

# Iterative algorithms for stabilizing solutions of game theoretic Riccati equations of stochastic control

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## Abstract

We propose two iterative procedures for the computation of the stabilizing solution of coupled generalized algebraic Riccati equations with indefinite quadratic term - such equations appear in the solution of stochastic  $H_\infty$  control problems. The method used in this paper is a generalization of the defect correction method introduced in [17] and extends to this framework the algorithm derived in [16] for game-theoretic (deterministic) algebraic Riccati equations.

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# 1 Introduction

Consider the system of coupled algebraic Riccati equations:

$$A_0^T(i)X(i) + X(i)A_0(i) + \sum_{k=1}^r A_k^T(i)X(i)A_k(i) + \sum_{j=1}^N q_{ij}X(j) - X(i)(B_2(i)B_2^T(i) - \gamma^{-2}B_1(i)B_1^T(i))X(i) + C^T(i)C(i) = 0, \quad 1 \leq i \leq N. \quad (1.1)$$

This kind of system of coupled Riccati equations appears in connection with the solution of  $H_\infty$  control problems with level of attenuation  $\gamma > 0$  for the following controlled system:

$$dx(t) = [A_0(\eta_t)x(t) + B_1(\eta_t)v(t) + B_2(\eta_t)u(t)]dt + \sum_{k=1}^r A_k(\eta_t)x(t)dw_k(t) \quad (1.2)$$

with the cost functional

$$J(u) = E \left[ \int_0^{t_f} x^T(t)C^T(\eta_t)C(\eta_t)x(t) + |u(t)|^2 - \gamma^2|v(t)|^2 \right] dt \quad (1.3)$$

where  $\{\eta_t\}_{t \geq 0}$  is a right continuous Markov process on a given probability space  $(\Omega, \mathcal{F}, \mathcal{P})$  and the state space  $\mathcal{D} = \{1, 2, 3, \dots, N\}$  and the transition probability matrix  $P(t) = e^{Qt}$ ,  $t \geq 0$ ; here  $\{w_k(t)\}_{t \geq 0}$  are scalar Wiener processes. In (1.2), (1.3)  $x(t) \in \mathbf{R}^n$  is the state vector,  $u(t) \in \mathbf{R}^{m_2}$  is the input control and  $v(t) \in \mathbf{R}^{m_1}$  is the vector of the exogenous disturbances. In the case  $\mathcal{D} = \{1\}$  the corresponding stochastic  $H_\infty$  control problem was investigated in [7] and [14, 19]. For the case  $A_k(i) = 0, 1 \leq k \leq N, i \in \mathcal{D}$  we mention [8] and [18]. The general case of systems subject to both multiplicative white noise and Markovian jumping was considered in [9, 13] and in the discrete-time case in [4]. The theory of discrete-time Markovian jump linear systems has been nicely summarized in [3].

In few words, the  $H_\infty$  control problem described by (1.1)-(1.3) asks to find a control  $\tilde{u}(t)$  such that  $J(\tilde{u}) < 0$  for arbitrary exogenous disturbances  $v(t)$ . In the case of finite time horizon a solution of the  $H_\infty$  control problem described by (1.2)-(1.3) is given by

$$u(t) = -B_2^T(\eta_t)X(t, \eta_t)x(t), \quad 0 \leq t \leq t_f \quad (1.4)$$

while if  $t_f = \infty$ ,  $u(t)$  is given by

$$u(t) = -B_2^T(\eta_t)\tilde{X}(\eta_t)x(t). \quad (1.5)$$

In (1.4) the function  $X(\cdot, i)$  denotes the solution of the terminal value problem:

$$\frac{d}{dt}X(t, i) + R_i(X(t)) = 0 \quad (1.6)$$

$X(t_f, i) = 0, 1 \leq i \leq N$ , moreover, in (1.5),  $\tilde{X}(i), 1 \leq i \leq N$  is the stabilizing solution of (1.1).

In (1.6)  $R_i(X)$  is defined by the left hand side of (1.1). The solution of the terminal value problem (1.6) may be computed using any numerical procedure for the solution of a Cauchy problem. The attenuation level  $\gamma > 0$  is chosen such that the solution  $X(\cdot, i)$  to be well defined on the whole interval  $[0, t_f]$ .

