Baer's extension problem for multidimensional linear systems

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Abstract. Within an algebraic analysis approach, the purpose of this paper is to constructively solve the following problem: given two fixed multidimensional linear systems $\mathcal{B}_1$ and $\mathcal{B}_2$, parametrize the multidimensional linear systems $\mathcal{B}$ which contain $\mathcal{B}_1$ as a subsystem and satisfy that $\mathcal{B}/\mathcal{B}_1$ is isomorphic to $\mathcal{B}_2$. In particular, we parametrize the equivalence classes of multidimensional linear systems $\mathcal{B}$ which admit a fixed parametrizable subsystem $\mathcal{B}_p$ and satisfy that $\mathcal{B}/\mathcal{B}_p$ is isomorphic to a fixed autonomous system $\mathcal{B}_a$.

Keywords. Multidimensional linear systems, behavioural approach, Baer extensions, differential time-delay systems, constructive algebra, module theory.

1 Introduction

A well-known result due to R. E. Kalman states that any time-invariant 1-D linear system defined by a state-space representation can be decomposed into the direct sum of its controllable (i.e., parametrizable) and autonomous subsystems ([11]). Within the behavioural approach, this result was extended by J. C. Willems to time-invariant polynomial linear systems ([16]). Using an algebraic analysis approach, M. Fliess generalized this result in [10] to time-varying linear systems of ordinary differential equations whose coefficients belong to a differential field. However, it is well-known that this result does not admit a generalization for multidimensional linear systems.

In the recent works [20, 21], we constructively characterized when a multidimensional linear system $\mathcal{B}$ is isomorphic to $\mathcal{B}_2$ as a subsystem and satisfies $\mathcal{B}/\mathcal{B}_1$ isomorphic to $\mathcal{B}_a$.
mensional linear system can be decomposed into a direct sum of its parametrizable subsystem and the system formed by its autonomous elements. The corresponding algorithm was implemented in the library OreModules ([6, 7]) and illustrated by different explicit examples. Moreover, we applied these results to the *Monge problem* which questions the existence of parametrizations of the solutions of multidimensional linear systems and to optimal control and variational problems ([20, 21]).

Within an algebraic analysis approach, we constructively solve here the more general problem consisting in parametrizing all the multidimensional linear systems $C$ whose parametrizable subsystems are isomorphic to a given parametrizable system $B_p$ and such that $C/B_p$ are isomorphic to a given autonomous system $B_a$, i.e., $C/B_p \cong B_a$. In particular, $B_p$ (resp., $B_a$) can be chosen as the parametrizable subsystem (resp., the system formed by the autonomous elements) of a multidimensional linear system $B$. Solving this last problem allows us to parametrize all the multidimensional linear systems which have the same parametrizable subsystem and autonomous system as $B$. We then show how that result allows us to find again those obtained in [20, 21]. Our results are based on the important concept of *Baer extensions* developed in homological algebra and its connections with the extension abelian group $\text{ext}_D^1(M, N)$ ([5, 12, 23]). This problem was pointed out to us by S. Shankar (Chennai Mathematical Institute) ([24]). We would like to thank him.

The plan of the paper is the following one: In Section 2, we recall Baer’s interpretation of the elements of the abelian group $\text{ext}_D^1(M, N)$ in terms of equivalence classes of extensions of $N$ by $M$. In Section 3, we explicitly characterize $\text{ext}_D^1(M, N)$ as an abelian group, which allows us in Section 4 to parametrize the equivalence classes of multidimensional linear systems $B$ which admit as a subsystem the system $B_1$ defined by $M$ and satisfy that $B/B_1$ are isomorphic to the system $B_2$ defined by $N$. In Section 5, the previous results are applied to the particular situation where $N = t(P)$ is the torsion left $D$-submodule of a given finitely presented left $D$-module $P$ and $M = P/t(P)$. We finally explain how to find again the results of [20, 21].

In what follows, we refer to [6, 13, 15, 18, 19, 25] and the references therein for the concepts relevant to the module-theoretic approach to systems theory.

## 2 Baer extensions and Baer sums

We refer to [5, 12, 23] for the classical definitions of a complex and an exact sequence.

Let us first introduce the concept of *Baer extensions* which will play an important role in what follows.

