1. INTRODUCTION

Observability is an important and fundamental issue in control theory. Observability is related to the possibility of distinguishing between distinct state trajectories using only measured outputs, and is clearly necessary for the existence of state observers for estimating the state from measured outputs.

Two basic papers on observability of nonlinear systems were first reported by Kostyukovskii [1], [2]. According to Kostyukovskii [1], the system is observable if there is a one-to-one correspondence between the state trajectories and the observed output trajectories. Griffith and Kumar [3] proved that the sufficient condition for observability given by Kostyukovskii [1] is incorrect. For the past two to three decades, differential geometry has proved to be a powerful tool in the analysis of nonlinear systems. Brockett [4] considered an autonomous system defined on a differentiable manifold and addressed issues related to controllability and observability. In 1977, Hermann-Krener [5] proved the well known rank condition for local observability of nonlinear systems. Even in the analytic case, the Hermann-Krener condition is only sufficient. Moreover, when the condition fails, it does not indicate which states are indistinguishable from a given state.

The objective of this paper is to obtain results that explicitly characterize sets of states that produce identical output-time histories. For this purpose, we introduce the extended observation space in Section 4, and show that the set of states indistinguishable from a given state is the intersection of level sets of functions from the extended observation space, containing the given state. We give alternative characterizations of the set of states indistinguishable from a given state using the annihilator of the extended observation space. Specifically, we show that every point in the closure of the orbit of the annihilator of the extended observation space through a given state is indistinguishable from the given state. Furthermore, under additional assumptions, the orbit of the annihilator of the extended observation space through a given state is the connected component of the set of states indistinguishable from that state. We also show that sets of indistinguishable states are related to prolongations of the annihilator of the extended observation space as defined in [6]. In Section 5, we explore the relationship between the extended observation space and the observation space considered in the literature. In the case of an analytic system the closure of the extended observation space and closure of the observation space under pointwise limits as well as their annihilators are essentially the same. In Section 6, as an application we consider the reduced attitude dynamics of an unactuated inertially symmetric rigid body, and identify its sets of indistinguishable states. We begin by presenting the necessary mathematical preliminaries in Section 2.

2. PRELIMINARIES

Let $\mathcal{M}$ be a smooth $n$-dimensional manifold. Given $\mathcal{U} \subseteq \mathcal{M}$ and $z \in \mathcal{U}$, we let $\text{Con}_z(\mathcal{U})$ and $\text{cl}(\mathcal{U})$ denote the connected component of $\mathcal{U}$ containing $z$ and the closure of $\mathcal{U}$, respectively.

We denote the set of all smooth real-valued functions defined on $\mathcal{M}$ by $C^\infty(\mathcal{M})$, and the set of all smooth vector fields defined on $\mathcal{M}$ by $\mathcal{V}(\mathcal{M})$. If $\mathcal{M}$ is a real-analytic manifold, then we denote the set of all real-analytic, real-valued functions defined on $\mathcal{M}$ by $C^\omega(\mathcal{M})$. Given $\mathcal{A} \subseteq C^\infty(\mathcal{M})$, we let $\bar{\mathcal{A}}$ denote the smallest subset of $C^\infty(\mathcal{M})$ that contains $\mathcal{A}$ and is closed under pointwise limits in $C^\infty(\mathcal{M})$, that is, $\mathcal{A}$ is the smallest set that contains every smooth function in $\mathcal{A}$ that is a pointwise limit of a sequence of functions in $\mathcal{A}$.

