Asymptotic Pseudo State Observers

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1 Introduction

Already in 1964 Luenberger constructed an asymptotic state estimator or observer for observable Kalman state systems. We consider the same problem for (more general) Rosenbrock systems or polynomial matrix models. The results include necessary and sufficient conditions for the existence of an input/output behavior with proper transfer matrix, called an asymptotic pseudo state observer, whose output is asymptotically equal to the Rosenbrock system's pseudo state if the input of the observer is formed by the input and output of the Rosenbrock system. Moreover, an algorithm for the construction of one and then indeed many stable observers is given.

Vidyasagar [3, p. 149] already studied pseudo state observers in context with two-parameter compensators where, however, the systems are described by their transfer matrices and not as I/O behaviors and where the autonomous parts of the behaviors are not investigated in detail.

We also discuss the existence and construction of asymptotic input observers and asymptotic output controllers. A predecessor of these results is the work of Wolovich [4, pp. 161 - 177]. We present an algorithm which checks whether a given input/output behavior, the plant, admits an asymptotic input observer, and, if so, constructs all possible observers. Such an observer is another input/output system which uses the plant's output as input to produce an approximation of the plant's input as output. Analogous results are obtained for the output controllers. For the observer and controller input/output behaviors, properties such as properness and stability are taken into account as was already the case in Wolovich's book.

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2 Technical Preparations

Definition 1 (I/O behavior). *Let $\mathcal{F}$ be the signal space of all real-valued $\mathcal{C}^\infty$-functions on $\mathbb{R}$. Alternatively, $\mathcal{F}$ may also be the set of distributions on $\mathbb{R}$.*
The set of all solutions $\binom{y(t)}{u(t)} \in \mathcal{F}^{p+m}$ of a system of differential equations of the form

$$P\left(\tfrac{d}{dt}\right) y(t) = Q\left(\tfrac{d}{dt}\right) u(t)$$

where $P \in \mathbb{R}[s]^{p\times p}$, $\det(P) \neq 0$, and $Q \in \mathbb{R}[s]^{p\times m}$ is called input/output behavior*, or, shorter,* I/O behavior *or* I/O system*. The vector $u(t)$ can be chosen arbitrarily and is therefore called the system's* input*, the vector $y(t)$ is referred to as* output*. The matrix*

$$H := P^{-1}Q \in \mathbb{R}(s)^{p\times m}$$

is called the behavior's transfer matrix*.*

Remark 2. For any I/O behavior

$$\mathcal{B} = \left\{ \begin{pmatrix} y(t) \\ u(t) \end{pmatrix} \in \mathcal{F}^{p+m};\ P\left(\tfrac{d}{dt}\right) y(t) = Q\left(\tfrac{d}{dt}\right) u(t) \right\}$$

it is possible to construct matrices $A_1 \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times m}$, $C_1 \in \mathbb{R}^{p\times n}$ and $D_1 \in \mathbb{R}[s]^{p\times m}$ such that the equations

$$\begin{aligned} x'(t) &= A_1 x(t) + B_1 u(t), \\ y(t) &= C_1 x(t) + D_1\left(\tfrac{d}{dt}\right) u(t) \end{aligned}$$

represent a *Kalman realization* of the behavior, i.e., there is an isomorphism

$$\left\{ \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \in \mathcal{F}^{n+m};\ x'(t) = A_1 x(t) + B_1 u(t) \right\} \cong \mathcal{B},$$
$$\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \mapsto \begin{pmatrix} C_1 x(t) + D_1\left(\frac{d}{dt}\right) u(t) \\ u(t) \end{pmatrix} =: \begin{pmatrix} y(t) \\ u(t) \end{pmatrix}.$$

If the transfer matrix $H = P^{-1}Q$ of the I/O behavior is *proper* (i.e., for all the entries of H the degree of the numerator is smaller than or equal to the degree of the denominator), then the matrix D_1 in the Kalman realization is also a constant matrix. In this case the system can be built without use of differentiators. This property is very desirable in practice. We will call behaviors with proper transfer matrix *proper* behaviors.

