text{for } i = 1, 2,\\
Y_i &= (\mathbf{H}^1_{\Gamma_i}(\Omega_i))^* \times (\mathbf{H}^{1/2}(\Gamma_0))^* \times (\mathbf{H}^1_{\Gamma_i}(\Omega_i))^* \times (\mathbf{H}^{1/2}(\Gamma_0))^* \quad \text{for } i = 1, 2,\\
X &= X_1 \times X_2\\
Y &= Y_1 \times Y_2\\
Z &= \mathbf{L}^{3/2}(\Omega_1) \times \mathbf{L}^2(\Gamma_0) \times \mathbf{L}^{3/2}(\Omega_1) \times \mathbf{L}^2(\Gamma_0) \times \mathbf{L}^{3/2}(\Omega_2) \times \mathbf{L}^2(\Gamma_0) \times \mathbf{L}^{3/2}(\Omega_2) \times \mathbf{L}^2(\Gamma_0)\\
X^h &= \mathbf{W}^h_1 \times S^h_1 \times \mathbf{W}^h_1 \times S^h_1 \times \mathbf{W}^h_2 \times S^h_2 \times \mathbf{W}^h_2 \times S^h_2.
\end{aligned}$$

Let the operator $T \in \mathcal{L}(Y : X)$ be defined by

$$T(\boldsymbol{\xi}_1, \boldsymbol{\theta}_1, \boldsymbol{\eta}_1, \boldsymbol{\gamma}_1, \boldsymbol{\xi}_2, \boldsymbol{\theta}_2, \boldsymbol{\eta}_2, \boldsymbol{\gamma}_2) = (\mathbf{u}_1, p_1, \boldsymbol{\lambda}_1, \beta_1, \mathbf{u}_2, p_2, \boldsymbol{\lambda}_2, \beta_2) \tag{4.80}$$

if and only if, for $i = 1, 2$,

$$a_i(\mathbf{u}_i, \mathbf{v}) + b_i(\mathbf{v}, p_i) = (\boldsymbol{\xi}_i, \mathbf{v})_{\Omega_i} + (\boldsymbol{\theta}_i, \mathbf{v})_{\Gamma_0} \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_i}(\Omega_i), \tag{4.81}$$

$$b_i(\mathbf{u}_i, q) = 0 \quad \forall q \in L^2(\Omega_i), \tag{4.82}$$

$$a_i(\boldsymbol{\lambda}_i, \mathbf{v}) + b_i(\mathbf{v}, \beta_i) = (\boldsymbol{\eta}_i, \mathbf{v})_{\Omega_i} + (\boldsymbol{\gamma}_i, \mathbf{v}))_{\Gamma_0} \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_i}(\Omega_i), \tag{4.83}$$

and

$$b_i(\boldsymbol{\lambda}_i, q) = 0 \quad \forall q \in L^2(\Omega_i). \tag{4.84}$$

Similarly, the discrete operator $T^h \in \mathcal{L}(Y, X^h)$ is defined as follows: for $(\boldsymbol{\xi}_1, \boldsymbol{\theta}_1, \boldsymbol{\eta}_1, \boldsymbol{\gamma}_1, \boldsymbol{\xi}_2, \boldsymbol{\theta}_2, \boldsymbol{\eta}_2, \boldsymbol{\gamma}_2) \in Y$ and $(\mathbf{u}^h_1, p^h_1, \boldsymbol{\lambda}^h_1, \beta^h_1, \mathbf{u}^h_2, p^h_2, \boldsymbol{\lambda}^h_2, \beta^h_2) \in X^h$,

$$T^h(\boldsymbol{\xi}_1, \boldsymbol{\theta}_1, \boldsymbol{\eta}_1, \boldsymbol{\gamma}_1, \boldsymbol{\xi}_2, \boldsymbol{\theta}_2, \boldsymbol{\eta}_2, \boldsymbol{\gamma}_2) = (\mathbf{u}^h_1, p^h_1, \boldsymbol{\lambda}^h_1, \beta^h_1, \mathbf{u}^h_2, p^h_2, \boldsymbol{\lambda}^h_2, \beta^h_2) \tag{4.85}$$

if and only if, for $i = 1, 2$,

$$a_i(\mathbf{u}^h_i, \mathbf{v}^h) + b_i(\mathbf{v}^h, p^h_i) = (\boldsymbol{\xi}_i, \mathbf{v}^h)_{\Omega_i} + (\boldsymbol{\theta}_i, \mathbf{v}^h)_{\Gamma_0} \quad \forall \mathbf{v}^h \in \mathbf{W}^h_i, \tag{4.86}$$

$$b_i(\mathbf{u}^h_i, q^h) = 0 \quad \forall q^h \in S^h_i, \tag{4.87}$$

$$a_i(\boldsymbol{\lambda}^h_i, \mathbf{v}^h) + b_i(\mathbf{v}^h, \beta^h_i) = (\boldsymbol{\eta}_i, \mathbf{v}^h)_{\Omega_i} + (\boldsymbol{\gamma}_i, \mathbf{v}^h)_{\Gamma_0} \quad \forall \mathbf{v}^h \in \mathbf{W}^h_i, \tag{4.88}$$

and

$$b_i(\boldsymbol{\lambda}^h_i, q^h) = 0 \quad \forall q^h \in S^h_i. \tag{4.89}$$