Given $X \in \mathcal{V}(\mathcal{M})$, $t \in \mathbb{R}$ and $z \in \mathcal{M}$, we let $\phi^X: (t, z) \rightarrow \phi^X_t(z)$ denote the flow of $X$. The vector field $X$ is complete if its flow is defined on all of $\mathbb{R} \times \mathcal{M}$. A set $\mathcal{U} \subseteq \mathcal{M}$ is invariant under $X \in \mathcal{V}(\mathcal{M})$ if $\phi^X_t(\mathcal{U}) = \mathcal{U}$ for every $t \in \mathbb{R}$. Given $\mathcal{B} \subseteq \mathcal{V}(\mathcal{M})$ and $z \in \mathcal{M}$, we let $\mathcal{B}(z)$ denote the smallest subset of $\mathcal{M}$ that contains $z$ and is invariant under every $X \in \mathcal{B}$. Since every connected component of an invariant set is invariant, it follows that $\mathcal{B}(z)$ is connected for every $\mathcal{B} \subseteq \mathcal{V}(\mathcal{M})$ and $z \in \mathcal{M}$. It can be shown that, for every $z \in \mathcal{M}$ and $\mathcal{B} \subseteq \mathcal{V}(\mathcal{M})$, $\mathcal{B}(z)$ is precisely the orbit of the set of vector fields $\mathcal{B}$ through $z$. Given $\mathcal{U} \subseteq \mathcal{M}$ and $\mathcal{B} \subseteq \mathcal{V}(\mathcal{M})$, we let $\mathcal{B}(\mathcal{U}) = \bigcup_{z \in \mathcal{U}} \mathcal{B}(z)$. 

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Given $X \in \mathcal{V}(\mathcal{M})$ and $\gamma \in C^\infty(\mathcal{M})$, we let $L_X \gamma$ denote the Lie derivative of $\gamma$ with respect to $X$. Given $X \in \mathcal{V}(\mathcal{M})$ and $\gamma \in C^\infty(\mathcal{M})$ we denote Lie derivatives $L_X^k \gamma = L_X^{k-1} \gamma$ for every $k > 0$.

Given $z \in \mathcal{M}$ and $A \subseteq C^\infty(\mathcal{M})$, we let Level$_A(z)$ denote the intersection of level sets of functions from $A$ containing $z$, that is, Level$_A(z) = \{w \in M: \kappa(w) = \kappa(z) \text{ for every } \kappa \in A\}$. Since every level set of every function in $C^\infty(\mathcal{M})$ is closed, it follows that Level$_A(z)$ is closed. Given $A \subseteq C^\infty(\mathcal{M})$, the annihilator of $A$ is the set $A^\perp = \{X \in \mathcal{V}(\mathcal{M}): L_X \gamma = 0 \text{ for every } \gamma \in A\}$, that is, $A^\perp$ is the set of all vector fields in $\mathcal{V}(\mathcal{M})$ along which every function in $A$ has zero Lie derivative. It can be shown that, for every $A \subseteq C^\infty(\mathcal{M})$, $A^\perp = (\overline{A})^\perp$.

A distribution on $\mathcal{M}$ is a mapping $\Delta$ which assigns to every point $z \in \mathcal{M}$ a linear subspace of the tangent space to $\mathcal{M}$ at $z$. Given $z \in \mathcal{M}$, the dimension of the distribution at $z$ is the dimension of the linear subspace $\Delta(z)$ and is denoted by $\dim(\Delta(z))$. Given $B \subseteq \mathcal{V}(\mathcal{M})$, we let $\Delta_B$ denote the distribution spanned by $B$, that is, for every $z \in \mathcal{M}$, $\Delta_B(z) = \text{span}\{X(z): X \in B\}$.

Given $z \in \mathcal{M}$ and $B \subseteq \mathcal{V}(\mathcal{M})$, the first prolongation $B^1(z)$ of $B$ under the set of vector fields $B$ is the set of all $z' \in \mathcal{M}$ such that there exist sequences $\{z_m\}_m = 1$ and $\{z'_m\}_m = 1$ in $\mathcal{M}$ with $z_m \to z$, $z'_m \to z'$ and $z'_m \in B(z_m)$ for all $m$ [6]. More precisely, $B^1(z)$ is the intersection of all sets of the form $\text{cl}(\{B^m(z)\})$, where $\mathcal{U} \subseteq \mathcal{M}$ is an open neighborhood of $z$. Given $\mathcal{V} \subseteq \mathcal{M}$, the first prolongation of the set $\mathcal{V}$ is $B^1(\mathcal{V}) = \bigcup\{B^1(z): z \in \mathcal{V}\}$. Given $z \in \mathcal{M}$, we denote $B^k(z) = B^1(B^{k-1}(z))$ for every $k > 1$.

We denote the Euclidean norm on $\mathbb{R}^3$ by $\| \cdot \|$, and the unit sphere in $\mathbb{R}^3$ by $S^2 \overset{\text{def}}{=} \{x \in \mathbb{R}^3: \|x\| = 1\}$.

