Global Pinning Controllability of Complex Networks*

M. Porfiri† *and M. di Bernardo*‡

1 Introduction

Pinning-control has been proposed in the literature [7, 16, 12, 3, 13, 17, 11] as a viable strategy to control lattices and networks of coupled dynamical systems onto some desired common reference trajectory. The general idea behind pinning control is to use a feedback control input on just a limited subset of the whole dynamical system, that is, to actively control only a few network nodes (the so-called pinned sites or reference sites). Specifically, a direct control action is active only on such pinned nodes, and is propagated to the rest of the network through the coupling among the oscillators, represented by edges in the network.

It has been shown that such a strategy is effective in taming the dynamics of networks of linear and nonlinear oscillators with different topological features. Some proofs of local asymptotic stability have been given in the case where the desired asymptotic trajectory is an equilibrium point [12] for the dynamical system. A particularly challenging open problem is to provide sufficient conditions for global pinning-controllability of complex networks. A strategy for the numerical exploration of pinning-controllability based on the master-stability function is presented in [15].

The aim of this paper is to establish sufficient conditions for global pinning-controllability of a generic network of oscillators to some desired solution, not necessarily an equilibrium point. We consider a complex dynamical system comprised of a number of identical chaotic oscillators coupled via a time-invariant bidirectional

*The first author was supported by the National Science Foundation via Grant CMMI-0745753.
†Department of Mechanical and Aerospace Engineering, Polytechnic University, Brooklyn, NY 11201
‡Department of Systems and Computer Science, University of Naples Federico II, Naples 80125, Italy, and Department of Engineering Mathematics, University of Bristol, Bristol BS8 1TR, UK

communication network. The reference trajectory is generated by an exogenous oscillator identical to the network oscillators. Pinned nodes in the network are coupled to the exogenous node through a linear state feedback. Pinned nodes may be viewed as slave systems of the reference node, that consequently behaves as a master oscillator. The oscillators' network is globally pinning-controlled if the oscillators globally synchronize, that is, if for any initial condition the oscillators asymptotically track the reference trajectory. Note that, in contrast to the synchronization problem often studied in the literature, see for example [2] and references therein, such reference trajectory is supposed to be known and chosen to achieve some desired control objective. We cast the global synchronization problem into a global asymptotic stability problem by describing the system time evolution in terms of the error dynamics. By using Lyapunov-stability theory and algebraic graph theory, we establish sufficient conditions for global synchronization. The conditions derived in this paper involve the network topological structure, the dynamics of an individual oscillator, and the structure of the state feedback control action. One of the main contributions of this work is to establish a simple algebraic condition for global pinning controllability based on the fraction of pinned nodes, the relative network algebraic connectivity, the coupling strength among the network oscillators, and the gain of the feedback control action. We show that as the fraction of pinned nodes increases, smaller feedback gains are needed for global pinning-control. We also prove that, for a connected network, even for a limited number of pinned nodes, global pinning-controllability can be achieved by properly selecting the coupling strength and the feedback gain.

Our notation throughout is standard. $\mathbb{Z}^+$ refers to the set of nonnegative integers. $\|\cdot\|$ refers to the Euclidean norm in $\mathbb{R}^m$ or corresponding induced norm in $\mathbb{R}^{m\times m}$, with $m \in \mathbb{Z}^+$. The vector in $\mathbb{R}^m$ that consists of all unit entries is denoted $1_m = [1, \dots, 1]^T$, and $\otimes$ is the standard Kronecker product. I_m is the $m \times m$ identity matrix. The symmetric part of a matrix $B \in \mathbb{R}^{m\times m}$ is indicated with sym$B$, that is, $\mathrm{sym}B = \frac{1}{2}(B + B^T)$. The smallest and largest eigenvalues of a symmetric matrix $A \in \mathbb{R}^{m\times m}$ are indicated with $\lambda_{min}(A)$ and $\lambda_{max}(A)$, respectively. The algebraic spectrum of A, say $\{\lambda_i(A)\}_{i=1}^m$, is ordered so that $\lambda_{min}(A) = \lambda_1(A) \le \lambda_2(A) \le \cdots \le \lambda_{m-1}(A) \le \lambda_m(A) = \lambda_{max}(A)$. The unit norm eigenvector of A corresponding to $\lambda_i(A)$ is called $v_i(A)$.