Remark 3. For any matrix $H \in \mathbb{R}(s)^{p\times m}$ there exists a unique controllable I/O system

$$\mathcal{B} = \left\{ \begin{pmatrix} y(t) \\ u(t) \end{pmatrix} \in \mathcal{F}^{p+m};\ P\left(\tfrac{d}{dt}\right) y(t) = Q\left(\tfrac{d}{dt}\right) u(t) \right\}$$

with transfer matrix H, called the *controllable realization* of H.
The matrix $(P, -Q)$ describing the system's equations can be computed as universal left annihilator of $d \cdot \binom{H}{\mathrm{id}_m}$ where d is a common divisor of all of the denominators

of entries of H.
A universal left annihilator of a matrix $M \in \mathbb{R}[s]^{k\times l}$ with Smith form $S = UMV$ are the last $k - \mathrm{rank}(M)$ rows of U.

Definition 4. *An I/O behavior*

$$\mathcal{B} = \left\{ \begin{pmatrix} y(t) \\ u(t) \end{pmatrix} \in \mathcal{F}^{p+m};\ P\left(\tfrac{d}{dt}\right) y(t) = Q\left(\tfrac{d}{dt}\right) u(t) \right\}$$

is called stable if its autonomous part

$$\mathcal{B}_0 := \left\{ y(t) \in \mathcal{F}^{p};\ P\left(\tfrac{d}{dt}\right) y(t) = 0 \right\}$$

is asymptotically stable, i.e., if all possible outputs to the input zero tend to zero for $t \to \infty$. In consequence, all possible outputs to the same input are asymptotically equal if the system is stable.

Definition 5. *We define the set*

$$T := \{t \in \mathbb{R}[s];\ \textit{ all zeroes of } t \textit{ have negative real part}\}$$

and the quotient ring $\mathbb{R}[s]_T$ of stable rational functions, i.e.,

$$\mathbb{R}[s]_T = \left\{ \tfrac{f(s)}{g(s)} \in \mathbb{R}(s);\ g(s) \in T \right\}.$$

Let $\mathcal{S}$ denote the ring of all proper stable rational functions:

$$\mathcal{S} := \left\{ \tfrac{f(s)}{g(s)} \in \mathbb{R}(s);\ \deg(f(s)) \leq \deg(g(s)) \textit{ and } \textit{all zeroes of } g(s) \textit{ have negative real part} \right\}.$$

3 Asymptotic Pseudo State Observers

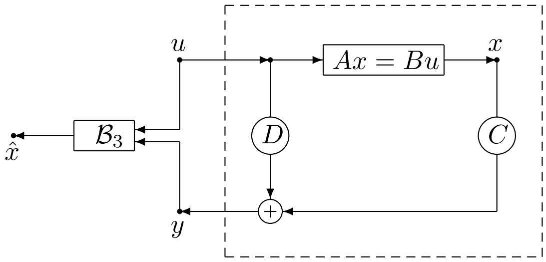
Description of the problem

Definition 6. *A* Rosenbrock system *or* polynomial matrix model *is a generalization of a Kalman system, given by equations of the form*

$$\begin{aligned} A\left(\tfrac{d}{dt}\right) x(t) &= B\left(\tfrac{d}{dt}\right) u(t), \\ y(t) &= C\left(\tfrac{d}{dt}\right) x(t) + D\left(\tfrac{d}{dt}\right) u(t) \end{aligned}$$

where $A \in \mathbb{R}[s]^{n\times n}$, $B \in \mathbb{R}[s]^{n\times m}$, $C \in \mathbb{R}[s]^{p\times n}$ and $D \in \mathbb{R}[s]^{p\times m}$ are polynomial matrices and $\det(A) \neq 0$. *We will always assume that the transfer matrices $H_1 := A^{-1}B$ and $H_2 := CA^{-1}B + D$ are proper.*
The system's input $u(t)$, *its* output $y(t)$ *and the* pseudo state $x(t)$ *are again vectors in the module $\mathcal{F}$ of $\mathcal{C}^\infty$-functions (or, more generally, distributions) on $\mathbb{R}$.*

Input and output of a Rosenbrock system are normally known, or they can be measured. That is, in general, not true for the pseudo state vector. It may however be important to know the pseudo state vector or at least an approximation to it, since this knowledge is, for example, necessary for stabilizing the system via feedback. Consequently, it is an interesting question if it is possible to reconstruct a Rosenbrock system's pseudo state from its input and output. More detailed, we are looking for a new I/O behavior $\mathcal{B}_3$ whose output is asymptotically equal to the desired pseudo state vector if input and output of the Rosenbrock system are used as input.
The given Rosenbrock system and the desired "pseudo state observer system" are shown in the following picture.