3. Observability

Consider the nonlinear control system
\begin{align*}
\dot{x}(t) &= F(x(t)) + G_1(x(t))u_1(t) + \cdots + G_p(x(t))u_p(t), \\
y(t) &= h(x(t)),
\end{align*}
defined on an $n$-dimensional manifold $\mathcal{M}$, where $F, G_1, \ldots, G_p \in \mathcal{V}(\mathcal{M})$ are complete vector fields on $\mathcal{M}$, the input vector $u = [u_1 \cdots u_p]^T$ takes values in a compact subset $\Omega$ of $\mathbb{R}^p$ containing zero in its interior, and the output function $h: \mathcal{M} \to \mathbb{R}^q$ has components in $C^\infty(\mathcal{M})$. Let $\mathcal{D}$ denote the set of vector fields generated by the system (1) by using constant control inputs taking values in $\Omega$, that is, $\mathcal{D} = \{F + G_1 u_1 + \cdots + G_p u_p : [u_1 \cdots u_p]^T \in \Omega\}$. Every state trajectory of the system (1) can be approximated well by state trajectories generated by using piecewise constant control inputs [7, Lem. 1]. Therefore, without loss of generality we restrict, an admissible input $u$ to be a piecewise constant function $u: \mathbb{R} \to \Omega$. Note that the state trajectories of the system (1) generated by piecewise constant control inputs are piecewise smooth concatenations of integral curves of vector fields in $\mathcal{D}$.

Two states $z, z' \in \mathcal{M}$ are indistinguishable for the system (1) if, for every admissible input $u$, the solutions of (1) satisfying the initial conditions $x(0) = z$ and $x(0) = z'$ produce identical output-time histories. Note that indistinguishability is an equivalence relation. For every $z \in \mathcal{M}$, let $\mathcal{I}(z) \subseteq \mathcal{M}$ denote the set of all states that are indistinguishable from $z$.

The system (1) is observable at $z \in \mathcal{M}$ if $\mathcal{I}(z) = \{z\}$, and is observable if $\mathcal{I}(z) = \{z\}$ for every $z \in \mathcal{M}$. The system (1) is locally observable at $z \in \mathcal{M}$ if $z$ is an isolated point of $\mathcal{I}(z)$. The system is locally observable if it is locally observable at every $z \in \mathcal{M}$. Clearly, observability implies local observability. In the following section we introduce the extended observation space and investigate its relationship with sets of indistinguishable states.

4. Extended Observation Space

The extended observation space $\mathcal{O}_e$ associated with the input-output system (1) is the smallest linear subspace of $C^\infty(\mathcal{M})$ that contains the output functions $h_1, \ldots, h_q$ and that is closed under composition with the flows of the vector fields in $\mathcal{D}$. More precisely, $\mathcal{O}_e$ is the smallest linear subspace $\mathcal{C} \subseteq C^\infty(\mathcal{M})$ containing $h_1, \ldots, h_q$ such that $\kappa \circ \phi^X_t \in \mathcal{C}$ for every $\kappa \in \mathcal{C}$, $t \geq 0$ and $X \in \mathcal{D}$. It can be easily shown that
\[
\mathcal{O}_e = \text{span}\left\{h_i \circ \phi^X_{t_r} \circ \cdots \circ \phi^X_{t_1} : r > 0, X_1, \ldots, X_r \in \mathcal{D}, t_1, \ldots, t_r \geq 0 \text{ and } i = 1, \ldots, q\right\}.
\]

Our next result gives the relationship between the set of states indistinguishable from a given state and the functions from the extended observation space $\mathcal{O}_e$. It states that the set of states indistinguishable from a given state is the intersection of level sets of functions from the extended observation space $\mathcal{O}_e$ or equivalently, its closure $\overline{\mathcal{O}_e}$ under pointwise limits, containing the given state.

**Proposition 4.1.** Suppose $z \in \mathcal{M}$. Then $\mathcal{I}(z) = \text{Level}_{\mathcal{O}_e}(z) = \text{Level}_{\overline{\mathcal{O}_e}}(z)$.
The following result gives a sufficient condition for local observability of the system (1). The proof relies on Proposition 4.1.

**Proposition 4.2.** Suppose the dimension of the span of the differentials of functions from $\mathcal{O}_e$ at $z \in M$ is $n$. Then the system (1) is locally observable at $z \in M$.

