

Diffusive Feedback, Dissipativeness and Synchronization of Coupled Oscillators

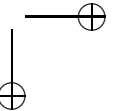
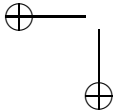
*Vl. Răsvan, Daniela Danciu and D. Popescu**

1 Introduction. Synchronization and synchronization models

Synchronization turns to be a rather intuitive notion provided there exists a minimal knowledge about oscillators and oscillatory behavior. According to an expert [10] early theory arose (at least partially) from the theory of oscillations: in the theory of dynamical systems, in control theory or in any theory dealing with transients, any system that has some dynamics is called an oscillator and any transient process that ends in some steady state (equilibrium, limit cycle etc.) is sometimes called *oscillation* (it seems that this terminology goes up to Lagrange). Genuine oscillations are in fact oscillatory steady state solutions e.g. periodic or almost periodic trajectories whose geometric image is a limit cycle in the first case and a toric manifold in the second one.

A. The oscillatory steady state may be induced by an external oscillatory signal source or may be generated internally by the oscillatory system in the absence of external oscillatory sources. The first case is usually called *forced oscillation* while the second one is called *self-sustained oscillation*. The forced oscillatory behavior is sometimes called *synchronization with the external oscillator* generating the oscillatory signal. This is due to the fact that mainly in linear cases to each external signal there corresponds a steady state which is of the same type as the external signal, being periodic (almost periodic) if the external signal is periodic (almost periodic) with the same harmonic contents; moreover, if the free (unforced) system

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is exponentially stable, this property is inherited by the steady state solution of the forced system. But a similar property is met for a broad class of nonlinear systems having sector restricted nonlinearities [2, 12] what strengthens the motivation for calling this behavior - synchronization.

The present paper will be nevertheless concerned with the synchronization of the other class of oscillators - the self sustained oscillators. As known, it is understood by self sustained oscillations existence of an oscillatory (e.g. periodic or almost periodic) steady state under the circumstance that the system (sometimes called oscillator) is subject to non-oscillatory external sources.

Motivated by problems in physics, engineering, biology and medicine [1], there exists now a renewed interest in the dynamics induced by the coupling of local oscillators on (possibly periodic) lattices. In such systems the dichotomy *synchronization/chaotic behavior* is basic unlike in standard control systems where the other dichotomy (stability/instability) is more important [10].

The synchronization of self sustained (self excited) oscillations is mainly concerned with periodic motions for which each oscillator oscillates in the same way except (possibly) for a phase shift (the locking phenomenon). To be more specific consider the basic local oscillators to be described by an ordinary differential equation of standard type for self sustained oscillation generation (van der Pol, Liénard, Duffing) on a uniform periodic lattice on the real line, each oscillator interacting only with its near neighbor

$$\ddot{y}_k + f(y_k)\dot{y}_k + g(y_k) - \rho(y_{k+1} - 2y_k + y_{k-1}) = 0, \quad k = 1, \dots, N \pmod{N} \quad (1)$$

with $\rho > 0$ - a coupling constant; from the lattice periodicity we deduce that $y_0 = y_N$, $y_{N+1} = y_1$ hence system (1) is well defined. With respect to this standard case we are able to define several notions and problems.

1° If each local oscillator has a limit cycle i.e. a periodic solution, it is required to find conditions on the coupling factor $\rho > 0$ in order to obtain

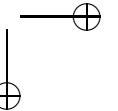
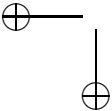
$$y_i(t) - y_j(t) \rightarrow 0, \quad t \rightarrow \infty \quad \forall i, j$$

2° Assuming the autonomous oscillators have a globally asymptotically stable equilibrium at the origin, find conditions on $\rho > 0$ in order that a common oscillatory solution should exist and be asymptotically stable. This is the so called *Turing problem* since it has been formulated by S. Smale for a cell model in biology (the so-called chemostat of Turing).

