

The Zero Module of Noncommutative Multidimensional Linear Systems and its Connection with State Space Models (Extended Abstract)

Nevio Carpanese *

* *P.O. Box 802, 35122 Padova Italy (e-mail:
nevio.carpanese@unipd.it).*

Abstract: Non commutative multidimensional systems are considered in this paper. The zero module of an input/output multidimensional system is defined under this framework and related to the output nulling controlled invariant subspace associated to the state space realization of the system. The non-commuting framework allows to copy Wyman-Sain geometrical poles-zeros characterization from 1D to nD systems.

1. INTRODUCTION

From the earlier papers [3] and [4] to the survey [1] it has been clarified that in multidimensional system theory there are no clear characterization of minimal state model realization apart from the non-commutative case. The investigation of the 1D-geometrical characterization of pole and zero module is obtained under minimal realization of the system under study ([9]). In this paper we are concerned with the connection between the pole-zero modules and controlled invariant subspaces of the underlying state-space realization for the noncommuting multidimensional case.

Multidimensional models received great attention, prospective applications ranging from formal languages, automata theory towards to more recently linear fractional transformation ([2]) promoted deep understanding of Noncommuting Multidimensional(NM) linear systems structure and modeling, starting from the seminal papers [3] and [4] where NM-system state space realization has been studied to the complete full structured NM-system approach developed in [1] a complete picture of the classical system-theoretic properties has been developed. The fact that the NM-systems is evolving over a free semigroup gives a clear statement of the minimal realization problem in terms of structural properties despite the commuting case. This explains why in the commuting case multidimensional systems are studied with behavioral or module-theoretic approach, the main references devoted to study of pole-zero structure for behavioral approach are [8] and [10] whereas for module-theoretic approach [6] and [7].

2. BACKGROUND NOTATIONS AND RESULTS

As previously explained we consider systems defined over non commutative rings, we now set up the usual notations. Given $z_1, z_2, \dots, z_n$ items let $\langle z_1, z_2, \dots, z_n \rangle$ denote the set of strings of finite length obtained by concatenation of the items $z_1, z_2, \dots, z_n$, the concatenation turns $\langle z_1, z_2, \dots, z_n \rangle$ into a semigroup with the empty string acting as the unity. The set of non-commutative polyno-

mials and non-commutative power series over the field k are denoted by $k\langle z_1, z_2, \dots, z_n \rangle$ and $k\langle\langle z_1, z_2, \dots, z_n \rangle\rangle$ respectively. These two sets become non-commutative rings with the usual sum and product definition. Another useful ring is the set of non-commutative Laurent power series denoted by $k\langle\langle z_1, z_2, \dots, z_n, z_1^{-1}, z_2^{-1}, \dots, z_n^{-1} \rangle\rangle$. In order to give the notion of non-commutative rational power series we recall that a ring R is said to be *rationally closed* if $r \in R$ implies $r^{-1} \in R$. Now we can define $k\langle\langle z_1, z_2, \dots, z_n \rangle\rangle$, the ring of rational non-commutative power series, as the smallest subring of $k\langle\langle z_1, z_2, \dots, z_n \rangle\rangle$ rationally closed containing $k\langle z_1, z_2, \dots, z_n \rangle$.

The rationality of a formal power series H can be characterized in terms of realizations or in terms of Hankel matrix. A realization of a strictly proper $H \in k\langle\langle z_1, z_2, \dots, z_n \rangle\rangle$ can be obtained with the Fornasini-Marchesini (FM) system or the Givone-Roesser (GR) system or generally speaking with the framework of [1].

