

Duffing mechanical oscillator: algebraic reconstruction

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Abstract:

This paper concerns algebraic parameter identification of the Duffing Mechanical Oscillator. The approach followed here guarantees accurate parameter identification as well as easy numerical implementation of the corresponding algorithm. Robustness against high frequency output measurement noise is taken into account via the inclusion of a suitable invariant filter. The system identification method is based on iterated convolutions and depends on the availability of one measurable system output. Exact formulae for the unknown parameters are provided, assuming unavailability of the system initial conditions.

Keywords: Mechanical oscillators, Duffing system, chaotic dynamics, algebraic reconstruction, parameter identification, nonlinear dynamical systems.

1 Introduction

The reconstruction of chaotic behaviors in dynamical systems concerns a wide range of scientific and engineering problems (neural science, chemistry, micromechanics, meteorological modeling, and sismology, are some typical fields involved in this domain of research; see for instance [18, 10, 4, 27, 33] and the references therein). Partial knowledge of the concerned dynamical system (mainly due to unavailability of some variables or parameters) makes the reconstruction of chaotic attractors a difficult task; for the pioneering works see for instance [32], [25], and [28]. Two main approaches are taken into account when considering the partial knowledge case of the reconstruction problem: the first one is based on embedding timeseries of observed variables into a phase space; the state of the dynamical system is then

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obtained via time delayed values of a single observed state (see for instance [24, 5, 19, 1, 26, 2, 9]). The second approach, which is followed in this paper, is based on control theoretic ideas, mainly concerning systems identification (see for instance [23, 7, 22, 6, 30, 31]).

This paper presents an algebraic methodology for the reconstruction of chaotic behaviors of the Duffing mechanical oscillator, when considering partial knowledge of the state of the system. The algebraic approach discussed here (which follows the ideas developed in [11] and [17]) supposes measurability of the oscillator position and is based on the proved fact that the unknown system parameters are linearly identifiable (see [15]). Simple algebraic procedures (which do not require the knowledge of the system initial conditions) are provided for the exact computation of the unknown system parameters. Robustness against unavoidable output measurement noise (of unknown statistics) is guaranteed by the proposed algebraic methodology.

The rest of the paper is organized as follows:

Section 2 describes the Duffing mechanical oscillator, while Section 3 concerns the formulation of the partial knowledge reconstruction problem. Section 4 deals with the algebraic solution, which is illustrated in Section 5 via computer-based simulations. The paper concludes in Section 6 with some final remarks.

2 Duffing mechanical oscillator

The Duffing nonlinear system is a classical oscillators first published by Duffing in 1918 [16]. The model, which was originally introduced to describe the large amplitude vibration modes of a steel beam subjected to periodic forces, is represented as follows:

$$\begin{cases} \dot{x}(t) = v(t), \\ \dot{v}(t) = -pv(t) - p_1x(t) - p_2x^3(t) + A \cos(\omega t), \end{cases} \quad (1)$$

where:

- $[x(\cdot), v(\cdot)] \in \mathbb{R}^2$ denotes the state of the system, with $x(\cdot)$ corresponding to the position of the oscillator and $v(\cdot)$ corresponding to the velocity;
- A denotes the magnitude of the forcing function;
- ω denotes the forcing frequency;
- p denotes the damping coefficient;
- p_1 and p_2 denote the stiffness constants associated to the nonlinear spring.

Remark 1. *It is well-known that the motion of (1) presents periodic and chaotic behaviors when the parameters belongs to the neighborhood of $p = 0.4$, $p_1 = 1.1$, $p_2 = -1$, $\omega = 0.8$, and:*

$$\begin{array}{lll} A = 0.622 \text{ (period 1);} & A = 1.498 \text{ (period 2);} & A = 1.8 \text{ (chaotic);} \\ A = 2.1 \text{ (chaotic);} & A = 2.3 \text{ (period 1);} & A = 7 \text{ (period 1).} \end{array}$$

See for instance [20].

