Symmetric Symmetrizers and Some Applications

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1 Symmetric Symmetrizers

The Left Symmetric Symmetrizer (L-SySy) of a square matrix $A \in \mathbb{R}^{n \times n}$ is a matrix $X$ satisfying

\[ X' = X \quad \text{symmetry} \quad (1) \]
\[XA = A'X \quad \text{symmetrizer} \quad (2) \]

The problem of finding symmetric symmetrizers arises in various problems of applied linear algebra and linear system theory. Obviously, we can also define a Right Symmetric Symmetrizer (R-SySy) of a matrix. Moreover if $X$ is an L-SySy of $A$, then $X$ is a R-SySy of $A'$. This perhaps should make a case for the more palindromic acronym “SySyS". In this paper their construction and some of their properties and applications are explored.

Example:
Consider the matrix $A$ in the observability canonical form

\[ A := \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}. \]

All symmetrizers of $A$ are given by

\[ X = y \begin{bmatrix} -a_2x & a_2 - a_1a_2x \\ a_2 - a_1a_2x & a_1 + (a_2 - a_1^2)x \end{bmatrix}, \]

where $x$ and $y$ are arbitrary, augmented with the set

\[ X = y \begin{bmatrix} 0 & a_1 \\ a_1 & a_2 \end{bmatrix}, \]

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for arbitrary $y$.

Note that the set of symmetrizers of matrix $A$ is a subspace of the set of symmetric matrices. In the above example, this set is spanned by

\[
\left\{ \begin{bmatrix} a_2^2 & a_1a_2 \\ a_1a_2 & a_1^2 - a_2 \end{bmatrix}, \begin{bmatrix} 0 & a_2 \\ a_2 & -a_1 \end{bmatrix} \right\}.
\]

Note that the first matrix has determinant $-a_2$, therefore if $a_2 < 0$, this symmetrizer is positive definite, and indefinite if $a_2 > 0$. The second symmetrizer has a negative determinant. Does a positive definite symmetrizer exist in the case that $a_2 > 0$? Consider a linear combination

\[
\begin{bmatrix} a_2^2 & a_1a_2 \\ a_1a_2 & a_1^2 - a_2 \end{bmatrix} + t \begin{bmatrix} 0 & a_2 \\ a_2 & -a_1 \end{bmatrix}.
\]

The characteristic polynomial of the resulting symmetrizer is

\[s^2 + (-a_2^2 + a_2 - a_1^2 + a_1t)s - a_2^3a_1t - a_2^2t^2.
\]

It follows that the symmetrizer is positive definite if a real $t$ exists such that

\[
\begin{align*}
-a_2^2 + a_2 - a_1^2 + a_1t < 0 & \quad (3) \\
-a_2^3 - 3a_2^2a_1t - a_2^2t^2 < 0 & \quad (4)
\end{align*}
\]

It is easily shown that, if $9a_1^2 - 4a_2 < 0$, a positive definite symmetrizer does not exist.

**Theorem 1.** If $A$ has a purely real spectrum with all its eigenvalues either non-negative or non-positive, then a positive semi-definite symmetric symmetrizer exist.

**Proof.** Let $\lambda_i > 0$ be the eigenvalues of $A$, possibly of multiplicity higher than one. Then the corresponding left-eigenvectors may be assumed real. From

\[v'_i A = \lambda_i v'_i;
\]

we obtain

\[\sum_i v_i v'_i A = \sum_i \lambda_i v_i v'_i.
\]

Hence, letting $X = X' = \sum_i v_i v'_i$, we get $XA = A'X$. \qed

It is clear from the example that the condition is not necessary.

**Theorem 2.** If $A$ has a symmetric symmetrizer $X$, then generically the only other symmetrizers of the form $ZX$ are scalar multiples of $X$.