B. It is possible to generalize the above notions and problems as follows. Since in dimension 2 the limit cycle is compact and lies in a compact set which is also invariant with respect to system's solutions i.e. in a compact global attractor, the periodic oscillation will be replaced by the existence of the above mentioned compact attractor of the n -dimensional system

$$\dot{z}_k = f_k(z_k), \quad k = 1, \dots, N \quad (2)$$

If the linearly coupled system is considered



$$\dot{y} = A(r)y + f(y) \quad (3)$$

where $y = (y_1 \dots y_N)^T$, $f = (f_1 \dots f_N)^T$, r is some vector parameter and A a coupling matrix, we call synchronization the case when the global attractor of the coupled system belongs to the “diagonal” in $\mathbb{R}^{nN}$: $y_1 = y_2 = \dots y_N$ provided z belongs to the attractor; its definition will imply

$$y_i(t) - z_j(t) \rightarrow 0, \quad t \rightarrow \infty \quad \forall i, j$$

C. Following the papers of Hale [4, 5, 6] we note that in any discussion of synchronization dissipation plays a very important role. If each subsystem (2) has a compact global attractor, then a natural dissipative mechanism should exist in each subsystem. When the subsystems are coupled *via* $A(r)$, existence of the overall attractor will impose some constraints on A among which it is mentioned the condition on strong dissipation.

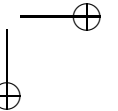
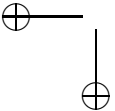
The present paper is concerned mainly with dissipativeness of the coupled oscillatory system (or network) and with its application to some systems the authors view as *benchmark*. Consequently the paper is organized as follows. First a review of the notions of dissipativeness is presented following the survey of the first author [13]. It appears that dissipativeness in the sense of Levinson is most suited for the synchronization problem; on the other hand this property may be obtained using the second method of Liapunov and the properties of the system for large deviations what will send to the dissipation inequality of System Theory. The a dissipativeness analysis is performed for two types of local oscillators with tunnel diode and capacitor - the first one which has a coil (inductance) to define the oscillatory circuit and the second one with a LLTL (lossless transmission line) instead. The main mathematical tool is the Liapunov function(al) that arises from the stored electromagnetic energy of the circuits. With the same function in mind, but for the interconnected system of the (1) type we discuss again dissipativeness. Finally some considerations are made for the general case matrix of (3)

2 Basic dissipativeness notions for dynamical systems

A. Classical(standard) dissipativeness is the notion which occurs in Lagrangian Mechanics: let $q \in \Omega \subseteq \mathbb{R}^n$ be the vector of generalized Lagrangian coordinates, $\dot{q} \in \mathbb{R}^n$ be the vector of the generalized speeds. Let $T : I \times \Omega \times \mathbb{R}^n \mapsto \mathbb{R}$, where $I = (\tau, +\infty)$, $\tau \in \mathbb{R}$, be the *kinetic energy* which is a second degree polynomial in the speed vector $\dot{q}$:

$$T(t, q, \dot{q}) = \frac{1}{2} \dot{q}^T A(t, q) \dot{q} + b(t, q)^T \dot{q} + d(t, q) \quad (4)$$

where $A(t, q)$ is a $n \times n$ matrix, $b(t, q)$ a n -vector and $d(t, q)$ a scalar, all of them $\mathcal{C}^1$ on $I \times \Omega$. Denote by $\Pi : I \times \mathbb{R}^n \mapsto \mathbb{R}$ the potential energy supposed a $\mathcal{C}^1$ function and by $Q : I \times \Omega \times \mathbb{R}^n \mapsto \mathbb{R}^n$ the Lagrangian (non-potential) forces vector supposed also $\mathcal{C}^1$ with respect to all its arguments. The system is described by the Euler-Lagrange equations



$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} + Q(t, q, \dot{q}) = 0 \quad (5)$$

with $L = T + \Pi$.