The FM-realization of $H \in k\langle\langle z_1, \dots, z_n \rangle\rangle$ exists if there are suitable matrices $C, B_1, \dots, B_n, A_1, \dots, A_n$ such that:

$$H = C(I - z_1 A_1 - \dots - z_n A_n)^{-1}(z_1 B_1 + \dots + z_n B_n)$$

whereas the GR-realization exists if there are suitable matrices $C_1, \dots, C_n, B_1, \dots, B_n, A_{11}, \dots, A_{ij}$ such that:

$$H = [C_1 \dots C_n] \left(\begin{bmatrix} I_1 & & \\ & \ddots & \\ & & I_n \end{bmatrix} - \begin{bmatrix} z_1 A_{11} & \dots & z_1 A_{1n} \\ \vdots & & \vdots \\ z_n A_{n1} & \dots & z_n A_{nn} \end{bmatrix} \right)^{-1} \begin{bmatrix} z_1 B_1 \\ \vdots \\ z_n B_n \end{bmatrix}$$

The Hankel matrix ($\mathbb{H}$) for a formal power series $H \in k\langle\langle z_1, \dots, z_n \rangle\rangle$ is a bi-infinite matrix with row indexed by word $h \in \langle z_1, \dots, z_n \rangle$ and column indexed by word $k \in \langle z_1, \dots, z_n \rangle$ and the corresponding entry given by (H, hk) , the coefficient of H relative to the power hk . The Hankel matrix has an operatorial meaning if we define system equation $y = Hu$ where the input u has a finite support lying in $\langle\langle z_1, \dots, z_n \rangle\rangle$. In [1] the general structured multidimensional linear system is studied, namely with evolution on a free semigroup obtained from a general graph, and equivalence between rational H , the finiteness

of the rank of the corresponding $\mathbb{H}$ and the existence of finite dimensional realization is established.

Summarizing the rationality of a formal power series H can be characterized in terms of realizations or in terms of Hankel matrix as in the following theorem.

Theorem 1. (1) $H \in k\langle\langle z_1, z_2 \rangle\rangle$ iff there exists finite dimensional realizations.
(2) $H \in k\langle\langle z_1, z_2 \rangle\rangle$ iff its Hankel matrix has finite rank.

As is well known([9]), the minimal state space characterization for a strictly proper transfer 1D function $G \in \mathbb{R}(z)$ is given by the module

$$X := \frac{k[z^{-1}]}{G^{-1}(k[z^{-1}]) \cap k[z^{-1}]},$$

which is the algebraic formulation of the Nerode equivalence. For the nD case the minimal state X of a realization of a strictly proper $H \in k\langle\langle z_1, \dots, z_n \rangle\rangle$ accordingly with [1][Th. 11.1] is defined by (up to isomorphisms)

$$X := \frac{k\langle\langle z_1^{-1}, \dots, z_n^{-1} \rangle\rangle \setminus k}{H^{-1}(k\langle\langle z_1, \dots, z_n^{-1} \rangle\rangle \setminus k\langle\langle z_1, \dots, z_n \rangle\rangle) \cap k\langle\langle z_1^{-1}, \dots, z_n^{-1} \rangle\rangle \setminus k}$$

Indeed one can recognize that to X belong any $u \in k\langle\langle z_1^{-1}, \dots, z_n^{-1} \rangle\rangle \setminus k$ (hence with compact support in the quarterplane represented by $\langle z_1^{-1}, \dots, z_n^{-1} \rangle$) modulo those having the same $y = Hu$ anywhere except in the quarterplane represented by $\langle z_1, \dots, z_n \rangle$. Namely an item of X is the set of inputs u which have the same $y = Hu$ with respect to the “quarterplane” $\langle z_1, \dots, z_n \rangle$, in other terms the set of input of the Nerode’s equivalence.