**Proof.** Since $XA$ is symmetric it has an eigen decomposition $XA = U\Lambda U'$ where $U$ is orthogonal and $\Lambda$ real diagonal (but not necessarily positive). Any matrix of the form $Z = U\Omega U'$ where $U$ is an arbitrary real diagonal matrix satisfies

\[ZX A = U\Omega U' U A U' = U(\Omega \Lambda)U' = A'(ZX)'.
\]
so that \( ZX \) is also a symmetrizer of \( A \). It is however not itself symmetric in general. But \( ZX \), with both \( Z \) and \( X = V\Sigma V' \) symmetric, is symmetric if
\[
U\Omega U'V\Sigma V' = V\Sigma V'U\Omega U'.
\]
or, equivalently if
\[
\Omega U'V\Sigma V'U = U'V\Sigma V'U\Omega.
\]
Since \( W = U'V \) is also orthogonal, this means
\[
\Omega(W\Sigma W') = (W\Sigma W')\Omega,
\]
implying \((\Omega_{ii} - \Omega_{jj})(W\Sigma W')_{ij} = 0\). If all entries of \( W\Sigma W' \) are nonzero, the diagonal matrix \( \Omega \) can only be a multiple of the identity matrix. In turn this implies that generically \( ZX \) is a symmetric symmetrizer in addition to \( X \) only if \( Z \) is a multiple of \( I \).

In this paper we are interested in determining all \( \text{SySy's} \) of a given matrix, and determining when a definite \( \text{SySy} \) may exist. In the next section, we consider the construction of a \( \text{SySy} \) for an arbitrary matrix and discuss some properties. The remaining sections look at some applications of \( \text{SySy's} \): System Identification in Section 3, Delay Systems in Section 4 and Poly-Systems (flocks) in Section 5.

### 2 Construction and Properties of \( \text{SySy's} \)

Given \( A \) consider for arbitrary \( b \in \mathbb{R}^{n \times 1} \) and \( c \in \mathbb{R}^{1 \times n} \) the siso system \((A, b, c)\). If it is minimal, then its Hankel matrix \( H \) has full rank. We note that not for every matrix \( A \) vectors \( b \) and \( c \) can be found that makes it minimal. A trivial counter example is given by the null matrix.

**Theorem 3.** If \( A \) is simple, then it has a symmetric symmetrizer \( X \), given by
\[
X = R^{-T}(A, b)O(A, c).
\]

**Proof.** We assume that \( b \) and \( c \) are such that \((A, b, c)\) is minimal. The Hankel matrix factors as \( H(A, b, c) = O(A, c)R(A, b) \). Since for a siso system the Hankel matrix is symmetric, we have \( O(A, c)R(A, b) = R'(A, b)O'(A, c) \) Hence, the matrix \( X(b, c) = R^{-T}(A, b)O(A, c) \) is symmetric. It is also a symmetrizer, since \( X(b, c)A = R^{-T}(A, b)O(A, c)A \) and \( A'X'(b, c) = A'O'(A, c)R^{-1}(A, b) \). But the shifted Hankel matrix \( \hat{H} \) is symmetric as well and is given by \( \hat{H} = O(A, c)AR(A, b) \). Again, \( O(A, c)AR(A, b) = R'(A, b)A'O'(A, c) \), and by minimality,
\[
R^{-T}(A, b)O(A, c)A = A'O^{-T}(A, c)R'(A, b),
\]
thus \( A'X(b, c) = X(b, c)A \).  

In system theory, the matrix \( X \) satisfying the above property is known as the Bezoutian [4]. In the above, we have parameterized the \( \text{SySy} \) by \( b \) and \( c \), which seems
to indicate that there are 2n degrees of freedom. We know from the introductory example that this is not the case. The given parameters are therefore redundant.

**Theorem 4.** There are n degrees of freedom for SySy’s.

**Proof.** Assume that $(A, b)$ is in the reachability canonical form, then $b$ is fixed while the $c$ vector contains now all free parameters. Furthermore, we can always transform the given matrix to this form. 

Note that the SySy given by the theorem is not necessarily positive definite. We already know that a positive definite one may not exist. This prompts the following question: “If $A$ is known to have a positive definite SySy. How can it be constructed?” (i.e., in view of the above theorem, how does the choice of $b$ lead to the pos. def one?).

### 2.1 Relation to Nonsymmetric Symmetrizers

Given an arbitrary matrix $A$, its transpose, $A'$ is a Symmetrizer, but it is obviously not symmetric in general. We also note that $A'AA'$ is a symmetrizer. In fact all matrices of the form $(a_i \in \mathbb{R})$

$$M(A; a) = \sum_{i=0}^{n} a_i (A'A)^{n-i}A'$$

are symmetrizers. Without loss of generality, we normalize $a_0 = 1$. We now sketch some problems we found interesting:

**Interesting Problem 1:** Can we find an $a$ such that $M(A; a) = M'(A; a)$? If not in general, for which class of $A$ can it be done?