Let $\Lambda$ be a positive compact interval containg $1/\nu$ and define the nonlinear operator $G : \Lambda \times X \to Y$ as follows: For $(\boldsymbol{\xi}_1, \boldsymbol{\theta}_1, \boldsymbol{\eta}_1, \boldsymbol{\gamma}_1, \boldsymbol{\xi}_2, \boldsymbol{\theta}_2, \boldsymbol{\eta}_2, \boldsymbol{\gamma}_2) \in Y$ and $(1/\nu, (\mathbf{u}_1, p_1, \boldsymbol{\lambda}_1, \beta_1, \mathbf{u}_2, p_2, \boldsymbol{\lambda}_2, \beta_2)) \in \Lambda \times X$

$$G(1/\nu, (\mathbf{u}_1, p_1, \boldsymbol{\lambda}_1, \beta_1, \mathbf{u}_2, p_2, \boldsymbol{\lambda}_2, \beta_2)) = (\boldsymbol{\xi}_1, \boldsymbol{\theta}_1, \boldsymbol{\eta}_1, \boldsymbol{\gamma}_1, \boldsymbol{\xi}_2, \boldsymbol{\theta}_2, \boldsymbol{\eta}_2, \boldsymbol{\gamma}_2) \tag{4.90}$$

if and only if, for $i = 1, 2$,

$$\begin{aligned}
(\boldsymbol{\xi}_i, \mathbf{v})_{\Omega_i} &= \frac{1}{\nu} c_i(\mathbf{u}_i, \mathbf{u}_i, \mathbf{v}) - \frac{1}{\nu}(\mathbf{f}, \mathbf{v})_{\Omega_i} \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_i}(\Omega_i), && (4.91)\\
(\boldsymbol{\theta}_i, \mathbf{v})_{\Gamma_0} &= \frac{1}{\nu}\frac{(-1)^i}{\delta}(-\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2, \mathbf{v})_{\Gamma_0} \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_i}(\Omega_i), && (4.92)\\
(\boldsymbol{\eta}_i, \mathbf{v})_{\Omega_i} &= \frac{1}{\nu} c_i(\mathbf{v}, \mathbf{u}_i, \boldsymbol{\lambda}_i) + \frac{1}{\nu} c_i(\mathbf{u}_i, \mathbf{v}, \boldsymbol{\lambda}_i) \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_i}(\Omega_i) && (4.93)
\end{aligned}$$

and

$$(\boldsymbol{\gamma}_i, \mathbf{v})_{\Gamma_0} = \frac{1}{\nu}(-1)^i(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{v})_{\Gamma_0} \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_i}(\Omega_i). \tag{4.94}$$

Then, the optimality system (4.14)-(4.17) and (4.53)-(4.55) with $\mathbf{g}$ given by (4.55) is equivalent to

$$(\mathbf{u}_1, \frac{1}{\nu}p_1, \boldsymbol{\lambda}_1, \frac{1}{\nu}\beta_1, \mathbf{u}_2, \frac{1}{\nu}p_2, \boldsymbol{\lambda}_2, \frac{1}{\nu}\beta_2) + TG(\frac{1}{\nu}, (\mathbf{u}_1, \frac{1}{\nu}p_1, \boldsymbol{\lambda}_1, \frac{1}{\nu}\beta_1, \mathbf{u}_2, \frac{1}{\nu}p_2, \boldsymbol{\lambda}_2, \frac{1}{\nu}\beta_2)) = 0 \tag{4.95}$$

and the discrete optimality system (4.70)-(4.71) is also expressed as the operator equation

$$(\mathbf{u}^h_1, \frac{1}{\nu}p^h_1, \boldsymbol{\lambda}^h_1, \frac{1}{\nu}\beta^h_1, \mathbf{u}^h_2, \frac{1}{\nu}p^h_2, \boldsymbol{\lambda}^h_2, \frac{1}{\nu}\beta^h_2) + T^hG(\delta, (\mathbf{u}^h_1, \frac{1}{\nu}p^h_1, \boldsymbol{\lambda}^h_1, \frac{1}{\nu}\beta^h_1, \mathbf{u}^h_2, \frac{1}{\nu}p^h_2, \boldsymbol{\lambda}^h_2, \frac{1}{\nu}\beta^h_2)) = 0. \tag{4.96}$$

Thus, the optimality system (4.14)-(4.17), (4.53)-(4.55) and the discretized optimality system (4.70)-(4.73) can be cast into the forms (4.74) and (4.75), respectively. For simplicity of the notation, we set $\alpha = 1/\nu$ in the next theorem.

**Theorem 4.5** *Assume $\Lambda$ is a compact interval of $R$ and that $\{(\alpha, \varphi(\alpha) = (\mathbf{u}_1, p_1, \boldsymbol{\lambda}_1, \beta_1, \mathbf{u}_2, p_2, \boldsymbol{\lambda}_2, \beta_2) : \alpha \in \Lambda\}$ is a branch of nonsingular solutions of the optimality system (4.14)-(4.17) and (4.53)-(4.55). Also assume that $\mathbf{W}^h_i$, $S^h_i$ satisfy the conditions (4.67)-(4.69). Then there exists a neighborhood $\mathcal{O}$ of the origin in $X$ and, for $h \le h_0$ small enough, a unique branch $\{(\alpha, \varphi^h(\alpha) = (\mathbf{u}^h_1, p^h_1, \boldsymbol{\lambda}^h_1, \beta^h_1, \mathbf{u}^h_2, p^h_2, \boldsymbol{\lambda}^h_2, \beta^h_2) : \alpha \in \Lambda\}$ of solutions of the discrete optimality system (4.70)-(4.73) such that $\varphi^h(\alpha) - \varphi(\alpha) \in \mathcal{O}$ for all $\alpha \in \Lambda$. Moreover,*