The computation of the stabilizing solution  $\tilde{X}$  of (1.1)- in particular the formulation of sufficient conditions for the existence and convergence of an approximating sequence - is more complicated than the computation of the solution of problem (1.6). Unlike in the deterministic framework in the stochastic case, the alternative methods for the solution of (1.6) as the methods based on invariant subspaces cannot be used. That is why it is expected that the techniques based on iterative procedures are more adequate in the stochastic case. Unfortunately the iterations based on the Newton-Kantorovich algorithm cannot be directly applied to compute the stabilizing solution of (1.1) due to the indefinite sign of the quadratic part of that equation.

In this work, we propose an iterative procedure to compute the stabilizing solution of (1.1) starting from the ideas used for the first time in connection with perturbed or linearly coupled Riccati equations arising in controlled systems subject to Markovian jumping in [2, 1]. The algorithm proposed in our paper is a modified defect correction method (see [17]) and extends to the present framework the algorithm derived in [16] for the game-theoretic Riccati equations in deterministic case. To motivate the interest for the computation of the stabilizing solution of the game theoretic Riccati equations we mention that the solution of the  $H_\infty$  problem given by a control of type (1.5) based on the stabilizing solution of (1.1) has some additional useful properties. For example, a such control minimizes the effect of the additive white noise perturbations (see [6, 8, 10]). Hence, the stabilizing solution of (1.1) allows us to construct the solution of some mixed  $H_2/H_\infty$  control problems in the case of full-state measurements.

## 2 Some Preliminaries

Equation (1.1) is written as:

$$\begin{aligned} & A^T(i)X(i) + X(i)A(i) + \Pi_i(X) \\ & -X(i)[B_2(i)B_2^T(i) - \gamma^{-2}B_1(i)B_1^T(i)]X(i) + C^T(i)C(i) = 0 \end{aligned} \quad (2.1)$$

where

$$A(i) = A_0(i) + \frac{1}{2}q_{ii}I_n, \quad X = (X(1), X(2), \dots, X(N)) \text{ and}$$

$$\Pi_i(X) = \sum_{k=1}^r A_k^T(i)X(i)A_k(i) + \sum_{j \neq i} q_{ij}X(j).$$

We recall (see [5]) that the elements of the transition rates matrix  $Q$  satisfy  $q_{ij} \geq 0$  if  $i \neq j$  and  $\sum_{j=1}^N q_{ij} = 0$ ,  $1 \leq i \leq N$ . Thus we have  $\Pi_i(X) \geq 0$ ,  $1 \leq i \leq N$  if  $X(j) \geq 0$ ,  $1 \leq j \leq N$ .

If  $\mathcal{S}_n \subset \mathbf{R}^{n \times n}$  is the linear subspace of  $n \times n$  symmetric matrices then, by definition  $\mathcal{S}_n^N = \mathcal{S}_n \oplus \mathcal{S}_n \oplus \dots \oplus \mathcal{S}_n$ . In the following  $\mathcal{L}_{A,\Pi} : \mathcal{S}_n^N \rightarrow \mathcal{S}_n^N$  stands for the perturbed Lyapunov operator defined by

$$(\mathcal{L}_{A,\Pi} X)(i) = A^T(i)X(i) + X(i)A(i) + \Pi_i(X), \quad 1 \leq i \leq N \quad (2.2)$$

for all  $X = (X(1), X(2), \dots, X(N)) \in \mathcal{S}_n^N$ .

**Definition 2.1** A solution  $\tilde{X} = (\tilde{X}(1), \tilde{X}(2), \dots, \tilde{X}(N))$  of equation (1.1) is called stabilizing solution if the zero state equilibrium of the closed-loop system

$$dx(t) = [A_0(\eta_t) + B_1(\eta_t)\tilde{F}_1(\eta_t) + B_2(\eta_t)\tilde{F}_2(\eta_t)]x(t)dt + \sum_{k=1}^r A_k(\eta_t)X(t)dw_k(t)$$

is exponentially stable in mean square (ESMS for shortness), where

$$\begin{aligned} \tilde{F}_1(i) &= \gamma^{-2}B_1^T(i)\tilde{X}(i) \\ \tilde{F}_2(i) &= -B_2^T(i)\tilde{X}(i), \quad 1 \leq i \leq N. \end{aligned} \quad (2.3)$$

Below we shall use some results and notations from stochastic control that can be found in [13]. The next result provides a deterministic characterization of the concept of a stabilizing solution.

