Structure preserving model reduction of port-Hamiltonian systems

***R. Polyuga* * *and A. J. van der Schaft* †**

Abstract

It is shown that by use of the Kalman-decomposition an uncontrollable and/or unobservable port-Hamiltonian system is reduced to a controllable/observable system that inherits a port-Hamiltonian structure. Energy and co-energy variable representations for port-Hamiltonian systems are discussed and the reduction procedures are used for both representations. These exact reduction procedures motivate two approximate reduction procedures structure preserving for a general port-Hamiltonian system in scattering representation, Effort- and Flow-constraint reduction methods.

Keywords: Energy, co-energy variable representations, port-Hamiltonian systems, Effort-constraint, Flow-constraint, model reduction.

1 Introduction

Port-based network modeling of physical systems leads directly to their representation as port-Hamiltonian systems which are, if the Hamiltonian is non-negative, an important class of passive state-space systems. At the same time network modeling of physical systems often leads to high-dimensional dynamical models. Large state-space dimensions are obtained as well if distributed-parameter models are spatially discretized. Therefore an important issue concerns model reduction of these high-dimensional systems, both for analysis and control. The goal of this work is to show that the specific model reduction techniques of linear port-Hamiltonian systems preserve the port-Hamiltonian structure, and, as a consequence, passivity.

Port-Hamiltonian systems are endowed with more structure than just passivity. Other important issues like interconnection between port-Hamiltonian systems and energy dissipation are also reflected by the port-Hamiltonian structure. In section 2 we provide a brief overview of linear port-Hamiltonian systems. General theory on port-Hamiltonian systems can be found in [10]. We will show by applying the Kalman-decomposition in section 3 that the reduction of the dynamics of an uncontrollable/unobservable linear port-Hamiltonian system to a dynamics on the reachability/observability subspace preserves the port-Hamiltonian structure. This result holds both for energy and co-energy variable representations of linear port-Hamiltonian systems. The co-energy variable representation of port-Hamiltonian systems is considered in section 4. It is

*Institute for Mathematics and Computing Science, University of Groningen, P.O.Box 407, 9700 AK Groningen, The Netherlands, tel: +31-50-3633972, e-mail: r.polyuga@math.rug.nl

†Institute for Mathematics and Computing Science, University of Groningen, P.O.Box 407, 9700 AK Groningen, The Netherlands, tel: +31-50-3633731/3379, e-mail: a.j.van.der.schaft@math.rug.nl

shown in section 4 that the reduced models in the co-energy coordinates take a somewhat "dual" form to the reduced models obtained in the standard energy coordinates.

Within the systems and control literature a popular and elegant tool for model reduction is balancing, going back to [7]. One favorable property of model reduction based on balancing, as compared with other techniques such as modal analysis, is that the approximation of the dynamical system is explicitly based on its input-output properties. Standard open-loop balancing assumes that the system is asymptotically stable. Therefore this type of balancing cannot be directly applied to lossless port-Hamiltonian systems. In order to overcome this difficulty it is useful to switch to scattering representation presented in section 5, see also [11]. We will apply Effort- and Flow-constraint methods of model reduction in section 6 to linear port-Hamiltonian systems in scattering representation and show that the reduced-order models are again port-Hamiltonian.

2 Linear port-Hamiltonian systems

Port-based network modeling of physical systems leads to their representation as port-Hamiltonian systems (see e.g. [5], [8]). In the linear case, and in the absence of algebraic constraints, port-Hamiltonian systems take the form (see [2], [10], [11])

$$\begin{aligned} \dot{x} &= (J-R)Qx + Bu \\ y &= B^T Qx \end{aligned} \tag{1}$$

with $H(x) = \frac{1}{2}x^T Qx$ the *total energy* (Hamiltonian), $Q = Q^T \geq 0$ the *energy* matrix and $R = R^T \geq 0$ the *dissipation* matrix. The matrices $J = -J^T$ and B specify the *interconnection* structure of the system. By skew-symmetry of J and since R is positive semidefinite it immediately follows that

$$\frac{d}{dt}\frac{1}{2}x^T Qx \;=\; u^T y - x^T QRQx \;\leq\; u^T y \tag{2}$$

Thus if $Q \geq 0$ (and the Hamiltonian is non-negative) any port-Hamiltonian system is *passive* (see [11], [14]). The state variables $x \in \mathbb{R}^n$ are also called *energy* variables, since the total energy $H(x)$ is expressed as a function of these variables. Furthermore, the variables $u \in \mathbb{R}^m, y \in \mathbb{R}^m$ are called *power* variables, since their product $u^T y$ equals the power supplied to the system.