The Lagrangian forces are called *dissipative* if the following inequality holds

$$Q(t, q, \dot{q})\dot{q} \geq 0, \quad \forall (t, q, \dot{q}) \in I \times \Omega \times \mathbb{R}^n \quad (6)$$

Obviously the above scalar product represents the power developed by the Lagrangian forces. If additionally there exists some comparison (Kamke-Massera) function – well known in stability theory – $\alpha : \mathbb{R}_+ \mapsto \mathbb{R}_+$ which is continuous, non-decreasing with $\alpha(0) = 0$, such that $Q(t, q, \dot{q})\dot{q} \geq \alpha(|\dot{q}|)$ then the forces are called *totally dissipative*. The dissipative forces which are linear with respect to $\dot{q}$ are called forces of viscous friction; an example of such forces is given by those derived from the Rayleigh function $\mathcal{R}(t, q, \dot{q}) = \dot{q}^T R(t, q)\dot{q}$; the Rayleigh matrix R is positive definite for all its arguments: $\mathcal{R}(t, q, \dot{q}) \geq \alpha|\dot{q}|^2$, $\alpha > 0$.

B. Dissipativeness in the sense of Levinson as qualitative property is important in the theory of differential equations, especially in the study of self-sustained and forced nonlinear oscillations. The standard framework is again the framework given by autonomous

$$\dot{x} = f(x) \quad (7)$$

or even non-autonomous differential equations

$$\dot{x} = f(t, x) \quad (8)$$

We may state, following N. Levinson

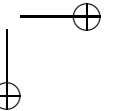
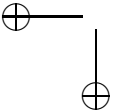
Definition 1. *System (8) is called dissipative if there exists some $R > 0$ such that*

$$\overline{\lim}_{t \rightarrow \infty} |x(t; t_0, x_0)| < R \quad (9)$$

for any solution $x(t; t_0, x_0)$ of (8).

Some comments are necessary here as previously. This qualitative property, called also *ultimate boundedness* means that all system's trajectories will eventually enter a ball of radius R and will remain there. Various boundedness properties have been considered by Yoshizawa in his fundamental books [19, 20] and their role in establishing existence of self-sustained and forced oscillations has been pointed out there. The basic results on dissipativeness in the sense of Levinson are also due to Yoshizawa [19] but we shall give here only a very special but useful results in the study of systems' dynamics

Theorem 1. *(Barbashin and Krasovskii) Let $V(x)$ be a Liapunov function in the generalized sense introduced previously, defined for $|x| \geq \rho_1$ and with the additional*



properties: *i)* it is radially unbounded i.e. $\lim_{|x| \rightarrow \infty} V(x) = +\infty$; *ii)* it is non-increasing along those solutions such that $|x(t)| \geq \rho_1$; *iii)* $\dot{V}^*(t) = (d/dt)V(x(t)) \neq 0$ for $|x(t)| \geq \rho_1$. Then system (7) is dissipative in the sense of Levinson.

C. Dissipativeness in the sense of System Theory arises from Circuit Theory in Electrical Engineering; some further development led V.M. Popov to his hyperstability theory that could be seen as creative interaction of dissipativeness in the Circuit Theory sense and absolute stability. Later J.C. Willems [17] published the cycle of two papers that spread this concept throughout the world; finally dissipativeness became not only a systems analysis tool but also a systems synthesis one. Here we shall describe the framework introduced in [17, 8]. The system is described by a nonlinear finite dimensional model which is affine in the input

$$\begin{cases} \dot{x} = f(x) + G(x)u(t) \\ y = h(x) + J(x)u(t) \end{cases} \quad (10)$$

with $\dim x = n$, $\dim y = p$, $\dim u = m$, $f : \mathbb{R}^n \mapsto \mathbb{R}^n$, $h : \mathbb{R}^n \mapsto \mathbb{R}^m$ and G, J are matrices of appropriate dimensions. It is assumed for all function entries to be smooth enough, also $f(0) = 0$, $h(0) = 0$.

Definition 2. *System (10) is called dissipative with supply rate $w(u, y)$ if there exists a state function $V : \mathbb{R}^n \mapsto \mathbb{R}_+$ called storage function such that the following dissipation inequality holds*

$$V(x(t)) \leq V(x_0) + \int_0^t w(u(\tau), y(\tau)) d\tau \quad (11)$$

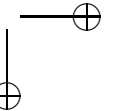
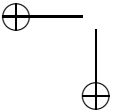
where $x(t)$ is the trajectory of (10) corresponding to the initial state x_0 and to the input (control) signal $u : (0, t) \mapsto \mathbb{R}^m$.