**Proposition 2.1** ([13]) *The following are equivalent:*

- i)  $\tilde{X} = (\tilde{X}(1), \tilde{X}(2), \dots, \tilde{X}(N))$  is the stabilizing solution of (1.1)
- ii) The eigenvalues of the perturbed Lyapunov operator  $\mathcal{L}_{A+B\tilde{F},\Pi}$  are located in the half planes  $\mathbf{C}^- = \{z \in \mathbf{C} | \text{Re}z < 0\}$ ,  $\mathcal{L}_{A+B\tilde{F},\Pi}$  being defined as in (2.2) for  $A(i) + B_1(i)\tilde{F}_1(i) + B_2(i)\tilde{F}_2(i)$  instead of  $A(i)$ .

For a scalar  $\gamma > 0$  we denote by  $\mathcal{A}(\gamma)$  the set of state feedback gains  $F = (F(1), F(2), \dots, F(N))$ ,  $F(i) \in \mathbf{R}^{m_2 \times n}$  with the property that the following system of Riccati inequality

$$\begin{aligned} (A(i) + B_2(i)F(i))^T Y(i) + Y(i)(A(i) + B_2(i)F(i)) + \Pi_i(Y) \\ + \gamma^{-2}Y(i)B_1(i)B_1^T(i)Y(i) + C^T(i)C(i) + F^T(i)F(i) < 0, \end{aligned} \quad (2.4)$$

has a solution  $Y = (Y(1), Y(2), \dots, Y(N))$  with  $Y(i) > 0$ ,  $1 \leq i \leq N$ . Based on Theorem 10 in Chapter 6 in [13] one can derive that  $F \in \mathcal{A}(\gamma)$  if and only if the control  $u(t) = F(\eta_t)x(t)$ , is a solution of the  $H_\infty$  control problem described by (1.2)-(1.3) and the attenuation level  $\gamma$ .

Consider the linear stochastic system

$$dx(t) = A_0(\eta_t)x(t) + \sum_{k=1}^r A_k(\eta_t)x(t)dw_k(t), y(t) = C(\eta_t)x(t)$$

obtained from (1.2)-(1.3) ignoring the inputs  $u(t)$  and  $v(t)$ . Reasoning as in [7] and [8] one obtains the following result which provides conditions which guarantee the

existence of the stabilizing solution of (1.1). Also, some properties of the stabilizing solution which characterizes it in the set of all nonnegative solutions of (1.1), are displayed.

**Theorem 2.2** *Assume that the system (2.5) is stochastically detectable. Then the following statements are equivalent:*

- i)  $\mathcal{A}(\gamma)$  is not empty;
- ii) *The system of coupled algebraic Riccati equations (1.1) has a stabilizing solution  $\tilde{X} = (\tilde{X}(1), \tilde{X}(2), \dots, \tilde{X}(N))$ ,  $\tilde{X}(i) \geq 0$ ,  $1 \leq i \leq N$ .*

Moreover this stabilizing solution has the following properties:

- ( $\alpha$ )  $\tilde{X}(i) \leq \hat{X}(i)$ ,  $1 \leq i \leq N$  for any solution  $\hat{X} = (\hat{X}(1), \hat{X}(2), \dots, \hat{X}(N))$  of (1.1) with  $\hat{X}(i) \geq 0$ ,  $1 \leq i \leq N$ .
- ( $\beta$ ) For any  $F = (F(1), F(2), \dots, F(N)) \in \mathcal{A}(\gamma)$   $\tilde{X}(i) \leq Y^F(i)$ ,  $1 \leq i \leq N$ , where  $Y^F = (Y^F(1), Y^F(2), \dots, Y^F(N))$  is the stabilizing solution of

$$\begin{aligned} & (A(i) + B_2(i)F(i))^T Y(i) + Y(i)(A(i) + B_2(i)F(i)) + \Pi_i(Y) \\ & + \gamma^{-2} Y(i) B_1(i) B_1^T(i) Y(i) + C^T(i) C(i) + F^T(i) F(i) = 0, 1 \leq i \leq N. \end{aligned}$$

### 3 The main results

The next result extends to this general framework the result of Lemma 2 in [16].