We are concerned with the zero module of 2D systems, for the 1D case the zero module, retaining the assumption of strictly proper function G , is given by ([9]) $Z(G) := \frac{Z_1}{Z_2}$ where

$$Z_1 := \frac{G^{-1}(\mathbb{R}[z^{-1}])}{G^{-1}(\mathbb{R}[z^{-1}]) \cap \mathbb{R}[z^{-1}]}$$

and

$$Z_2 := \frac{\ker G}{\ker G \cap \mathbb{R}[z^{-1}]}$$

The projection p of these modules over the state space X defined previously gives the following commutative and exact diagram:

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ Z_2 & \rightarrow & R^* \\ \downarrow & & \downarrow \\ Z_1 & \rightarrow & V^* \\ \downarrow & & \downarrow \\ Z(H) & \rightarrow & V^*/R^* \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

Where R^* and V^* are the maximum reachable subspace contained in $\text{Ker}C$ and the maximum output nulling invariant subspace relative to a minimal realization of H respectively, namely $H = C(zI - A)^{-1}B$, $R^* = R(A, B, \text{Ker}C)$, $V^* = V(A, B, \ker C)$.

3. GEOMETRICAL CHARACTERIZATION FOR ZEROS OF THE 2D STATE SPACE SYSTEMS

For the sake of clarity we develop our characterization for NM-systems in the 2D case and when needed we consider 2D state space realization in the Fornasini-Marchesini form. We consider the Input/Output structure given by

$$\begin{bmatrix} I & -H \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} = 0$$

where the “signal” space for u and y is $k\langle\langle z_1, z_2, z_1^{-1}, z_2^{-1} \rangle\rangle$ and $H \in k\langle\langle z_1, z_2 \rangle\rangle$ and strictly causal i.e. $(H, z_1^0 z_2^0) = 0$. Using the characterization given by theorem 1 we can write $y(z_1, z_2) = p(z_1, z_2)q(z_1, z_2)^{-1}u(z_1, z_2)$ with $p, q \in \mathbb{R}\langle\langle z_1, z_2 \rangle\rangle$.

To develop our calculations we need to assume that $u \in k\langle\langle z_1, z_2, z_1^{-1}, z_2^{-1} \rangle\rangle$ has $(u, w) = 0$ for all strings $w \in \langle z_1, z_2, z_1^{-1}, z_2^{-1} \rangle$ which contain negative indexes beyond a prescribed bound, i.e. a compact support as for those u contained in the first quarter-plane $u \in k\langle\langle z_1, z_2 \rangle\rangle$. A state map for H can be obtained with a state-space realization as in the Fornasini-Marchesini model namely (A_1, A_2, B_1, B_2, C) such that $H = C \sum_{k=0}^{+\infty} (A_1 z_1 + A_2 z_2)^k (B_1 z_1 + B_2 z_2)$. The local state space associated is given by:

$$X :=$$

$$\frac{k\langle\langle z_1^{-1} z_2^{-1} \rangle\rangle \setminus k}{H^{-1}(k\langle\langle z_1, z_2, z_1^{-1}, z_2^{-1} \rangle\rangle \setminus k\langle\langle z_1, z_2 \rangle\rangle) \cap k\langle\langle z_1^{-1} z_2^{-1} \rangle\rangle \setminus k}$$

from (1) and strict causality we remark that y has the following recursive structure:

$$\{(y, z)z \mid z \in \langle z_1, z_2 \rangle w\} = \{(y, z)z \mid z \in \langle z_1, z_2 \rangle w z_1^{-1}\} z_1 \cup \{(y, z)z \mid z \in \langle z_1, z_2 \rangle w z_2^{-1}\} z_2 \cup \{(H, z_1)(u, z_1^{-1} w)z_1\} \cup \{(H, z_2)(u, z_2^{-1} w)z_2\}.$$

From this it follows the recursive structure of the Nerode’s equivalence classes. We define the following actions on X

$$\begin{aligned} A_1 : X &\rightarrow X & x &\mapsto x z_1^{-1}; \\ A_2 : X &\rightarrow X & x &\mapsto x z_2^{-1} \end{aligned}$$

and two maps from input to state space:

$$\begin{aligned} B_1 : k &\rightarrow X & u &\mapsto [u z_1^{-1}]; \\ B_2 : k &\rightarrow X & u &\mapsto [u z_2^{-1}] \end{aligned}$$

where $[\cdot]$ denotes the equivalence class associated to $\cdot$. Finally the map from X to the output space is denoted $C : X \rightarrow k$ and a projection application $\hat{B} : k\langle\langle z_1^{-1}, z_2^{-1} \rangle\rangle \setminus k \rightarrow X$ given by $\hat{B}(u) = [u]$.