### Definition 1 ([5, 12, 23])

We have the following definitions:

1. Let $M$ and $N$ be two left $D$-modules. An *extension of $N$ by $M$* is an exact sequence $e$ of left $D$-modules of the form:

\[ e : 0 \rightarrow N \xrightarrow{f} E \xrightarrow{g} M \rightarrow 0. \]  

### (1)
2. Two extensions of \(N\) by \(M\), \(e_i : 0 \rightarrow N \xrightarrow{f_i} E_i \xrightarrow{g_i} M \rightarrow 0\), \(i = 1, 2\), are said to be equivalent, denoted by \(e_1 \sim e_2\), if there exists a \(D\)-isomorphism \(\phi : E_1 \rightarrow E_2\) such that we have the commutative exact diagram

\[
\begin{array}{c}
0 \rightarrow N \xrightarrow{f_1} E_1 \xrightarrow{g_1} M \rightarrow 0 \\
\| \downarrow \phi \| \end{array}
\]

\[
\begin{array}{c}
0 \rightarrow N \xrightarrow{f_2} E_2 \xrightarrow{g_2} M \rightarrow 0, \\
\end{array}
\]

or, equivalently, such that \(f_2 = \phi \circ f_1\) and \(g_1 = g_2 \circ \phi\) hold.

3. We denote by \([e]\) the equivalence class of the extension \(e\) for the equivalence relation \(\sim\). The set of all equivalence classes of extensions of \(N\) by \(M\) is denoted by \(e_D(M, N)\).

4. A short exact sequence of the form (1) is said to split if \(E \cong M \oplus N\), where \(\oplus\) (resp., \(\cong\)) denotes the direct sum (resp., that two modules are isomorphic).

Let us introduce the concept of Baer sum of two extensions ([5, 12, 23]).

**Definition 2 ([5]).** Let \(e_i : 0 \rightarrow N \xrightarrow{f_i} E_i \xrightarrow{g_i} M \rightarrow 0\), \(i = 1, 2\), be two extensions of \(N\) by \(M\) and let us define the following two \(D\)-morphisms:

\[
-f_1 \oplus f_2 : N \rightarrow E_1 \oplus E_2, \quad (g_1, -g_2) : E_1 \oplus E_2 \rightarrow M
\]

\[
n \mapsto (-f_1(n), f_2(n)), \quad (a_1, a_2) \mapsto g_1(a_1) - g_2(a_2).
\]

Then, the Baer sum of the extensions \(e_1\) and \(e_2\), denoted by \(e_1 + e_2\), is defined by the left \(D\)-module \(E_3 = \ker(g_1, -g_2)/\im(-f_1 \oplus f_2)\), i.e., by the short exact sequence

\[
0 \rightarrow N \xrightarrow{f_3} E_3 \xrightarrow{g_3} M \rightarrow 0,
\]

\[
n \mapsto \varpi(f_1(n), 0) = \varpi(0, f_2(n)), \quad \varpi(a_1, a_2) \mapsto g_1(a_1) = g_2(a_2)
\]

where \(\varpi : \ker(g_1, -g_2) \rightarrow E_3\) denotes the canonical projection onto \(E_3\).

We have the following classical but important result on extensions.

**Theorem 3 ([5, 12, 23]).** The set \(e_D(M, N)\) equipped with the Baer sum forms an abelian group: the equivalence class of the split short exact sequence

\[
0 \rightarrow N \xrightarrow{f_3} M \oplus N \xrightarrow{f_3} M \rightarrow 0
\]

defines the zero element of \(e_D(M, N)\) and the inverse of the equivalence class \([e]\) of (1) is defined by the equivalence class of the following two equivalent extensions:

\[
0 \rightarrow N \xrightarrow{-f_3} E \xrightarrow{g_3} M \rightarrow 0, \quad 0 \rightarrow N \xrightarrow{f_3} E \xrightarrow{-g_3} M \rightarrow 0.
\]
3 Computing extensions of finitely presented modules

In this section, we show how to compute the abelian group $\text{ext}_D^1(M, N)$, when $M$ and $N$ are two finitely generated left $D$-modules over a noetherian domain $D$ ([23]).

By assumption, the left $D$-module $M$ admits the finite free resolution

$$
\ldots \xrightarrow{R_3} D^{1 \times p_2} \xrightarrow{R_2} D^{1 \times p_1} \xrightarrow{R_1} D^{1 \times p_0} \xrightarrow{\eta} M \longrightarrow 0,
$$

namely, (2) is an exact sequence of left $D$-modules where $R_i \in D^{p_i \times p_{i-1}}$ and $(.R_i)(\lambda) = \lambda R_i$, for all $\lambda \in D^{1 \times p_i}$. Applying the contravariant left exact functor $\text{hom}_D(\cdot, N)$ to the complex $\ldots \xrightarrow{R_3} D^{1 \times p_2} \xrightarrow{R_2} D^{1 \times p_1} \xrightarrow{R_1} D^{1 \times p_0} \longrightarrow 0$, we obtain the following complex of abelian groups

$$
\ldots \xrightarrow{R_3} N^{p_2} \xrightarrow{R_2} N^{p_1} \xrightarrow{R_1} N^{p_0} \longrightarrow 0,
$$

where $(R_i)(\eta) = R_i \eta$, for all $\eta \in N^{p_{i-1}}$. For more details, see, e.g., [5, 12, 19, 23].