**Lemma 3.1** *Let  $X = (X(1), X(2), \dots, X(N))$ ,  $Z = (Z(1), Z(2), \dots, Z(N)) \in \mathcal{S}_n^N$  be such that*

$$\begin{aligned} & [A(i) + \gamma^{-2} B_1(i) B_1^T(i) X(i) - B_2(i) B_2^T(i) X(i)]^T Z(i) \\ & + Z(i) [A(i) + \gamma^{-2} B_1(i) B_1^T(i) X(i) - B_2(i) B_2^T(i) X(i)] \\ & + \Pi_i(Z) - Z(i) B_2(i) B_2^T(i) Z(i) + R_i(X) = 0. \end{aligned} \quad (3.1)$$

Let  $F_2 = (F_2(1), F_2(2), \dots, F_2(N)) \in \mathcal{A}(\gamma)$ ,  $Y^{F_2} = (Y^{F_2}(1), Y^{F_2}(2), \dots, Y^{F_2}(N))$  be the stabilizing solution of the corresponding equation (2.6).

Define

$$\begin{aligned} \check{A}(i) &= A(i) + \gamma^{-2} B_1(i) B_1^T(i) (X(i) + Z(i)) + B_2(i) F_2(i) \\ \hat{A}(i) &= A(i) + \gamma^{-2} B_1(i) B_1^T(i) X(i) + B_2(i) F_2(i), 1 \leq i \leq N. \end{aligned}$$

Then the following hold:

- i)  $Y^{F_2}(i) \geq X(i) + Z(i)$ ,  $1 \leq i \leq N$  if the eigenvalues of the Lyapunov operator  $\mathcal{L}_{\hat{A}, \Pi}$  are located in the half plane  $\mathbf{C}^-$ .
- ii) The eigenvalues of the Lyapunov operator  $\mathcal{L}_{\check{A}, \Pi}$  are in the half plane  $\mathbf{C}^-$  if  $Y^{F_2}(i) \geq X(i) + Z(i)$ ,  $1 \leq i \leq N$ .

**Proof.** Hint: *i*) Let  $U(i) = Y^{F_2}(i) - X(i) - Z(i), 1 \leq i \leq N$ . By direct computation one obtains that

$$\hat{A}^T(i)U(i) + U(i)\hat{A}(i) + \Pi_i(U) + W(i) = 0 \quad (3.2)$$

$1 \leq i \leq N$ , where  $W(i) = \gamma^{-2}[Y^{F_2}(i) - X(i)]B_1(i)B_1^T(i)[Y^{F_2}(i) - X(i)] + [F_2(i) + B_2^T(i)(X(i) + Z(i))]^T[F_2(i) + B_2^T(i)(X(i) + Z(i))] \geq 0$ .

Then assertion *i*) follows from Theorem 2.13(b) in [11] (see also Theorem 3.5 in [12]).

*ii*) Equation (3.2) may be written in the form

$$A^T(i)U(i) + U(i)\check{A}(i) + \Pi_i(U) + \mathcal{C}^T(i)\mathcal{C}(i) = 0 \quad (3.3)$$

$$\text{where } \mathcal{C}(i) = \begin{pmatrix} \gamma^{-1}B_1^T(i)U(i) \\ \gamma^{-1}B_1^T(i)Z(i) \\ F_2(i) + B_2^T(i)(X(i) + Z(i)) \end{pmatrix}$$

We conclude that  $(\mathcal{C}, \mathcal{L}_{\check{A}, \Pi})$  is detectable. Therefore conclusion follows applying Theorem 2.23 in [11].

The main results of this paper are contained in the next two theorems:

**Theorem 3.2** *Assume:*

*a) The system (2.5) is stochastically detectable.*

*b) The set  $\mathcal{A}(\gamma)$  is not empty.*

*Let us construct the sequences:  $\{X_k(i)\}_{k \geq 0}, \{Z_k(i)\}_{k \geq 0}, 1 \leq i \leq N$  as follows:*

$$X_0(i) = 0, 1 \leq i \leq N \quad (3.4)$$

$Z_k = (Z_k(1), Z_k(2), \dots, Z_k(N))$  is the stabilizing solution of the coupled algebraic Riccati equations:

$$M_k^T(i)Z_k(i) + Z_k(i)M_k(i) + \Pi_i(Z_k) - Z_k(i)B_2(i)B_2^T(i)Z_k(i) + R_i(X_k) = 0, \quad (3.5)$$

$$X_{k+1}(i) = Z_k(i) + X_k(i), \quad 1 \leq i \leq N \quad (3.6)$$

where

$$M_k(i) = A(i) + B_1(i)F_{1,k}(i) + B_2(i)F_{2,k}(i) \quad (3.7)$$

with

$$F_{1,k}(i) = \gamma^{-2}B_1^T(i)X_k(i) \quad (3.8)$$

and

$$F_{2,k}(i) = -B_2^T(i)X_k(i). \quad (3.9)$$

Under the considered assumptions, both sequences  $\{X_k(i)\}_{k \geq 0}, \{Z_k(i)\}_{k \geq 0}, 1 \leq i \leq N$  are well defined and convergent.