The following corollary of Proposition 4.2 follows by noting that $\mathcal{O}_e \subseteq \mathcal{O}_e$.

**Corollary 4.1.** Suppose the dimension of the span of the differentials of functions from $\mathcal{O}_e$ at $z \in M$ is $n$. Then the system (1) is locally observable at $z \in M$.

In the foregoing discussions, we explored the relationship between the set of states indistinguishable from a given state and the extended observation space. In practice, it is difficult to compute the set of states indistinguishable from a given state by using the characterization given in Proposition 4.1. Therefore, we focus on the annihilator of the extended observation space to provide an alternative characterization for the set of states indistinguishable from a given state. In this direction, we first present a partial result which states that, given a state $z \in M$, every point in the closure of the orbit of $z$ under the annihilator of the extended observation space is indistinguishable from the given state $z$.

**Proposition 4.3.** For every $z \in M$, $\text{cl}(\mathcal{O}_e^\perp(z)) \subseteq \text{Con}_z(\mathcal{I}(z))$.

Our next result gives sufficient conditions for $\mathcal{O}_e^\perp(z)$ to equal the connected component of $\mathcal{I}(z)$ containing $z \in M$. The first part of the result is a stronger version of Theorem 2.3.2 of [8].

**Proposition 4.4.** Let $z \in M$ Suppose there exist $k > 0$ and an open neighborhood $\mathcal{U} \subseteq M$ of $z$ such that $\dim(\Delta_{\mathcal{O}_e^\perp}(s)) = k$ for every $s \in \mathcal{U}$. Then $\text{Con}_z(\mathcal{I}(z) \cap \mathcal{U}) = \text{Con}_z(\mathcal{O}_e^\perp(z) \cap \mathcal{U})$. In addition, if $\mathcal{U}$ is invariant under every vector field in $\mathcal{O}_e^\perp$ and $\mathcal{O}_e^\perp(z)$ is closed, then $\text{Con}_z(\mathcal{I}(z)) = \mathcal{O}_e^\perp(z)$.

Next, we consider an example to illustrate the relationship between the set of states indistinguishable from a given state and the annihilator of the extended observation space. The example involves a smooth system introduced in [10].

**Example 4.1.** Consider the single-input nonlinear system described by

$$\dot{x}(t) = F(x(t)) + G_1(x(t))u_1(t), \quad y(t) = h(x(t)),$$

defined on the manifold $M = \mathbb{R}^2$ with state $x = [x_1 \ x_2]^T \in M$, the drift vector field given by $F(x) = \eta(x_1)e_2$, the control vector field $G_1(x) = e_1$, the input $u_1 \in \mathbb{R}$ and the output function $h : M \to \mathbb{R}$ given by $h(x) = x_2$, where $\eta \in C^\infty(\mathbb{R})$ is strictly monotonically increasing on $(0, \infty)$, and satisfies $\eta(\gamma) = 0$ for $\gamma \leq 0$ and $\eta(\gamma) > 0$ for $\gamma > 0$ with $e_1 = [1 \ 0]^T$ and $e_2 = [0 \ 1]^T$.

Let $\mathcal{I}(z)$ and $\mathcal{O}_e$ denote the set of states indistinguishable from $z \in M$ and the extended observation space for the system (3), respectively. By choosing $u_1 > 0$, it can be easily shown that, $\mathcal{I}(z) = \{z\}$, for every $z \in M$ and hence the system (3) is observable.

By using (2) it can be easily shown that, $\mathcal{O}_e = \text{span}\left\{h, \eta(x_1 + c_1), \int_0^t \eta(x_1 + c_2 + \tau)d\tau : c_1, c_2 \in \mathbb{R}, \ t \geq 0\right\}$.

A simple computation yields $\text{Level}_{\mathcal{O}_e}(z) = \{z\}$, $\mathcal{O}_e^\perp = \{0\}$ and $\mathcal{O}_e^\perp(z) = \{z\}$ for every $z \in M$. Note that for this example $\mathcal{I}(z) = \text{Level}_{\mathcal{O}_e}(z)$ and $\mathcal{I}(z) = \mathcal{O}_e^\perp(z)$ for every $z \in M$.