Definition 7. *Let a Rosenbrock system be given. An I/O behavior*

$$\mathcal{B}_3 = \left\{ \begin{pmatrix} \hat{x}(t) \\ y(t) \\ u(t) \end{pmatrix} \in \mathcal{F}^{n+p+m}; P_3\left(\tfrac{d}{dt}\right)\hat{x}(t) = (Q_y, Q_u)\left(\tfrac{d}{dt}\right)\begin{pmatrix} y(t) \\ u(t) \end{pmatrix} \right\}$$

with input $\begin{pmatrix} y(t) \\ u(t) \end{pmatrix}$ *and output* $\hat{x}(t)$ *is called* asymptotic observer of the pseudo state $x(t)$ *if the following condition is fulfilled:*
If $u(t)$ *denotes the input,* $y(t)$ *the output and* $x(t)$ *the pseudo state of the original system and* $\hat{x}(t)$ *is an output of* $\mathcal{B}_3$ *to* $\begin{pmatrix} y(t) \\ u(t) \end{pmatrix}$, *then*

$$\lim_{t\to\infty} (\hat{x}(t) - x(t)) = 0$$

i.e., $\hat{x}(t)$ *and the pseudo state* $x(t)$ *are asymptotically equal.*

Results

We consider the Rosenbrock system

$$A\left(\tfrac{d}{dt}\right)x(t) = B\left(\tfrac{d}{dt}\right)u(t),$$
$$y(t) = C\left(\tfrac{d}{dt}\right)x(t) + D\left(\tfrac{d}{dt}\right)u(t)$$

where we assume that the transfer matrices $H_1 = A^{-1}B$ and $H_2 = CA^{-1}B + D$ are proper.

Theorem 8. *If there exist matrices $X \in \mathbb{R}[s]_T^{n\times n}$ and $Y \in \mathcal{S}^{n\times p}$ such that*

$$(X, Y)\begin{pmatrix} A \\ C \end{pmatrix} = \mathrm{id}_n,$$

then there is a proper stable asymptotic observer of the pseudo state of the Rosenbrock system. In this case one such system can be constructed in the following way: Let (P_3, Q_3) be the controllable realization of (X, Y). Then the behavior

$$\mathcal{B}_3 := \left\{ \begin{pmatrix} \hat{x}(t) \\ y(t) \\ u(t) \end{pmatrix} \in \mathcal{F}^{n+p+m};\ P_3\left(\tfrac{d}{dt}\right)\hat{x}(t) = \right.$$
$$\left. = \left(P_3 Y, P_3\left(XB - YD\right)\right)\left(\tfrac{d}{dt}\right)\begin{pmatrix} y(t) \\ u(t) \end{pmatrix} \right\}$$

with transfer matrix $H_3 := (H_y, H_u) := (Y, XB - YD)$ is a valid proper stable asymptotic pseudo state observer.

Remark 9.

1. The assumption of the preceding theorem is slightly stronger than detectability (compare [1, Ch. 5.3.2]) of the Rosenbrock system, i.e., existence of a left inverse matrix of $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ in $\mathbb{R}[s]_T$.

2. A method for computing the controllable realization of a matrix was described in Remark 3.

3. Observe that the assumption of the preceding theorem is in particular satisfied if $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ has a left inverse in $\mathcal{S}$. In fact, the following lemma shows that these two conditions are equivalent for many common examples. The next theorem will show how to test whether a matrix has a left inverse in $\mathcal{S}$ in practice.

Lemma 10. *Assume that the studied Rosenbrock system is internally proper according to Vardulakis [2, Ch. 4.5], i.e., not only the transfer matrices H_1 and H_2, but also the matrices A^{-1} and CA^{-1} are proper. Internal properness is for instance fulfilled for Kalman systems.*
Then $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ has a left inverse (X, Y) with $X \in \mathbb{R}[s]_T^{n\times n}$ and $Y \in \mathcal{S}^{n\times p}$ (as required in the last theorem) if and only if it has a left inverse $(X, Y) \in \mathcal{S}^{n\times(n+p)}$.

Theorem 11. *Let*

$$\begin{pmatrix} E \\ 0 \end{pmatrix} = U\begin{pmatrix} A \\ C \end{pmatrix}V \quad \textit{where} \quad E := \begin{pmatrix} e_1 & & 0 \\ & \ddots & \\ 0 & & e_n \end{pmatrix}$$

be the Smith form of $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ with respect to $\mathcal{S}$, i.e., $U \in \mathrm{Gl}_{n+p}(\mathcal{S})$, $V \in \mathrm{Gl}_n(\mathcal{S})$, and $e_1|\cdots|e_n$ in $\mathcal{S}$. Then the following statements are equivalent:

1. $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ *has a left inverse* $(X, Y) \in \mathcal{S}^{n \times (n+p)}$,
2. $E^{-1} \in \mathcal{S}^{n \times n}$, *and*
3. $e_n^{-1} \in \mathcal{S}$.

