Computing the smallest fixed point of nonexpansive mappings arising in game theory and static analysis of programs

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Abstract

The problem of computing the smallest fixed point of a monotone map arises classically in the study of zero-sum repeated games. It also arises in static analysis of programs by abstract interpretation. In this context, the discount rate may be negative. We characterize the minimality of a fixed point in terms of the nonlinear spectral radius of a certain semidifferential. We apply this characterization to design a policy iteration algorithm, which applies to the case of finite state and action spaces. The algorithm returns a locally minimal fixed point, which turns out to be globally minimal when the discount rate is nonnegative.

Keywords: Game theory, policy iteration algorithm, negative discount, static analysis, nonexpansive mappings, semidifferentials.

1 Introduction

Zero-sum repeated games can be studied classically by means of dynamic programming or Shapley operators. When the state space is finite, such an operator is a

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map f from $\mathbb{R}^d$ to $\mathbb{R}^d$, where d is the number of states. Typically, the operator f can be written as:

$$f_i(x) = \min_{a \in A_i} \max_{b \in B_{i,a}} P_i^{a,b} x + r_i^{a,b}$$

Here, A_i represents the set of actions of Player I (Minimizer) in state i, $B_{i,a}$ represents the set of actions of Player II (Maximizer) in state i, when Player I has just played a (the information of both players is perfect), $r_i^{a,b}$ is an instantaneous payment from Player I to Player II, and $P_i^{a,b} = (P_{ij}^{ab})_j$ is a substochastic vector, giving the transition probabilities to the next state, as a function of the current state and of the actions of both players. The difference $1 - \sum_j P_{ij}^{a,b}$ gives the probability that the game terminates as a function of the current state and actions. The operator f will send $\mathbb{R}^d$ to $\mathbb{R}^d$ if for instance the instantaneous payments are bounded. We may consider the game in which the total payment is the expectation of the sum of the instantaneous payments of Player I to Player II, up to the time at which the game terminates. This includes the discounted case, in which for all i, $\sum_j P_{ij}^{a,b} = \alpha < 1$, for some discount factor α. Then, the fixed point of f is unique, and its ith-coordinate gives the value of the game when the initial state is i, see [1]. In more general situations [2], the value is known to be the largest (or dually, the smallest) fixed point of certain Shapley operators, and it is of interest to compute this value, a difficulty being that Shapley operators may have several fixed points.

The same problem appears in a totally different context. Static analysis of programs by abstract interpretation [3] is a technique to compute automatically invariants of programs, in order to prove them correct. The fixed point operators arising in static analysis include the Shapley operators of stochastic games as special cases. However, the "discount factor" may be larger than one, which is somehow unfamiliar from the game theoretic point of view. In this context, the existence of the smallest fixed point is guaranteed by Tarski-type fixed point arguments, and this fixed point is generally obtained by a monotone iteration (also called Kleene iteration) of the operator f. This method is often slow. Some accelerations based on "widening" and "narrowing" [4] are commonly used, which may lead to a loss of precision, since they only yield an upper bound of the minimal fixed point. Some of the authors introduced alternative algorithms based on policy iteration instead [5, 6], which are often faster and more accurate. However, the fixed point that is returned is not always the smallest one.

The purpose of the present work is to refine these policy iteration algorithms in order to reach the smallest fixed point of f even in degenerate situations. In order to so, the present work is partly inspired by [7], where the uniqueness (rather than the minimality) problem for the fixed point of a nonexpansive maps is studied using semidifferentiability techniques.

2 Basic notions

In this paper, we will work in $\mathbb{R}^d$ equipped with the sup-norm $\|\cdot\|$. We consider the natural partial order on $\mathbb{R}^d$ defined as: $x \leq y$ if for all i, $1 \leq i \leq d$, $x_i \leq y_i$ where x_i indicates the ith coordinate of x. We write $x < y$ when $x \leq y$ and there exists a

j such that $x_j < y_j$. We denote by $\mathbb{R}_+$ (resp. $\mathbb{R}_-$) the set of real nonnegative (resp. nonpositive) numbers. All vectors of $\mathbb{R}^d$ such that $f(x) = x$ are called fixed points of the map f. We denote by $\mathbf{Fix}(f)$ the set of fixed points of f.