Given system (1), the main objective of this work is to recover the parameter vector $\mathbf{q} = [p, p_1, p_2, A]$, when only the position $x(\cdot)$ is measured.

In the sequel it will be considered the following:

Assumption 1. *The parameter vector $\mathbf{q}$ belongs to a neighborhood ensuring that system (1) exhibits chaotic behavior.*

Remark 2. *The natural frequency of the oscillator (directly related to the periodic driving signal), i.e. ω , is given a priori. Assumption 1 is required to avoid singularities in the identification method.*

3 Problem Formulation

This section deals with the problem statement. Moreover, some identification-oriented algebraic properties of the Duffing system are recalled. The main algebraic property establishes that the unknown parameter vector of the Duffing system is linearly identifiable, which makes possible to recover the unknown parameter vector from the model-based identification process proposed in the next section.

The partial knowledge reconstruction problem is then defined as follows:

Problem 1. *Let the oscillator position $x(\cdot)$ of system (1) be known for every $t > 0$. Find the unknown parameter vector $\mathbf{q}$ and the oscillator velocity $v(\cdot)$.*

In what follows some identification-oriented algebraic properties of system (1) are recalled (see [8]).

Property 1. Algebraic Observability *Consider a smooth nonlinear system described by:*

$$\dot{X} = f(X, P), \quad Y = h(X), \quad (2)$$

where: $X = \{x_i\}_{i=1}^n \in \mathbb{R}^n$ denotes the state vector; $Y = \{y_i\}_{i=1}^m \in \mathbb{R}^m$ denotes the output vector; $h(\cdot)$ is a smooth vector function and $P \in \mathbb{R}^l$ is a constant parameter vector, with $l < n$. Let $Y^{(j)}$ denote the j -th time derivative of vector Y . Vector state X is said to be algebraically observable if it can be uniquely expressed as

$$X = \Phi(Y, \dots, Y^{(j)}, P), \quad (3)$$

for some integer j and for some smooth function Φ .

Property 2. Algebraic linear identifiability *Given an algebraically observable smooth nonlinear system (2), P is said to be algebraically linearly identifiable (with respect to the output vector Y) if it satisfies the following linear relation:*

$$\Omega_1(Y, \dots, Y^{(j)}) = \Omega_2(Y, \dots, Y^{(j)})P, \quad (4)$$

where $\Omega_1(\cdot)$ and $\Omega_2(\cdot)$ are, respectively, $n \times 1$ and $n \times n$ smooth matrices.

It is quite obvious that Duffing system (1) is algebraically observable with respect to the output $y = x$. Indeed, the state variables of the system can be rewritten as:

$$\begin{cases} v = \dot{y}, \\ \dot{v} = -p_2 v - p_1 y - p_2 y^3 + A \cos(\omega t). \end{cases} \quad (5)$$

Moreover, from the second relation of (5) it follows that:

$$\ddot{y} = [-\dot{y}, -y, -y^3, \cos(\omega t)] \mathbf{q}^T. \quad (6)$$

That is, $\mathbf{q} = [p, p_1, p_2, A]$ is linearly identifiable with respect to the selected output y .

Remark 3. *In system identification terms algebraic linear identifiability means that it is possible to perform exact non-asymptotic recovery of the unknown parameters vector (see [11]).*

Assuming that the parameter vector of an algebraic observable system is linearly identifiable, there are two different families of methods to recover the parameter vector. The first family includes the methods which estimate high order time derivatives applying (reduced order or high gain) state observers. The second family includes the methods which relies on the estimation of parametric iterative integrals of the measurable output. The main disadvantage of the first family is due to the presence of stationary and numerical errors, since the errors increases as the derivative order does it. The second family has not this disadvantage. The reconstruction methodology discussed here belongs to the second family, i.e. is based on the implementation of numerical integrators.