In this case, $V(E^{-1}, 0)U \in \mathcal{S}^{n \times (n+p)}$ *is one such left inverse, and the set of all left inverse matrices of* $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ *in* $\mathcal{S}$ *is given by the affine submodule*

$$V(E^{-1}, 0)U + \mathcal{S}^{n \times p} U^2$$

where U^2 *is a universal left annihilator of* $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ *in* $\mathcal{S}$ *(conf. Remark 3).*

Remark 12. Note that the ring $\mathcal{S}$ can be interpreted as quotient ring $\mathbb{R}[\sigma]_{T_1}$ where

$$\sigma := (s+1)^{-1} \quad \text{and} \quad T_1 := \left\{ \tfrac{t}{(s+1)^{\deg(t)}};\, t \in T \right\}.$$

Consequently, the Smith form of a matrix $M \in \mathbb{R}[s]^{k \times l}$ with respect to $\mathcal{S}$ can be computed in the following way: Consider M as a matrix in $\mathbb{R}(\sigma)$ via

$$\mathbb{R}[s] = \mathbb{R}[s+1] \subseteq \mathbb{R}(s+1) = \mathbb{R}((s+1)^{-1}) = \mathbb{R}(\sigma)$$

and compute the Smith form (with respect to $\mathbb{R}[\sigma]$). This is then also the Smith form with respect to $\mathcal{S} = \mathbb{R}[\sigma]_{T_1}$.

In Theorem 8, sufficient conditions for the existence of a proper asymptotic pseudo state observer behavior have been given. The following theorem will show that these conditions are in fact also necessary.

Theorem 13. *If there is a proper asymptotic observer*

$$\mathcal{B}_3 = \left\{ \begin{pmatrix} \hat{x}(t) \\ y(t) \\ u(t) \end{pmatrix} \in \mathcal{F}^{n+p+m};\, P_3\left(\tfrac{d}{dt}\right) \hat{x}(t) = (Q_y, Q_u)\left(\tfrac{d}{dt}\right) \begin{pmatrix} y(t) \\ u(t) \end{pmatrix} \right\}$$

of the pseudo state of a Rosenbrock system, then there exist matrices $X \in \mathbb{R}[s]_T^{n \times n}$, $Y \in \mathcal{S}^{n \times p}$ *such that*

$$\begin{aligned} (X, Y)\begin{pmatrix} A \\ C \end{pmatrix} &= \mathrm{id}_n, \\ Y &= H_y, \quad \text{and} \\ XB - YD &= H_u, \end{aligned}$$

where $H_3 := (H_y, H_u) := P_3^{-1}(Q_y, Q_u)$ *is the transfer matrix of* $\mathcal{B}_3$. *Moreover,* $\mathcal{B}_3$ *is stable.*

Corollary 14. *There is a proper asymptotic observer of the pseudo state of the Rosenbrock system*

$$\begin{aligned} A\left(\tfrac{d}{dt}\right) x(t) &= B\left(\tfrac{d}{dt}\right) u(t), \\ y(t) &= C\left(\tfrac{d}{dt}\right) x(t) + D\left(\tfrac{d}{dt}\right) u(t), \end{aligned}$$

(where the transfer matrices $H_1 := A^{-1}B$ and $H_2 := CA^{-1}B + D$ are assumed to be proper) if and only if there exist matrices $X \in \mathbb{R}[s]_T^{n\times n}$ and $Y \in \mathcal{S}^{n\times p}$ such that

$$XA + YC = \mathrm{id}_n\,.$$

The observer system is then automatically stable.

If we skip both the assumption that the given Rosenbrock system has proper transfer matrices and the requirement that the constructed observer system shall be proper, then the previous result has the following form:

Corollary 15. *There is an asymptotic observer of the pseudo state of the Rosenbrock system*

$$\begin{aligned} A\left(\tfrac{d}{dt}\right)x(t) &= B\left(\tfrac{d}{dt}\right)u(t),\\ y(t) &= C\left(\tfrac{d}{dt}\right)x(t) + D\left(\tfrac{d}{dt}\right)u(t), \end{aligned}$$

(where the transfer matrices H_1 and H_2 are not necessarily proper) if and only if there exists a matrix $(X, Y) \in \mathbb{R}[s]_T^{n\times(n+p)}$ such that

$$XA + YC = \mathrm{id}_n,$$

i.e., the Rosenbrock system is detectable.
The observer system is then automatically stable.