Applying the covariant right exact functor $D^m \otimes_D \cdot$ to the finite presentation (i.e., to the exact sequence) $D^{1 \times t} \xrightarrow{S} D^{1 \times s} \xrightarrow{\delta} N \longrightarrow 0$ of the left $D$-module $N$, and using the fact that $D^m$ is a free right $D$-module, and thus, a flat right $D$-module, we obtain the following exact sequence:

$$
D^{m \times t} \xrightarrow{S} D^{m \times s} \xrightarrow{\text{id}_m \otimes \delta} N^m \longrightarrow 0.
$$

For more details, see, e.g., [5, 12, 19, 23].

Using the notations $p = p_0$, $q = p_1$, $r = p_2$, $R = R_1$ and combining (3) and (4), we obtain the following commutative diagram of abelian groups with exact columns:

$$
\begin{array}{cccc}
0 & 0 & 0 & \\
N^q & R_2 & N^p & \\
\downarrow \text{id}_q \otimes \delta & \downarrow \text{id}_q \otimes \delta & \downarrow \text{id}_p \otimes \delta & \\
D^{q \times s} & R_2 & D^{p \times s} & \\
\downarrow S & \downarrow S & \downarrow S & \\
D^{q \times t} & R_2 & D^{p \times t} & \\
\end{array}
$$

Let us now introduce the abelian group $\text{ext}_D^1(M, N) = \ker_N(R_2) / \text{im}_N(R)$, where:

$$
\ker_N(R_2) = \{ \eta \in N^q \mid R_2 \eta = 0 \} = \{ \eta = (\text{id}_q \otimes \delta)(A) \mid A \in D^{p \times s} : R_2 \eta = 0 \},
$$

$$
\text{im}_N(R) = RN^p = \{ \eta = (\text{id}_q \otimes \delta)(A) \mid \exists B \in D^{p \times s} : \eta = R((\text{id}_p \otimes \delta)(B)) \}.
$$

From (5), we get $(R_2) \circ (\text{id}_q \otimes \delta) = (\text{id}_r \otimes \delta) \circ (R_2)$ and $(R) \circ (\text{id}_p \otimes \delta) = (\text{id}_q \otimes \delta) \circ (R)$. Hence, using the exactness of the columns of (5), we obtain:

$$
R_2((\text{id}_q \otimes \delta)(A)) = (\text{id}_r \otimes \delta)(R_2 A) = 0 \Leftrightarrow \exists B \in D^{q \times t} : R_2 A = B S.
$$
With the previous notations, we have:

**Lemma 4.**

\[(\text{id}_q \otimes \delta)(A) = R((\text{id}_p \otimes \delta)(X)) = (\text{id}_q \otimes \delta)(R X)\]
\[\Leftrightarrow (\text{id}_q \otimes \delta)(A - R X) = 0 \Leftrightarrow \exists Y \in D^{q \times t}: A = R X + Y S\]

Hence, we obtain the following results.

**Example 5.**

Let us consider the commutative polynomial ring
\[\mathbb{D}\]
over, if \(Z \in D^{p \times s}, \exists Y \in D^{q \times t}: A = R X + Y S\]

Moreover, if we define the abelian group
\[\Omega = \{A \in D^{q \times s} | \exists B \in D^{r \times t}: R_2 A = B S\}, \]
then we have the following isomorphism of abelian groups
\[(\text{ext}^{1}_{D}(M, N) = \ker N(R_2) / \text{im} N(R_1)) \xrightarrow{\iota} \Omega / (\text{R} D^{p \times s} + D^{q \times t} S), \]
where \(\rho : \ker N(R_2) \rightarrow \text{ext}^{1}_{D}(M, N) \) (resp., \(\varepsilon : \Omega \rightarrow \Omega \big/ (\text{R} D^{p \times s} + D^{q \times t} S)\))

denotes the canonical projection onto \(\text{ext}^{1}_{D}(M, N) \) (resp., \(\Omega \big/ (\text{R} D^{p \times s} + D^{q \times t} S)\)).

We let the reader check that \(\iota\) is well-defined and bijective ([22]).

We recall that the abelian group \(\text{ext}^{1}_{D}(M, N)\) characterizes the obstructions for the existence of \(\xi \in N^p\) satisfying the inhomogeneous linear system \(R \xi = \zeta\), where \(\zeta \in N^q\) satisfies the compatibility condition \(R_2 \zeta = 0\). In particular, the vanishing of \(\text{ext}^{1}_{D}(M, N)\) implies that \(R_2 \zeta = 0\) is a necessary and sufficient condition for the existence of \(\xi \in N^p\) satisfying \(R \xi = \zeta\). For more details, see [6, 7, 18, 19].

If \(\ker D(R) = 0\), i.e., \(R_2 = 0\), we then get \(\Omega = D^{p \times s}\). Another simple case is \(N = D^{1 \times s}, S = 0\), for which we have \(\Omega = \{A \in D^{q \times s} | R_2 A = 0\} \) (see [4]).