It can be easily verified that, given $z = [z_1 \ z_2]^T \in M$ and $c_1 \in \mathbb{R}$ with $c_1 > -z_1$, the differentials of the functions $h$ and $x \mapsto \eta(x_1 + c_1)$, both of which lie in $\mathcal{O}_e$, are linearly independent at $z$. By Corollary 4.1 it follows that the system (3) is locally observable at every $z \in M$ and hence the system (3) is locally observable.

The theory of prolongations has been widely used to study the stability of nonlinear dynamical systems [11]. In the following result, we use prolongations to particularly characterize the set of states indistinguishable from a given state. The following partial result states that, given $z \in M$, prolongations of $z$ under the annihilator of the extended observation space are contained in the set of states indistinguishable from $z$. Note that this result is stronger than Proposition 4.3, since the prolongation of a state under a given set of vector fields contains the orbit of the same set of vector fields through the given state.

**Proposition 4.5.** Suppose $z \in M$ and $w \in \mathcal{I}(z)$. Then $(\mathcal{O}_e^\perp)^k(w) \subseteq \mathcal{I}(z)$ for every $k > 0$. 


The results given in this section establish the relationship between the sets of indistinguishable states and the extended observation space. In the following section we explore the relationship between the extended observation space and the observation space considered in the literature.

5. Observation Space

The observation space $\mathcal{O}$ associated with the input-output system (1) is the smallest linear subspace of $\mathcal{C}^\infty(\mathcal{M})$ that contains the output functions $h_1, \ldots, h_q$ and that is closed under Lie derivatives with respect to the vector fields in $\{F,G_1,\ldots, G_p\}$. It can be shown that [9, Prop. 3.30]

$$\mathcal{O} = \text{span}\{L_{X_1}^k \cdots L_{X_r}^k h_i : r > 0, \ X_1,\ldots, X_r \in \mathcal{D}, \ k_1,\ldots, k_r \geq 0 \text{ and } i = 1,\ldots, q\}.$$ \hspace{1cm} (4)

Our next result relates $\overline{\mathcal{O}}$ and $\overline{\mathcal{O}_e}$.

**Proposition 5.1.** $\overline{\mathcal{O}} \subseteq \overline{\mathcal{O}_e}$.

The following corollary of Proposition 4.2 is the Hermann-Krener sufficient rank condition for the local observability of the system (1) [5, Thm. 3.1] and follows by noting that $\mathcal{O} \subseteq \overline{\mathcal{O}} \subseteq \overline{\mathcal{O}_e}$.

**Corollary 5.1.** Suppose the dimension of the span of the differentials of functions from $\mathcal{O}$ at $z \in \mathcal{M}$ is $n$. Then the system (1) is locally observable at $z \in \mathcal{M}$.

Next, we consider the system introduced in Example 4.1 to investigate the relationship between the set of indistinguishable states from a given state and the observation space.

**Example 5.1.** Let $\mathcal{O}$ denote the observation space associated with the system (3). By using (4) it can be easily shown that $\mathcal{O} = \text{span}\{h, \eta \circ \pi_1\}$, where $\pi_1 : \mathcal{M} \to \mathbb{R}$ is given by $\pi_1(x) = x_1$. Furthermore, $\mathcal{O}^\perp = \{\psi_0 : \psi \in \mathcal{C}^\infty(\mathcal{M}), \ \psi(x) = 0 \text{ for every } x \in \mathcal{M} \text{ such that } x_1 \geq 0\}$.

Consider $z = [z_1 \ z_2]^T \in \mathcal{M}$. It can be easily shown that $\text{Level}_\mathcal{O}(z) = \{z' \in \mathcal{M} : z'_2 = z_2 \text{ and } z'_1 \leq 0\}$ if $z_1 \leq 0$, while $\text{Level}_\mathcal{O}(z) = \{z\}$ if $z_1 > 0$. A simple computation yields, $\mathcal{O}^\perp(z) = \{z' \in \mathcal{M} : z'_2 = z_2 \text{ and } z'_1 < 0\}$ if $z_1 < 0$, while $\mathcal{O}^\perp(z) = \{z\}$ if $z_1 \geq 0$.