When the action spaces are finite, dynamic programming operators are not differentiable, and they may even have empty subdifferentials or superdifferentials. However, their local behavior can be analyzed by means of a non-linear analogue of the differential, the semidifferential. In order to define it, let us recall some basic notions concerning cones.

Definition 1. *A subset* $\mathbf{C}$ *of* $\mathbb{R}^d$ *is called a cone if for all* $\lambda \geq 0$ *and for all* $x \in \mathbf{C}$, $\lambda x \in \mathbf{C}$. *A cone* $\mathbf{C}$ *is said to be pointed if* $\mathbf{C} \cap -\mathbf{C} = \{0\}$.

Definition 2. *Let* $\mathbf{C}$ *be cone and* g *be a self-map on* $\mathbf{C}$. *The function* g *is said to be homogeneous (of degree one), if for all strictly positive real numbers* λ, $g(\lambda x) = \lambda g(x)$.

Closed convex pointed cones are precisely what we need to define spectral radii of homogeneous continuous self-maps g on $\mathbf{C}$.

Definition 3 (Spectral radius). *Let* $\mathbf{C}$ *be a closed convex pointed cone. Let* g *be a homogeneous continuous self-map on* $\mathbf{C}$. *We define the spectral radius* $\rho_{\mathbf{C}}(g)$ *to be the nonnegative number:*

$$\rho_{\mathbf{C}}(g) = \sup\{\lambda \geq 0 \mid \exists\, x \in \mathbf{C}\backslash\{0\}, g(x) = \lambda x\}$$

A vector $x \in \mathbf{C}\backslash\{0\}$ such that $g(x) = \lambda x$ is a *non-linear eigenvector* of g, and λ is the associated non-linear eigenvalue. The existence of non-linear eigenvectors is guaranteed by standard fixed point arguments [8].

We next recall the notion of semidifferential, see [9] for more background.

Definition 4 (Semidifferential). *Let* $u \in \mathbb{R}^d$ *and* f *be a self-map on* $\mathbb{R}^d$. *We say that* f *is semidifferentiable at* u *if there exists a homogeneous continuous map* g *on* $\mathbb{R}^d$ *and a neighborhood* $\mathcal{V}$ *of 0 such that for all* $h \in \mathcal{V}$:

$$f(u+h) = f(u) + g(h) + o(\|h\|)$$

We call g *the semidifferential of* f *at* u *and we note it* f'_u.

If f is semidifferentiable at u, we have for all $t > 0$ and for h in a small enough neighborhood of 0:

$$f(u+h) = f(u) + t f'_u(h) + o(t\|h\|)$$

This implies

$$f'_u(h) = \lim_{t \to 0^+} \frac{f(u+th) - f(u)}{t}$$

and the semidifferential coincides with the directional derivative of f at u in direction h (on the positive side). The following result shows that semidifferentiability requires the latter limit to be uniform in the direction h.

Proposition 5 (see [9, Theorem 7.21]). *Let f be a self-map on $\mathbb{R}^d$. Let u be in $\mathbb{R}^d$. f'_u, the semidifferential of f at u exists if and only if for all vectors h, the following limit exists:*

$$\lim_{\substack{t\to 0^+ \\ h'\to h}} \frac{f(u+th')-f(u)}{t}$$

3 Main Results

Definition 6. *Let $u \in \mathbf{Fix}(f)$. We say that u is a locally minimal fixed point if there is a neighborhood $\mathcal{V}$ of u such that for all $v \in \mathcal{V} \cap \mathbf{Fix}(f)$, $v \le u \implies v = u$.*

Now, for a semidifferentiable map f at u, we denote $\mathbf{Fix}_{|\mathbb{R}^d_-}(f'_u)$ the set of the negative fixed points of f'_u.

Theorem 7. *Let $u \in \mathbf{Fix}(f)$. Consider the following statements:*

1. *u is a locally minimal fixed point.*
2. *$\mathbf{Fix}_{|\mathbb{R}^d_-}(f'_u) = \{0\}$.*
3. *$\rho_{\mathbb{R}^d_-}(f'_u) < 1$.*

   Then $3 \implies 2 \implies 1$.