In the sequel we will often abbreviate $J - R$ to $F = J - R$. Clearly

$$F + F^T \;\leq\; 0 \tag{3}$$

while conversely any F satisfying (3) can be written as $J - R$ as above by decomposing F into its skew-symmetric and symmetric part

$$\begin{aligned} J &= \tfrac{1}{2}(F - F^T) \\ R &= -\tfrac{1}{2}(F + F^T) \end{aligned} \tag{4}$$

Two special cases of port-Hamiltonian systems correspond to either $R = 0$ or $J = 0$. In fact, if $R = 0$ (no internal energy dissipation) then the dissipation inequality (2) reduces to an equality

$$\frac{d}{dt}\frac{1}{2}x^T Qx \;=\; u^T y \tag{5}$$

In this case the transfer matrix $G(s) = B^T Q(sI - JQ)^{-1}B$ of the system satisfies

$$G(s) \;=\; -G^T(-s) \tag{6}$$

Conversely, any transfer matrix $G(s)$ satisfying $G(s) = -G^T(-s)$ can be shown to have a minimal realization

$$\begin{aligned} \dot{x} &= JQx + Bu \\ y &= B^T Qx \end{aligned} \tag{7}$$

with in fact Q being invertible.

The other special case corresponds to $J = 0$, in which case the system takes the form

$$\begin{aligned} \dot{x} &= -RQx + Bu \\ y &= B^T Qx \end{aligned} \tag{8}$$

with transfer matrix $G(s) = B^T Q(sI + RQ)^{-1} B$ satisfying

$$G(s) = G^T(s) \tag{9}$$

Conversely, any transfer matrix $G(s)$ satisfying (9) is represented by a minimal state-space representation (8) with Q invertible, where, however, R need not necessarily be positive semidefinite.

In these two special cases, either $R = 0$ or $J = 0$, there is a direct relationship between controllability and observability properties of the port-Hamiltonian system.

Proposition 2.1. *Consider a port-Hamiltonian system (7) or (8), and assume* $\det Q \neq 0$. *The system is* controllable *if and only if it is* observable, *while the unobservability subspace* $\mathcal{N}$ *is related to the reachability subspace* $\mathcal{R}$ *by*

$$\mathcal{N} = \mathcal{R}^{\perp} \tag{10}$$

with $\perp$ *denoting the orthogonal complement with respect to the (possibly indefinite) inner product defined by* Q.

Proof. For any port-Hamiltonian system (1) with $F = J - R$ we have

$$\begin{bmatrix} B^T Q \\ B^T QFQ \\ B^T QFQFQ \\ \vdots \end{bmatrix} = \left[\, B \mid F^T QB \mid F^T QF^T QB \mid \ldots \,\right]^T Q \tag{11}$$

Since the kernel of the matrix on the left-hand side defines the unobservability subspace, while on the right-hand side the image of the matrix preceding Q defines the reachability subspace if $F^T = F$ or $F^T = -F$, the assertion follows. □

Nevertheless, in general controllability and observability for a port-Hamiltonian system are not equivalent, as the following example shows.

Example 2.1. *Consider a port-Hamiltonian system*

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \tag{12}$$

corresponding to $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *and* $R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. *The system is observable but* not *controllable.*

3 The Kalman-decomposition of port-Hamiltonian systems

In this section we will show how an uncontrollable and/or unobservable port-Hamiltonian system is reduced to a controllable/observable system that is again port-Hamiltonian.