If \(D\) is a commutative ring and \(\otimes\) denotes the Kronecker product, then using the identity \(U V W = \text{row}(V)(U^T \otimes W)\), where \(\text{row}(V)\) is obtained by concatenating the rows of \(V\), we have \(\Omega / (\text{R} D^{p \times s} + D^{q \times t} S) \cong D^{1 \times u} Z / (D^{1 \times (p \times q) \times t}) X\), where
\[X = \left(\begin{array}{c}
R_T \otimes I_s \\
I_q \otimes S
\end{array}\right) \in D^{(p \times q) \times q}, \quad Y = \left(\begin{array}{c}
R_T \otimes I_s \\
I_r \otimes S
\end{array}\right) \in D^{(q \times r) \times r},
\]
and \(Z \in D^{u \times q}\) is defined by \(\ker D(Y) = D^{1 \times u} (Z - T)\) and \(T \in D^{u \times r}\). Moreover, if \(D\) is a polynomial ring over a computable field \(k\) (e.g., \(k = \mathbb{Q}, \mathbb{F}_p\)), then, using Gröbner or Janet bases, we can explicitly describe the \(D\)-module \(\text{ext}^{1}_{D}(M, N)\) by means of generators and relations ([2, 8]). For the implementations of the corresponding algorithms, see the packages homalg ([3, 2]) and OreMorphisms ([9]).

**Example 5.**}

Let us consider the commutative polynomial ring \(D = \mathbb{Q}(\alpha)[\partial, \delta]\) of differential time-delay operators, where \(\alpha \in \mathbb{R}\), and the following two matrices:
\[R = \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 + \delta^2 & -\alpha \partial \delta
\end{pmatrix} \in D^{2 \times 3}, \quad S = \begin{pmatrix}
\partial & -\partial \\
\partial \delta^2 & -\partial
\end{pmatrix} \in D^{2 \times 2}. \]
Let us define the $D$-modules $M = D^{1 \times 3}/(D^{1 \times 2} R)$ and $N = D^{1 \times 2}/(D^{1 \times 2} S)$. We have $R_2 = 0$, and thus, $\Omega = D^{2 \times 2}$, $\text{ext}^1_D(M, N) \cong D^{2 \times 2}/(R D^{3 \times 2} + D^{2 \times 2} S)$ and:

$$\text{ext}^1_D(M, N) \cong D^{1 \times 4}/ \left( D^{1 \times 10} \left( R^T \otimes I_2 \right) \right).$$ (11)

We denote by $L$ the matrix appearing in (11) and $\epsilon : D^{1 \times 4} \rightarrow P = D^{1 \times 4}/(D^{1 \times 10} L)$ the canonical projection onto $P$. Denoting by $v_i = \epsilon(g_i)$ the residue class in $P$ of the $i^{th}$ vector of the standard basis $\{g_i\}_{1 \leq i \leq 4}$ of $D^{1 \times 4}$, we obtain:

$$v_i = 0, \quad i = 1, 2, \quad (1 + \delta^2) v_i = 0, \quad i = 3, 4, \quad \partial v_i = 0, \quad i = 3, 4.$$

Hence, the $D$-module $P$ is generated by $v_3 = \epsilon((0, 0, 1, 0))$ and $v_4 = \epsilon((0, 0, 0, 1))$. Transforming back the row vectors $g_3$ and $g_4$ into $2 \times 2$ matrices, we obtain that the $D$-module $D^{2 \times 2}/(R D^{3 \times 2} + D^{2 \times 2} S)$ is generated by $\epsilon(A_1)$ and $\epsilon(A_2)$, where:

$$A_1 = \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right), \quad A_2 = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right).$$ (12)

It is a torsion $D$-module as we have $(1 + \delta^2) \epsilon(A_i) = 0$ and $\partial \epsilon(A_i) = 0, i = 1, 2$. Using (9), we obtain that the $\rho((\text{id}_2 \otimes \delta)(A_i))$'s generate the $D$-module $\text{ext}^1_D(M, N) = N^2/(R N^3)$ and satisfy $(1 + \delta^2) \rho((\text{id}_2 \otimes \delta)(A_i)) = 0$, $\partial \rho((\text{id}_2 \otimes \delta)(A_i)) = 0, i = 1, 2$.