Proof. Point 3 implies point 2 indeed. In order to show that 2 implies 1, assume that u is not a locally minimal fixed point. Then, there exists a sequence h_n of non-zero vectors in $\mathbb{R}^d_-$ tending to the zero vector such that $u + h_n$ is a fixed point of f. After replacing h_n by a subsequence, we may assume that $y_n := \|h_n\|^{-1} h_n$ has a limit y. Then, $\|y\| = 1$ and $y \in \mathbb{R}^d_-$. Writing $u + h_n = u + \|h_n\| y_n$, and using Prop. 5, we conclude that $y = f'_u(y)$, showing that f'_u has a non-zero fixed point in $\mathbb{R}^d_-$. □

In the previous theorem, there are no restrictive conditions on the map, but we only get a sufficient condition for the local minimality of a fixed point. We next consider the special case of piecewise affine maps.

Definition 8. *Let f be a map on $\mathbb{R}^d$. We say that f is piecewise affine if for all $j \in \{1, \cdots, d\}$ there exists finite sets A_j, $\{B_a\}_{a\in A_j}$ and a family $\{g_{a,b}\}_{(a,b)\in A_j \times B_a}$ of affine maps such that*

$$f_j = \min_{a\in A_j} \max_{b\in B_a} g_{a,b}$$

It is shown in [10] that the set of piecewise affine maps that we define is the same as the set of functions f for which there exists a family of convex closed sets with non empty interior which covers $\mathbb{R}^d$ and such that the restriction of f on each element of this family is affine.

Proposition 9. *Let f be a piecewise affine self-map on $\mathbb{R}^d$. Then f is semidifferentiable at all $u \in \mathbb{R}^d$. Moreover for all $u \in \mathbb{R}^d$:*

1. *Let $\bar{A}_j = \{a \in A_j \mid f_j(u) = \max_{b \in B_a} g_{a,b}(u)\}$ and $\bar{B}_a = \{\bar{b} \in B_a \mid g_{a,\bar{b}}(u) = \max_{b \in B_a} g_{a,b}(u)\}$, then*

$$(f'_u)_j = \min_{a \in \bar{A}_j} \max_{b \in \bar{B}_a} \nabla g_{a,b} \cdot$$

2. *There is neighborhood $\mathcal{V}$ of u such that, for all $u + h \in \mathcal{V}$, $f(u + h) = f(u) + f'_u(h)$.*

The first assertion may be deduced by applying the rule of the "differentiation" of a max see Exercise 10.27 of [9]. The following proof that we provide for the convenience of the reader, is a variation of this argument.

Proof. We set, for all $a \in A_j$, $g_a(x) = \max_{b \in B_a} g_{a,b}(x)$. By definition of $\bar{B}_a$, there exists a neighborhood $\mathcal{V}_a$ of u such that: $g_a(u+h) = \max_{b \in \bar{B}_a} g_{a,b}(u+h)$, for all h such that $u + h \in \mathcal{V}_a$. Since $g_{a,b}$ is affine, we have: $g_{a,b}(u+h) = g_{a,b}(u) + \nabla g_{a,b} \cdot h$. It follows that, for all $\bar{b} \in \bar{B}_a$:

$$g_a(u+h) = g_{a,\bar{b}}(u) + \max_{b \in \bar{B}_a} \nabla g_{a,b} \cdot h \tag{1}$$

By definition of $\bar{A}_j$, there exists a neighborhood of u, $\mathcal{V} \subset \bigcap_a \mathcal{V}_a$, such that: $f_j(u+h) = \min_{a \in \bar{A}_j} g_a(u+h)$ if $u + h \in \mathcal{V}$. Applying (1), we get: $f_j(u+h) = f_j(u) + \min_{a \in \bar{A}_j} \max_{b \in \bar{B}_a} \nabla g_{a,b} \cdot h$ if $u + h \in \mathcal{V}$, which shows the two assertions. $\square$

Corollary 10. *Let f be a piecewise affine map on $\mathbb{R}^d$ and let $u \in \mathbf{Fix}(f)$, then u is a locally minimal fixed point if and only $\mathbf{Fix}_{|\mathbb{R}^d_-}(f'_u) = \{0\}$.*

The second part of the previous proposition is the basis of a "descent" algorithm given in the section 4. If a fixed point u is not locally minimal, then there exists a strictly negative fixed point h for f'_u which may be thought of as a *descent direction* such that $u + h$ is a fixed point of f.

