Squaring–down descriptor systems

Cristian Oară[*], Şerban Sabău[†]

Abstract

We consider the problem of squaring–down an arbitrary $p \times m$ descriptor linear time–invariant system. Squaring–down consists in finding a pre– and a post–compensator such that the system is turned into a square invertible one. We consider static, dynamic and norm–preserving compensators that are able to keep unchanged the H^{∞} norm of the original system. All characterization are made by using generalized state–space realizations while the associated computations are performed by employing orthogonal transformations and standard reliable procedures for eigenvalue assignment.

1 Introduction

Given a multivariable system with unequal number of inputs and outputs, the first stage in many popular design schemes is to combine all outputs together into a new set of outputs and/or to combine all inputs into a new set of inputs such that the resulting system has an equal number of inputs and outputs and is in addition invertible. Even if the system to start with is square but not invertible, one needs to to reduce in the preliminary design step the number of inputs and outputs to make the resulting system square and invertible. This preliminary design step is called "squaring–down". The problem has been considered in the past three decades in a lot of publications receiving various theoretical solutions [2, 12, 4, 3, 10, 8, 9]. More recently, numerically–sound algorithms have been proposed in [1] following the theoretical solution in [8]. All solutions proposed so far deal with standard state–space systems that fulfill some additional hypotheses (strictly proper, proper, the input and output matrices in a state–space realization have full rank, etc) while the pre– and post–compensators are either static or dynamic. In the static case the new zeros cannot always be placed in desired locations and this brings important limitations in the subsequent design stages. However, in the dynamic case the newly introduced zeros can be placed arbitrary. In all available solutions the zeros at infinity are left unchanged.

In this paper we deal with the problem of squaring down a continuous–time time–invariant linear descriptor system G given by a generalized state–space realization

$$\begin{aligned} E\dot{x} &= Ax + Bu, \\ y &= Cx + Du, \end{aligned} \tag{1}$$

[*]Faculty of Automatic Control and Computers, University Polytechnica Bucharest, 34 Austrului, RO 73115, Bucharest, Romania, Tel/Fax: +40 21 3234 234, email: oara@riccati.pub.ro. His work has been supported by the Romanian National University Research Council (CNCSIS) under grant ID 814/2007.

[†]Electrical Engineering Dept. A.V. Williams Bldg., Rm. 2231, Univ. of Maryland, College Park, MD 20783, USA Fax : (301) 314-9920, Tel : (301) 405-1939, email: sserban@isr.umd.edu.

by using a postcompensator G_1 and a precompensator G_2 such that the resulting squared–down system $\widehat{G} = G_1 G G_2$ has an equal number of inputs and outputs and is invertible. Here G stands for the transfer function matrix of the system, $y = Gu$, given in terms of (1) by

$$G(s) = C(sE - A)^{-1}B + D =: \left[\begin{array}{c|c} A - sE & B \\ \hline C & D \end{array}\right],$$

$y \in \mathbb{R}^p$ is the output, $u \in \mathbb{R}^m$ is the input, $x \in \mathbb{R}^n$ is the (generalized) state, $A - sE$ is a regular pencil, i.e. it is square and $\det(A - sE) \not\equiv 0$, and the intervening matrices in (1) have appropiate dimensions.

We consider the general case in which p, m and the normal rank r of G are all arbitrary. As a generalized realization (1) exists for any arbitrary rational matrix G (even improper or polynomial) the approach is equivalent to transforming an arbitrary transfer function matrix into a square and invertible one by pre– and post–multiplication with two rational matrices of appropiate dimensions.

The problem at hand may be decomposed in two dual problems, one to find a postcompensator G_1 such that $G_1 G$ has full row rank and one to find a precompensator such that GG_2 has full column rank. For the rest of the paper we deal only with the postcompensation problem since the solution for the precompensation follows by duality.