If $D$ is a non-commutative ring, then $\text{ext}^1_D(M, N)$ is an abelian group, but not a left $D$-module. If $D$ is a $k$-algebra, where $k$ is a field contained in the center of $D$, then $\text{ext}^1_D(M, N)$ is a $k$-vector space. If $M$ and $N$ are two finite-dimensional $k$-vector spaces or two holonomic left modules over the $k$-algebra of differential operators with $k$-polynomial (resp., $k$-rational) coefficients (the so-called Weyl algebras $A_n(k)$ and $B_n(k)$), then we can compute a $k$-basis of $\text{ext}^1_D(M, N)$ (see [8] and the references therein). However, $\text{ext}^1_D(M, N)$ is generally an infinite-dimensional $k$-vector space. If $D$ is a non-commutative polynomial ring over which Gröbner or Janet bases exist (e.g., the Weyl algebras, certain classes of Ore algebras [6]), then we can compute the $k$-vector space formed by the matrices $A \in D^{n \times n}$ with a fixed order in the functional operators and a fixed degree (resp., fixed degrees) in the polynomial (resp., rational) coefficients which satisfy $R_2 A \in D^{n \times n}$. See [8] for more details and the package OreMorphisms ([9]) for an implementation.

4. **An explicit description of** $\text{ext}^1_D(M, N)$

The following theorem is an important result in homological algebra which can be traced back to the pioneering work of R. Baer ([1]).

**Theorem 6 ([5, 12, 23])**. Let $M$ and $N$ be two left $D$-modules. Then, the abelian groups $\text{ext}^1_D(M, N)$ and $\text{v}_D(M, N)$ are isomorphic.

The explicit description of $\text{ext}^1_D(M, N)$ – being proved by making Theorem 6 constructive for the interesting class of modules in systems theory – can be given now. For the sake of brevity, we refer to [22, Theorem 3] for the proof.
Theorem 7. Let $R \in D^{q \times p}$ and $S \in D^{t \times s}$ be two matrices with entries in $D$ and \( M = D^{1 \times p} / (D^{1 \times q} R) \) and $N = D^{1 \times s} / (D^{1 \times t} S)$ the left $D$-modules finitely presented by $R$ resp. $S$. Let us denote by $R_2 \in D^{r \times q}$ a matrix satisfying \( \ker_D(R) = D^{1 \times r} R_2 \). Then, every equivalence class of extensions of $N$ by $M$ is represented by

\[
e : 0 \rightarrow N \xrightarrow{\alpha} E \xrightarrow{\beta} M \rightarrow 0,
\]

where the left $D$-module $E$ is defined by

\[
D^{1 \times (q+r)} \xrightarrow{Q} D^{1 \times (p+s)} \xrightarrow{\phi} E \rightarrow 0, \quad Q = \begin{pmatrix} R & -T \\ 0 & S \end{pmatrix} \in D^{(q+r) \times (p+s)},
\]

and $T$ is a certain element of $\Omega = \{ A \in D^{q \times s} \mid \exists B \in D^{r \times t} : R_2 A = B S \}$.

Finally, the equivalence class $[e]$ only depends on the residue class $\varepsilon(T)$ of $T \in \Omega$ in $\Omega / (RD^{p \times s} + D^{r \times t} S) = \iota(\text{ext}_D^1(M, N))$, where $\iota$ is defined in (9).

Example 8. Let us consider again Example 5. Theorem 7 says there exist two non-trivial equivalence classes of extensions of $N$ by $M$ respectively defined by $E_1 = D^{1 \times 5} / \left(D^{1 \times 4} \begin{pmatrix} R & -T_1 \\ 0 & S \end{pmatrix} \right)$, where the matrices $R$ and $S$ are given by (10) and the matrices $T_1 = A_1$ and $T_2 = A_2$ by (12). Finally, the trivial extension of $N$ by $M$ (i.e., the split extension) is defined by the $D$-module $E_0$ where $T_0 = 0$.

Let $\mathcal{F}$ be a left $D$-module. Applying the contravariant left exact functor $\text{hom}_D(\cdot, \mathcal{F})$ to (13), we obtain the following results [22, Corollary 1].

Corollary 9. With the previous notations, we have the following results:

1. $\ker \mathcal{F}(S) \xrightarrow{\alpha^*} \ker \mathcal{F}(Q) \xrightarrow{\beta^*} \ker \mathcal{F}(R) \leftarrow 0$ is an exact sequence, where the $D$-morphism $\beta^*$ (resp., $\alpha^*$) is defined by $\beta^*(\xi) = (\xi^T \ 0^T)^T$, for all $\xi \in \ker \mathcal{F}(R)$ (resp., $\alpha^*(\eta) = \eta_2$, for all $\eta = (\eta_1^T \ \eta_2^T)^T$, $\eta_1 \in \mathcal{F}^p$ and $\eta_2 \in \mathcal{F}^s$).

2. If $\mathcal{F}$ is an injective left $D$-module ([23]), then we have the exact sequence:

\[
0 \leftarrow \ker \mathcal{F}(S) \xrightarrow{\alpha^*} \ker \mathcal{F}(Q) \xrightarrow{\beta^*} \ker \mathcal{F}(R) \leftarrow 0. \quad (15)
\]

Moreover, if $\mathcal{F}$ is cogenerator ([23]), then (15) is exact if and only if (13) is.