In order to pass from local minimality to global minimality, we shall need the following nonexpansiveness condition.

Definition 11. *Let f be a map on $\mathbb{R}^d$: f is nonexpansive (with respect to the sup-norm) if for all $x, y \in \mathbb{R}^d$, $\|f(x) - f(y)\| \leq \|x - y\|$.*

Proposition 12. *Let f be a nonexpansive map on $\mathbb{R}^d$ and let $u \in \mathbb{R}^d$, then $\mathbf{Fix}_{|\mathbb{R}^d_-}(f'_u) = \{0\}$ if and only if $\rho_{\mathbb{R}^d_-}(f'_u) < 1$.*

Proof. It suffices to show that $\rho_{\mathbb{R}^d_-}(f'_u) \leq 1$. Firstly, f'_u is nonexpansive because it is a pointwise limit of nonexpansive mappings. Assume by contradiction that $\rho_{\mathbb{R}^d_-}(f'_u) > 1$, so there exists $\mu > 1$ and $v \in \mathbb{R}^d_-$ such that $f'_u(v) = \mu v$, then

$\|f'_u(v) - f'_u(0)\| = \mu\|v\| > \|v - 0\|$ which contradicts the nonexpansiveness of f'_u. $\square$

The main result of this paper is the following theorem, which will allow us to check the global minimality of a fixed point.

Theorem 13. *Let f be a monotone nonexpansive self-map on $\mathbb{R}^d$. Let u be a fixed point of f. Then, u is locally minimal if and only if it is the smallest fixed point of f. If in addition f is piecewise affine, the following assertions are equivalent:*

1. *u is the smallest fixed point of f.*
2. *$\mathbf{Fix}_{|\mathbb{R}^d_-}(f'_u) = \{0\}$.*
3. *$\rho_{\mathbb{R}^d_-}(f'_u) < 1$.*

The proof of this theorem relies on the existence of a monotone and nonexpansive retract on the fixed point set of f. This idea was already used in [5]. The existence of nonexpansive retracts on the fixed point set is a classical topic in the theory of nonexpansive mappings, see Nussbaum [8]. In the present finite dimensional case, the result of the next lemma is an elementary one.

Lemma 14. *Let f be a nonexpansive monotone map on $\mathbb{R}^d$. Let u be in $\mathbf{Fix}(f)$. We suppose $\mathbf{Fix}(f) \neq \emptyset$. Then there is a nonlinear monotone and nonexpansive map P such that $P(\mathbb{R}^d) = \mathbf{Fix}(f)$ and $\mathbf{Fix}(P) = \mathbf{Fix}(f)$.*

Proof. Following the idea of [11], we shall construct the map P as follow: $P(x) = \lim_{k\to+\infty} f^k(y)$ where $y = \limsup_{l\to+\infty} f^l(x)$. Since f is nonexpansive and $u \in \mathbf{Fix}(f)$, $\{f^k(x)\}_{k\geq 0}$ is bounded for all $x \in \mathbb{R}^d$. We can now write, for all $x \in \mathbb{R}^d$, $Q(x) = \limsup_{k\to+\infty} f^k(x)$. Moreover, given $k \geq 0$, we have, for all $m \geq k$, $\sup_{n\geq k} f^n(x) \geq f^m(x)$ and since f is monotone, for all $m \geq k$, $f(\sup_{n\geq k} f^n(x)) \geq f(f^m(x))$ so, $f(\sup_{n\geq k} f^n(x)) \geq \sup_{m\geq k} f(f^m(x))$, we conclude, by taking the limit when k tends to $+\infty$ and using the continuity of f, $f(Q(x)) \geq Q(x)$, so $\{f^l(Q(x)\}_{k\geq 0}$ is an increasing sequence. Moreover, since f is nonexpansive, the limit $P(x) = \lim_{l\to+\infty} f^l(Q(x))$ is finite. Observe that the map P is monotone and nonexpansive since it is the pointwise limit of monotone and nonexpansive maps. Furthermore, $f(P(x)) = f(\lim f^l(Q(x)) = \lim f^{l+1}Q(x) = P(x)$ so $P(\mathbb{R}^d) \subset \mathbf{Fix}(f)$. Moreover, it is easy to see that P fixes every fixed point of f. It follows that P is a projector. $\square$