As in the 1D-case we can define a $k\langle\langle z_1^{-1}, z_2^{-1} \rangle\rangle$ -module ([9]) which gives the set of u such that y is zero in $k\langle\langle z_1, z_2 \rangle\rangle$

$$Z_1 :=$$

$$\frac{H^{-1}(k\langle\langle z_1, z_2, z_1^{-1}, z_2^{-1} \rangle\rangle \setminus k\langle\langle z_1, z_2 \rangle\rangle)}{H^{-1}(k\langle\langle z_1, z_2, z_1^{-1}, z_2^{-1} \rangle\rangle \setminus k\langle\langle z_1, z_2 \rangle\rangle) \cap (k\langle\langle z_1^{-1}, z_2^{-1} \rangle\rangle \setminus k)}$$

Since we have that

$$k\langle\langle z_1^{-1} z_2^{-1} \rangle\rangle \setminus k \subseteq H^{-1}(k\langle\langle z_1, z_2, z_1^{-1}, z_2^{-1} \rangle\rangle \setminus k\langle\langle z_1, z_2 \rangle\rangle),$$

hence $X \subseteq Z_1$ the projection $p_1 : Z_1 \rightarrow X$ makes sense.

We want to know which is the range of to the projection: $p_1(Z_1)$. In the 1D case we have the following result ([9]) $p_1(Z_1) \equiv V^*$ where V^* is the maximum output nulling

controlled invariant subspace. To expose the corresponding result for 2D systems we recur to systems defined over the ring $k[z]$, in fact considering the $k[z]$ module of Laurent formal power series $k_b^n[[z]]$ we can also consider columns with n entries in $k_b^n[[z]]$ to obtain another $k[z]$ module denoted by $k_b^n[[z]]$. We have the following definition.

Definition 2. A submodule of $\mathcal{V} \subseteq \mathbb{R}_b^n[[z]]$ is an output nulling invariant controlled subspace if

$$(A_1 + A_2 z)\mathcal{V} + (B_1 + B_2 z)\text{Range}k_b[[z]] \subseteq \mathcal{V} \quad (1)$$

and $\mathcal{V} \subseteq \text{Ker}C$ where $\text{Ker}C := \{\sum_{-\infty}^{+\infty} x_i z^i \mid x_i \in \text{Ker}C\}$

We define the projection P with $p : (X)^\mathbb{Z} \rightarrow X$ in the following way: $P(\sum_{-\infty}^{+\infty} x_i z^i) = x_0$. As in the 1D case let $\mathcal{V}^*$ denote the maximum output nulling invariant controlled subspace with respect to the inclusion, moreover we define $V^* = P(\mathcal{V}^*)$

Proposition 3. We have the following equivalence: $p_1(Z_1) \equiv V^*$.

Proof. For the sake of brevity we give only a sketch of the proof. The first part reduces to show that $p_1(Z_1) \subseteq V^*$. Given $z \in Z_1$ we consider a representative $u = u_{--} + \tilde{u}$ such that $u_{--} = k\langle z_1^{-1}, z_2^{-1} \rangle \setminus k$ and $\tilde{u} = k\langle \langle z_1, z_2, z_1^{-1}, z_2^{-1} \rangle \setminus (k\langle \langle z_1, z_2 \rangle \rangle \setminus k) \rangle$ we obtain $p_1(z) = p_1(u_{--})$ and $u_{--} \in \text{Ker}C$. If we compute the action of z_1^{-1} on $p_1(u_{--})$ we obtain