$$\begin{aligned}
\|\varphi^h(\alpha) - \varphi(\alpha)\|_X &= \sum_{i=1}^{2}(\|\mathbf{u}_i(\alpha) - \mathbf{u}^h_i(\alpha)\|_{1,\Omega_i} + \|p_i(\alpha) - p^h_i(\alpha)\|_{0,\Omega_i}\\
&\qquad + \|\boldsymbol{\lambda}_i(\alpha) - \boldsymbol{\lambda}^h_i(\alpha)\|_{1,\Omega_i} + \|\beta_i(\alpha) - \beta^h_i(\alpha)\|_{0,\Omega_i}) \to 0 \quad (4.97)
\end{aligned}$$

*as* $h \to 0$ *uniformly in* $\alpha \in \Lambda$. *If, in addition, the solution of the optimality system satisfies* $(\mathbf{u}_1(\alpha), p_1(\alpha), \boldsymbol{\lambda}_1(\alpha), \beta_1(\alpha), \mathbf{u}_2(\alpha), p_2(\alpha), \boldsymbol{\lambda}_2(\alpha), \beta_2(\alpha)) \in \mathbf{H}^{m+1}(\Omega_1) \times H^m(\Omega_1) \times \mathbf{H}^{m+1}(\Omega_1) \times H^m(\Omega_1) \times \mathbf{H}^{m+1}(\Omega_2) \times H^m(\Omega_2) \times \mathbf{H}^{m+1}(\Omega_2) \times H^m(\Omega_2)$ *for* $\alpha \in \Lambda$, *then there exists a constant* $C$, *independent of* $h$, *such that*

$$\sum_{i=1}^{2}(\|\mathbf{u}_i(\alpha) - \mathbf{u}_i^h(\alpha)\|_{1,\Omega_i} + \|p_i(\alpha) - p_i^h(\alpha)\|_{0,\Omega_i} + \|\boldsymbol{\lambda}_i(\alpha) - \boldsymbol{\lambda}_i^h(\alpha)\|_{1,\Omega_i} + \|\beta_i(\alpha) - \beta_i^h(\alpha)\|_{0,\Omega_i})$$

$$\leq Ch^m \sum_{i=1}^{2}(\|\mathbf{u}_i(\alpha)\|_{m+1,\Omega_i} + \|p_i(\alpha)\|_{m,\Omega_i} + \|\boldsymbol{\lambda}_i(\alpha)\|_{m+1,\Omega_i} + \|\beta_i(\alpha)\|_{m,\Omega_i}), \tag{4.98}$$

*uniformly in* $\alpha \in \Lambda$.

**Proof:** Since $G$ is a $C^\infty$ mapping from $X$ into $Y$ and $\Lambda$ is compact, $D^2G$ is bounded on all bounded sets of $\Lambda \times X$ by (4.21). The derivative of $G$, $D_\varphi G$ is defined by

$$D_\varphi G(\alpha, (\mathbf{u}_1, p_1, \boldsymbol{\lambda}_1, \beta_1, \mathbf{u}_2, p_2, \boldsymbol{\lambda}_2, \beta_2))(\mathbf{v}_1, q_1, \boldsymbol{\omega}_1, \psi_1, \mathbf{v}_2, q_2, \boldsymbol{\omega}_2, \psi_2)$$
$$= (\tilde{\boldsymbol{\xi}}_1, \tilde{\boldsymbol{\theta}}_1, \tilde{\boldsymbol{\eta}}_1, \tilde{\boldsymbol{\gamma}}_1, \tilde{\boldsymbol{\xi}}_2, \tilde{\boldsymbol{\theta}}_2, \tilde{\boldsymbol{\eta}}_2, \tilde{\boldsymbol{\gamma}}_2)$$

for $(\mathbf{v}_1, q_1, \boldsymbol{\omega}_1, \psi_1, \mathbf{v}_2, q_2, \boldsymbol{\omega}_2, \psi_2) \in X$ if and only if, for $i = 1, 2$,

$$\begin{aligned}
(\tilde{\boldsymbol{\xi}}_i, \overline{\mathbf{v}})_{\Omega_i} &= \alpha c_i(\mathbf{u}_i, \mathbf{v}_i, \overline{\mathbf{v}}) + \alpha c_i(\mathbf{v}_i, \mathbf{u}_i, \overline{\mathbf{v}}) \quad \forall \overline{\mathbf{v}} \in \mathbf{H}^1_{\Gamma_i}(\Omega_i), \\
(\tilde{\boldsymbol{\theta}}_i, \overline{\mathbf{v}})_{\Gamma_0} &= \alpha \frac{(-1)^i}{\delta}(-\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2, \overline{\mathbf{v}})_{\Gamma_0} \quad \forall \overline{\mathbf{v}} \in \mathbf{H}^1_{\Gamma_i}(\Omega_i), \\
(\tilde{\boldsymbol{\eta}}_i, \overline{\mathbf{v}})_{\Omega_i} &= \alpha c_i(\overline{\mathbf{v}}, \mathbf{v}_i, \boldsymbol{\lambda}_i) + \alpha c_i(\overline{\mathbf{v}}, \mathbf{u}_i, \boldsymbol{\omega}_i) + \alpha c_i(\mathbf{v}_i, \overline{\mathbf{v}}, \boldsymbol{\lambda}_i) + \alpha c_i(\mathbf{u}_i, \overline{\mathbf{v}}, \boldsymbol{\omega}_i) \quad \forall \overline{\mathbf{v}} \in \mathbf{H}^1_{\Gamma_i}(\Omega_i)
\end{aligned}$$