5 Applications to multidimensional systems theory

The purpose of this section is to parametrize all equivalence classes of multidimensional linear systems which have a fixed parametrizable subsystem and a fixed autonomous system. Let $R \in D^{r \times p}$ be a matrix with entries in a noetherian domain $D$. If $M = D^{1 \times p} / (D^{1 \times q} R)$ denotes the left $D$-module finitely presented by $R$, then $t(M) = \{ m \in M \mid \exists 0 \neq a \in D : \ a m = 0 \}$ is a left $D$-submodule of $M$ and we have the following canonical short exact sequence (see, e.g., [5, 12, 23]):

\[
0 \rightarrow t(M) \xrightarrow{\iota} M \xrightarrow{T} M/t(M) \rightarrow 0. \quad (16)
\]
An element of \( t(M) \) is called a torsion element of \( M \) and \( M \) is said to be torsion-free if \( t(M) = 0 \) and torsion if \( t(M) = M \) (see, e.g., [23]). Constructive results developed in [6, 7, 17] show that there exists a matrix \( R' \in \mathbb{D}^{q \times p} \) satisfying:

\[
t(M) = (D^{1 \times q} \ R')/(D^{1 \times q} \ R), \quad M/t(M) = D^{1 \times p}/(D^{1 \times q} \ R').
\]

If \( \mathcal{F} \) is an injective left \( D \)-module, applying the exact functor \( \text{hom}_D(\cdot, \mathcal{F}) \) to the exact sequence (16), we then get the exact sequence of abelian groups:

\[
0 \hookrightarrow \text{hom}_D(t(M), \mathcal{F}) \xleftarrow{\epsilon^*} \text{hom}_D(M, \mathcal{F}) \xrightarrow{\tau} \text{hom}_D(M/t(M), \mathcal{F}) \twoheadrightarrow 0.
\]

The linear system \( \ker F(R') = \{ \zeta \in \mathcal{F}^p \mid R' \zeta = 0 \} \cong \text{hom}_D(M/t(M), \mathcal{F}) \) is the parametrizable subsystem of \( \ker F(R) = \{ \eta \in \mathcal{F}^p \mid R \eta = 0 \} \cong \text{hom}_D(M, \mathcal{F}) \) as there always exists a matrix \( Q' \in \mathbb{D}^{p \times m} \) such that \( \ker F(R') = Q' \mathcal{F}^m \), i.e., any solution \( \eta \in \mathcal{F}^p \) of the system \( R' \eta = 0 \) has the form \( \eta = Q' \xi \) for a certain \( \xi \in \mathcal{F}^m \). For more details, see [6, 15, 17, 25]. For certain classes of multidimensional systems, \( \ker F(R') \) is also called the controllable subsystem of \( \ker F(R) \) (see, e.g., [6, 15, 17, 18, 25]).

If we denote by \( R'' \in D^{r \times q'} \) (resp., \( R'' \in D^{r' \times q''} \)) a matrix satisfying \( R = R'' R' \) (resp., \( \ker D_R(R') = D^{1 \times r'} \ R'' \)), then we have the following \( D \)-isomorphism ([8, 21]):

\[
t(M) \cong D^{1 \times q'/ \left( D^{1 \times (q+r')} \left( \begin{array}{cc} R'' & R_2' \end{array} \right) \right)}.
\]  

The autonomous system defined by \( \ker F((R''^T \ R_2'')^T, \) \( ) \cong \text{hom}_D(t(M), \mathcal{F}) \) satisfies:

\[
\ker F((R''^T \ R_2'')^T, \) \( ) \cong \ker F(R)/\tau^*(\ker F(R')).
\]

This last system will be called the autonomous quotient of the system \( \ker F(R) \).

If \( M \) and \( N \) are respectively a torsion-free and a torsion left \( D \)-module defined by two finite presentations, Theorem 7 parametrizes the equivalence classes of extensions of \( N \) by \( M \). Moreover, if \( \mathcal{F} \) is an injective left \( D \)-module, by Corollary 9, we then obtain the equivalence classes of systems admitting \( \text{hom}_D(M, \mathcal{F}) \) as a parametrizable subsystem and \( \text{hom}_D(N, \mathcal{F}) \) as autonomous quotient. If we consider the left \( D \)-module \( P = M \oplus N \), we then have \( t(P) \cong N \) and \( P/t(P) \cong M \) and the previous problem can be reduced to the case where we only consider the extensions of \( t(P) \) by \( P/t(P) \) for a finitely presented left \( D \)-module \( P \).