4 Model based on-line parameter vector estimation

4.1 Partial knowledge reconstruction in algebraic terms

Since the algebraic linear identifiability (with respect to y) of the parameter vector $\mathbf{q}$ of system (1), it is possible to recover it via iterative numerical integration from the measurable system output signal. The method proposed here is based on the solution of a linear matrix algebraic equation, being $\mathbf{q}$ the corresponding unknown vector. As the first step of the procedure both sides of (1) are multiplied by t^2 . The resulting expression is then integrated by parts where possible. Finally, additional time integrations are applied to get as many equations as necessary to obtain a linear system for the unknown linearly identifiable parameters.

Firstly, equation (1) is multiplied by t^2 and the resulting expression is then iteratively integrated twice. Thus:

$$\int^{(2)} t^2 \ddot{y}(t) = -p \int^{(2)} t^2 \dot{y}(t) - p_1 \int^{(2)} t^2 y(t) - p_2 \int^{(2)} t^2 y^3(t) + \int^{(2)} A \cos(\omega t) t^2. \quad (7)$$

Where the integral term $\int^{(m)} t^j x(t)$ is defined as follows:

$$\int^{(m)} t^j x(t) := \int_0^t \int_0^{\sigma_1} \dots \int_0^{\sigma_{m-1}} (\sigma_m)^j x(\sigma_m) d\sigma_m d\sigma_{m-1} \dots d\sigma_1.$$

Remark 4. The following facts hold (resulting from the application of the integration by parts method):

$$\int_0^t t^2 \ddot{y}(t) = t^2 y - 4 \int_0^t \sigma y d\sigma + 2 \int_0^t \int_0^\sigma y d\lambda d\sigma$$

and:

$$\int_0^t t^2 \dot{y}(t) = \int_0^t \sigma^2 y d\sigma - 2 \int_0^t \int_0^\sigma \lambda y d\lambda d\sigma.$$

So that equation (7) is equivalent to the following parametric iterative integration:

$$-pw_{11}(t) - p_1 w_{12}(t) - p_2 w_{13}(t) + Aw_{14}(t) = f_1(t), \quad (8)$$

where:

$$\begin{aligned} w_{11}(t) &= \int_0^t \sigma^2 y d\sigma - 2 \int_0^t \int_0^\sigma \lambda y d\lambda d\sigma; & w_{12}(t) &= \int_0^t t^2 y(t); \\ w_{13}(t) &= \int_0^t t^2 y^3(t); & w_{14}(t) &= \int_0^t A \cos(wt) t^2; \end{aligned}$$

and:

$$f_1(t) = t^2 y - 4 \int_0^t \sigma y d\sigma + 2 \int_0^t \int_0^\sigma y d\lambda d\sigma.$$

Remark 5. Since the quantities $w_{1n}(t)$, for $1 \leq n \leq 4$ and $f_1(t)$ depends directly on the measurable output y (they are also independent from the initial conditions), they can always be numerically estimated.

Integrating three more times Equation (8) with respect to t allows to obtain the following linear equation in the unknown parameter vector:

$$W(t)\mathbf{q}^T = \mathbf{f}(t), \quad (9)$$

where the elements of matrix $W(t)$ are generated as follows:

$$\begin{aligned} w_{21}(t) &= \int_0^t w_{11}(t); & w_{22}(t) &= \int_0^t w_{12}(t); & w_{23}(t) &= \int_0^t w_{13}(t); & w_{24}(t) &= \int_0^t w_{14}(t) \\ w_{31}(t) &= \int_0^t w_{21}(t); & w_{32}(t) &= \int_0^t w_{22}(t); & w_{33}(t) &= \int_0^t w_{23}(t); & w_{34}(t) &= \int_0^t w_{24}(t); \\ w_{41}(t) &= \int_0^t w_{31}(t); & w_{42}(t) &= \int_0^t w_{32}(t); & w_{43}(t) &= \int_0^t w_{33}(t); & w_{44}(t) &= \int_0^t w_{34}(t). \end{aligned} \quad (10)$$

Vector $\mathbf{f}(t)$ is given by:

$$f_2(t) = \int_0^t f_1(t); \quad f_3(t) = \int_0^t f_2(t); \quad f_4(t) = \int_0^t f_3(t). \quad (11)$$

Proposition 1. For any time t , after a small open time interval of the form $[0, \epsilon)$ with $\epsilon > 0$, the matrix $W(t)$ becomes invertible and $\mathbf{f}(t)$ is non zero.