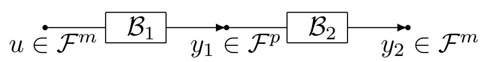
4 Other Asymptotic Observers and Controllers

Asymptotic Input Observers

We consider an input/output system

$$\mathcal{B}_1 := \left\{ \begin{pmatrix} y_1(t) \\ u(t) \end{pmatrix} \in \mathcal{F}^{p+m};\ P_1\left(\tfrac{d}{dt}\right)y_1(t) = Q_1\left(\tfrac{d}{dt}\right)u(t) \right\}$$

where $P_1 \in \mathbb{R}[s]^{p\times p}$, $\det(P_1) \neq 0$, and $Q_1 \in \mathbb{R}[s]^{p\times m}$. We don't assume that the transfer matrix $H_1 := P_1^{-1}Q_1 \in \mathbb{R}(s)^{p\times m}$ is proper. The goal in this section is to reconstruct – or at least approximate – the input u from the output y_1 if that is possible. More precisely: We are looking for an "*asymptotic input observer*", i.e., a second I/O system such that the output $y_2(t)$ of $\mathcal{B}_2$ is asymptotically equal to the input $u(t)$ of $\mathcal{B}_1$ if the output $y_1(t)$ of $\mathcal{B}_1$ is taken as input of $\mathcal{B}_2$. The relationship between $\mathcal{B}_1$ and $\mathcal{B}_2$ is visualized in the following picture:



Definition 16. *Let an I/O system $\mathcal{B}_1$ with input $u(t)$ and output $y_1(t)$ be given. A second I/O system*

$$\mathcal{B}_2 := \left\{ \begin{pmatrix} y_2(t) \\ y_1(t) \end{pmatrix} \in \mathcal{F}^{m+p};\ P_2\left(\tfrac{d}{dt}\right)y_2(t) = Q_2\left(\tfrac{d}{dt}\right)y_1(t) \right\}$$

with input $y_1(t)$ *and output* $y_2(t)$ *is called an* asymptotic input observer *of* $\mathcal{B}_1$ *if*

$$\lim_{t\to\infty}(y_2(t)-u(t))=0$$

whenever $y_1(t)$ *is an output of the original system* $\mathcal{B}_1$ *to the input* $u(t)$ *and* $y_2(t)$ *is an output of the observer system* B_2 *to the input* $y_1(t)$.

Theorem 17. *Let*

$$\mathcal{B}_1 := \left\{ \begin{pmatrix} y_1(t) \\ u(t) \end{pmatrix} \in \mathcal{F}^{p+m};\ P_1\left(\tfrac{d}{dt}\right) y_1(t) = Q_1\left(\tfrac{d}{dt}\right) u(t) \right\}$$

be an I/O system with transfer matrix $H_1 = P_1^{-1}Q_1$. *Then the following two statements are equivalent:*

1. *The system* $\mathcal{B}_1$ *has a proper asymptotic input observer* $\mathcal{B}_2$.
2. *There exists a matrix* $H_2 \in \mathcal{S}^{m\times p}$ *such that*
   (a) $H_2H_1 = \mathrm{id}_m$ *and*
   (b) $H_2P_1^{-1} \in \mathbb{R}[s]_T^{m\times p}$.

If these conditions are fulfilled, then $\mathcal{B}_2$ *is automatically stable.*
If condition (2) is fulfilled and $H_2 \in \mathcal{S}^{m\times p}$ *is such a left inverse of* H_1, *then the controllable realization* $\mathcal{B}_2$ *of* H_2 *is a proper asymptotic input observer system.*