Observe that (11) is very much alike to a Liapunov inequality, the main problem being similar: guessing a storage function. We have to add here another problem, which is not present within the second (direct) method of Liapunov – association of a supply rate. The generalized system “dynamics plus supply rate” of the dissipativeness theory is very much alike to the pair “dynamics plus integral index” introduced by V.M. Popov in his hyperstability theoretical kernel and even earlier, when deducing the famous frequency domain inequality.

3 Dissipativeness of the tunnel diode circuits

A. The standard oscillator with tunnel diode is given together with the nonlinear diode characteristic in fig.1

Obviously this dynamical system of second order may have one or three equilibria. In the case of a single equilibrium this equilibrium can be made globally asymptotically stable hence synchronization will reduce to the “trivial” problem of the global asymptotic stability of the interconnected system.



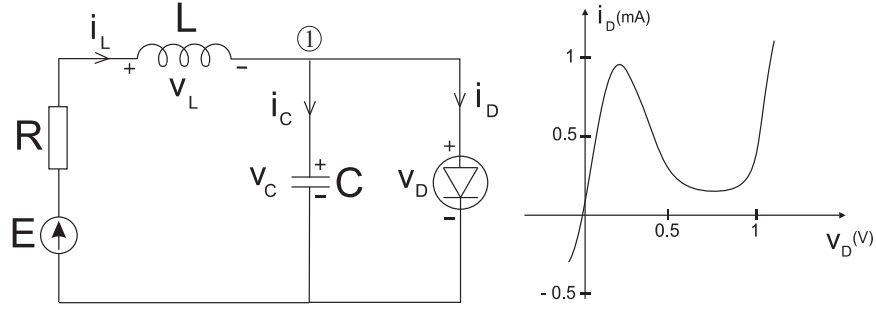


Figure 1. Local oscillator (standard LC circuit).

The case of three equilibria is more interesting for several reasons :i) the system is unlike the very standard ones (Liénard, van der Pol, Duffing) which have a single equilibrium; ii) the equilibrium Q_2 in the “middle” of the “load line” (fig.2) is a saddle point while the other two are either both focuses (most probable) or both nodes ; consequently they are all inside the limit cycle according to Poincaré theorem; iii) if the system in deviations with respect to the saddle point is considered, the diode characteristic becomes a $\mathcal{S}$ -like function for which self sustained oscillations are most likely.

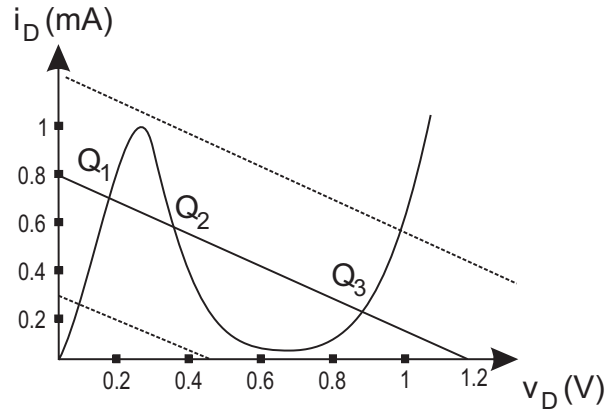
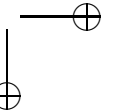
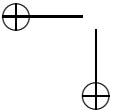


Figure 2. Equilibria of the tunnel diode oscillator

Consider therefore the oscillator equations

$$\begin{aligned} C \frac{dv}{dt} &= -f(v) + i \\ L \frac{di}{dt} &= -v - R_0 i + E \end{aligned} \quad (12)$$

and let (V_*, I_*) be the saddle point i.e. $f'(V_*) < 0$. Denoting $\iota = i - I_*$, $\nu = v - V_*$



we obtain the equations in deviations

$$\begin{aligned} C \frac{d\nu}{dt} &= -[f(\nu + V_*) - f(V_*)] + \iota \\ L \frac{d\iota}{dt} &= -\nu - R_0 \iota \end{aligned} \quad (13)$$