2 Problem Statement

We consider a dynamical system consisting of N identical oscillators interconnected pairwise via a time-constant information network. Graph theory is used to model the oscillators' communication network, with vertices representing individual oscillators and edges representing active communication links. All communication is bidirectional, so that the corresponding graph is undirected. In addition, all communication links are equal, so that the corresponding graph is unweighted or unity-weighted. A connected network is one in which a path exists between each pair of vertices in the graph. We extensively use the Laplacian matrix for characterizing the effects of the communication network. The Laplacian is defined as

$L = D - \mathcal{A}$, where $\mathcal{A}$ is the adjacency matrix for the graph and D is the degree matrix. The adjacency matrix $\mathcal{A}$ is a zero-one matrix whose entries are nonzero when they correspond to interconnected nodes. The degree matrix D is a diagonal matrix whose diagonal entries equal the number of edges incident with the corresponding nodes. Defined in this manner, L is symmetric and positive semidefinite. In addition, the vector 1_N is in the null space of L. The total number of zero eigenvalues of L equals the number of connected components of the graph [6].

The time evolution of the i^{th} oscillator is described by

$$\dot{x}_i(t) = f(x_i(t)) - \sigma B \sum_{j=1}^{N} l_{ij} x_j(t) + u_i(t) \tag{1}$$
$$x_i(t_0) = x_{i0}, \quad i = 1, \dots N, \quad t \geq t_0$$

where $t_0 \in \mathbb{R}$ is the initial time; $x_i(t) \in \mathbb{R}^n$ is the state of the i^{th} oscillator; $x_{i0} \in \mathbb{R}^n$ is the initial condition of the i^{th} oscillator; $f : \mathbb{R}^n \to \mathbb{R}^n$ describes the oscillators' individual dynamics; $B \in \mathbb{R}^{n \times n}$ is the inner linking matrix that describes the coupling between the states of coupled oscillators; $\sigma > 0$ is a control parameter that partially assigns coupling strength between oscillators; $u_i(t)$ is the control input to oscillator i; and scalars l_{ij}'s are the elements of the graph Laplacian L.

We further assume that the control inputs are applied only to selected pinned nodes in the network, and that they are generated by a sole state linear feedback law with respect to a reference trajectory $s(t)$, satisfying an individual oscillator's dynamics, that is, $\dot{s}(t) = f(s(t))$. Thus, we set $u_i(t) = p_i K e_i(t)$, where p_i is equal to one for pinned nodes and zero otherwise, and K is the feedback gain matrix. We indicate with r the number of pinned nodes, that is, the number of nonzero diagonal entries of $P := \text{Diag}[p_1, p_2, \dots, p_N]$. The vector e_i describes the error of the oscillator i in tracking the reference signal s, that is $e_i(t) = s(t) - x_i(t)$.

We collect all the states of the system in the nN dimensional vector $x(t) = \left[x_1^T(t), \dots, x_N^T(t)\right]^T \in \mathbb{R}^{nN}$, and we rewrite the system of equations (1) using the Kronecker product as

$$\dot{x}(t) = [1_N \otimes f](x(t)) - \sigma L \otimes B x(t) + P \otimes K(1_N \otimes s(t) - x(t)) \tag{2}$$

where $[1_N \otimes f]\,(x(t)) = \left[f^T(x_1(t)), \dots, f^T(x_N(t))\right]^T$ and $P = \text{Diag}\,[p_1, \dots, p_N]$. We note that the matrix P is diagonal and positive semidefinite.