Example 18. We consider the case $p = m = 1$, i.e., P_1 and Q_1 are polynomials, $P_1 \neq 0$, and $H_1 = \frac{Q_1}{P_1} = \frac{Q_{1\mathrm{cont}}}{P_{1\mathrm{cont}}}$ (where $\gcd(P_{1\mathrm{cont}}, Q_{1\mathrm{cont}}) = 1$) is a rational function. If H_2 shall satisfy condition (2) of the previous theorem, (2a) yields $H_2 = \frac{P_1}{Q_1} = \frac{P_{1\mathrm{cont}}}{Q_{1\mathrm{cont}}}$. $H_2 \in \mathcal{S}$ implies

$$\deg(P_1) \le \deg(Q_1) \quad \text{and} \quad Q_{1\mathrm{cont}} \in T.$$

Condition (2b) means that $H_2P_1^{-1} = \frac{1}{Q_1}$ shall be contained in $\mathbb{R}[s]_T$, i.e.,

$$Q_1 \in T.$$

Note that in this case $H_2P_1^{-1}$ is even contained in $\mathcal{S}$.
So we get the result that $\mathcal{B}_1$ admits a proper asymptotic input observer system if and only if

$$\deg(P_1) \le \deg(Q_1) \quad \text{and} \quad Q_1 \in T$$

($Q_{1\mathrm{cont}} = \frac{Q_1}{\gcd(P_1,Q_1)}$ is then automatically also contained in T).
In this case the controllable realization

$$\mathcal{B}_2 := \left\{ \begin{pmatrix} y_2(t) \\ y_1(t) \end{pmatrix} \in \mathcal{F}^{1+1};\ Q_{1\mathrm{cont}}\left(\tfrac{d}{dt}\right) y_2(t) = P_{1\mathrm{cont}}\left(\tfrac{d}{dt}\right) y_1(t) \right\}$$

of H_2 is one such observer.

Remark 19.

1. The system of the serial connection of first $\mathcal{B}_1$ and then $\mathcal{B}_2$ is stable if and only if both of the systems $\mathcal{B}_1$ and $\mathcal{B}_2$ are stable.

2. It is not so easy to check in practice if condition (2) in Theorem 17 is satisfied. However, if the original system $\mathcal{B}_1$ is stable, then any matrix $H_2 \in \mathcal{S}^{m\times p}$ fulfilling $H_2H_1 = \mathrm{id}_m$ does automatically satisfy condition (2b). Hence, we only have to check the existence of a left inverse matrix of H_1 in $\mathcal{S}$ in this case, and that can be achieved using Theorem 11. This theorem yields in addition all possible left inverse matrices in $\mathcal{S}$, and for each of these left inverses, the controllable realization (conf. Remark 3) is a proper stable asymptotic input observer system.

3. If we don't want to assume that $\mathcal{B}_1$ is stable, we may also use the following considerations: Condition (2) is in particular fulfilled if there is a matrix $H_2 \in \mathcal{S}^{m\times p}$ such that $H_2H_1 = \mathrm{id}_m$ and $H_2P_1^{-1} \in \mathcal{S}^{m\times p}$. The next theorem provides an algorithm for checking whether this condition is fulfilled.

Theorem 20. *Consider an I/O system*

$$\mathcal{B}_1 := \left\{ \begin{pmatrix} y_1(t) \\ u(t) \end{pmatrix} \in \mathcal{F}^{p+m};\ P_1\left(\tfrac{d}{dt}\right) y_1(t) = Q_1\left(\tfrac{d}{dt}\right) u(t) \right\}$$

with transfer matrix $H_1 = P_1^{-1}Q_1$ *and* $p \geq m$. *Let*

$$\begin{pmatrix} E \\ 0 \end{pmatrix} = XH_1Y \quad \textit{where} \quad E := \begin{pmatrix} e_1 & & 0 \\ & \ddots & \\ 0 & & e_m \end{pmatrix}$$

be the Smith form of H_1 *with respect to* $\mathcal{S}$, *and let*

$$\begin{pmatrix} F \\ 0 \end{pmatrix} = U \begin{pmatrix} P_1X^{-1} \\ 0, \mathrm{id}_{p-m} \end{pmatrix} V \quad \textit{where} \quad F := \begin{pmatrix} f_1 & & 0 \\ & \ddots & \\ 0 & & f_p \end{pmatrix}$$

be the Smith form of $\begin{pmatrix} P_1X^{-1} \\ 0, \mathrm{id}_{p-m} \end{pmatrix}$ *with respect to* $\mathcal{S}$.
Then the following statements are equivalent:

1. *There is a matrix* $H_2 \in \mathcal{S}^{m\times p}$ *such that*

   (a) $H_2H_1 = \mathrm{id}_m$ *and*

   (b) $H_2P_1^{-1} \in \mathcal{S}^{m\times p}$.