with the origin as saddle point. It is not difficult to see that if we consider $\underline{V}$, V_* , $\bar{V}$ the three solutions of $f(v) = I_*$ then we have $\underline{\nu} = \underline{V} - V_* < 0$, $\bar{\nu} = \bar{V} - V_* > 0$ and $\nu g(\nu) > 0$ for $\nu < \underline{\nu}$, $\nu > \bar{\nu}$, where $g(\nu) = R_0(f(\nu + V_*) - f(V_*))$. The sector condition $\nu g(\nu) > 0$ which is fulfilled for sufficiently large deviations obviously suggests a possible dissipativeness property. Taking the approach of [14] we define the Liapunov function

$$\mathcal{V}(\nu, \iota) = \frac{1}{2}(C\nu^2 + L\iota^2) \quad (14)$$

whose derivative is

$$\mathcal{W}(\nu, \iota) = -\frac{1}{R_0}\nu g(\nu) - R_0\iota^2 \quad (15)$$

and which is negative definite provided $\nu \notin (\underline{\nu}, \bar{\nu})$. We may repeat the arguments of Yoshizawa [19] or of the more recent contribution [14] to obtain dissipativeness in the sense of Levinson.

B. Consider now the tunnel diode oscillator containing a lossless transmission line (fig.3). Its equations are as follows

$$\begin{aligned} L_s \frac{\partial i}{\partial t} &= -\frac{\partial v}{\partial \lambda}, \quad C_s \frac{\partial v}{\partial t} = -\frac{\partial i}{\partial \lambda}, \quad 0 \leq \lambda \leq 1 \\ E &= v(0, t) + R_0 i(0, t) \\ -C \frac{d}{dt} v(1, t) &= -i(1, t) + f(v(1, t)) \end{aligned} \quad (16)$$

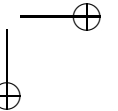
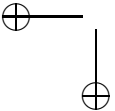
For constant E we obtain the same equilibria as in the previous case since $v(\lambda, t) \equiv \text{const}$, $i(\lambda, t) \equiv \text{const}$ and only the d.c. (direct current) equations remain

$$E = R_0 I + V, \quad I = f(V) \quad (17)$$

These equations are as in the previous case; therefore we may have again the three equilibria on the “load” line, (V_*, I_*) being the equilibrium in the middle. All considerations concerning the properties of $f(v)$ and $g(\nu)$ remain valid. If we introduce the deviations

$$\iota(\lambda, t) = i(\lambda, t) - I_*, \quad \nu(\lambda, t) = v(\lambda, t) - V_* \quad (18)$$

the following system in deviations is obtained



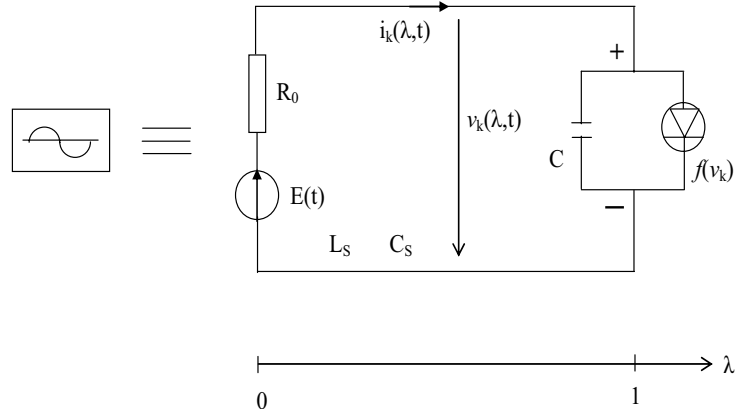


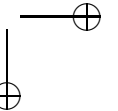
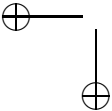
Figure 3. *Local oscillator (Nagumo - Shimura circuit)*

$$\begin{aligned}
 L_s \frac{\partial \iota}{\partial t} &= -\frac{\partial \nu}{\partial \lambda}, \quad C_s \frac{\partial \nu}{\partial t} = -\frac{\partial \iota}{\partial \lambda}, \quad 0 \leq \lambda \leq 1 \\
 \nu(0, t) + R_0 \iota(0, t) &= 0 \\
 -C \frac{d}{dt} \nu(1, t) &= -\iota(1, t) + \frac{1}{R_0} g(\nu(1, t))
 \end{aligned} \tag{19}$$