3.1 Reduction to a controllable port-Hamiltonian system

Consider a port-Hamiltonian system which is not controllable. Take linear coordinates $x = (x_1, x_2)^T$ such that the upper part of

$$
\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \\
y &= \begin{bmatrix} B_1^T & B_2^T \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{aligned}
\tag{13}
$$

is the reachability subspace $\mathcal{R}$. By invariance of $\mathcal{R}$ (see e.g. [9]) this implies

$$
\begin{aligned}
F_{21}Q_{11} + F_{22}Q_{21} &= 0 \\
B_2 &= 0
\end{aligned}
\tag{14}
$$

It follows that the dynamics restricted to $\mathcal{R}$ is given as

$$
\begin{aligned}
\dot{x}_1 &= (F_{11}Q_{11} + F_{12}Q_{21})x_1 + B_1 u \\
y &= B_1^T Q_{11} x_1
\end{aligned}
\tag{15}
$$

Now let us assume that F_{22} in invertible. Then from the first equation in (14) we may solve for Q_{21} as $Q_{21} = -F_{22}^{-1}F_{21}Q_{11}$. Substitution in (15) yields

$$
\begin{aligned}
\dot{x}_1 &= (F_{11} - F_{12}F_{22}^{-1}F_{21})Q_{11}x_1 + B_1 u \\
y &= B_1^T Q_{11} x_1
\end{aligned}
\tag{16}
$$

which is again a port-Hamiltonian system. Indeed, $F + F^T \leq 0$ implies that the Schur complement $\bar{F} = F_{11} - F_{12}F_{22}^{-1}F_{21}$ satisfies $\bar{F} + \bar{F}^T \leq 0$.

Remark 3.1. *Note that $\bar{F}$ is skew-symmetric if F is skew-symmetric, and is symmetric if F is symmetric.*

3.2 Reduction to an observable port-Hamiltonian system

Consider again a port-Hamiltonian system (1) and suppose the system is not observable. Then there exist coordinates $x = (x_1, x_2)^T$ such that the *lower* part of (13) is the unobservability subspace $\mathcal{N}$. By invariance of $\mathcal{N}$ (see again [9]) it follows that

$$
\begin{aligned}
F_{11}Q_{12} + F_{12}Q_{22} &= 0 \\
B_1^T Q_{12} + B_2^T Q_{22} &= 0
\end{aligned}
\tag{17}
$$

Then the dynamics on the quotient space $\mathcal{X}/\mathcal{N}$ is

$$
\begin{aligned}
\dot{x}_1 &= (F_{11}Q_{11} + F_{12}Q_{21})x_1 + B_1 u \\
y &= B_1^T Q_{11} x_1 + B_2^T Q_{21} x_1
\end{aligned}
\tag{18}
$$

Assuming invertibility of Q_{22} it follows from (17) that $F_{12} = -F_{11}Q_{12}Q_{22}^{-1}$ and $B_2^T = -B_1^T Q_{12}Q_{22}^{-1}$. Substitution in (18) yields

$$
\begin{aligned}
\dot{x}_1 &= F_{11}(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1 + B_1 u \\
y &= B_1^T (Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1
\end{aligned}
\tag{19}
$$

which is again a port-Hamiltonian system with Hamiltonian $\bar{H} = \frac{1}{2}x_1^T(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1$.

Remark 3.2. *Note that $(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21}) \geq 0$ if $Q \geq 0$.*

3.3 The Kalman-decomposition

It is well known that a linear system $\dot{x} = Ax + Bu, y = Cx$ can be represented in a suitable basis as (see [9], [13])

$$A = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ 0 & A_{22} & 0 & 0 \\ A_{31} & A_{32} & A_{33} & A_{34} \\ 0 & A_{42} & 0 & A_{44} \end{bmatrix}, \ B = \begin{bmatrix} B_1 \\ 0 \\ B_3 \\ 0 \end{bmatrix}, \ C = \begin{bmatrix} C_1^T \\ C_2^T \\ 0 \\ 0 \end{bmatrix}^T$$

with $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3 \times \mathcal{X}_4$, where

$$\begin{aligned} \mathcal{N} &= \mathcal{X}_3 \times \mathcal{X}_4 \\ \mathcal{R} &= \mathcal{X}_1 \times \mathcal{X}_3 \end{aligned} \tag{20}$$