2. (a) $\mathrm{rank}(H_1) = m$ *and* $e_m^{-1} \in \mathcal{S}$ *and*

   (b) $\frac{V_{ij}}{e_i f_j} \in \mathcal{S}$ *for* $1 \leq i \leq m$ *and* $1 \leq j \leq p$.

If these conditions are fulfilled and $R \in \mathcal{S}^{m\times p}$ *is defined by* $R_{ij} := \frac{V_{ij}}{e_i f_j}$ *, then the set of all matrices* H_2 *satisfying condition (1) is*

$$Y(E^{-1},0)X - YRU^{12}X^2 + \mathcal{S}^{m\times(p-m)}U^{22}X^2$$

where

$$U =: \begin{pmatrix} U^{11} & U^{12} \\ U^{21} & U^{22} \end{pmatrix} \in \mathcal{S}^{(p+(p-m))\times(p+(p-m))} \quad \text{and} \quad X =: \begin{pmatrix} X^1 \\ X^2 \end{pmatrix} \in \mathcal{S}^{(m+(p-m))\times p}.$$

If we skip the requirement that the constructed input observers system shall be proper, we get the following result:

Theorem 21. *Let*

$$\mathcal{B}_1 := \left\{ \begin{pmatrix} y_1(t) \\ u(t) \end{pmatrix} \in \mathcal{F}^{p+m};\ P_1\left(\tfrac{d}{dt}\right) y_1(t) = Q_1\left(\tfrac{d}{dt}\right) u(t) \right\}$$

be an I/O system with transfer matrix $H_1 = P_1^{-1}Q_1$*. Then the following three statements are equivalent:*

1. *The system* $\mathcal{B}_1$ *has an asymptotic input observer* $\mathcal{B}_2$*.*
2. *There exists a* $H_2 \in \mathbb{R}[s]_T^{m\times p}$ *such that*
   (a) $H_2H_1 = \mathrm{id}_m$ *and*
   (b) $H_2P_1^{-1} \in \mathbb{R}[s]_T^{m\times p}$.
3. *There exists a* $Z \in \mathbb{R}[s]_T^{m\times p}$ *such that* $ZQ_1 = \mathrm{id}_m$*.*

If these conditions are fulfilled, then $\mathcal{B}_2$ *is automatically stable.*
If condition (2) is fulfilled and $H_2 \in \mathbb{R}[s]_T^{m\times p}$ *is such a left inverse of* H_1*, then the controllable realization* $\mathcal{B}_2$ *of* H_2 *is an asymptotic input observer system.*

Remark 22. Note that the existence of $Z \in \mathbb{R}[s]_T^{m\times p}$ with $ZQ_1 = \mathrm{id}_m$ as required in condition (3) can be checked using the same method as described in Theorem 11, using the ring $\mathbb{R}[s]_T$ instead of $\mathcal{S}$ in this case. Theorem 11 does also indicate how to find the set of all Z satisfying condition (3). For any of these matrices Z, the matrix $H_2 := ZP_1$ fulfills condition (2) and hence gives rise to an asymptotic input observer.

Asymptotic Output Controllers

The problem we are going to study in this section will turn out to be very similar to the one of an asymptotic input observer.

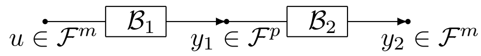
This time, we start with an input/output behavior

$$\mathcal{B}_2 = \left\{ \begin{pmatrix} y_2(t) \\ y_1(t) \end{pmatrix} \in \mathcal{F}^{m+p};\ P_2\left(\tfrac{d}{dt}\right) y_2(t) = Q_2\left(\tfrac{d}{dt}\right) y_1(t) \right\}$$

where $P_2 \in \mathbb{R}[s]^{m\times m}$, $\det(P_2) \neq 0$, and $Q_2 \in \mathbb{R}[s]^{m\times p}$. The goal is to construct – if possible – an I/O system

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} y_1(t) \\ u(t) \end{pmatrix} \in \mathcal{F}^{p+m};\ P_1\left(\tfrac{d}{dt}\right) y_1(t) = Q_1\left(\tfrac{d}{dt}\right) u(t) \right\}$$

such that for the serial connection of first $\mathcal{B}_1$ and then $\mathcal{B}_2$ the output $y_2(t)$ of $\mathcal{B}_2$ is asymptotically equal to any input $u(t)$ of the behavior $\mathcal{B}_1$. This means that, if such a system $\mathcal{B}_1$ does exist, it is possible to control the output of the given behavior $\mathcal{B}_2$. Therefore, $\mathcal{B}_1$ is called an "asymptotic output controller" of $\mathcal{B}_2$. Note that the situation is, as the following picture shows, indeed very similar to the situation described in the section on asymptotic input observers. But this time, the behavior $\mathcal{B}_2$ is given, and the behavior $\mathcal{B}_1$ is the one we wish to construct.