**Example:**

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}.$$  

We find

$$M(A; a) = \begin{bmatrix} 2+a_1 & 2 \\ 6+a_1 & 2a_1 \end{bmatrix}.$$  

Clearly, letting $a_1 = -4$ gives a way to do so. So it is interesting to scrutinize this a bit further as it may give a general method to compute SySy’s. This opens another interesting problem:

**Interesting Problem 2:** Can we find an $a$ such that $M(A; a) = M'(A; a) > 0$?  

We already know the answer: it cannot be done in general. If not in general, for which class of $A$ can it be done?

**Partial answer:** If $A$ has the SVD: $A = U\Omega V'$, then $M(A; a)$ is of the form

$$M = Va(\Omega^2)\Omega U' = Vb(\Omega)U'$$
for some polynomial \( b \). The problem boils down to checking conditions on \( U \) and \( V \) (hence \( A \)) such that \( V \Delta U' \) is symmetric for some diagonal matrix \( \Delta \). That’s easy:

\[
(U'V)\Delta = \Delta(V'U),
\]

with \( W = U'V \) a constructible orthogonal matrix, we need to find \( \Delta \) diagonal such that \( W\Delta = \Delta W' \). It gives \( W_{ij}\delta_j = \delta_iW_{ji} \). What next? Note that we’re asking: “When does an orthogonal matrix have a diagonal symmetrizer?”

**Conjecture:** \( \Delta \) must be a sign matrix if \( W \) is nontrivial.

If correct, then the Interesting Problem 2 is solved by finding \( b \) such that \( b(\Omega) = \Delta \).

The following is readily shown:

**Theorem 5.** If \( X \) is a SySy for \( A \), then so are all of \( p(A)X \), where \( p \) is an arbitrary polynomial.

Of course, by virtue of the Cayley-Hamilton theorem it suffices to consider polynomials of degree \( n - 1 \).

### 2.2 Properties of SysSy's

We denote by the SySy\((A)\) the property of being a SySy for \( A \), i.e., \( AX = XA' \).

1. If \( X \) is a SySy\((A)\), then \( X^{-1} \) is a SySy\((A')\).
   
   Indeed, \( AX = Y = XA' \) implies \( X^{-1}A = X^{-1}YX^{-1} = A'X^{-1} \).

2. If \( (P, Q) \) is a Lyapunov pair for \( A \), then \( (XPX, XQX) \) is a Lyapunov pair for \( A' \).
   
   **Proof:**

   \[
   AP + PA' + Q = 0
   
   X^{-1}APX^{-1} + X^{-1}PA'X^{-1} + X^{-1}QX^{-1} = 0
   
   A'X^{-1}PX^{-1} + X^{-1}PX^{-1}A + X^{-1}QX^{-1} = 0
   
   A'(P'P + PA + Q) = 0.
   \]

3. If \( X \) is SySy\((A)\), then \( Y = AX \) is SySy\((A)\).
4. \( \{X, AX, A^2X, \ldots, A^{n-1}X\} \) are SySy\((A)\).
5. \( \{a(A)X \} \) where \( a(p) \in \mathbb{R}^{n \times n}[p] \) are SySy\((A)\).
6. All SySy\((A)\) have the same signature.
   
   **Proof:** If \( a(p) = a_1p^{n-1} + a_2p^{n-2} + \cdots + a_{n-1}p + a_n \), then

   \[
a(A)X = \sum a_{2i}A^iX(A')^i + \sum a_{2i+1}A^iY(A')^i.
   \]
7. All SySy(A) are of the above polynomial form, i.e., if $AZ = ZA'$, then $Z = z(A)X$ for some polynomial $z$.

Proof: $Z = Z'$ has $n(n + 1)/2$ degrees of freedom. $AZ = ZA'$ provides us with $n(n - 1)/2$ unknowns. Hence, there are at most $n$ degrees of freedom.

8. SySy’s and similarity:
   If $AX = XA'$, i.e., $X \in \text{SySy}(A)$, then $TXT' \in \text{SySy}(TAT^{-1})$.

9. If $A = \Lambda$ is diagonal and simple, then $\Lambda X = X\Lambda \Rightarrow (\lambda_i - \lambda_j)X_{ij} = 0$. Hence, $X$ is diagonal. Consequently, all SySy’s are diagonal. Since $I$ is SySy($\Lambda$), $a(\Lambda)$ is SySy($\Lambda$).
   What happens if $A$ is not simple? Example:

   $$\Lambda = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

   Then
   $$\Lambda X = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x_1 & x \\ x & x_2 \end{bmatrix} = \begin{bmatrix} \lambda x_1 + x & \lambda x + x_2 \\ \lambda x & 0 \end{bmatrix}$$

   so that $X$ is a SySy iff $x_2 = 0$. Thus
   $$X = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow Y = \lambda X + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

2.3 Open Problem

If $X$ is indefinite, give conditions on $A$ and $X$ for the existence of a polynomial $a(\cdot)$ such that $a(A)X(A')$ is definite.