and

$$(\tilde{\boldsymbol{\gamma}}_i, \overline{\mathbf{v}})_{\Gamma_0} = (-1)^i \alpha(\mathbf{v}_1 - \mathbf{v}_2, \overline{\mathbf{v}})_{\Gamma_0} \quad \forall \overline{\mathbf{v}} \in \mathbf{H}^1_{\Gamma_i}(\Omega_i).$$

Thus, $D_\varphi G(\alpha, (\mathbf{u}_1, p_1, \boldsymbol{\lambda}_1, \beta_1, \mathbf{u}_2, p_2, \boldsymbol{\lambda}_2, \beta_2)) \in \mathcal{L}(X, Y)$. On the other hand, since $(\mathbf{u}_1, p_1, \boldsymbol{\lambda}_1, \beta_1, \mathbf{u}_2, p_2, \boldsymbol{\lambda}_2, \beta_2) \in X$ and $(\mathbf{v}_1, q_1, \boldsymbol{\omega}_1, \psi_1, \mathbf{v}_2, q_2, \boldsymbol{\omega}_2, \psi_2) \in X$, by the Sobolev imbedding theorem, $\mathbf{u}_i$, $\mathbf{v}_i$, $\boldsymbol{\lambda}_i$ and $\boldsymbol{\omega}_i \in \mathbf{L}^6(\Omega_i)$, for $i = 1, 2$, $\partial \mathbf{u}_i / \partial x_j$, $\partial \mathbf{v}_i / \partial x_j$, $\partial \boldsymbol{\lambda}_i / \partial x_j$ and $\partial \boldsymbol{\omega}_i / \partial x_j \in L^2(\Omega_i)$ for $i, j = 1, 2$ and $\boldsymbol{\omega}_i|_{\Gamma_0} \in \mathbf{L}^2(\Gamma_0)$. Then it follows that $(\tilde{\boldsymbol{\xi}}_1, \tilde{\boldsymbol{\theta}}_1, \tilde{\boldsymbol{\eta}}_1, \tilde{\boldsymbol{\gamma}}_1, \tilde{\boldsymbol{\xi}}_2, \tilde{\boldsymbol{\theta}}_2, \tilde{\boldsymbol{\eta}}_2, \tilde{\boldsymbol{\gamma}}_2) \in Z$, therefore,

$$D_\varphi G(\alpha, (\mathbf{u}_1, p_1, \boldsymbol{\lambda}_1, \beta_1, \mathbf{u}_2, p_2, \boldsymbol{\lambda}_2, \beta_2)) \in \mathcal{L}(X, Z).$$

Note that the imbedding $Z \subset Y$ is compact.

Next, we turn to the approximation properties of the operator $T^h$. It can be shown using well-known results for the Stokes equations that if $(\overline{\mathbf{u}}_1, \overline{p}_1, \overline{\boldsymbol{\lambda}}_1, \overline{\beta}_1, \overline{\mathbf{u}}_2, \overline{p}_2, \overline{\boldsymbol{\lambda}}_2, \overline{\beta}_2)$ and $(\overline{\mathbf{u}}_1^h, \overline{p}_1^h, \overline{\boldsymbol{\lambda}}_1^h, \overline{\beta}_1^h, \overline{\mathbf{u}}_2^h, \overline{p}_2^h, \overline{\boldsymbol{\lambda}}_2^h, \overline{\beta}_2^h)$ are the solutions of (4.81)-(4.84) and (4.86)-(4.89), then

$$\sum_{i=1}^{2}(\|\overline{\mathbf{u}}_i - \overline{\mathbf{u}}_i^h\|_{1,\Omega_i} + \|\overline{p}_i - \overline{p}_i^h\|_{0,\Omega_i} + \|\overline{\boldsymbol{\lambda}}_i - \overline{\boldsymbol{\lambda}}_i^h\|_{1,\Omega_i} + \|\overline{\beta}_i - \overline{\beta}_i^h\|_{0,\Omega_i}) \to 0.$$