3 Global Pinning-Controllability

We collect the error components $e_i(t)$ into the error vector $e(t) = 1_N \otimes s(t) - x(t)$ and formulate the control problem defined in (2) as an asymptotic stability problem about the origin for the system

$$\dot{e}(t) = 1_N \otimes f(s(t)) - [1_N \otimes f](x(t)) - \sigma L \otimes B e(t) - P \otimes K e(t) \tag{3}$$

where we have used the fact that 1_N is in the null space of the Laplacian, that is, $L 1_N = 0$. Following [9], we define the matrix function $F_{\xi,\widetilde{\xi}}$ for any $\xi, \widetilde{\xi} \in \mathbb{R}^n$ by

$$f(\xi) - f(\widetilde{\xi}) = F_{\xi,\widetilde{\xi}}(\xi - \widetilde{\xi}) \tag{4}$$

We assume that $F_{\xi,\widetilde{\xi}}$ is bounded in $\mathbb{R}^n \times \mathbb{R}^n$, that is, we assume that there exists a positive constant α such that for any $\xi, \widetilde{\xi} \in \mathbb{R}^n$

$$\|F_{\xi,\widetilde{\xi}}\| \leq \alpha \tag{5}$$

Condition (5) applies to a large variety of chaotic oscillators, see for example [9] and the examples therein. We use condition (5) to enforce global synchronization of the oscillator network to the reference trajectory. Whereas milder conditions can be used to assess local synchronization, see for example [15], in this paper we retain the inherent system nonlinearities and we focus on global synchronization.

Using equation (4), the error dynamics (3) is rewritten as

$$\dot{e}(t) = \mathcal{F}(e(t), t)e(t) - (\sigma L \otimes B + P \otimes K)e(t) \tag{6}$$

where

$$\mathcal{F}(e(t), t) = \mathrm{Diag}\left[F_{s(t),s(t)-e_1(t)}, \ldots, F_{s(t),s(t)-e_N(t)}\right] \tag{7}$$

Definition 1. *We say that (1) is globally pinning-controllable if the error dynamical system in (6) is globally asymptotically stable about the origin.*

Global asymptotic stability of (6) is here studied using Lyapunov stability criteria, see for example [10]. The following proposition establishes a sufficient condition for global stability.

Proposition 2. *If the feedback gain matrix K, the inner linking matrix B, and the coupling strength σ are chosen such that for every $t \geq t_0$, and for every $y_1, \ldots, y_N \in \mathbb{R}^n$*

$$\lambda_i(y, t) < -\mu, \quad i = 1, \ldots, nN \tag{8}$$

where $y = \left[y_1^T, \ldots, y_N^T\right]^T$, $\mu > 0$, $\{\lambda_i(y,t)\}_{i=1}^{nN}$ are the eigenvalues of the matrix $H(y,t)$ defined by

$$H(y, t) = \mathcal{D}(y, t) - 2(\sigma L \otimes \mathrm{sym}QB + P \otimes \mathrm{sym}QK) \tag{9}$$

with Q positive definite symmetric matrix in $\mathbb{R}^{n\times n}$, and

$$\mathcal{D}(y, t) = 2\mathrm{Diag}[\mathrm{sym}QF_{s(t),s(t)-y_1}, \ldots, \mathrm{sym}QF_{s(t),s(t)-y_N}] \tag{10}$$

Then, the dynamical system (6) is globally exponentially stable about the origin, implying that the network described by (1) is globally pinning-controllable.

Proof. We choose a quadratic candidate Lyapunov function

$$V(e) = e^T(I_N \otimes Q)e \tag{11}$$

where $Q \in \mathbb{R}^{n\times n}$ is symmetric and positive definite. The time derivative of $V(e(t))$ along the trajectory of the error dynamical system (6) is

$$\dot{V}(e(t)) = (\dot{e}^T(t)(I_N \otimes Q)e(t) + e^T(t)(I_N \otimes Q)\dot{e}(t)) = e^T(t)H(e(t), t)e(t) \tag{12}$$