For the postcompensation problem we will give a class of solutions that feature the nice property to have any desired poles and/or zeros. The postcompensator is constructed by using a preliminary spectral decomposition of the system pencil achieved by using orthogonal transformations gaining therefore benefits in terms of numerical reliability of the proposed solution. Furthermore, the postcompensator has two free parameters that can be tuned to assign any desired poles and/or zeros to the solution. Particular choices of the tuning parameters lead to various solutions of the problem: static compensators (without the possibility to assign always the new zeros of the squared–down system) or dynamic compensators (that are able to assign the new zeros of the squared–down system). The usual benefits of classical squaring–down schemes like phase–minimality, stabilizability, preserving of infinite zero structure, etc are recovered by the proposed solution.

The present work extends and improves existing results in two ways. On one side, it is more general than other available results (see for example [8, 9, 10, 1] and the references therein) as it works with an arbitrary (possibly improper) system. Even for proper systems it relaxes the usual requirements that the B and C matrices are of full rank. On the other side, the use of a preliminary spectral decomposition which is based solely on orthogonal transformations shortcuts and streamlines the theoretical results while improves the numerical soundness of the overall solution.

2 Preliminaries

We introduce here some definitions and notations. For a system given by a generalized state–space (or descriptor) realization (1) we define the *pole pencil* $A - sE$ and the *system pencil* $\left[\begin{array}{cc} A - sE & B \\ C & D \end{array}\right]$. The dimension n of the square matrices A and E is called the *order of the realization* (1). We use $\Lambda(A - sE)$ to denote the union of generalized eigenvalues of the regular pencil $A - sE$ (finite and infinite, multiplicities counting). The realization (1) is called *irreducible* if (see [13]) it all its finite poles are controllable and observable, i.e. rank $\left[\begin{array}{cc} A-sE & B \end{array}\right] = n, \forall s \in \mathbb{C}$ and rank $\left[\begin{array}{c} A-sE \\ C \end{array}\right] = n, \quad \forall s \in \mathbb{C}$, respectively, and its infinite poles are controllable and observable, i.e. rank $\left[\begin{array}{cc} E & B \end{array}\right] = n$ and rank $\left[\begin{array}{c} E \\ C \end{array}\right] = n$, respectively. We say the pair $(A-sE, B)$ is stabilizable if it is controllable for all poles (finite or infinite) in $\overline{\mathbb{C}}^+$ (the closure of the open right–half plane). Any rational matrix has an irreducible realization which can be obtained from an arbitrary realization by using solely orthogonal transformations (see for example [11]). Irreducibility together with $A\ker(E) \subseteq \mathrm{Im}(E)$ imply the minimality of the realization (1) (i.e., the least possible order). However, even for a minimal realization (1) having order n we have in general that the McMillan degree of G, denoted $\delta(G)$, satisfies $\delta(G) \leq n$, with strict inequality unless G is proper. The Weierstrass and Kronecker canonical forms of the pole and system pencil of an irreducible realization play an important role in the sequel as they are in one–to–one correspondence with the Smith–McMillan form and the minimal indices to the left and right of G (for details see [11]). We denote by $\mathcal{Z}(G(s))$ the union of zeros (finite and infinite, multiplicities counting) of the rational matrix G.

3 Spectral decomposition of the system pencil

We start with a preparatory result that outlines the needed spectral decomposition of the system pencil of the original system (1).

Theorem 3.1. *Let a generalized state–space system given by a stabilizable realization (1) and having transfer function matrix G. Then there exist two orthogonal matrices Q and Z, such that*

$$\left[\begin{array}{cc} I & O \\ O & Q^T \end{array}\right] \left[\begin{array}{c|c} A-sE & B \\ \hline C & D \end{array}\right] Z$$

$$= \left[\begin{array}{cccc} A_{rz}-sE_{rz} & B_1-sF_1 & B_2-sF_2 & B_3-sF_3 \\ O & A_\ell - sE_\ell & B_\ell & B_{\ell n}-sF_{\ell n} \\ O & O & O & B_n \\ \hline O & C_{\ell 1} & D_\ell & D_1 \\ O & C_{\ell 2} & O & D_2 \end{array}\right] \qquad (2)$$

where

(a) $\mathcal{Z}(G(s)) = \Lambda(A_{rz}-sE_{rz})$ and $A_{rz}-sE_{rz}$ is of full row rank for all $s \notin \mathcal{Z}(G(s))$.