Proof of Theorem 13. Suppose that u is a locally minimal fixed point but not the smallest fixed point. Then, there is fixed point v such that $\inf(v, u) < u$. For all scalars $t \geq 0$, define $\omega_t := \inf(v + t, u)$. Let us take P as in Lemma 14. Since P is nonexpansive for the sup-norm, $\|P(v + t) - P(v)\| \leq t$ for all $t \geq 0$ and so $P(v + t) \leq P(v) + t$. Using the monotonicity of P, we deduce that $P(\omega_t) \leq \inf(P(v + t), P(u)) \leq \inf(P(v) + t, P(u)) = \omega_t$. Let $t_0 = \inf\{t \geq 0 \mid \omega_t = u\}$. Then, for $0 < t < t_0$, $P(\omega_t)$ is a fixed point of f, which is such that $P(\omega_t) < u$.

Since P is continuous, $P(\omega_t)$ tends to $P(\omega_{t_0}) = P(u) = u$ as t tends to t_0^-, which contradicts the local minimality of u. Hence, u is the smallest fixed point. Define by $1'$ the property that u is locally minimal fixed point. We just showed $1 \Leftrightarrow 1'$, by Theorem 7, we had $3 \implies 2 \implies 1'$. By Corollary 10, $1' \implies 2$ and by Prop 12 we got $2 \implies 3$. $\square$

4 A policy iteration algorithm to compute the smallest fixed point

The previous results justify the following policy iteration algorithm which returns the smallest fixed point of a nonexpansive monotone piecewise affine map. Assume that f is a map from $\mathbb{R}^d$ to $\mathbb{R}^d$, every coordinate of which is given by

$$f_j(x) = \inf_{a \in A_j} f_a(x) \tag{2}$$

where A_j is finite and every f_a is a supremum of monotone nonexpansive affine maps. A strategy π is a map from $\{1, \cdots, d\}$ to $\mathcal{A} = \bigcup_{1 \le j \le d} A_j$ such that $\pi(j) \in A_j$. We define $f^\pi = (f_{\pi(1)}, \cdots, f_{\pi(j)}, \cdots, f_{\pi(d)})$. The algorithm, needs two oracles. Oracle 1 returns the smallest fixed point in $\mathbb{R}^d$ of a map f^π. Oracle 2 checks whether the restriction of f'_u over the convex cone $\mathbb{R}^d_-$ has a spectral radius equal to 1, and if this is the case, returns a vector $h \in \mathbb{R}^d_- \backslash \{0\}$ such that $f'_u(h) = h$. We discuss below the implementation of these oracles for subclasses of maps.

Algorithm 4.1
INPUT: f in the form (2). *OUTPUT*: The smallest fixed point of f in $\mathbb{R}^d$.
INIT: Select a strategy π^0, $k = 0$.
VALUE DETERMINATION ($\mathbf{D_k}$): Call Oracle 1 to compute the smallest fixed point u^k of f^{π^k}.
POLICY IMPROVEMENT ($\mathbf{I_k^1}$): If $f(u^k) < u^k$ define π^{k+1} such that $f(u^k) = f^{\pi^{k+1}}(u^k)$ and go to Step (D_{k+1}).
POLICY IMPROVEMENT ($\mathbf{I_k^2}$): If $f(u^k) = u^k$, call Oracle 2 to compute $\alpha_k := \rho(f'_{u^k})$. If $\alpha_k < 1$, returns u^k, which is the smallest fixed point. If $\alpha_k = 1$, take $h \in \mathbb{R}^d_- \backslash \{0\}$ such that $f'_{u^k}(h) = h$. Define $\pi^{k+1}(j)$ to be any action a which attains the minimum in $(f'_{u^k})_j(h) = \min_{a \in \tilde{A}_j} (f'_a)_{u^k}(h)$ where $\bar{A}_j = \{a \mid f_a(u^k) = f(u^k)\}$. Then go to (D_{k+1}).