Since H is a symmetric matrix, all its eigenvalues $\{\lambda_i\}_{i=1}^{nN}$ are real. By imposing condition (8), we find

$$\dot{V}(e(t)) < -\mu\|e(t)\|^2 \tag{13}$$

Equation (13) implies global exponential stability of (6), see for example [10]. Therefore, (1) is globally pinning-controllable. □

In what follows, we specialize Proposition 2 for the case where the inner linking matrix B and feedback gain matrix K are related by the following condition

$$\text{sym}QK = \kappa\text{sym}QB \tag{14}$$

for some κ positive constant and Q positive definite symmetric matrix in $\mathbb{R}^{n\times n}$. Condition (14) implies that the control action on the pinned nodes replicates the structure of the inner linking among the oscillators. In most of the literature on pinning-control, the stronger condition $B = \kappa K$ is used, see for example [3, 11, 15]. Our claims reported in what follows do not require direct relations between the skew parts of the matrices QB and QK.

3.1 Selection of Pinned Nodes

The following claim builds on Proposition 2 to establish a sufficient condition for global-pinning controllability that involves the network topology, the location of pinned-nodes, the coupling strength, and the feedback gain.

Corollary 3. *If, for some Q positive definite symmetric matrix in $\mathbb{R}^{n\times n}$, condition (14) is satisfied,* symQB *is a positive definite matrix and*

$$\lambda_{min}(\sigma L + \kappa P)\lambda_{min}(\text{sym}QB) > \alpha\|Q\|, \tag{15}$$

where the positive constant α satisfies (5), then (1) is globally pinning-controllable.

Proof. We choose the candidate Lyapunov function in (11). Using condition (14), the symmetric matrix $H(y,t)$ in (9) becomes

$$H(y,t) = \mathcal{D}(y,t) - 2(\sigma L + \kappa P) \otimes (\text{sym}QB) \tag{16}$$

where $\mathcal{D}$ is defined in (10). The largest eigenvalue of $H(y,t)$ can be bounded using Weyl's inequality, see for example Theorem 8.4.11 of [1],

$$\lambda_{max}(H(y,t)) \leq \lambda_{max}(\mathcal{D}(y,t)) + \lambda_{max}(-2(\sigma L + \kappa P) \otimes (\text{sym}QB)). \tag{17}$$

The first summand in the right hand side of equation (17) can be bounded as follows

$$\lambda_{max}(\mathcal{D}(y,t)) \leq \max_{i=1,\dots N} |\lambda_i(\mathcal{D}(y,t))| \leq \|\mathcal{D}(y,t)\|. \tag{18}$$

Substituting (10) into (18), and using condition (5), we find

$$\lambda_{max}(\mathcal{D}(y,t)) \leq 2\alpha\|Q\|, \tag{19}$$

for any $y \in R^{nN}$.

Now, we consider the second term in (17). Since symQB is positive definite and $\sigma L + \kappa P$ is positive semidefinite, being the sum of two positive semidefinite matrices, using Proposition 7.1.10 of [1], we find

$$\lambda_{max}(-2(\sigma L + \kappa P) \otimes (\mathrm{sym}QB)) = -2\lambda_{min}(\sigma L + \kappa P)\lambda_{min}(\mathrm{sym}QB). \tag{20}$$

Using (19) and (20) in equation (17), we find that under condition (15) the hypotheses of Proposition 2 are satisfied with

$$\mu = -2\alpha\|Q\| + 2\lambda_{min}(\sigma L + \kappa P)\lambda_{min}(\mathrm{sym}QB). \tag{21}$$

$\square$

We refer to the positive parameter μ in (21) as the synchronization strength. Indeed, from equation (13), we note that $\mu/\|Q\|$ is an upper bound for the exponential rate of $V(e(t))$, where the Lyapunov function is defined in (11). Corollary 3 can be used to quantitatively investigate the optimal selection of pinned nodes. Indeed, the quantity $\lambda_{min}(\sigma L + \kappa P)$ depends not only on the feedback gain κ, but also on the location of pinned nodes through the matrix P. An optimization problem similar to those analyzed in [5] for maximizing the algebraic connectivity of graphs can be formulated.