(b) E_ℓ, D_ℓ and B_n are invertible, the pair $(A_\ell - sE_\ell, B_\ell)$ is stabilizable and

$$\left[\begin{array}{c} A_\ell - B_\ell D_\ell^{-1} C_{\ell 1} + sE_\ell \\ C_{\ell 2} \end{array}\right]$$

has full column rank $\forall s \in \mathbb{C}$.

This result is a slight variation of Theorem III.1 in [5] or [6]. The matrices Q and Z can be constructed by a numerically–sound algorithm (see for example [7]). The theorem extracts from the original system (1) a proper subsystem (E_ℓ is invertible)

$$G_s(s) = \left[\begin{array}{c|c} A_\ell - sE_\ell & B_\ell \\ \hline C_{\ell 1} & D_\ell \\ C_{\ell 2} & O \end{array}\right]$$

which has no zeros, no right minimal indices and a number of left minimal indices equal to the dimension of $A_\ell - sE_\ell$. The results in next section shows that it is enough to solve the squaring–down problem for this subsystem in order to get the solution for the original one. Notice also the quite relaxed hypotheses on the realization: stabilizability only.

4 Squaring–down

We start with a static solution to the postcompensation squaring–down problem, i.e., a solution in which the postcompensator is a constant matrix D_1 such that D_1G has full row–rank. A dual result can further be applied to D_1G to get a constant D_2 such that D_1GD_2 is square and invertible.

Theorem 4.1. *Let (1) be a generalized state–space system fulfilling the assumptions in Theorem 3.1. Let Q and Z be two orthogonal matrices for which (2) holds. A class of static solutions to the postcompensation squaring–down problem is given by*

$$D_1 = \left[\begin{array}{cc} I_r & D_x \end{array}\right] Q^T \tag{3}$$

where D_x is an arbitrary matrix of dimension $r \times (p-r)$ and r is the normal rank of G. The poles of the squared–down system $\widehat{G}(s)$ are among the poles of $G(s)$ while its zeros are included in the union of the zeros of $G(s)$ with

$$\Lambda(A_\ell - B_\ell D_\ell^{-1}(C_{\ell 1} + D_x C_{\ell 2}) - sE_\ell). \tag{4}$$

The identity matrix in the static solution (3) preserves the zeros structure at infinity of G in the squared–down system $\widehat{G}$. The solution (3) shows that the resulting squared–down system has a number of zeros which are among the set (4). As assigning the eigenvalues of $A_\ell - B_\ell D_\ell^{-1}(C_{\ell 1} + D_x C_{\ell 2}) - sE_\ell$ by an appropriate choice of D_x is equivalent with a pole assignment problem by output feedback it is clear that in general we can not place the new zeros arbitrarily. Since placing the new zeros in the open left–half plane is a requirement of subsequent design problems we are lead to dynamic compensators.