Proof. Suppose that $W(t)$ is singular on an open interval of the form (t_0, t_1) with $t_1 > t_0$. That is, $\det W(t) = 0$, for all $t \in (t_0, t_1)$, implies the existence of constants $\{k_1, k_2, k_3, k_4\}$ different from zero such that:

$$0 = k_1 \dot{y}(t) + k_2 y(t) + k_3 y^3(t) + k_4 \cos(wt); \quad t_0 \leq t \leq t_1, \quad (12)$$

where y is also a solution of the chaotic system (1). Note that, if $k_1 = 0$ it follows that:

$$k_2 y + k_3 y^3 + k_4 \cos(wt) = 0,$$

which is not possible because of Assumption 1. So that system (12) is equivalent to:

$$\begin{cases} \dot{z} = w; \\ \dot{y} = -(k_2 y + k_3 y^3 + k_4 \cos(z))/k_1. \end{cases}$$

Then, according to the Poincare-Bendixon Theorem, y can not exhibit a chaotic behavior, since y satisfies the above second order differential equation. This fact contradicts Assumption 1. Consequently, constant k_i , for $i = 1 \dots, 4$, is equal to zero, which concludes the proof. ■

Remark 6. Due to the fact that $W(t)$ is, initially, an ill-conditional matrix, it would be convenient let elapse a time interval long enough until matrix $W(t)$ become invertible. And then, in order to avoid singularities following simple rule is applied: If $r(W(t)) < k$ (where k is sufficiently large) then compute the inverse of $W(t)$; otherwise, compute the inverse of $W(t - h)$ until the condition holds, where h is a sample time.

Finally, the computation of the parameter vector $\mathbf{q}$, allows for the state $v(\cdot)$ to be easily estimated. To accomplish this, we multiply both sides of (5) by t and integrate once with respect to t . Then:

$$v(t) = \frac{1}{t} \left(-p(ty - \int y) - p_1 \int ty - p_2 \int ty^3 + A \int t \cos(wt) + y(t) - y(0) \right),$$

which is well defined for all $t > 0$.

Remark 7. Note that the definite integration starting from $t = 0$ can be changed to any new initial value $t_0 > 0$. That is, the integration process may be re-started at any instant of time, as desired, or needed.

4.2 Invariant Filtering

This subsection deals with the partial knowledge reconstruction problem when the measurable output $y(\cdot)$ is corrupted by additive noise. Moreover, it is assumed that the statistics of the noise are unknown. Then:

$$y(t) = y_a(t) + \alpha(t),$$

where: $y(\cdot)$ is the measurable output; $y_a(\cdot)$ is the actual position; α is the unknown high frequency noise. In order to enhance the performance of the proposed algebraic reconstruction method, simple invariant filters will be considered. This kind of filters has been already presented and justified in [11] (see also the highly interesting papers [13] and [14]).

Thus, the time integration operator is applied to relation(9), i.e.:

$$\int^{(k)} W(t) \mathbf{q}^T = \int^{(k)} \mathbf{f}(t), t > \epsilon > 0, \quad (13)$$

where k is the number of nested integrations needed to eliminate the undesirable noise effect.

Remark 8. *It is evident that the performance of filter (13) improves when k increases. However, it is not convenient to take great values for k , because of the computation required by the corresponding nested integration process.*

Remark 9. *The nested integration operation works in the filtering process (13) computing average values from both sides of (13). Thus, the parameter vector $\mathbf{q}$ is obtained via a simple matrix inversion (see for instance [12]).*

5 Numerical Simulations

We performe some numerical simulations on MatlabTM in order to evaluate the effectiveness of the proposed algebraic parameters identification technique for the Duffing Mechanical Oscillator (we use the Runge-Kutta integration algorithm, with an integration step equal to 0.001).