We further note that, according to equation (15), simultaneously increasing the coupling strength σ and the pinning-control strength κ positively affects the network synchronization. If condition (5) is relaxed, synchronization can be lost as σ and κ are simultaneously increased above threshold values determined using the so-called master stability function, see for example [15].

3.2 Effects of Network Topology

The following claim builds on Corollary 3 to establish a simpler sufficient condition for global pinning-controllability that takes into account only the connectivity of the communication network, the number of pinned nodes, the coupling strength, and the feedback gain.

Corollary 4. *If for some Q positive definite symmetric matrix in $\mathbb{R}^{n\times n}$ condition (14) is satisfied and* symQB *is a positive definite matrix, and the feedback gain κ satisfies*

$$\frac{\sigma\kappa\left(\frac{\lambda_2(L)}{N}\right)}{\sigma\left(\frac{\lambda_2(L)}{r}\right) + \kappa} > \alpha\|Q\|\frac{1}{\lambda_{min}(\mathrm{sym}QB)} \tag{22}$$

then (1) is globally pinning-controllable.

Proof. We show that under the conditions of the claim, inequality (15) is satisfied. To this aim, we use the the recent perturbation bounds for eigenvalues of [8].

Without lack of generality, we assume that the first r nodes are pinned-controlled. First, by applying Weyl's inequality we find that

$$\lambda_{min}(\sigma L + \kappa P) \geq \sum_{i=1}^{r} \lambda_{min}\left(\frac{\sigma}{r}L + \kappa \pi_i \pi_i^T\right). \tag{23}$$

Next, by applying Theorem 2.1 in [8] to $\frac{\sigma}{r}L + \kappa\pi_i\pi_i^T$, we obtain

$$\lambda_{min}\left(\frac{\sigma}{r}L + \kappa \pi_i \pi_i^T\right) \geq \phi(\sigma, \kappa) \tag{24}$$

where the positive function ϕ is defined by

$$\phi(\sigma, \kappa) = \frac{1}{2}\left(\frac{\sigma}{r}\lambda_2(L) + \kappa - \sqrt{\left(\frac{\sigma}{r}\lambda_2(L) + \kappa\right)^2 - \frac{4\sigma\kappa}{rN}\lambda_2(L)}\right) \tag{25}$$

Here, we have used the fact that $\|\pi_i\| = 1$, that $\lambda_{min}(L) = 0$, and that $v_1(L) = \frac{1}{\sqrt{N}}1_N$. We note that for positive σ and κ

$$\phi(\sigma, \kappa) \geq \frac{\sigma\kappa\lambda_2(L)}{(\sigma\lambda_2(L) + r\kappa)N}. \tag{26}$$

Substituting (24) into (23), and using (26), we find

$$\lambda_{min}(\sigma L + \kappa P) \geq \frac{\sigma\kappa\left(\frac{\lambda_2(L)}{N}\right)}{\sigma\left(\frac{\lambda_2(L)}{r}\right) + \kappa}. \tag{27}$$

Under condition (22), Corollary 3 applies, and the claim follows. $\square$

Corollary 4 does not provide indications on how to select the most appropriate pinning nodes for achieving global-pinning controllability. Nevertheless, it does provide useful insights into the influence of the network topology on global pinning-controllability. To better illustrate the influence of the network topology, we name the left hand side of equation (22) $\Phi(\sigma, \kappa)$, that is, we set

$$\Phi(\sigma, \kappa) = \frac{\sigma\kappa\left(\frac{\lambda_2(L)}{N}\right)}{\sigma\left(\frac{\lambda_2(L)}{r}\right) + \kappa}. \tag{28}$$