Writing out

$$\begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ 0 & A_{22} & 0 & 0 \\ A_{31} & A_{32} & A_{33} & A_{34} \\ 0 & A_{42} & 0 & A_{44} \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} \\ Q_{21} & Q_{22} & Q_{23} & Q_{24} \\ Q_{31} & Q_{32} & Q_{33} & Q_{34} \\ Q_{41} & Q_{42} & Q_{43} & Q_{44} \end{bmatrix}$$

this implies that the blocks of the F and Q matrix satisfy

$$\begin{array}{l} (a)\ F_{11}Q_{13} + F_{12}Q_{23} + F_{13}Q_{33} + F_{14}Q_{43} = 0 \\ (b)\ F_{11}Q_{14} + F_{12}Q_{24} + F_{13}Q_{34} + F_{14}Q_{44} = 0 \leftarrow \\ (c)\ F_{21}Q_{11} + F_{22}Q_{21} + F_{23}Q_{31} + F_{24}Q_{41} = 0 \\ (d)\ F_{21}Q_{13} + F_{22}Q_{23} + F_{23}Q_{33} + F_{24}Q_{43} = 0 \\ (e)\ F_{21}Q_{14} + F_{22}Q_{24} + F_{23}Q_{34} + F_{24}Q_{44} = 0 \leftarrow \\ (f)\ F_{41}Q_{11} + F_{42}Q_{21} + F_{43}Q_{31} + F_{44}Q_{41} = 0 \leftarrow \\ (g)\ F_{41}Q_{13} + F_{42}Q_{23} + F_{43}Q_{33} + F_{44}Q_{43} = 0 \leftarrow \end{array} \tag{21}$$

and similarly by writing out

$$\begin{bmatrix} B_1^T & 0 & B_3^T & 0 \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} \\ Q_{21} & Q_{22} & Q_{23} & Q_{24} \\ Q_{31} & Q_{32} & Q_{33} & Q_{34} \\ Q_{41} & Q_{42} & Q_{43} & Q_{44} \end{bmatrix} = \begin{bmatrix} C_1 & C_2 & 0 & 0 \end{bmatrix}$$

we obtain

$$\begin{array}{l} B_1^T Q_{13} + B_3^T Q_{33} = 0 \\ B_1^T Q_{14} + B_3^T Q_{34} = 0 \end{array} \tag{22}$$

The resulting dynamics on $\mathcal{X}_1$ (the part of the system that is both controllable and observable) can be identified in port-Hamiltonian form, by combining the previous two reduction schemes corresponding to controllability and observability. Indeed, application of Section 3.2 yields the following observable system on $\mathcal{X}_1 \times \mathcal{X}_2$

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \bar{Q} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} B_1^T & 0 \end{bmatrix} \bar{Q} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \tag{23}$$

where

$$\bar{Q} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} - \begin{bmatrix} Q_{13} & Q_{14} \\ Q_{23} & Q_{24} \end{bmatrix} \begin{bmatrix} Q_{33} & Q_{34} \\ Q_{43} & Q_{44} \end{bmatrix}^{-1} \begin{bmatrix} Q_{31} & Q_{32} \\ Q_{41} & Q_{42} \end{bmatrix} \tag{24}$$

Next, application of Section 3.1 to (23) yields the following port-Hamiltonian description of the dynamics on $\mathcal{X}_1$

$$\begin{array}{rcl} \dot{x}_1 & = & (F_{11} - F_{12}F_{22}^{-1}F_{21})\bar{Q}_{11}x_1 + B_1 u \\ y & = & B_1^T \bar{Q}_{11} x_1 \end{array} \tag{25}$$