We may apply now the standard techniques for such systems by associating a system of functional differential equations composed of a delay-differential system coupled to a difference equation with continuous time, sometimes called *shift equation* [15]; in more recent engineering publications this is also called *delay-differential algebraic system of equations (DDAE)*. We shall follow the procedures of [11] by introducing first the Riemann invariants, also called forward and backward waves

$$\begin{aligned}
 \nu(\lambda, t) &= u^1(\lambda, t) + u^2(\lambda, t) \\
 \iota(\lambda, t) &= \sqrt{\frac{C_s}{L_s}} (u^1(\lambda, t) - u^2(\lambda, t))
 \end{aligned} \tag{20}$$

to obtain the transformed system



$$\begin{aligned}
\frac{\partial u^1}{\partial t} + \frac{1}{\sqrt{L_s C_s}} \frac{\partial u^1}{\partial \lambda} &= 0 \quad , \quad \frac{\partial u^2}{\partial t} - \frac{1}{\sqrt{L_s C_s}} \frac{\partial u^2}{\partial \lambda} = 0 \\
\left(1 + R_0 \sqrt{\frac{C_s}{L_s}}\right) u^1(0, t) + \left(1 - R_0 \sqrt{\frac{C_s}{L_s}}\right) u^2(0, t) &= 0 \\
C \frac{d\nu}{dt} &= -\frac{1}{R_0} g(\nu) + \sqrt{\frac{C_s}{L_s}} (u^1(\lambda, t) - u^2(\lambda, t)) \\
u^1(1, t) + u^2(1, t) &= \nu(t)
\end{aligned} \tag{21}$$

The next step, according to (*op.cit.*) is to integrate along the characteristics; there are two families of characteristics - sometimes also called forward and backward

$$t_1(\lambda; 0, t) = \lambda \sqrt{L_s C_s} + t \quad , \quad t_2(\lambda; 1, t) = (1 - \lambda) \sqrt{L_s C_s} + t \tag{22}$$

If the solutions of (21) are considered along these characteristics - the forward wave along the forward (monotonically increasing) characteristics and the backward wave along the backward one - then

$$u^1(1, t + \sqrt{L_s C_s}) = u^1(0, t) \quad , \quad u^2(0, t + \sqrt{L_s C_s}) = u^2(1, t) \tag{23}$$

Denoting

$$\eta^1(t) = u^1(1, t + \sqrt{L_s C_s}) \quad , \quad \eta^2(t) = u^2(0, t + \sqrt{L_s C_s}) \tag{24}$$

the system of functional differential equations is obtained. To further simplify the writing, the following notations are made

$$T = C \sqrt{\frac{L_s}{C_s}} \quad , \quad \tau = 2 \sqrt{L_s C_s} \quad , \quad \delta_0 = \left(R_0 \sqrt{\frac{L_s}{C_s}}\right)^{-1} \quad , \quad \rho_0 = \frac{1 - \delta_0^{-1}}{1 + \delta_0^{-1}}$$

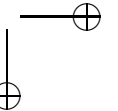
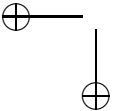
Finally, after eliminating $\eta^1(t)$ we obtain the system

$$\begin{aligned}
T \frac{d\nu}{dt} &= -\nu - \delta_0 g(\nu) - 2\rho_0 \eta^2(t - \tau) \\
\eta^2(t) &= \rho_0 \eta^2(t - \tau) + \nu(t)
\end{aligned} \tag{25}$$

Our purpose is to obtain dissipativeness conditions for this system using a Liapunov functional. Following e.g. Infante [9] and Hale [3] we consider

$$\mathcal{V}(\xi, \phi(\cdot)) = \frac{1}{2} T \xi^2 + \beta \int_{-\tau}^0 \phi^2(\theta) d\theta \quad , \quad \beta > 0 \tag{26}$$

We write down this positive definite functional along the solutions of (25) as



$$\mathcal{V}^*(t) = \frac{1}{2}T\nu^2(t) + \beta \int_{-\tau}^0 (\eta^2(t+\theta))^2 d\theta \quad (27)$$

which is differentiated along the solutions of (25) to obtain the “derivative function”

$$\mathcal{W}(\xi, \phi(\cdot)) = -\delta_0 g(\xi)\xi - (1-\beta)\xi^2 - 2\rho_0(1-\beta)\xi\phi(-\tau) - \beta(1-\rho_0)\phi^2(-\tau) \quad (28)$$