Theorem 4.2. *Let (1) be a generalized state–space system fulfilling the assumptions in Theorem 3.1. Let Q and Z be two orthogonal matrices for which (2) holds. A class of dynamic solutions to the postcompensation squaring–down problem is given by*

$$G_1(s) = \left[\begin{array}{c|cc} A_\ell - sE_\ell - B_\ell D_\ell^{-1} C_{\ell 1} - K C_{\ell 2} & B_\ell D_\ell^{-1} & K \\ \hline F & I_r & O \end{array}\right] Q^T \tag{5}$$

where r is the normal rank of G, F and K are arbitrary matrices of appropriate dimensions. The poles of the squared–down system $\widehat{G}(s)$ are among the poles of $G(s)$ while its zeros are included in the union of the zeros of $G(s)$ with

$$\Lambda(A_\ell + B_\ell D_\ell^{-1}(C_{\ell 1} + F) - sE_\ell). \tag{6}$$

The previous theorem shows that squaring–down can be performed by a postcompensator having any desired poles, for example stable poles. To choose the poles of the postcompensator we solve for K the eigenvalue assignment problem $A_\ell - sE_\ell - B_\ell D_\ell^{-1} C_{\ell 1} - KC_{\ell 2}$. This has always a solution irrespective of the desired location of the poles due to (b) of Theorem 3.1. Moreover, the new zeros introduced in the squared–down system can also be freely assigned in the left–half plane while the zeros at infinity of the original system are preserved unchanged. To choose the zeros of the squared–down system we solve for F the eigenvalue assignment problem $\Lambda(A_\ell + B_\ell D_\ell^{-1}(C_{\ell 1} + F) - sE_\ell) \subset \mathbb{C}^-$. This again has a solution irrespective of the desired location in $\mathbb{C}^-$ of the zeros due to (b) of Theorem 3.1. All these properties are extremely desirable in the subsequent steps of the design.

Remark 4.3. *Notice that (3) with $D_x = 0$ can be obtained directly from (5). One can combine the static solution with $D_x \neq 0$ and the dynamic solution to obtain a minimum McMillan degree postcompensator G_1 which assigns the new zeros in $\mathbb{C}^-$. This would imply to assign through the matrix F only those zeros that can not be assigned by solving the output feedback problem (4).*

Finally, we give a solution to the postcompensation squaring–down problem that apart from introducing zeros in the left–half plane is able to preserve the H^∞ norm of the original system. This is a highly desirable property when it comes to solving robustness problems.

Theorem 4.4. *Let (1) be a generalized state–space system fulfilling the assumptions in Theorem 3.1. Let Q and Z be two orthogonal matrices for which (2) holds. The continuous–time algebraic Riccati equation*

$$A_\ell^T X E_\ell + E_\ell^T X A_\ell - (E_\ell^T X B_\ell + C_{\ell 1}^T D_\ell)(D_\ell^T D_\ell)^{-1}(B_\ell^T X E_\ell + D_\ell^T C_{\ell 1}) + C_\ell^T C_\ell = 0 \quad (7)$$

has a stabilizing symmetric positive definite solution X_s such that $\Lambda(A_\ell + B_\ell F_s - sE_\ell) \subset \mathbb{C}^-$, where

$$F_s := -(D_\ell^T D_\ell)^{-1}(B_\ell^T X_s E_\ell + D_\ell^T C_{\ell 1}) \quad (8)$$

is the stabilizing Riccati feedback and $C_\ell^T := \begin{bmatrix} C_{\ell 1}^T & C_{\ell 2}^T \end{bmatrix}$. A norm–preserving solution to the postcompensation squaring–down problem is given by

$$G_1(s) = \left[\begin{array}{c|cc} A_\ell - sE_\ell - B_\ell D_\ell^{-1} C_{\ell 1} + X_s^{-1} E_\ell^{-T} C_{\ell 2}^T C_{\ell 2} & B_\ell D_\ell^{-1} & -X_s^{-1} E_\ell^{-T} C_{\ell 2}^T \\ \hline -(C_{\ell 1} + D_\ell F_s) & I & O \end{array}\right] Q^T. \quad (9)$$

The poles of the squared–down system $\widehat{G}$ are among the poles of $G(s)$ while the zeros are included in the union of the zeros of $G(s)$ with

$$\Lambda(A_\ell + B_\ell F_s - sE_\ell) \subset \mathbb{C}^-. \quad (10)$$

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