It can be easily verified that, the differentials of the functions $h$ and $\eta \circ \pi_1$, both of which lie in $\mathcal{O}$, are linearly independent at every $z \in \mathcal{M}$ with $z_1 > 0$. By Corollary 5.1 it follows that the system (3) is locally observable at every $z = [z_1 \ z_2]^T \in \mathcal{M}$ with $z_1 > 0$. On the other hand, the differentials of the functions $h$ and $\eta \circ \pi_1$ are not linearly independent at every $z = [z_1 \ z_2]^T \in \mathcal{M}$ with $z_1 \leq 0$. Therefore, we cannot conclude about the local observability of the system (3) at $z$ with $z_1 \leq 0$. In Example 4.1 we have seen that, for the system (3), $\mathcal{I}(z) = \{z\}$ for every $z \in \mathcal{M}$. Note that in this example, for every $z \in \mathcal{M}$, $\mathcal{I}(z) \neq \text{Level}_\mathcal{O}(z)$ and $\mathcal{I}(z) \neq \mathcal{O}^\perp(z)$, where as in Example 4.1, $\mathcal{I}(z) = \text{Level}_\mathcal{O}(z) = \mathcal{O}^\perp(z)$. From examples 4.1 and 4.2, it is evident that the sets of indistinguishable states are related to the extended observation space, which is not true with the observation space. We conclude that, in general, the observation space is not related to the set of states indistinguishable from a given state in the same way that the extended observation space is.

Note that the computation of elements of the observation space only involves computing Lie derivatives of the output functions, while computation of elements in the extended observation space involves the relatively more difficult computation of flows of vector fields in $\mathcal{D}$. However, the observation space is not related to the set of indistinguishable states in general. Our next result, however, shows that in the case of an analytic system the closures $\overline{\mathcal{O}}$ and $\overline{\mathcal{O}_e}$ as well as the annihilators $\mathcal{O}^\perp$ and $\mathcal{O}_e^\perp$ are the same. Therefore, in the case of an analytic system it is sufficient to compute the observation space and its annihilator to characterize the set of states indistinguishable from a given state.

**Proposition 5.2.** If $\mathcal{M}$ is a real-analytic manifold, the vector fields $F, G_1, \ldots, G_p$ are analytic vector fields and the output functions $h_i \in \mathcal{C}^\omega(\mathcal{M})$ for every $i = 1,\ldots, q$, then $\overline{\mathcal{O}} = \overline{\mathcal{O}_e}$ and $\mathcal{O}^\perp = \mathcal{O}_e^\perp$.

6. Observability of Reduced Attitude Dynamics of an Inertially Symmetric Rigid Body Without Actuators

In this section we characterize the set of states indistinguishable from a given state of the reduced attitude dynamics of an inertially symmetric rigid body without any actuators. Let $b \in \mathbb{R}^3$ denote the inertial components
of an inertially fixed vector $b$. The instantaneous body components $p(t) \in \mathbb{R}^3$ of $b$ satisfy
\[ \dot{p}(t) = p(t) \times \omega(t), \tag{5} \]
where $\omega(t) \in \mathbb{R}^3$ is the instantaneous vector of body components of the angular velocity of the spacecraft body frame relative to the inertial frame. Note that equation (5) implies that $\|p(\cdot)\|$ is constant. For an inertially symmetric rigid body without any actuators, the rotational dynamics equation is given by
\[ \dot{\omega}(t) = 0. \tag{6} \]

Typical attitude sensors measure a body component of a vector whose inertial components are known. Assuming that the rigid body carries a sensor that measures one of the body components of the vector $b$, the output equation is given by
\[ y(t) = e_1^T p(t), \tag{7} \]
where $e_1 \in \mathbb{R}^3$ is the unit vector along the axis of the sensor.

Equations (5)-(7) define a control system of the form (1) on the manifold $\mathcal{N} \overset{\text{def}}{=} S^2 \times \mathbb{R}^3$ with state $x = (p, \omega) \in \mathcal{N}$, drift vector field $F(x) = (p \times \omega, 0)$ and output function $h(x) = e_1^T p$. Note that the control vector field $F$ and the output function $h$ are analytic. Hence by Proposition 5.2 it is sufficient to compute the observation space in order to study the observability of the reduced attitude dynamics (5)-(7).

Let $\mathcal{O}_0$ denote the observation space associated with the system (5)-(7). Our next result characterizes $\mathcal{O}_0$ and $\mathcal{O}_0^\perp$ in terms of a few low order Lie derivatives of the output function.