To show the high performance of the algorithm, we first simulate the free of noise system fixing the parameters at $p = 0.4$, $p_1 = -1.3$, $p_2 = 1.2$, $w = 1$ and $A = 1.7$; the initial conditions were set to be $x(0) = 2$ and $\dot{x}(0) = -1$. In a second numerical experiment we add noise to the measured variable (in this case the parameters of the system and the initial conditions do not change from the noise free experiment. In both cases we force the singular number matrix, $r(W)$, to be less than 2000.

In Figures 1 and 2 we present the actual and the estimated parameter vector $\mathbf{q}$. From Figs. 1 and 2 we can see that the reconstruction method recovers the parameters of the system, almost perfectly, after $t > 2.3$ [seconds].

6 Final remarks

In this work we presented an algebraic approach for the non-asymptotic, however fast and accurate, estimation of the unknown parameters of the Duffing Mechanical Oscillator, solely from the knowledge of the available output, which is the oscillator position. This approach provides a direct formula that computes the unknown parameters. The formula consists of iterated integral convolutions of the available measurable signal and the straightforward avoidance of an initial computational

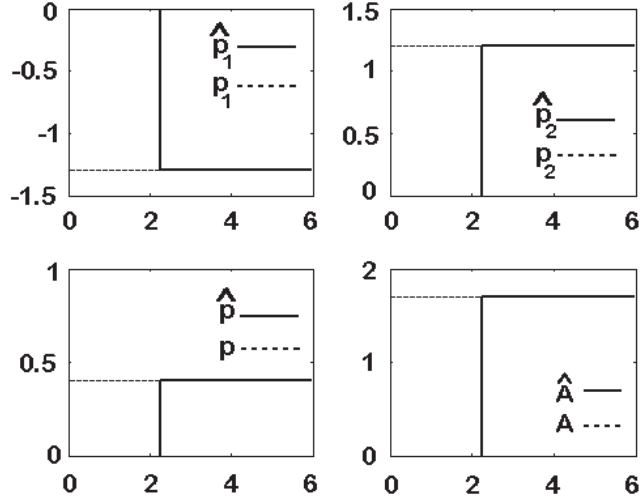


Figure 1. Estimation of the unknown parameters free of noise, where their actual values were set as $p = 0.4$, $p_1 = -1.3$, $p_2 = 1.2$ and $A = 1.7$.

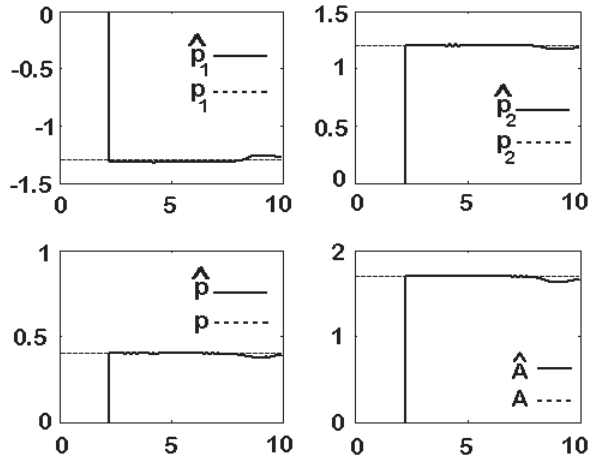


Figure 2. Estimation of the unknown parameters when the measurable output is perturbed by high frequency noise, where their actual values were set as $p = 0.4$, $p_1 = -1.3$, $p_2 = 1.2$ and $A = 1.7$.

singularity. We show that the parameters of the Duffing system are linearly identifiable, therefore a linear system can be built for the on-line estimation of the unknown parameters.

We emphasize that *ad-hoc* algebraic operations on the differential equations of the output allows the proposed method to be independent of the initial conditions of the underlying nonlinear dynamical system. Using a method, here referred as invariant filtering, we also were able to estimate the parameter values of the concerned system when noise, represented by computer generated stochastic processes, was considered on the output measurements.

Finally, we presented and discussed computer simulations to validate the high efficiency and accuracy of the proposed method.

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