See [52]. Hence, (4.77) is satisfied and (4.97) holds. Since the imbedding $Z \subset Y$ is compact, (4.78) follows from (4.77). Also we can have the approximation result

$$\|(T - T^h)G(\alpha, \varphi(\alpha))\|_X \leq Ch^m \sum_{i=1}^{2}(\|\mathbf{u}_i\|_{m+1,\Omega_i} + \|p_i\|_{m,\Omega_i} + \|\boldsymbol{\lambda}_i\|_{m+1,\Omega_i} + \|\beta_i\|_{m,\Omega_i})$$

for $(\mathbf{u}_1, p_1, \boldsymbol{\lambda}_1, \beta_1, \mathbf{u}_2, p_2, \boldsymbol{\lambda}_2, \beta_2) \in \mathbf{H}^{m+1}(\Omega_1) \times H^m(\Omega_1) \times \mathbf{H}^{m+1}(\Omega_1) \times H^m(\Omega_1) \times \mathbf{H}^{m+1}(\Omega_2) \times H^m(\Omega_2) \times \mathbf{H}^{m+1}(\Omega_2) \times H^m(\Omega_2)$, which is obtained from the following results for Stokes equation:

$$\begin{aligned}\sum_{i=1}^{2}&(\|\overline{\mathbf{u}}_i - \overline{\mathbf{u}}_i^h\|_{1,\Omega_i} + \|\overline{p}_i - \overline{p}_i^h\|_{0,\Omega_i} + \|\overline{\boldsymbol{\lambda}}_i - \overline{\boldsymbol{\lambda}}_i^h\|_{1,\Omega_i} + \|\overline{\beta}_i - \overline{\beta}_i^h\|_{0,\Omega_i}) \\ &\leq Ch^m \sum_{i=1}^{2}(\|\overline{\mathbf{u}}_i\|_{m+1,\Omega_i} + \|\overline{p}_i\|_{m,\Omega_i} + \|\overline{\boldsymbol{\lambda}}_i\|_{m+1,\Omega_i} + \|\overline{\beta}_i\|_{m,\Omega_i}),\end{aligned}$$

where $C$ is independent of $h$. Then (4.98) follows from (4.79). □

## 4.5 A gradient method

In this section we study the gradient method (2.45) presented in Chapter 2 to solve the coupled optimality system (4.14)-(4.17) and (4.53)-(4.55). Combining (2.45) with (4.66) yields, for $n = 1, 2, \ldots,$

$$g^{(n+1)} = (1-\alpha)g^{(n)} - \frac{\alpha}{\delta}({\lambda_1}^{(n)} - {\lambda_2}^{(n)}) \tag{4.99}$$

where ${\lambda_1}^{(n)}$, ${\lambda_2}^{(n)}$ are determined from (4.53)-(4.54). In summary, the algorithm is given as follows.

**Algorithm 4.5**

1. *Choose* $g^{(0)}$.
2. *For* $n = 1, 2, 3, \ldots$,
   a. *compute* $\mathbf{u}_1{}^{(n)}$, $p_1^{(n)}$, $\mathbf{u}_2{}^{(n)}$ *and* $p_2^{(n)}$ *by*

$$a_1(\mathbf{u}_1{}^{(n)}, \mathbf{v}) + c_1(\mathbf{u}_1^{(n)}, \mathbf{u}_1^{(n)}, \mathbf{v}) + b_1(\mathbf{v}, p_1^{(n)}) = (\mathbf{f}_1, \mathbf{v})_{\Omega_1} + (\mathbf{g}^{(n)}, \mathbf{v})_{\Gamma_0} \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_1}(\Omega_1),$$

$$b_1(\mathbf{u}_1^{(n)}, q) = 0 \quad \forall q \in L^2(\Omega_1),$$

$$a_2(\mathbf{u}_2{}^{(n)}, \mathbf{v}) + c_2(\mathbf{u}_2^{(n)}, \mathbf{u}_2^{(n)}, \mathbf{v}) + b_2(\mathbf{v}, p_2^{(n)}) = (\mathbf{f}_2, \mathbf{v})_{\Omega_2} - (\mathbf{g}^{(n)}, \mathbf{v})_{\Gamma_0} \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_2}(\Omega_2)$$

   *and*

$$b_2(\mathbf{u}_2^{(n)}, q) = 0 \quad \forall q \in L^2(\Omega_2),$$

   b. *compute* $\boldsymbol{\lambda}_1{}^{(n)}$, $\beta_1^{(n)}$, $\boldsymbol{\lambda}_2{}^{(n)}$ *and* $\beta_2^{(n)}$ *by*

$$\begin{aligned} a_1(\boldsymbol{\lambda}_1{}^{(n)}, \boldsymbol{\xi}) + c_1(\boldsymbol{\xi}, \mathbf{u}_1^{(n)}, \boldsymbol{\lambda}_1^{(n)}) + c_1(\mathbf{u}_1^{(n)}, \boldsymbol{\xi}, \boldsymbol{\lambda}_1^{(n)}) + b_1(\boldsymbol{\xi}, \beta_1^{(n)}) \\ = (\mathbf{u}_1^{(n)} - \mathbf{u}_2^{(n)}, \boldsymbol{\xi})_{\Gamma_0} \quad \forall \boldsymbol{\xi} \in \mathbf{H}^1_{\Gamma_1}(\Omega_1), \end{aligned}$$

$$b_1(\boldsymbol{\lambda}_1^{(n)}, \boldsymbol{\eta}) = 0 \quad \forall \boldsymbol{\eta} \in L^2(\Omega_1),$$

$$\begin{aligned} a_2(\boldsymbol{\lambda}_2{}^{(n)}, \boldsymbol{\xi}) + c_2(\boldsymbol{\xi}, \mathbf{u}_2^{(n)}, \boldsymbol{\lambda}_2^{(n)}) + c_2(\mathbf{u}_2^{(n)}, \boldsymbol{\xi}, \boldsymbol{\lambda}_2^{(n)}) + b_2(\boldsymbol{\xi}, \beta_2^{(n)}) \\ = -(\mathbf{u}_1^{(n)} - \mathbf{u}_2^{(n)}, \boldsymbol{\xi})_{\Gamma_0} \quad \forall \boldsymbol{\xi} \in \mathbf{H}^1_{\Gamma_2}(\Omega_2), \end{aligned}$$