$$\begin{aligned} \tilde{B}(u_{--} z_1^{-1}) = \\ p_1(u z_1^{-1}) - B_1 u(0, 0) - B_2 u(1, -1) - A_2 x(1, -1) \end{aligned}$$

moreover

$$\begin{aligned} A_1 x(0, 0) = \\ \tilde{B}(u_{--} z_1^{-1}) = w_1 - (B_1 u(0, 0) + B_2 u(1, -1) + A_2 x(1, -1)) \end{aligned}$$

with $w_1 \in p_1(Z_1)$, a similar result can be obtained for $A_2 x(0, 0)$. Summarizing for a representative of $z \in Z_1$ namely u this gives u_{--} such that $p_1(u_{--}) \in \text{Ker}C$ and

$$A_1 p(u_{--}) + A_2 x_2 + B_1 u_1 + B_2 u_2 \in p(Z_1) \quad (2)$$

and

$$A_2 p(u_{--}) + A_1 x_2 + B_1 u_3 + B_2 u_1 \in p(Z_1) \quad (3)$$

for appropriate u_1, u_2, u_3, x_1, x_2 . This pair of relations can be written for all u lying on the portion of the plane $\{(k, -k) \mid k \in \mathbb{Z}\}$ thus giving (1).

The second part reduces to show that $\mathcal{V}^* \subseteq p_1(Z_1)$. It is not hard to show that for $v \in V^*$, $v \in X$ one can find $u \in k\langle \langle z_1, z_2, z_1^{-1}, z_2^{-1} \rangle \rangle$ such that $p_1(u_{--}) = v$ and $[u] \in Z_1$. Consider $v \in V^*$ then there's a sequence $\dots, v_{-1}, v, v_1, v_2, \dots$ such that $\sum_{-\infty}^{+\infty} v_i z^i + v \in \mathcal{V}^*$, define $x(-i, i) = v_i$ and $x(0, 0) = v$ and $X_0 = \sum_{-\infty}^{+\infty} x(-i, i) z^i$ then compute $X_1(z) = (A_1 + A_0 z)X_0(z) + (B_1 + B_2 z)U_0(z)$ with $X_1(z) \in \mathcal{V}^*$ for a suitable $U_0(z) \in k_b[[z]]$. In this way we obtain the input $u(-i, i)$ with $i \in \mathbb{Z}$. Since $X_1(z) \in \mathcal{V}^*$ this procedure can be repeated to obtain $X_2(z) \in \mathcal{V}^*$ with $U_1(z) \in k_b[[z]]$, this $U_1(z)$ gives $u(-i + 1, i)$, $i \in \mathbb{Z}$. With this inductive algorithm we obtain the $u \in k\langle \langle z_1, z_2, z_1^{-1}, z_2^{-1} \rangle \rangle$ needed.

□

In order to complete the study of the zero module we need to consider as for 1D systems the module Z_2 defined by:

$$Z_2 := \frac{\text{Ker}H}{\text{Ker}H \cap (k\langle z_1^{-1}, z_2^{-1} \rangle \setminus k)}$$

namely the output nulling inputs in $\langle \langle z_1, z_2, z_1^{-1}, z_2^{-1} \rangle \rangle$ eventually modulo those lying in $k\langle z_1^{-1}, z_2^{-1} \rangle \setminus k$. Since $\text{Ker}H \cup (k\langle z_1^{-1}, z_2^{-1} \rangle \setminus k) \subseteq H^{-1}(k\langle \langle z_1, z_2, z_1^{-1}, z_2^{-1} \rangle \rangle \setminus \mathbb{R}\langle \langle z_1, z_2 \rangle \rangle)$ we define the projection $p_2 : Z_2 \rightarrow X$ and we seek the range of $p_2(Z_2)$. As for Z_1 we need to recur to the 1D system defined over the ring $k[z]$, we make reference to the geometrical objects perviously defined in [5].