Let \( L \in D^{m \times l} \) be a matrix with entries in a noetherian domain \( D \) and let us consider the finitely presented left \( D \)-module \( P = D^{1 \times l}/(D^{1 \times m} \ L) \). As shown in [6, 18] and implemented in [7], computing the left \( D \)-module \( \text{ext}_D^1(N, D) \), where \( N = D^m/(L \ D^1) \), gives us a matrix \( L' \in D^{m' \times l} \) satisfying:

\[
\begin{cases}
  t(P) = (D^{1 \times m'} \ L')/(D^{1 \times m} \ L), \\
p/t(P) = D^{1 \times l}/(D^{1 \times m'} \ L').
\end{cases}
\]  

We denote by \( \epsilon : D^{1 \times m} \rightarrow P \) (resp., \( \epsilon' : D^{1 \times m} \rightarrow P/t(P) \)) the canonical projection onto \( P \) (resp., \( P/t(P) \)). In particular, we have the relation \( \epsilon' = \tau \circ \epsilon \), where \( \tau \) denotes the canonical projection \( P \rightarrow P/t(P) \) (see (16) with \( M = P \)).
Corollary 10. Every class of extensions of $t(P)$ by $P/t(P)$ is defined by means of the left $D$-module $E = D^{1 \times (l + m')}/(D^{1 \times (m' + m + n')} Q)$, where $Q$ has the form

$$Q = \begin{pmatrix} L' & -T \\ 0 & L'' \end{pmatrix} \in D^{(m'+m+n') \times (l+m')}$$  \hspace{1cm} (19)$$

(with $L''$ (resp., $L'_2$) playing the role of $R''$ (resp., $R'_2$) in (17)) and $T$ is an element of the abelian group:

$$\Omega = \left\{ A \in D^{m' \times m'} \mid \exists B \in D^{n' \times (m+n')} : L'_2 A = B \begin{pmatrix} L'' \\ L'_2 \end{pmatrix} \right\}.$$  \hspace{1cm} (20)$$

Finally, the equivalence classes of the extensions of $t(P)$ by $P/t(P)$ only depend on the residue classes $\varepsilon(T)$ in the following abelian group where $i$ as defined in (9):

$$\Omega \left( L' D^{l \times m'} + D^{m' \times (m+n')} \begin{pmatrix} L'' \\ L'_2 \end{pmatrix} \right) = \varepsilon(t_D(P/t(P), t(P))).$$  \hspace{1cm} (21)$$

If $\mathcal{F}$ is an injective left $D$-module and $\ker \mathcal{F}(L) \cong \text{hom}_D(P, \mathcal{F})$, then Corollaries 9 and 10 give a parametrization of the equivalence classes of linear systems $\ker \mathcal{F}(Q) \cong \text{hom}_D(E, \mathcal{F})$ which admit $\ker \mathcal{F}(L')$ as a parametrizable subsystem and $\ker \mathcal{F}(L'' L'_2^T)$ as an autonomous quotient.

Example 11. Let us consider the differential time-delay system ([14])

$$\begin{cases} \dot{y}_1(t) - \dot{y}_2(t - 2h) + \alpha \dot{y}_3(t - h) = 0, \\
\dot{y}_1(t - 2h) - \dot{y}_2(t) + \alpha \dot{y}_3(t - h) = 0,
\end{cases}$$  \hspace{1cm} (22)$$

where $\alpha \in \mathbb{R}$ and $h$ is a strictly positive real number. We denote by $D = \mathbb{Q}(\alpha) [\partial, \delta]$ the commutative polynomial ring of differential time-delay operators, the matrix

$$L = \begin{pmatrix} \partial & -\partial \delta^2 & \alpha \partial^2 \delta \\
-\partial \delta^2 & \partial & \alpha \partial^2 \delta \end{pmatrix} \in D^{2 \times 3},$$

and the $D$-module $P = D^{1 \times 3}/(D^{1 \times 2} \ L)$. Using a constructive algorithm developed in [6, 17] and implemented in [7], we get $L' = R \in D^{2 \times 1}$ defined by (10). We can check that $\ker_D(L') = 0$ and $L = L'' L'$, where $L'' = S \in D^{2 \times 2}$ is defined by (10). Hence, we obtain $t(P) \cong D^{1 \times 2}/(D^{1 \times 2} \ L'')$. Now, the equivalence classes of extensions of $t(P)$ by $P/t(P)$ are in 1-1 correspondence with the elements of the $D$-module $\text{ext}^3_D(P/t(P), t(P))$. Using Examples 5 and 8, we obtain that the two non-trivial equivalence classes of extensions are defined by the $D$-modules $E_1$ and $E_2$ given in Example 8. They respectively correspond to the following systems:

$$\begin{cases} z_1(t) + z_2(t) = 0, \\
z_2(t) + z_2(t - 2h) \\
-(\alpha \ z_3(t - h) + z_4(t)) = 0, \\
\dot{z}_4(t) - \dot{z}_5(t) = 0, \\
\dot{z}_4(t - 2h) - \dot{z}_5(t) = 0,
\end{cases}$$