   *and*

$$b_2(\boldsymbol{\lambda}_2^{(n)}, \boldsymbol{\eta}) = 0 \quad \forall \boldsymbol{\eta} \in L^2(\Omega_2),$$

   c. *compute* $\mathbf{g}^{(n)}$ *by*

$$\mathbf{g}^{(n)} = (1 - \alpha)\mathbf{g}^{(n-1)} - \frac{\alpha}{\delta}(\boldsymbol{\lambda}_1{}^{(n)} - \boldsymbol{\lambda}_2{}^{(n)}).$$

## 4.6 Nonlinear least squares approach

In this section we reconsider the minimization problem (4.32)-(4.34) from a point of view of nonlinear least squares problem. Define the nonlinear operator $F : \mathbf{L}^2(\Gamma_0) \to \mathbf{L}^2(\Gamma_0) \times \mathbf{L}^2(\Gamma_0)$ by

$$F(\mathbf{g}) = \begin{pmatrix} (\mathbf{u}_1 - \mathbf{u}_2)|_{\Gamma_0} \\ \sqrt{\delta}\, \mathbf{g} \end{pmatrix},$$

where $\mathbf{u}_1$ and $\mathbf{u}_2$ are the solutions of (4.4)-(4.7) and (4.8)-(4.11), respectively. Then, (4.13) can be written as

$$\mathcal{J}_\delta(\mathbf{g}) = \frac{1}{2}\|F(\mathbf{g})\|^2_{0,\Gamma_0} \tag{4.100}$$

and the nonlinear least squares problem we consider is to

$$\textit{seek } \mathbf{g} \in \mathbf{L}^2(\Gamma_0) \textit{ which minimizes } (4.100). \tag{4.101}$$

We can linearize $F(\mathbf{g})$ using the Fréchet derivative of $F(\cdot)$ at $\bar{\mathbf{g}}$, $F'(\bar{\mathbf{g}})$, by

$$F(\mathbf{g}) = F(\bar{\mathbf{g}}) + F'(\bar{\mathbf{g}})(\mathbf{g} - \bar{\mathbf{g}}) + o(\|\mathbf{g} - \bar{\mathbf{g}}\|^2)$$

so that solutions of the nonlinear least squares problem can be obtained by solving the linear least squares problem

$$\min_{\mathbf{h}\in\mathbf{L}^2(\Gamma_0)} \frac{1}{2}\|F(\bar{\mathbf{g}}) + F'(\bar{\mathbf{g}})\mathbf{h}\|^2_{0,\Gamma_0} \tag{4.102}$$

successively, where $\mathbf{h} = \mathbf{g} - \bar{\mathbf{g}}$. Hence, starting with arbitrary $\mathbf{g}^{(0)}$ we can find a sequence $\{\mathbf{g}^{(n)}\}$ obtained by $\mathbf{g}^{(n)} = \mathbf{g}^{(n-1)} + \mathbf{h}^{(n)}$, where $\mathbf{h}^{(n)}$ is a solution of the linear least squares problem (4.102).

The Fréchet derivative $F'(\bar{\mathbf{g}})(\cdot) : \mathbf{L}^2(\Gamma_0) \to \mathbf{L}^2(\Gamma_0) \times \mathbf{L}^2(\Gamma_0)$ for $\bar{\mathbf{g}} \in \mathbf{L}^2(\Gamma_0)$ is defined by

$$F'(\bar{\mathbf{g}})(\mathbf{h}) = \begin{pmatrix} (\mathbf{w}_1 - \mathbf{w}_2)|_{\Gamma_0} \\ \sqrt{\delta}\,\mathbf{h} \end{pmatrix},$$

where $\mathbf{w}_1$ and $\mathbf{w}_2$ are the solutions of

$$\nu a_i(\mathbf{w}_i, \mathbf{v}) + c_i(\mathbf{w}_i, \mathbf{u}_i, \mathbf{v}) + c_i(\mathbf{u}_i, \mathbf{w}_i, \mathbf{v}) + b_i(\mathbf{v}, t_i) = (-1)^{i+1}(\mathbf{h}, \mathbf{v})_{\Gamma_0} \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_i}(\Omega_i) \tag{4.103}$$

and

$$b_i(\mathbf{w}_i, q) = 0 \quad \forall q \in L^2(\Omega_i) \tag{4.104}$$

for $i = 1, 2$. The $\mathbf{u}_i$ in (4.103) is the solution of (4.14)-(4.17) with $\mathbf{g}$ replaced by $\bar{\mathbf{g}}$.