Since $0 < \rho_0 < 1$ the quadratic form in $\xi, \phi(-\tau)$ is strictly positive definite provided $\beta > 0$ is chosen as below

$$\frac{\rho_0^2}{1-\rho_0+\rho_0^2} < \beta < 1 \quad (29)$$

Taking into account the assumption on $g(\xi)$ ultimate boundedness is again obtained using the arguments of Yoshizawa [19].

4 Dissipativeness of the array of tunnel diode oscillators

Consider the structure of oscillators in ring interconnection (fig.4). This structure occurs e.g. in [18] as well as in the papers of Hale [4, 5, 6] and belongs to the class of diffusive couplings among local oscillators. We already described this coupling in (1). In the following we shall consider this diffusive coupling structure applied to the two kinds of local oscillators.

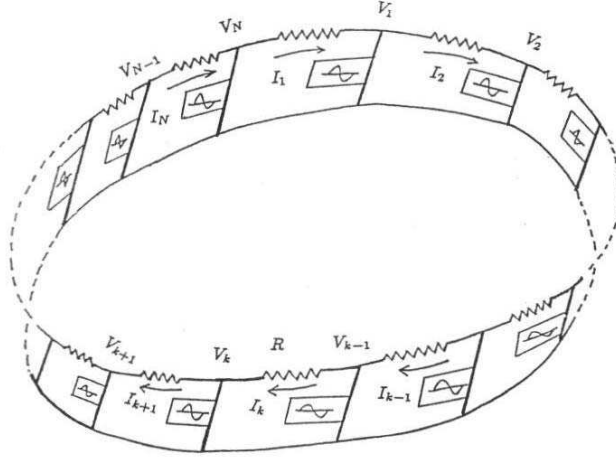
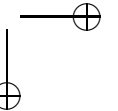
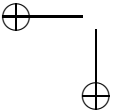


Figure 4. Ring interconnection structure



A. Consider equations (12) as describing the local oscillators and the couplings as in fig.4 i.e. of the type in (1). We deduce

$$\begin{aligned} C \frac{dv_k}{dt} &= -f(v_k) + i_k + \frac{1}{R}(v_{k+1} - 2v_k + v_{k-1}), \quad k = 1, \dots, N \pmod{N} \\ L \frac{di_k}{dt} &= -v_k - R_0 i_k + E \end{aligned} \quad (30)$$

If the equilibria are to be computed, it is easily found that the system in the deviations

$$\nu_k = v_k - V_{*k}, \quad \iota_k = i_k - I_{*k} \quad (31)$$

is

$$\begin{aligned} C \frac{d\nu_k}{dt} &= -\frac{1}{R_0}g(\nu_k) + \iota_k + \frac{1}{R}(\nu_{k+1} - 2\nu_k + \nu_{k-1}), \quad k = 1, \dots, N \pmod{N} \\ L \frac{d\iota_k}{dt} &= -\nu_k - R_0 \iota_k \end{aligned} \quad (32)$$

with $g(\nu)$ as previously. Associate now the energies stored in the local oscillators

$$\mathcal{V}(\nu_k, \iota_k) = \frac{1}{2}(C\nu_k^2 + L\iota_k^2) \quad (33)$$

which are obviously identical to (14) and consider the overall Liapunov function

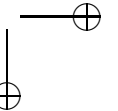
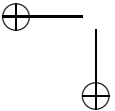
$$\tilde{\mathcal{V}}(\nu_1, \dots, \nu_N; \iota_1, \dots, \iota_N) = \sum_1^N \mathcal{V}(\nu_k, \iota_k) \quad (34)$$

If differentiated along the solutions of (32) we obtain

$$\begin{aligned} \tilde{\mathcal{W}}(\nu_1, \dots, \nu_N; \iota_1, \dots, \iota_N) &= -R_0 \sum_1^N [i_k^2 + \frac{1}{R_0} \nu_k g(\nu_k)] + \\ &\quad + \frac{1}{R} \sum_1^N [\nu_k (\nu_{k+1} - \nu_k) + \nu_k (\nu_{k-1} - \nu_k)] \\ &\leq -R_0 \sum_1^N [i_k^2 + \frac{1}{R_0} \nu_k g(\nu_k)] \end{aligned} \quad (35)$$

the last inequality following from Rearrangement Inequality No.368 [7], see also [16]; as previously this inequality will give dissipativeness of (32).