Moreover  $\lim_{k \rightarrow \infty} X_k(i) = \tilde{X}(i)$  and  $\lim_{k \rightarrow \infty} Z_k(i) = 0, 1 \leq i \leq N$ , where  $\tilde{X} = (\tilde{X}(1), \tilde{X}(2), \dots, \tilde{X}(N))$  is the stabilizing solution of (1.1).

**Proof.** Hint: For  $k = 0$  (3.5) becomes

$$A^T(i)Z_0(i) + Z_0(i)A(i) + \Pi_i(Z_0) - Z_0(i)B_2(i)B_2^T(i)Z_0(i) + C^T(i)C(i) = 0, \quad 1 \leq i \leq N. \quad (3.10)$$

From assumption *b*) one deduces that  $(\mathcal{L}_{A,\Pi}, B_2)$  is stabilizable while from assumption *a*) one deduces that  $(C, \mathcal{L}_{A,\Pi})$  is detectable. Applying Theorem 17 in Chapter 4 [13] one deduces that equation (3.10) has a unique stabilizing solution  $Z_0 = (Z_0(1), Z_0(2), \dots, Z_0(N)), Z_0(i) \geq 0, 1 \leq i \leq N$ .

The fact that  $Z_0$  is the stabilizing solution of (3.1) means that the eigenvalues of the Lyapunov perturbed operator  $\mathcal{L}_{A+B_2\tilde{F}_2, \Pi}$ , are in the half plane  $\mathbf{C}^-$  where  $\tilde{F}_2 = -B_2^T(i)Z_0(i), 1 \leq i \leq N$ .

Thus we have seen that  $X_0(i)$  and  $Z_0(i), 1 \leq i \leq N$  are well defined. To show that  $X_k(i), Z_k(i), 1 \leq i \leq N$  are well defined for  $k \geq 1$ , one proves iteratively the following items:

$\alpha_k$ )  $(\mathcal{L}_{A+B_1F_{1,k}, \Pi}, B_2)$  is stabilizable,  $F_{1,k}$  being as in (3.8).

$\beta_k$ )  $R_i(X_{k+1}) = \gamma^{-2}Z_k(i)B_1(i)B_1^T(i)Z_k(i)$ .

$\gamma_k$ ) The eigenvalues of the Lyapunov operator  $\mathcal{L}_{A+B_1F_{1,k}+B_2F_{2,k+1}, \Pi}$  are located in the half plane  $\mathbf{C}^-$ .

$\delta_k$ )  $X_k(i) \leq X_{k+1}(i) \leq \tilde{X}(i), 1 \leq i \leq N$ .

From  $\delta_k$ ) we conclude that the sequences  $\{X_k(i)\}_{k \geq 0}, 1 \leq i \leq N$  are convergent.

Let

$$X_\infty(i) = \lim_{k \rightarrow \infty} X_k(i), \quad 1 \leq i \leq N. \quad (3.11)$$

Further we have  $\lim_{k \rightarrow \infty} Z_{k+1}(i) = \lim_{k \rightarrow \infty} X_{k+1}(i) - X_k(i) = 0$ .

Taking the limit for  $k \rightarrow \infty$  in (3.5)-(3.9) we deduce that  $X_\infty = (X_\infty(1), X_\infty(2), \dots, X_\infty(N))$  is a solution of (1.1). Additionally, from  $(\delta_k)$  we have  $0 \leq X_\infty(i) \leq \tilde{X}(i), 1 \leq i \leq N$ . On the other hand from Theorem 2.2 it follows that  $\tilde{X}(i) \leq X_\infty(i), 1 \leq i \leq N$ . Hence  $X_\infty(i) = \tilde{X}(i), 1 \leq i \leq N$ , and thus the proof is complete.