It is necessary to define the adjoint operator of $F'(\bar{\mathbf{g}})$ in order to solve the linear least squares problem (4.102). We define $(F'(\bar{\mathbf{g}}))^*(\cdot) : \mathbf{L}^2(\Gamma_0) \times \mathbf{L}^2(\Gamma_0) \to \mathbf{L}^2(\Gamma_0)$ by

$$(F'(\bar{\mathbf{g}}))^* \begin{pmatrix} \mathbf{r} \\ \mathbf{s} \end{pmatrix} = (\mathbf{z}_1 - \mathbf{z}_2)|_{\Gamma_0} + \sqrt{\delta}\,\mathbf{s}$$

where $\mathbf{z}_1$ and $\mathbf{z}_2$ are the solutions of

$$\nu a_i(\mathbf{z}_i, \mathbf{v}) + c_i(\mathbf{u}_i, \mathbf{v}, \mathbf{z}_i) + c_i(\mathbf{v}, \mathbf{u}_i, \mathbf{z}_i) + b_i(\mathbf{v}, \boldsymbol{\eta}_i) = (-1)^{i+1}(\mathbf{r}, \mathbf{v})_{\Gamma_0} \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_i}(\Omega_i) \tag{4.105}$$

and

$$b_i(\mathbf{z}_i, q) = 0 \quad \forall q \in L^2(\Omega_i) \tag{4.106}$$

for $i = 1, 2$. Again, $\mathbf{u}_i$ in (4.105) is the solution of (4.14)-(4.17) with the replacement of $\mathbf{g}$ by $\bar{\mathbf{g}}$.

**Theorem 4.6** *If $\bar{\mathbf{g}}$ is close to the optimal $\mathbf{g}$ so that (4.28) is satisfied, $F'(\bar{\mathbf{g}})(\cdot)$ has a closed range. Hence, the solution of the linear least squares problem (4.102) can be obtained by solving the normal equation*

$$F'(\bar{\mathbf{g}})^* F(\bar{\mathbf{g}}) = -F'(\bar{\mathbf{g}})^* F'(\bar{\mathbf{g}})(\mathbf{h}). \tag{4.107}$$

**Proof:** Let $\{(\mathbf{x}^k, \mathbf{y}^k)^T\} \subset R(F'(\bar{\mathbf{g}}))$ converge to $(\hat{\mathbf{x}}, \hat{\mathbf{y}})^T$. Then there exist $\mathbf{w}_i^k \in \mathbf{H}^1_{\Gamma_i}(\Omega_i)$ and $\mathbf{h}^k \in \mathbf{L}^2(\Gamma_0)$ for $i = 1, 2$, such that $\mathbf{x}^k = \mathbf{w}_1^k - \mathbf{w}_2^k$, where $\mathbf{w}_i^k$ is the solution of (4.103)-(4.104) with $\mathbf{h}$ replaced by $\mathbf{h}^k$, and $\sqrt{\delta}\mathbf{h}^k = \mathbf{y}^k$. Clearly, $\mathbf{h}^k = \mathbf{y}^k/\sqrt{\delta} \to \hat{\mathbf{y}}/\sqrt{\delta}$ and by the passing to the limit we have $\hat{\mathbf{x}} = \hat{\mathbf{w}}_1 - \hat{\mathbf{w}}_2$, where $\hat{\mathbf{w}}_i$ for $i = 1, 2$, satisfy (4.103)-(4.104) with $\mathbf{h}$ replaced by $\hat{\mathbf{y}}/\sqrt{\delta}$. Hence, $F'(\bar{\mathbf{g}})(\hat{\mathbf{y}}/\sqrt{\delta}) = (\hat{\mathbf{x}}, \hat{\mathbf{y}})^T$ and $F'(\mathbf{g})(\cdot)$ has a closed range. The other result follows from [36, Thm. 2.1.1]. □

Again, we use the conjugate gradient method, Algorithm 2.2, to solve the linear least squares problem (4.102), and convergence of iterates generated by the algorithm can be seen similarly to the Poisson equation problem; see Theorem 2.6, Lemma 2.3 and Theorem 2.7. Thus, the nonlinear least squares problem (4.101) can be solved using the following algorithm, which is known as a Gauss-Newton method.

**Algorithm 4.6**

```
1. Choose $\mathbf{g}^{(0)}$.
2. For $n = 1, 2, 3, \cdots$,
    a. compute $\mathbf{h}^{(n)}$ by the conjugate gradient algorithm 2.2 such that
        $\min_{\mathbf{h}} \frac{1}{2}(F'(\mathbf{g}^{(n-1)})(\mathbf{h}^{(n)}), \mathbf{h}^{(n)})_{\Gamma_0} + (\mathbf{b}, \mathbf{h}^{(n)})_{\Gamma_0}$,
    b. set $\mathbf{g}^{(n)} = \mathbf{g}^{(n-1)} + \mathbf{h}^{(n)}$.
```

## 4.7 Numerical results

Let the domain $\Omega$ be the rectangle $\{(x,y) : 0 < x < 1, 0 < y < 0.5\}$. $\Omega$ is divided into two parts $\Omega_1$, $\Omega_2$ such that $\Omega_1 = \{(x,y) : 0 < x < 0.5, 0 < y < 0.5\}$ and $\Omega_2 = \{(x,y) : 0.5 < x < 1, 0 < y < 0.5\}$ with the interface $\Gamma_0 = \{(x,y) : x = 0.5, 0 < y < 0.5\}$. The finite element spaces ${W_i}^h$, ${S_i}^h$ for $i = 1, 2$, were chosen to consist of the usual continuous, piecewise quadratic and linear polynomials on triangular meshes, respectively, which are known to satisfy the stability condition (4.69). We adjusted the data $\mathbf{f}$ in (4.1) so that the Navier-Stokes equation (4.1)-(4.2) has the exact solution

$$\begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} sin(\pi x)y(y-0.5) \\ sin(x)(x-1)ycos(\pi y) \\ cos(2\pi x)y(y-0.5) \end{pmatrix}$$