Example:

$$X = \begin{bmatrix} 1 & -2 \\ -2 & \gamma \end{bmatrix}, \quad A = \begin{bmatrix} \gamma & 1 \\ 1 & \gamma \end{bmatrix},$$

then

$$AXA' = \begin{bmatrix} -2 & \gamma^2 \\ \gamma^2 & \gamma^2 \end{bmatrix}$$

and

$$\alpha X + \beta AXA' = \begin{bmatrix} \alpha - 2\beta & -2\alpha + \beta \gamma^2 \\ -2\alpha + \beta \gamma^2 & \alpha - 2\beta \end{bmatrix}.$$

The conditions lead to $\alpha > 2\beta$ and $\beta \gamma^2 > 2\alpha$. Hence, $\gamma^2 > 4$.

2.4 Feedback - Rank one modification

Given that $X$ is SySy($A$), what are the elements of SySy($A - bk'$)?

Let us assume that the pair ($A, b$) is reachable, and let $R$ be the reachability matrix. Consider now without loss of generality, for some $Z = Z'$ to be determined. the equation

$$(A - bk')(X + RZR') = (X + RZR')(A' - kb').$$
It follows that (if $Y = AX$) $Y - bk'(X + RZR') + ARZR'$ must be symmetric. The second matrix has rank one, thus symmetry can only be preserved if the third matrix has rank one. Thus, $Z$ is of the form $Z = qq'$. Symmetry requires that

$$bk'(X + Rqq'R') = -Rqq'R'A.$$ 

Thus $b$ is parallel to $Rq$, and $q$ is parallel to $e_1$. It follows that for some scalar $\zeta$

$$k'(X + Ret_1'e_1'R') = -e_1'R'A'.$$

This gives

$$Xk = \zeta(k'b) = -\zeta Ab.$$

Set $k'b = \beta$, then

$$k\beta = -\zeta X^{-1}(\beta b + Ab), \quad (5)$$

and

$$\beta + \beta\zeta(b'X^{-1}b) + \zeta(b'X^{-1}Ab) = 0. \quad (6)$$

There is thus one degree of freedom. We conclude with:

**Theorem 6.** If $X = SySy(A)$, then $X + \zeta\beta bb'$ is $SySy(A - bk')$, where $\zeta$, and $k$ relate to $\beta$ as shown in (5) and (6).

We now proceed next to some applications:

### 3 An Identification Problem

The following problem was posed as a special type of identification problem, relevant in certain multiple robot coordination applications [2]. Given matrix $L$ and $c$, and an unknown diagonal matrix $\Gamma$. Given observations of the autonomous system

$$\dot{x} = \Gamma Lx, \quad y = cx$$

determine the diagonal matrix $\Gamma$. See also [5]

**Solution:**

From knowledge of $y(\cdot)$ we have all its derivatives at zero available. Thus, let $y_i$ be the derivative of $y(t)$ at $t = 0$. Form the Hankel matrix and the shifted Hankel matrix of these derivatives

$$Y = \begin{bmatrix} y_0 & y_1 & \cdots & y_{n-1} \\ y_1 & y_2 & & y_n \\ \vdots & & \ddots & \vdots \\ y_{n-1} & y_n & \cdots & y_{2n-2} \end{bmatrix}, \quad \tilde{Y} = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_2 & y_3 & \cdots & y_{n+1} \\ \vdots & & \ddots & \vdots \\ y_n & y_{n+1} & \cdots & y_{2n-1} \end{bmatrix}$$

We can factor $Y$ and $\tilde{Y}$ as: $Y = O[x_0, \cdots, x_{n-1}]$, $\tilde{Y} = OA[x_0, \cdots, x_{n-1}]$, where $x_0, \cdots, x_{n-1}$ is the state sequence for an arbitrary similar realization. Consider
the similar system given in the observability canonical form. We know that the observability matrix in this form is the identity (for siso) systems. Hence, we get

\[ A = \bar{Y}Y^{-1} \]