Definition 23. *Let an I/O system $\mathcal{B}_2$ with input $y_1(t)$ and output $y_2(t)$ as above be given.*
A second I/O system

$$\mathcal{B}_1 := \left\{ \begin{pmatrix} y_1(t) \\ u(t) \end{pmatrix} \in \mathcal{F}^{p+m};\ P_1\left(\tfrac{d}{dt}\right) y_1(t) = Q_1\left(\tfrac{d}{dt}\right) u(t) \right\}$$

with input $u(t)$ and output $y_1(t)$ is called an asymptotic output controller *of $\mathcal{B}_2$ if*

$$\lim_{t\to\infty} (y_2(t) - u(t)) = 0$$

whenever $y_1(t)$ is an output of the controller system $\mathcal{B}_1$ to the input $u(t)$ and $y_2(t)$ is an output of the original system B_2 to the input $y_1(t)$.

Theorem 24. *Let*

$$\mathcal{B}_2 := \left\{ \begin{pmatrix} y_2(t) \\ y_1(t) \end{pmatrix} \in \mathcal{F}^{m+p};\ P_2\left(\tfrac{d}{dt}\right) y_2(t) = Q_2\left(\tfrac{d}{dt}\right) y_1(t) \right\}$$

be an input/output behavior with transfer matrix $H_2 = P_2^{-1}Q_2 \in \mathbb{R}(s)^{m\times p}$. Then the following two statements are equivalent:

1. *There is a proper stable asymptotic output controller $\mathcal{B}_1$ of the behavior $\mathcal{B}_2$.*
2. *$\mathcal{B}_2$ is stable and the transfer matrix H_2 has a right inverse $H_1 \in \mathcal{S}^{p\times m}$.*

If these conditions are fulfilled, the behavior of the serial connection of first $\mathcal{B}_1$ and then $\mathcal{B}_2$ is stable as well.
If condition (2) is satisfied, then one proper stable asymptotic output controller system $\mathcal{B}_1$ can be constructed as the controllable realization of H_1.

Remark 25.

1. Note that Theorem 11 can easily be modified for testing the existence of right inverse matrices instead of left inverse matrices.
2. The following theorem is a variation of the previous one without the requirement that the constructed output controller system shall be proper.

Theorem 26. *Let*

$$\mathcal{B}_2 := \left\{ \begin{pmatrix} y_2(t) \\ y_1(t) \end{pmatrix} \in \mathcal{F}^{m+p};\ P_2\left(\tfrac{d}{dt}\right) y_2(t) = Q_2\left(\tfrac{d}{dt}\right) y_1(t) \right\}$$

be an input/output behavior with transfer matrix $H_2 = P_2^{-1}Q_2 \in \mathbb{R}(s)^{m\times p}$. *Then the following two statements are equivalent:*

1. *There is a stable asymptotic output controller* $\mathcal{B}_1$ *of the behavior* $\mathcal{B}_2$.
2. $\mathcal{B}_2$ *is stable and the transfer matrix* H_2 *has a right inverse* $H_1 \in \mathbb{R}[s]_T^{p\times m}$.

If these conditions are fulfilled, the behavior of the serial connection of first $\mathcal{B}_1$ *and then* $\mathcal{B}_2$ *is stable as well.*
If condition (2) is satisfied, then one stable asymptotic output controller system $\mathcal{B}_1$ *can be constructed as the controllable realization of* H_1.

Remark

All presented results are valid in a more general context: Instead of $\mathbb{R}[s]$ one may consider the polynomial algebra $F[s]$ over an arbitrary field F. The set T can be replaced by any multiplicatively closed saturated set of nonzero polynomials in $F[s]$, and the signal space by an arbitrary injective $F[s]$-cogenerator $\mathcal{F}$. Asymptotic stability of an autonomous behavior is then replaced by T-autonomy where an autonomous behavior is called T-autonomous if there is some $t \in T$ such that $tw = 0$ for all trajectories w in the behavior.

In particular, the cases of other stability regions and of discrete systems are included in the more general framework.

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