**Proposition 6.1.** For the system (5)-(7), $\mathcal{O}_0 = \text{span}\{h, L_F h, L_{F1}^\perp h, \mu^j L_{Fj}^\perp h, k = 1, 2 \text{ and } j = 1, 2, \ldots\}$ and $\mathcal{O}_0^\perp = \{X \in \mathcal{V}(\mathcal{N}): L_X \kappa = 0, \kappa \in \{h, \nu, \mu, L_F h, L_{F1}^\perp h, \} \}$, where $\nu: \mathcal{N} \to \mathbb{R}$ and $\mu: \mathcal{N} \to \mathbb{R}$ are given by $\nu(x) = \|p\|^2$ and $\mu(x) = \|\omega\|^2$, respectively, for every $x = (p, \omega) \in \mathcal{N}$.

Consider $x = (p, \omega) \in \mathcal{N}$. Define $\zeta \in \mathcal{V}(\mathcal{N})$ by
\[ \zeta(x) \overset{\text{def}}{=} \begin{bmatrix} e_1 \times p \\ e_1 \times \omega \end{bmatrix} \tag{8} \]
and let $\mathcal{G}$ denote the set of nonsingular points of the vector field $\zeta$, that is, $\mathcal{G} = \{x \in \mathcal{N}: \zeta(x) \neq 0\}$. It is easy to check that $\mathcal{N} \setminus \mathcal{G}$ is closed and hence $\mathcal{G}$ is open. Further, $\mathcal{G}$ is dense in $\mathcal{N}$. The following result states that the subspace $\Delta_{\mathcal{O}_0}(x)$ is spanned by the vector $\zeta(x)$ for every $x \in \mathcal{N}$.

**Proposition 6.2.** For every $x \in \mathcal{N}$, $\Delta_{\mathcal{O}_0}(x) = \text{span}\{\zeta(x)\}$.

The following result states that on the set $\mathcal{G}$, $\mathcal{O}_0^\perp$ is the module generated by the vector field $\zeta$.

**Proposition 6.3.** For every $X \in \mathcal{O}_0^\perp$, there exists a smooth function $\beta \in C^\infty(\mathcal{G})$ such that $X(x) = \beta(x) \zeta(x)$ for every $x \in \mathcal{G}$.

Let $\mathcal{I}_0(z)$ denote the set of states indistinguishable from the given state $z \in \mathcal{N}$ for the system (5)-(7). Our next result makes use of Proposition 4.4 to show that the set of states indistinguishable from a nonsingular point $z$ of the vector field $\zeta$ is precisely the orbit of $\mathcal{O}_0^\perp$ through $z$. In other words, two states $z, z' \in \mathcal{G}$ are indistinguishable for the system (5)-(7) if and only if they are related by a rotation about the sensor axis $e_1$.

**Proposition 6.4.** Consider $z = (p, \omega) \in \mathcal{N}$. Then
\[ \mathcal{I}_0(z) = \begin{cases} \{(e^{\theta (e_1 \times )} p, e^{\theta (e_1 \times )} \omega) : \theta \in \mathbb{R}\} & \text{if } p \times \omega \neq 0 \\ \{(e^{\theta (e_1 \times )} p, \alpha e^{\theta (e_1 \times )} \omega) : \theta, \alpha \in \mathbb{R}\} & \text{if } p \times \omega = 0. \end{cases} \]

From Proposition 6.4 it is clear that the system (5)-(7) is not observable.

7. **Conclusion**

While existing sufficient conditions for observability are stated in terms of the observation space, examples show that in general, the observation space does not contain sufficient information to determine sets of indistinguishable states. We have introduced the extended observation space and shown that sets of indistinguishable states are the intersections of level sets of functions from the extended observation space. Since the extended observation space
essentially contains the observation space, the Hermann-Krener sufficient condition follows as a corollary. Under additional conditions, connected components of sets of indistinguishable states are orbits of the annihilator of the extended observation space. In the analytic case, the extended observation space and the observation space are essentially the same. To illustrate the results, we have applied them to the reduced attitude dynamics of an unactuated rigid body carrying a single attitude sensor, and shown that, for a dense set of initial conditions, the set of states indistinguishable from a given initial condition are obtained by rotating the given initial condition about the sensor axis.

REFERENCES