**Remark 3.1** a) From  $\beta_k$ ) we see that a possible test to stop the iterative process is

$$\lambda_{max}[\gamma^{-2}Z_k(i)B_1(i)B_1^T(i)Z_k(i)] < \varepsilon \quad (3.12)$$

$1 \leq i \leq N$  for a prescribed level of accuracy  $\varepsilon > 0$ .

b) From (3.5) we infer that at each step  $k, Z_k$  is the stabilizing solution of a system of coupled algebraic Riccati equations of stochastic control with defined sign. To

find the stabilizing solution of a such kind of generalized algebraic Riccati equations one can use the iterative procedure given in section 4.6 in [13] or in [15].

The next theorem provides another iterative procedure to obtain the stabilizing solution of (1.1). In this procedure, at each step  $k$ ,  $Z_k(i)$ ,  $1 \leq i \leq N$  are obtained as stabilizing solutions of  $N$ -uncoupled standard algebraic Riccati equations.

**Theorem 3.3** *Assume that the assumptions of Theorem 3.2 are fulfilled. We construct the sequences*

$$\{X_k(i)\}_{k \geq 0}, \{Z_k(i)\}_{k \geq 0}, \tag{3.13}$$

$1 \leq i \leq N$  as follows:

$$X_0(i) = 0, \tag{3.14}$$

$1 \leq i \leq N$ .  $Z_k(i)$  is the stabilizing solution of the standard algebraic Riccati equation

$$M_k^T(i)Z_k(i) + Z_k(i)M_k(i) - Z_k(i)B_2(i)B_2^T(i) + R_i(X_k) = 0, \tag{3.15}$$

$$X_{k+1}(i) = Z_k(i) + X_k(i), \tag{3.16}$$

$1 \leq i \leq N$ .  $M_k(i)$  being as in (3.7)-(3.9).

Under the considered assumptions the sequences  $\{X_k\}_{k \geq 0}, \{Z_k(i)\}_{k \geq 0}, 1 \leq i \leq N$  are convergent and  $\lim_{k \rightarrow \infty} X_k(i) = \tilde{X}(i), \lim_{k \rightarrow \infty} Z_k(i) = 0, 1 \leq i \leq N$ ,  $\tilde{X} = (\tilde{X}(1), \dots, \tilde{X}(N))$  being the stabilizing solution of (1.1).

**Proof.** The proof follows the same ideas as in the proof of Theorem 3.2 from above and Theorem 3 in [16]. The details are omitted for shortness. However we mention that at each step, one obtains:

$$R_i(X_{k+1}) = \gamma^{-2}Z_k(i)B_1(i)B_1^T(i)Z_k(i) + \Pi_i(Z_k). \tag{3.17}$$

This allows us to suggest the following test to stop the iterative process (3.13)-(3.15):

$$\lambda_{max}[\gamma^{-2}Z_k(i)B_1(i)B_1^T(i)Z_k(i) + \Pi_i(Z_k)] < \varepsilon \tag{3.18}$$

$1 \leq i \leq N$ , for a given  $\varepsilon > 0$ .

Comparing (3.17) and (3.12) one deduces that the algorithm proposed in Theorem 3.3 seems to be slower convergent than the one described in Theorem 3.2. A detailed discussion concerning the convergence rate of the two iterative procedures will be made in another paper.

## 4 Illustrative example

Consider the following numerical example for  $n = 2$ ,  $N = 2$ ,  $r = 1$  and  $Q = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ :

$$\begin{aligned} A_0(1) &= \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}; A_0(2) = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}; A_1(1) = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}; \\ A_1(2) &= \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}; B_1(1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}; B_1(2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \\ B_2(1) &= \begin{bmatrix} -1 \\ 1 \end{bmatrix}; B_2(2) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}; C^T(1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; C^T(2) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \end{aligned}$$

Then the iterative procedure proposed in Theorem 3.2 gives for  $\gamma = 100$  and  $\varepsilon = 10^{-8}$  the stabilizing solution  $X(1) = \begin{bmatrix} 9.5536 & 13.7815 \\ 13.7815 & 25.0974 \end{bmatrix}$ ,  $X(2) = \begin{bmatrix} 8.4178 & 16.5128 \\ 16.5128 & 44.1219 \end{bmatrix}$  after 3 iterations.

The same solution, for the same accuracy level was obtained applying the alternative procedure given in Theorem 3.3, after 52 iterations.

## 5 Concluding remarks

The paper presents two iterative methods to compute the stabilizing solution of a class of game theoretic algebraic coupled Riccati equations arising in the optimal control of linear stochastic systems subject both to multiplicative white noise and Markovian jumping. It is proved that these procedures are always convergent if the stabilizing solution exists.

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