Definition 4. A submodule X of $k_b[[z]]$ is the global reachable state submodule if it is $(A_1 + A_2 z)$ invariant: $A_1 + A_2 z)X$ and contains the range of the $k_b[[z]] \rightarrow k_b^n[[z]]$ operator defined by $u \mapsto (B_1 + B_2 z)u$.

As remarked in [5] the global reachable state is a finite generated submodule, the projection $P(X)$ gives C_s , the linear space of achievable states with zero initialization set and finite support input string defined in [1].

Definition 5. The output nulling maximum reachable subspace $\mathcal{R}^*$ is defined by $\mathcal{R}^* := X \cap \text{Ker}C$ where as previously $\text{Ker}C := \{\sum_{-\infty}^{+\infty} x_i z^i \mid x_i \in \text{Ker}C\}$.

We define $R^* = P(\mathcal{R}^*)$ to state the following result.

Proposition 6. We have the following equivalence: $p_2(Z_2) \equiv R^*$.

Proof. The proof is very similar to the previous one to which we make reference, if $z \in Z_2$ again we take a representative $u \in \text{Ker}H$ of z and split it in two terms $u = u_{--} + \tilde{u}$ as before. By reasoning in the same way as we did for $z \in Z_1$ we obtain two equations similar to (2) and (5):

$$A_1 p(u_{--}) + A_2 x_2 + B_1 u_1 + B_2 u_2 \in p(Z_2) \quad (4)$$

and

$$A_2 p(u_{--}) + A_1 x_2 + B_1 u_3 + B_2 u_1 \in p(Z_2) \quad (5)$$

for appropriate u_1, u_2, u_3, x_1, x_2 . Summarizing the projection of $z \in Z_2$ over X is $p_2(u_{--}) \in \text{Ker}C$ and (4), (5) can be written for all local states $x(k, -k)$ in the following way $(A_1 + A_2 z)X(z) + (B_1 + B_2 z)U(z) \in \mathcal{R}$ where $X(z) = \sum_{k=-\infty}^{+\infty} x(k, -k)z^k$ and $U(z) = \sum_{k=-\infty}^{+\infty} u(k, -k)z^k$. As a consequence $p_2(Z_2) \subseteq R^*$.

The second part consider the opposite inclusion namely $R^* \subseteq p_2(Z_2)$. Now for $v \in R^*$, $v \in X$ one can find $u \in \mathbb{R}\langle \langle z_1, z_2, z_1^{-1}, z_2^{-1} \rangle \rangle$ such that $p_2(u_{--}) = v$ and $[u] \in Z_2$. This gives rise to a global state $X_0 = \sum_{-\infty}^{+\infty} x(-i, i)z^i$ where $x(-i, i) = v_i$ and $x(0, 0) = v$ come from $P(\dots, v_{-1}, v, v_1, v_2, \dots) = v \in R^*$. We compute $X_1(z) = (A_1 + A_0 z)X_0(z) + (B_1 + B_2 z)U_0(z)$ and $X_1(z) \in \mathcal{R}^*$ because $X_0(k)$ belongs to a submodule $(\mathcal{R}^*)$ maximum $A_1 + A_2 z)$ invariant containing $\text{Range}k_b[[z]]$. Thus we have that $R^* \subseteq p_2(Z_2)$.

□

Summarizing the results contained in these two propositions by keeping the notations used we have obtained the following commutative diagram.

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
Z_2 & \xrightarrow{p_2} & R^* \\
\downarrow & & \downarrow \\
Z_1 & \xrightarrow{p_1} & V^* \\
\downarrow & & \downarrow \\
Z_1/Z_2 & \xrightarrow{p} & V^*/R^* \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Remark 7. The results presented in this section can be easily extended from the 2D to the nD framework with no great effort apart from introducing further heavy notations.

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