This matrix is similar to \( \Gamma L \), and is of the form

\[
A = \begin{bmatrix}
0 & 1 & \cdots & 0 \\
0 & 0 & \ddots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-a_n & -a_{n-1} & \cdots & -a_1
\end{bmatrix}
\]

Thus, for some invertible \( T \),

\[ TAT^{-1} = \Gamma L. \]

We know \( L \) and \( A \), and need to determine \( T \) and \( \Gamma \). Equivalently,

\[ LTAT^{-1} = L\Gamma L \]

is symmetric, and thus

\[ LTAT^{-1} = T^{-T}A'T'L', \]

and again

\[ T'LTA = A'T'L'T, \]

Since for the Laplacian, \( L = L' \), the above implies that \( T'LT \) is a SySy for \( A \). hence it can be constructed via the Bezoutian method in Section 1.2. So, in principle, we have \( X = T'LT \). If \( L \) is known, this gives \( T \).

### 3.1 Solution of \( A = T'BT \)

Let \( A \) and \( B \) be symmetric. Unless they both have the same signature, a solution to \( A = T'BT \) does not exist. Hence let \( \Sigma \) be their common signature. There exist matrices \( U \) and \( V \) such that

\[ A = U\Sigma U', \quad B = V\Sigma V'. \tag{7} \tag{8} \]

**Theorem 7.** If \( A \) and \( B \) have the same signature \( \Sigma \), then the solutions of \( A = T'BT \) are given by

\[ T = VW\Sigma V'U'. \]

where \( W\Sigma \) is an arbitrary \( \Sigma \)-orthogonal matrix, i.e., \( W\Sigma W\Sigma' = \Sigma \).

**Proof.** Indeed, if \( T \) belongs to the above class, then

\[ U = T'VW\Sigma. \]
Hence, \( T = VW\Sigma U' \). It is readily verified that indeed
\[
T'BT = UW\Sigma U'V'BVW\Sigma U' \\
= UW\Sigma V'V\Sigma V'BW\Sigma U' \\
= UW\Sigma \Sigma U' \\
= U\Sigma U' \\
= A.
\]

Conversely, to show that there are no other solutions in \( GL_n(\mathbb{R}) \), consider two arbitrary solutions \( T \) and \( S \). It follows from \( A = T'BT = S'BS \) that \( X = ST^{-1} \) must satisfy \( B = X'BX \). By the decomposition of \( B \), this implies \( V\Sigma V' = X'V\Sigma V'X \).

Hence \( Y = V'XV \) satisfies
\[
\Sigma = V'X'V\Sigma V'XV = Y'\Sigma Y.
\]

Since \( \Sigma \) is a signature matrix, the only solutions for \( Y \) are also signature matrices are the \( \Sigma \)-orthogonal matrices, \( W\Sigma \). Consequently, \( X = VW\Sigma V' \) and \( S = VW\Sigma V'T = VW\Sigma \Omega U' \). which belongs again to the same class.

### 4 Application to Delay Systems

A sufficient condition for asymptotic stability of the delayed system
\[
\dot{x}(t) = Ax(t) + Bx(t - \tau)
\]
regardless of the value of \( \tau > 0 \) is that there exists a triple of positive definite symmetric matrices \((P, Q, R)\) such that
\[
A'P + PA + Q + P BQ^{-1}B'P + R = 0. \quad (9)
\]

It is obvious that a necessary condition for the existence of a triple \((P, Q, R)\) satisfying this Riccati equation is that \( A \) is Hurwitz. It is also known that if \( \tau \to \infty \) this condition is necessary for asymptotic stability for all values of the delay [7].

**Definitions:**

1. If the delay system is asymptotically stable for some \( \tau \) and \( A, B \), we say that the triple \((A, B, \tau)\) is asymptotically stable. A NASC is that all roots of the transcendental equation, \( sI - A - Be^{-s\tau} = 0 \), are in the open left hand plane.

2. If \((A, B)\) satisfies the above sufficient Riccati equation condition, the system is called Riccati-stable.

**Conjectures:**

1. If \((A, B, \tau)\) is asymptotically stable and \((A, b)\) reachable, with \( A\) non-Hurwitz, then there exist a gain \( k \) such that \((A - bk, B, \tau)\) is asymptotically stable, with \( A - bk \) Hurwitz.
2. If \((A, B, \tau)\) is asymptotically stable and \((A, b)\) reachable, with \(A\) non-Hurwitz, then there exist a gain \(k\) such that \((A - bk, B)\) is Ricatti stable. with \(A - bk\) Hurwitz.