We also introduce the so-called *network relative connectivity* $\chi = \lambda_2(L)/N$ and the *fraction of pinned nodes* $\rho = r/N$. The network relative connectivity χ is a nonnegative parameter bounded by 1, since $\lambda_2(L)$ is less than or equal to the minimum vertex degree in the graph [4]. In terms of these parameters, (28) can be conveniently rewritten as the twice the harmonic mean of $\chi\sigma$ and $\rho\kappa$, that is

$$\Phi(\sigma, \kappa) = \left(\frac{1}{\rho\kappa} + \frac{1}{\sigma\chi}\right)^{-1}. \tag{29}$$

Equation (29) shows that as the fraction of pinned nodes increases along with the network relative connectivity, smaller feedback gains and coupling strengths may be sufficient for global pinning-control. In addition, the network relative connectivity is greater than zero if and only if the graph is connected. Therefore, a necessary condition for applying Corollary 4 is that the graph is connected. Nevertheless, connectedness may not be a sufficient condition for pinning-controlling the network according to Corollary 4. Indeed, once the coupling strength σ among the oscillators is prescribed, the largest value that $\Phi(\sigma,\kappa)$ can achieve in the limit of $\kappa \to \infty$ is equal to $\sigma\chi$. Thus, if $\lambda_{min}(\text{sym}QB)\sigma\chi \leq \alpha\|Q\|$, Corollary 4 does not provide a finite value for the feedback gain κ for global pinning-controllability. On the other hand, Corollary 4 establishes that if the network topology is connected then global pinning-controllability can be always made possible by increasing the coupling strength among the oscillators σ. This implies that even a single pinned node may be sufficient to synchronize the whole network onto the reference trajectory, provided that the oscillators are strongly coupled.

If the conditions of Corollary 4 are fulfilled, the synchronization strength μ in equation (21 is bounded by

$$\mu \leq -2\alpha\|Q\| + 2\Phi(\sigma,\kappa)\lambda_{min}(\text{sym}QB), \tag{30}$$

where we used equation (27) and definition (28). Equation (30) shows the significance of the network relative connectivity, and of the fraction of pinned nodes on the synchronization strength.

4 Illustration

To illustrate global pinning-controllability of oscillators coupled through a communication network, we consider a set of N chaotic Chua's oscillator, see for example [14]. The state x_i of the i^{th} oscillator is comprised of three components, that is $n = 3$. We write $x_i = [\xi_{i1}, \xi_{i2}, \xi_{i3}]^T$, where ξ_{i1}, ξ_{i2}, and ξ_{i3} are scalar quantities.

For a Chua's oscillator the nonlinear function $f(\xi)$ in (1) is given by

$$f(\xi) = \begin{bmatrix} a(\xi_2 - \xi_1 - h(\xi_1)) \\ \xi_1 - \xi_2 + \xi_3 \\ -b\xi_2 \end{bmatrix} \tag{31}$$

where $\xi = [\xi_1, \xi_2, \xi_3]^T \in \mathbb{R}^n$, $a > 0$, $b > 0$, and the nonlinear scalar function h has the form

$$h(\xi_1) = m_1\xi_1 + \frac{1}{2}(m_0 - m_1)(|\xi_1 + 1| - |\xi_1 - 1|) \tag{32}$$

with $m_0 < m_1 < 0$. We define

$$h(\xi_1) - h(\widetilde{\xi}_1) = w_{\xi_1,\widetilde{\xi}_1}(\xi_1 - \widetilde{\xi}_1) \tag{33}$$

where $w_{\xi_1,\widetilde{\xi}_1}$ depends on ξ_1 and $\widetilde{\xi}_1$ and is bounded by $m_0 \leq w_{\xi_1,\widetilde{\xi}_1} \leq m_1$, see for example [9]. Therefore the matrix function $F_{\xi,\tilde{\xi}}$ in (4) can be expressed as

$$F_{\xi,\tilde{\xi}} = A + M(w_{\xi_1,\widetilde{\xi}_1}) \tag{34}$$

where

$$A = \begin{bmatrix} -a & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{bmatrix}, \quad M(w_{\xi_1,\tilde{\xi}_1}) = \begin{bmatrix} -aw_{\xi_1,\tilde{\xi}_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{35}$$