with $\nu = 1$. Although we assumed that div $\mathbf{u} = 0$ for the analyses, it turns out that the *div-free* condition is not necessary for numerical experiments. The example we chose for the numerical test has zero boundary conditions, but the divergence of $\mathbf{u}$ is not zero. Hence, we replaced

$$a_i(\mathbf{u},\mathbf{v}) = \frac{1}{2}\int_{\Omega_i} (\nabla\mathbf{u} + (\nabla\mathbf{u})^T) : (\nabla\mathbf{v} + (\nabla\mathbf{v})^T)\, d\Omega \quad \forall \mathbf{u},\mathbf{v} \in \mathbf{H}^1_{\Gamma_i}(\Omega_i)$$

by

$$a_i(\mathbf{u},\mathbf{v}) = \frac{1}{2}\int_{\Omega_i} (\nabla\mathbf{u} + (\nabla\mathbf{u})^T) : (\nabla\mathbf{v} + (\nabla\mathbf{v})^T)\, d\Omega_i - \frac{1}{2\nu}\int_{\Omega_i} (\nabla\cdot\mathbf{u}_i)\,\mathbf{u}_i\cdot\mathbf{v}\, d\Omega_i$$

for all $\mathbf{u},\mathbf{v} \in \mathbf{H}^1_{\Gamma_i}(\Omega_i)$, which is the only change for the case that div $\mathbf{u} \neq 0$; see [44].

For the gradient method, the step size was fixed as 55 for the penalty parameter $\delta = 10^{-7}$, i.e., $\alpha$ was chosen to be $5.5 \times 10^{-6}$ and we used the stopping criterion defined by

$$\|\text{ change in successive value of } \mathbf{u}\| < 10^{-9}$$

with the maximum number of iteration of 500. For the Gauss-Newton method, we used the same penalty parameter $\delta = 10^{-7}$ and the stopping criterion defined by

$$\|F'(x)^* F(x)\| < 10^{-8}.$$

Errors and rates of convergence for the gradient method and the Gauss-Newton method are presented in Table 4.1 and Table 4.2, respectively. It was observed that the iterates obtained by using the gradient method converge very slowly. Even 500 iterations are not enough for the fine mesh 1/32 and the slow convergence might result from the non-uniqueness of the pressure $p$ satisfying (4.1)-(4.2). However, the Gauss-Newton method yields smaller errors and better rates for fine meshes with only 4 iterations, even though linear least squares problems need to be solved on each iteration.

Table 4.1: errors and rates by the gradient method for $\delta = 10^{-7}$

| $h$ | no. of iter. | $L^2$ error (u) | $L^2$ error (v) | $H^1$ error (u) | $H^1$ error (v) |
|---|---|---|---|---|---|
| $\frac{1}{4}$ | 292 | $6.633 \cdot 10^{-4}$ | $7.170 \cdot 10^{-4}$ | $2.213 \cdot 10^{-2}$ | $1.933 \cdot 10^{-2}$ |
| rate | | 3.20 | 3.46 | 2.18 | 2.34 |
| $\frac{1}{8}$ | 500 | $7.212 \cdot 10^{-5}$ | $6.529 \cdot 10^{-5}$ | $4.867 \cdot 10^{-3}$ | $3.814 \cdot 10^{-3}$ |
| rate | | 3.15 | 3.27 | 2.05 | 2.09 |
| $\frac{1}{16}$ | 500 | $8.143 \cdot 10^{-6}$ | $6.775 \cdot 10^{-6}$ | $1.176 \cdot 10^{-3}$ | $8.941 \cdot 10^{-4}$ |
| rate | | 2.75 | 2.81 | 1.88 | 2.00 |
| $\frac{1}{32}$ | 500 | $1.208 \cdot 10^{-6}$ | $9.634 \cdot 10^{-7}$ | $3.185 \cdot 10^{-4}$ | $2.229 \cdot 10^{-4}$ |

Table 4.2: errors and rates by the Gauss-Newton method for $\delta = 10^{-7}$

| $h$ | no. of GN iter. | $L^2$ error (u) | $L^2$ error (v) | $H^1$ error (u) | $H^1$ error (v) |
|---|---|---|---|---|---|
| $\frac{1}{4}$ | 4 | $6.633 \cdot 10^{-4}$ | $7.170 \cdot 10^{-4}$ | $2.213 \cdot 10^{-2}$ | $1.933 \cdot 10^{-2}$ |
| rate | | 3.21 | 3.45 | 2.19 | 2.34 |
| $\frac{1}{8}$ | 3 | $7.182 \cdot 10^{-5}$ | $6.535 \cdot 10^{-5}$ | $4.850 \cdot 10^{-3}$ | $3.815 \cdot 10^{-3}$ |
| rate | | 3.16 | 3.27 | 2.05 | 2.09 |
| $\frac{1}{16}$ | 4 | $8.051 \cdot 10^{-6}$ | $6.776 \cdot 10^{-6}$ | $1.168 \cdot 10^{-4}$ | $8.957 \cdot 10^{-4}$ |
| rate | | 3.03 | 3.09 | 2.00 | 2.03 |
| $\frac{1}{32}$ | 4 | $9.835 \cdot 10^{-7}$ | $7.934 \cdot 10^{-7}$ | $2.915 \cdot 10^{-4}$ | $2.200 \cdot 10^{-4}$ |

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