$$\begin{cases} z_1(t) + z_2(t) = 0, \\
z_2(t) + z_2(t - 2h) \\
-(\alpha \ z_3(t - h) + z_5(t)) = 0, \\
\dot{z}_4(t) - \dot{z}_5(t) = 0, \\
\dot{z}_4(t - 2h) - \dot{z}_5(t) = 0.
\end{cases}$$
The trivial class of extensions of \( t(P) \) by \( P/t(P) \) can be defined by the system:

\[
\begin{align*}
    z_1(t) + z_2(t) &= 0, \\
    z_2(t) + z_2(t - 2h) - \alpha \dot{z}_3(t - h) &= 0, \\
    \dot{z}_4 - \dot{z}_5(t) &= 0, \\
    \dot{z}_4(t - 2h) - \dot{z}_5(t) &= 0.
\end{align*}
\]

Hence, the three systems admit the same parametrizable subsystem and the same autonomous quotient as (22).

**Remark 12.** The matrix \( Q \) defined by (19) with \( T = I_{m'} \in \Omega \) was used in [20, 21] to parametrize the \( \mathcal{F} \)-solutions of the system \( \ker \mathcal{F}(L') \) in terms of the \( \mathcal{F} \)-solutions of \( \ker \mathcal{F}(L') \) and \( \ker \mathcal{F}((L')^T) \). We first need to solve the following autonomous homogeneous linear system \( \ker \mathcal{F}((L')^T) \) corresponding to \( \text{hom}_D(t(P), \mathcal{F}) \):

\[
\begin{align*}
    L'' \theta &= 0, \\
    L'_2 \theta &= 0.
\end{align*}
\]

Then, we need to solve the inhomogeneous system \( L' \eta = \theta \), i.e., find a particular solution \( \eta^* \in \mathcal{F}' \) of \( L' \eta^* = \theta \) and the general solution of the homogeneous system \( L' \eta = 0 \) associated with \( \text{hom}_D(P/t(P), \mathcal{F}) \). As the subsystem \( \text{hom}_D(P/t(P), \mathcal{F}) \) of \( \text{hom}_D(P/t(P), \mathcal{F}) \) is parametrizable, we can compute a matrix \( Q' \in D'^{l \times k} \) satisfying \( \ker \mathcal{F}(L') = Q' \mathcal{F}^k \) whenever \( \mathcal{F} \) is an injective left \( D \)-module ([6, 15, 19, 25]). Then, the solution of \( L \eta = 0 \) has the form \( \eta = \eta^* + Q' \xi \), for arbitrary \( \xi \in \mathcal{F}^k \). We refer to [21] for applications to variational and optimal control problems.

Next, we have a direct consequence of Remark 12. For more details, see [22].

**Proposition 13.** The exact sequence \( 0 \rightarrow t(P) \xrightarrow{\epsilon} P \xrightarrow{\pi} P/t(P) \rightarrow 0 \) splits iff \( \epsilon(I_{m'}) = 0 \), i.e., iff there exist \( X \in D'^{l \times m'} \), \( Y \in D'^{m' \times m} \) and \( Z \in D'^{m \times n'} \) satisfying:

\[
I_{m'} = L' X + Y L'' + Z L'_2 \quad \Leftrightarrow \quad L' = L' X L' = Y L.
\]

**Remark 14.** As shown in [20, 21], Proposition 13 gives a particular solution \( \eta^* \in \mathcal{F}' \) of the inhomogeneous system \( L' \eta = \theta \), where \( \theta \in \mathcal{F}' \) is a general solution of the system (23): using (24), we get \( \theta = L' X \theta + Y L'' \theta + Z' L'_2 \theta = L' (X \theta) \) as \( \theta \) satisfies (23). If \( \mathcal{F} \) is an injective left \( D \)-module, using Remark 12, we then obtain that the elements of \( \ker \mathcal{F}(L) \) have the form \( \eta = X \theta + Q' \xi \), for all \( \xi \in \mathcal{F}^k \).

The left \( D \)-module \( P/t(P) = D'^{l \times l}/(D'^{l \times m'} L') \) is stably free, i.e., satisfies \( P/t(P) \oplus D'^{l \times s} \cong D'^{l \times r} \) for non-negative integers \( r \) and \( s \) ([23]), iff there exists \( X \in D'^{l \times m'} \) such that \( L' X L' = L' ([17]) \). Hence, if \( P/t(P) \) is stably free, then (24) holds with \( Y = 0 \). In particular, if \( D = \mathbb{k}[t][\partial] \) is the Weyl algebra (\( k \) a field of characteristic 0) or a left principal ideal domain (e.g., \( K[\partial] \), \( K \) a differential field), then every torsion-free left \( D \)-module is stably free and, in particular, \( P/t(P) \) for any finitely presented left \( D \)-module \( P \). Hence, we find again Kalman’s result ([11]) and its different generalizations ([10, 16]) described in the introduction.
Bibliography