The norm of the matrix $M(w_{\xi_1,\tilde{\xi}_1})$ is bounded by $a|m_1|$. Therefore, using the triangle inequality, the norm of the matrix $F_{\xi,\tilde{\xi}}$ is bounded by $\|A\| + a|m_1|$, and the constant α in equation (5) may be chosen to be equal to $\|A\| + a|m_1|$. Despite being very simple, this bound may be relatively conservative and consequently may yield conservative estimates for the coupling strength σ and the feedback gain κ needed for global pinning-controllability, see for example equation (22). Therefore, we numerically determine the positive bounding constant α in (5) through

$$\alpha = \max_{\epsilon \in [m_0, m_1]} \|A + M(\epsilon)\| \tag{36}$$

Following [9], we select $a = 9.81$, $b = 13.441$, $m_0 = -1.217$, and $m_1 = -0.648$. Substituting these numerical constants in equation (36), we find $\alpha = 17.98$. We also assume that the matrices B, K, and Q are equal to the identity matrix. Assuming that $B = K = I_3$ implies that beyond fulfilling equation (14), we further impose that the internal coupling and the pinning-control act in the same way on all the oscillators' states.

We consider a set of $N = 50$ representative Chua oscillators behaving chaotically. Oscillators are coupled through a random graph where the probability of existence of each edge is equal to 0.2. We ascertained that the graph is connected and we computed its algebraic connectivity $\lambda_2(L)$ to be 3.70. Therefore, the relative connectivity χ is equal to $3.70/50 = 0.074$.

Assuming that the coupling strength σ is equal to 50 and that only 5 nodes are pinned, that is, $r = 5$, Corollary 4 does not yield a finite value for the feedback gain κ to guarantee global pinning-controllability of the network, since $\sigma\chi \leq \alpha$. In order to apply Corollary 4, σ has to be made far larger than 50, approximately 250. Using Corollary 3, we numerically find that the network is globally pinning-controllable for $\kappa = 250$ if the nodes with highest degree are pinned. Degrees of pinned nodes are 18, 16, 14, 14, and 14. In this case, $\lambda_{min}(\sigma L + \kappa P) = 18.44$ that is larger than α. In Figure 1, we report $\lambda_{min}(\sigma L + \kappa P)$ for a wide range of coupling strength σ and feedback gain κ, in case of pinning nodes with highest degree. Figure 1 shows that global pinning-controllability can be obtained with lower values of feedback gain κ, by increasing the coupling strength among the oscillators σ. Through numerical experiments we also find that by randomly selecting pinned nodes in the network, the quantity $\lambda_{min}(\sigma L + \kappa P)$ generally decreases with respect to the pinned nodes' section described above.

Figure 2 shows the time evolution of the error dynamics $\|e(t)\|$. We note the exponential rate of decay of the error dynamics.

5 Conclusions

We have defined the concept of global pinning-controllability, and we have found conditions to guarantee such a property in general networks of dynamical systems.

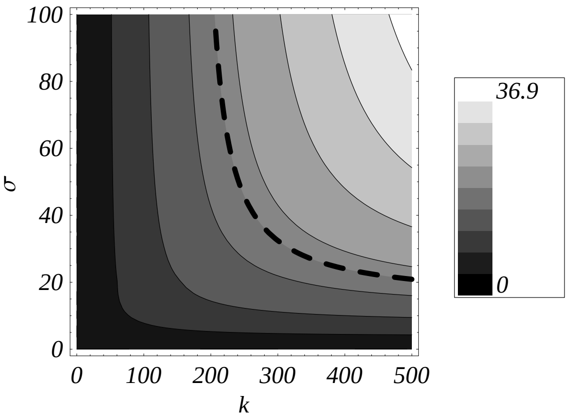

Figure 1. *Plot of* $\lambda_{min}(\sigma L + \kappa P)$ *as a function of* σ *and* κ*, in case of pinning nodes with highest degree. Dashed line represents the contour* $\lambda_{min}(\sigma L + \kappa P) = \alpha$*. In the top-right region global pinning-controllability is achieved.*