To show that the algorithm terminates, it suffices to check that the sequence $u^0, u^1, \cdots$ is strictly decreasing, because the corresponding policies must be distinct and the number of policies is finite. If an improvement of type I_k^1 arises, then, the proof of the Theorem 3 from [6] shows that $u^{k+1} < u^k$. If an improvement of type I_k^2 arises, then by Prop 10 assertion 2, $f^{\pi^{k+1}}(u^k + th) = u^k + th$ for $t > 0$ small enough. It follows that $u^{k+1} \le u^k + th < u^k$.

The following proposition identifies a situation where Oracle 1 can be implemented by solving a linear programming problem.

Proposition 15. *Let g be a monotone nonexpansive map that is the supremum of finitely many affine maps. Assume, furthermore, that g has a smallest fixed point in $\mathbb{R}^d$. Then this fixed point is the unique optimal solution of the linear program:* $\min\{\sum_{1\leq i\leq d} x_i \mid x \in \mathbb{R}^d, g(x) \leq x\}$.

Proof. Let $\bar{x}$ denote the smallest fixed point. The vector $\bar{x}$ is clearly feasible. If x is an another feasible point, consider $y = \lim_k g^k(x)$. Since g is monotone, nonexpansive and has a fixed point, $g^k(x)$ is a bounded decreasing sequence so $y \leq x$ and y is a finite fixed point of g so $y \geq \bar{x}$. We get $\sum_i \bar{x}_i \leq \sum_i y_i \leq \sum_i x_i$. Since this holds for all feasible x, it follows that $\bar{x}$ is an optimal solution. If x is an arbitrary optimal solution, we must have $\sum_i \bar{x}_i = \sum_i x_i$ and we deduce from $y \leq x$ and $y \geq \bar{x}$ that $\bar{x} \leq x$. It follows that $\bar{x} = x$. □

The implementation of Oracle 2 raises the issue of computing the spectral radius. Let g be a monotone, homogeneous and continuous self-map of $\mathbb{R}^d_-$. It is known that:

$$\rho_{\mathbb{R}^d_-}(g) = \inf_{x \in \mathrm{int}(\mathbb{R}^d_-)} \sup_{1\leq i\leq d} \frac{g_i(x)}{x_i} \tag{3}$$

$$\rho_{\mathbb{R}^d_-}(g) = \sup_{x \in \mathbb{R}^{\mathbf{d}}_-} \limsup_{k\to+\infty} \|g^k(x)\|^{\frac{1}{k}} \tag{4}$$

The first equality, which is a generalization of the Collatz-Wielandt property in Perron-Frobenius theory, follows from a result of Nussbaum [12] Theorem 3.1. The second characterization is shown by Mallet-Paret and Nussbaum in [13] under more general conditions. We deduce, for any vector $x \in \mathrm{int}(\mathbb{R}^d_-)$:

$$\rho_{\mathbb{R}^d_-}(g) \leq \left(\sup_{1\leq i\leq d} \frac{g_i^k(x)}{x_i}\right)^{\frac{1}{k}} . \tag{5}$$

Moreover, the latter upper bound converges to $\rho_{\mathbb{R}^d_-}(g)$ as k tends to infinity. This yields an obvious method to check whether $\rho_{\mathbb{R}^d_-}(g) < 1$, which consists in computing the upper bound in (5) for successive values of k as long as the upper bound is not smaller than 1. This algorithm will not stop when $\rho_{\mathbb{R}^d_-}(g) = 1$. However, we describe a simple situation where this idea leads to a terminating algorithm.

Definition 16. *We call a homogeneous min-max function of a vector $h \in \mathbb{R}^d$ a term in the grammar:* $X \mapsto \min(X, X), \max(X, X), h_1, \cdots, h_n, 0$.