B. We consider now equations (16) as describing the local oscillators and the diffusive couplings as previously; it follows



$$\begin{aligned}
L_s \frac{\partial i_k}{\partial t} &= -\frac{\partial v_k}{\partial \lambda}, \quad C_s \frac{\partial v_k}{\partial t} = -\frac{\partial i_k}{\partial \lambda}, \quad 0 \leq \lambda \leq 1 \\
E &= v_k(0, t) + R_0 i_k(0, t) \\
-C \frac{d}{dt} v_k(1, t) &= -i_k(1, t) + f(v_k(1, t)) - \frac{1}{R}(v_{k+1} - 2v_k + v_{k-1}) \\
&k = 1, \dots, N \pmod{N}
\end{aligned} \tag{36}$$

Since $v_k(\lambda, t) \equiv \text{const}$, $i_k(\lambda, t) \equiv \text{const}$ at equilibrium, the same equilibria as previously are obtained for $E \equiv \text{const}$. The same manipulation as in the previous section will give the following system

$$\begin{aligned}
T \frac{d\nu_k}{dt} &= -\nu_k - \delta_0 g(\nu) - 2\rho_0 \eta^2(t - \tau) + \delta_c \delta_0 (\nu_{k+1} - 2\nu_k + \nu_{k-1}) \\
\eta_k^2(t) &= \rho_0 \eta_k^2(t - \tau) + \nu_k(t)
\end{aligned} \tag{37}$$

with the same notations as previously and with $\delta_c = R_0/R$ - a coupling coefficient. We take for the local systems the same Liapunov functional defined by (26) and the Liapunov functional for the overall system as

$$\tilde{\mathcal{V}}(\xi_1, \dots, \xi_N; \phi_1(\cdot), \dots, \phi_N(\cdot)) = \sum_1^N \mathcal{V}(\xi_k, \phi_k(\cdot)) \tag{38}$$

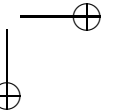
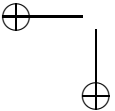
If differentiated along the solutions of (37) and after using once more Rearrangement Inequality No. 368 [7] we obtain

$$\tilde{\mathcal{W}}(\xi_1, \dots, \xi_N; \phi_1(\cdot), \dots, \phi_N(\cdot)) = \sum_1^N \mathcal{W}(\xi_k, \phi_k(\cdot)) \tag{39}$$

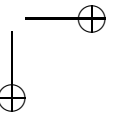
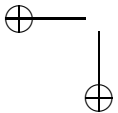
Consequently dissipativeness of the subsystems implies dissipativeness of the overall coupled system.

5 Conclusion:the next steps

This paper started from the basic notions concerning synchronization and dissipativeness as dynamical systems properties. Among dissipativeness notions, dissipativeness in the sense of Levinson turned out to be the most suitable for this type of problems. The paper deals next with a ring of tunnel diode local oscillators - the standard LC oscillator and the high frequency Nagumo-Shimura oscillator where the coil is replaced by a lossless LC transmission line). Both for the local oscillators and for the coupled systems Liapunov function(al)s are shown to ensure dissipativeness in the sense of Levinson. This property ensures the existence of a global compact attractor of the system what corresponds to synchronization in the sense of [4, 5, 6].

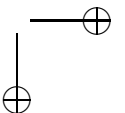
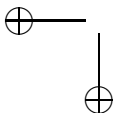


Our point of view is that the structure considered in this paper may be viewed as a genuine benchmark problem for synchronization topics. Among the next steps in this research we consider analysis of the structure based on the more general coupling (3). Another one is concerned with existence of periodic oscillations and therefore, with bifurcations (local or global)



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