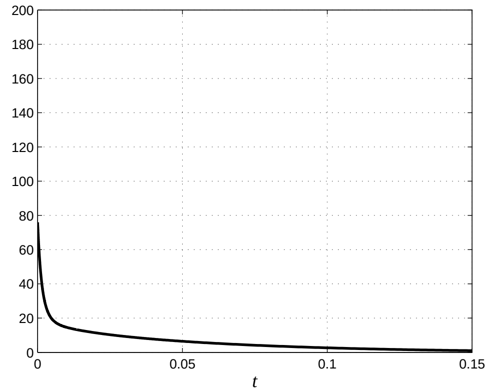

Figure 2. *Time evolution of the error dynamics.*

The influence of the structural properties of the oscillator network, the location and number of pinned nodes, and the individual oscillators' dynamics on global pinning-controllability has been investigated. A manageable formula for assessing pinning-controllability from the fraction of pinned nodes and the network algebraic connectivity has been proposed. Numerical examples were used to illustrate the viability of the conditions presented in the paper. In particular, we considered networks of identical Chua's oscillators, often used in the literature as a testbed problem for control and synchronization of networks.

Bibliography

[1] D. S. Bernstein, *Matrix Mathematics*, Princeton University Press, Princeton, NJ, 2005.

[2] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D. Hwang, *Complex networks: Structure and dynamics*, Physics Reports, 424 (2006), pp. 175–308.

[3] T. Chen, X. Liu, and W. Lu, *Pinning complex networks by a single controller*, IEEE Transactions on Circuits and Systems I, 54 (2007), pp. 1317–1326.

[4] M. Friedler, *Algebraic connectivity of graphs*, Czechoslovak Mathematical Journal, 23 (1973), pp. 298–305.

[5] A. Ghosh and S. Boyd, *Growing well-connected graphs*, in Proceedings of the 45th IEEE Conference on Decision and Control, San Diego, CA, USA, December 2006, pp. 6605–6611.

[6] C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer-Verlag, New York, NY, 2001.

[7] R. Grigoriev, M. Cross, and H. Schuste, *Pinning control of spatiotemporal chaos*, Physical Review Letters, 79 (1997), pp. 2795–2798.

[8] I. C. G. Ipsen and B. Nadler, *Refined perturbation bounds for eigenvalues of hermitian and non-hermitian matrices*, Unpublished, Samsi Technical Report no. 2007-2, (2007).

[9] G. P. Jiang, W. K. S. Tang, and G. Chen, *A simple global synchronization criterion for coupled chaotic system*, Chaos, Solitons & Fractals, 15 (2003), pp. 925–935.

[10] H. K. Khalil, *Nonlinear Systems*, Prentice Hall, Upper Saddle River, NJ, 3rd ed., 2002.

[11] R. Li, Z. Duan, and G. Chen, *Cost and effects of pinning control for network synchronization*, (2007).

[12] X. Li, X. Wang, and G. Chen, *Pinning a complex dynamical network to its equilibrium*, IEEE Transactions on Circuits and Systems-I, 51 (2004), pp. 2074–2087.

[13] W. Lu, *Adaptive dynamical networks via neighborhood information: Synchronization and pinning control*, Chaos, 17 (2007), p. 023122.

[14] N. F. Rulkov, *Images of synchronized chaos: Experiments with circuits*, Chaos, 6 (1996), pp. 262–279.

[15] F. Sorrentino, M. di Bernardo, F. Garofalo, and G. Chen, *Controllability of complex networks via pinning*, Physical Review E, 75 (2007), p. 046103.

[16] X. Wang and G. Chen, *Pinning control of scale-free dynamical networks*, Physica A, 310 (2002), pp. 521–531.

[17] J. Xiang and G. Chen, *On the V-stability of complex dynamical networks*, Automatica, 43 (2007), pp. 1049–1057.