For instance, the term $\min(h_1, \max(h_2, h_3, 0))$ is produced by this grammar. More generally, we shall say that a map from $\mathbb{R}^d$ to $\mathbb{R}^d$ is a homogeneous min-max map if its coordinates are the form of Def 16. This definition is inspired by the min-max functions considered by Gunawardena [14] and Olsder [15]. The terms of this form comprise the semidifferentials of the min-max map considered there. For simple classes of programs, like the one we shall consider in the next section, the semidifferential at any fixed point turns out to be a homogeneous min-max map. In this case, the spectral radius can be computed efficiently by using to the following integrity argument: $g(\mathbb{Z}^d) \subset \mathbb{Z}^d$.

Proposition 17. *Let g be a monotone homogeneous min-max nonexpansive map on $\mathbb{R}^d$, then $\rho_{\mathbb{R}^d_-}(g) \in \{0,1\}$. Moreover $\rho_{\mathbb{R}^d_-}(g) = 0$ iff $\lim_{k\to+\infty} g^k(-e) = 0$, where e is the vector all the entries of which are equal to 1. The latter limit is reached in at most d steps.*

Proof. Since g is nonexpansive and monotone $g(-e) \geq -e$. We deduce that $\{g^k(-e)\}_{k\geq 0}$ is a nondecreasing sequence bounded from above by 0. Moreover, g preserves the set of integral vectors. So this sequence converges in at most d steps to $b \in \mathbb{Z}^d_-$. Let us suppose that $b = 0$. For all $h \in \mathbb{R}^d_-$, there exists $t \geq 0$ such that, $0 \geq h \geq t(-e)$, since g is homogeneous and monotone, $0 \geq g^k(h) \geq tg^k(-e) = 0$ for all $k \geq d$ which implies that $\rho_{\mathbb{R}^d_-}(g) = 0$. If $b < 0$ we have $g(b) = b \neq 0$, and so $\rho_{\mathbb{R}^d_-} \geq 1$, and since g is nonexpansive, we also have $\rho_{\mathbb{R}^d_-}(g) \leq 1$. $\square$

5 Example

We next illustrate our results on an example from static analysis. We take a simple but interesting program with nested loops (Figure 1). From this program, we create semantic equations on the lattice of intervals [3] (Figure 2) that describe the outer approximations of the sets of values that program variables can take, for all possible executions. For instance, at control point 4, the value of variable y can come either from point 3 or point 5 (hence the union operator), as long as the condition $y \geq 5$ is satisfied (hence the intersection operator). An interval I is written as $[-i^-, i^+]$ in order to get fixed point equations of monotone maps in i^i and i^+. The equations we derive on bounds are monotone piecewise affine maps, to which we can apply our methods.

```
int x,int y,
x=[0,2];y=[10,15]      //1
while (x<=y) {         //2
  x=x+1;               //3
  while (5<=y) {       //4
    y=y-1;             //5
  }                    //6
}                      //7
```

Figure 1. *A simple C program*

$$
\begin{aligned}
(x_1, y_1) &= ([0,2],[10,15]) \\
x_2 &= (x_1 \cup x_6) \cap [-\infty, (y_1 \cup y_6)^+] \\
y_2 &= (y_1 \cup y_6) \cap [(x_1 \cup x_6)^-, +\infty] \\
(x_3, y_3) &= (x_2 + [1,1], y_2) \\
(x_4, y_4) &= (x_3, (y_3 \cup y_5) \cap [5, +\infty]) \\
(x_5, y_5) &= (x_4, y_4 + [-1,-1]) \\
(x_6, y_6) &= (x_5, (y_3 \cup y_5) \cap [-\infty, 4]) \\
x_7 &= (x_1 \cup x_6) \cap [(y_1 \cup y_6)^- + 1, +\infty] \\
y_7 &= (y_1 \cup y_6) \cap [-\infty, (x_1 \cup x_6)^+ - 1]
\end{aligned}
$$