Note that the second conjecture is more daring as - if true - it implies asymptotic stability for any delay.

### 4.1 SySy and Ricatti stability

The following sufficient condition for Riccati stability was obtained:

**Theorem 8.** Let \(A\) and \(B\) have full rank. If \(X > 0\) is a SySy\((A)\) with \(A'X = -Y = XA\), and \(0 < AY^{-1}A' < BY^{-1}B'\), then the pair \((A, B)\) is Ricatti stable. 

**Proof.** Since \(X \in \text{SySy}(A)\), there exists \(XA = -Y = A'X\). Since \(A\) and \(X\) have full rank by assumption, \(Y\) has full rank too. Consider now \((XA + Y)(A'X + Y) = 0\). Expanding,

\[
A'X + XA + XAY^{-1}A'X + Y = 0
\]

Let now \(Q \overset{\text{def}}{=} B'YA^{-1}B\), which has full rank. It follows that \(BQ^{-1}B' = AY^{-1}A'\). Thus \(A'X + XA + XAY^{-1}B'X + Y = 0\). Set \(R = Y - Q = B'(B^{-1}YA^{-1}B - A'Y A^{-1})B\). By the hypothesis, \(R > 0\), hence the Riccati condition (9) holds for \((P = X, Q, R)\), thus proving the statement. 

### 5 Application to Weakly Coupled Poly-Systems

A weakly coupled poly-system (WCPS) is a system of the form

\[
\dot{x}_i = Ax_i + bcu, \quad i = 1, \ldots, N
\]

\[
u = f_{\text{Sym}}(x_1, \ldots, x_N)
\]

For instance, thinking of \(c\) as a coupling by the mean field gives \(c = \gamma 1'x\), and

\[
\dot{x}_i = Ax_i + \gamma 1' x.
\]

The WCPS system is

\[
\frac{d}{dt} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} A & \vdots & \vdots \\ \vdots & \ddots & \vdots \\ \vdots & \vdots & A \end{bmatrix} + \begin{bmatrix} b \\ \vdots \\ b \end{bmatrix} \left[ h, \ldots, h \right] \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}.
\]

Introducing Kronecker products,

\[
\dot{x} = (I \otimes A + \gamma 11' \otimes bh)x
\]
Consider now nonuniform rate changes to
\[ \dot{x}_i = \Gamma A x_i + b c f_{\text{Sym}}(x), \quad i = 1, \ldots, N. \] (13)

Thus
\[ \dot{x} = (\Gamma \otimes A + \gamma 11' \otimes bh)x \]

Under what conditions will this dynamic matrix be positive semi-definite? We need nonnegativity for all \( x \) and \( y \) of:
\[
(x' \otimes y')(\Gamma \otimes A + \gamma 11' \otimes bh)(x \otimes y)
= (x' \otimes y'A + \gamma x'11'x \otimes y'bh)(x \otimes y)
= ||x||^2 y'Ay + \gamma (x'1)(y'bh). 
\]

Clearly, \( A \) should be positive definite, and
\[
\forall y: \ y'Ay - \gamma N |\ y'bhy| > 0. 
\]

Equivalently,
\[
A - \gamma N bh > 0. 
\]

5.1 Delayed WCPS

We assume that there is a delay in the feedback of the mean field (due to communication, and or computation). Now,
\[ \dot{x}(t) = \Gamma \otimes A x(t) + \gamma 11' \otimes bh x(t - \tau). \]

By applying the Riccati stability criterion and properties of Kronecker products, interesting relations are obtained [3] (they will be reported elsewhere). These results are very relevant to control with capacity constrained and delayed feedback channels (as for instance in networked control) [1].

6 Conclusions

The notion of a symmetric symmetrizer (SySy) was discussed. Some properties of SySy’s were shown, and a characterization of the set of all SySy’s of a given matrix was given. We posed the question of characterizing the existence of positive definite definite SySy’s. While showing that the latter may not always exist, we gave examples that many interesting problems may be solved if a definite SySy can be found. Some of these applications are in identifying a convergence speed for flocks obeying weighted Laplacian dynamics, and in characterizing the Riccati stability for time delay systems.
Bibliography