Figure 2. *Its abstract semantic equations in intervals*

The monotone nonexpansive piecewise affine map f for the bounds of these intervals is:

$$f\begin{pmatrix} x \\ y \end{pmatrix} = f\begin{pmatrix} x_2^- \\ x_2^+ \\ x_7^- \\ x_7^+ \\ y_2^- \\ y_2^+ \\ y_4^- \\ y_4^+ \\ y_6^- \\ y_6^+ \\ y_7^- \\ y_7^+ \end{pmatrix} = \left(\begin{array}{ccc} 0 & \vee & (x_2^- - 1) \\ 2 \vee (x_2^+ + 1) & \wedge & \underline{15 \vee y_6^+} \\ 0 \vee (x_2^- - 1) & \wedge & \underline{(-10 \vee y_6^-) - 1} \\ 0 & \vee & (x_2^+ + 1) \\ \underline{0 \vee (x_2^- - 1)} & \wedge & -10 \vee y_6^- \\ 15 & \vee & y_6^+ \\ y_2^- \vee (y_4^- + 1) & \wedge & \underline{-5} \\ y_2^+ & \vee & y_4^+ - 1 \\ y_2^- & \vee & y_4^- + 1 \\ y_2^+ \vee (y_4^+ - 1) & \wedge & \underline{4} \\ -10 & \vee & y_6^- \\ 15 \vee y_6^+ & \wedge & \underline{(2 \vee (x_2^+ + 1)) - 1} \end{array}\right)$$

In the equations for the intervals x_2, y_2, y_4, y_6, x_7 and y_7, an intersection appears, which gives a min ($\wedge$) in the corresponding coordinate of f. Choosing a policy is the same as replacing every minimum of terms by one of the terms, which yields a simpler "minimum-free" non-linear map, which can be interpreted as the dynamic programming operator of a one-player problem. The underlined terms represent the initial policy. We next illustrate the Algorithm 4. From this policy, we compute the smallest fixed point and we check whether this fixed point is a fixed point of f, see [5] for details.

We find $(\bar{x}, \bar{y}) = (0, 15, -1, 16, 0, 15, -5, 15, 0, 4, 0, 15)^\intercal$: it is a fixed point of f, and so policy iteration terminates in one step. Now, we use our method to determine if $(\bar{x}, \bar{y})$ is the smallest fixed point. First, we calculate the semidifferential at $(\bar{x}, \bar{y})$ in the direction $(\delta x, \delta y)$, using the first point of Prop 9:

$$f'_{(\bar{x},\bar{y})}(\delta\bar{x}, \delta\bar{y})^\intercal \quad = \quad \left(0, 0, \delta\bar{y}_6^-, \delta\bar{x}_2^+, 0 \wedge \delta\bar{y}_6^-, 0, 0, \delta\bar{y}_2^+, \delta\bar{y}_2^-, 0, \delta\bar{y}_6^-, 0 \wedge \delta\bar{x}_2^+\right)^\intercal$$

By using Prop 17, computing in three steps a fixed point for $f'_{(\bar{x},\bar{y})}$ that we call $h = (0, 0, -1, 0, -1, 0, 0, 0, -1, 0, -1, 0)^\intercal$. By this fixed point we choose a new policy. We only change the fifth coordinate of the previous policy, we replace $0 \vee (x_2^- - 1)$ by $-10 \vee y_6^-$ and we compute the smallest fixed point of this new policy.

We find a new fixed point $(\tilde{u}, \tilde{v}) = (0, 15, -5, 16, -4, 15, -5, 15, -4, 4, -4, 15)^\intercal$ for f. The semidifferential at $(\tilde{u}, \tilde{v})$ is then:

$$f'_{(\tilde{u},\tilde{v})}(\delta\tilde{u}, \delta\tilde{v})^\intercal \quad = \quad \left(0, 0, \delta\tilde{v}_6^-, \delta\tilde{u}_2^+, \delta\tilde{v}_6^-, 0, 0, \delta\tilde{v}_2^+, \delta\tilde{v}_2^- \vee \delta\tilde{v}_4^-, 0, \delta\tilde{v}_6^-, 0 \wedge \delta\tilde{u}_2^+\right)^\intercal$$

By Prop 17, we find $\rho_{\mathbb{R}^d_-}(f'_{(\tilde{u},\tilde{v})}) = 0$, we conclude that $(\tilde{u}, \tilde{v})$ is the smallest fixed point of f.

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