Further analysis (using a well-known matrix inversion formula) yields

$$\begin{array}{rcl} \bar{Q}_{11} & = & Q_{11} - Q_{13}(Q_{33} - Q_{34}Q_{44}^{-1}Q_{43})^{-1}Q_{31} + Q_{14}Q_{44}^{-1}Q_{43}(Q_{33} - Q_{34}Q_{44}^{-1}Q_{43})^{-1}Q_{31} + \\ & & Q_{13}Q_{33}^{-1}Q_{34}(Q_{44} - Q_{43}Q_{33}^{-1}Q_{34})^{-1}Q_{41} - Q_{14}(Q_{44} - Q_{43}Q_{33}^{-1}Q_{34})^{-1}Q_{41} \end{array} \tag{26}$$

leading to a port-Hamiltonian description

$$\begin{array}{rcl} \dot{x}_1 & = & (F_{11} - F_{12}F_{22}^{-1}F_{21})\bar{Q}_{11}x_1 + B_1 u \\ y & = & B_1^T \bar{Q}_{11} x_1 \end{array} \tag{27}$$

having the same transfer matrix as the original system (1).

Remark 3.3. *By first applying the procedure of Section 3.1 and then applying the procedure of Section 3.2 we obtain a different, but equivalent, port-Hamiltonian formulation.*

4 The co-energy variable representation

In this section we throughout assume that the matrix Q is *invertible*. This means that

$$e = Qx \tag{28}$$

is a valid coordinate transformation, and the port-Hamiltonian system (1) in these new coordinates takes the form

$$\begin{array}{rcl} \dot{e} & = & QFe + QBu, \quad F = J - R \\ y & = & B^T e \end{array} \tag{29}$$

Since $e = Qx = \frac{\partial H}{\partial x}(x)$, with $H(x) = \frac{1}{2}x^T Q x$ the energy, the variables e are usually called the *co-energy* variables.

Example 4.1. *Consider the LC-circuit in Figure 1, with q the charge on the capacitor and ϕ_1, ϕ_2 the fluxes over the inductors L_1, L_2 correspondingly. The energy (in the case of a linear capacitor and inductors) is given as*

$$H(q, \phi_1, \phi_2) = \frac{1}{2C}q^2 + \frac{1}{2L_1}\phi_1^2 + \frac{1}{2L_2}\phi_2^2 \tag{30}$$

and $x = (q, \phi_1, \phi_2)$ are the energy variables, in which the system takes the port-Hamiltonian form

$$\begin{array}{rcl} \begin{bmatrix} \dot{q} \\ \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix} & = & \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{q}{C} \\ \frac{\phi_1}{L_1} \\ \frac{\phi_2}{L_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \\ y & = & \phi_1 / L_1 \end{array} \tag{31}$$

with u, y being the voltage across and the current through the voltage source. The co-energy variables $e = [\ \frac{q}{C} \frac{\phi_1}{L_1} \frac{\phi_2}{L_2}\ ]^T = [\ V_C I_{L1} I_{L2}\ ]^T$ are the voltage over the capacitor and the currents through the inductors, leading to the following form of the dynamics

$$\begin{array}{rcl} \begin{bmatrix} \dot{V}_C \\ \dot{I}_{L1} \\ \dot{I}_{L2} \end{bmatrix} & = & \begin{bmatrix} \frac{1}{C} & 0 & 0 \\ 0 & \frac{1}{L_1} & 0 \\ 0 & 0 & \frac{1}{L_2} \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_C \\ I_{L1} \\ I_{L2} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L_1} \\ 0 \end{bmatrix} u \\ y & = & I_{L1} \end{array} \tag{32}$$

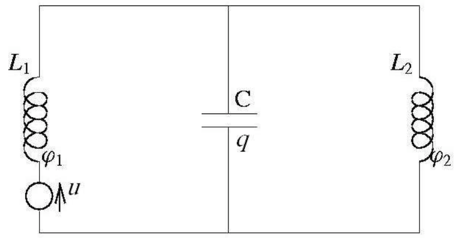

Figure 1. *LC-circuit*

Example 4.2. *Consider a mass-damper-spring system*

$$\begin{aligned} \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} &= (\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}) \begin{bmatrix} kq \\ \frac{p}{m} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \frac{p}{m} \end{aligned} \tag{33}$$

with energy $H(q,p) = \frac{1}{2}kq^2 + \frac{1}{2m}p^2$ *(potential and kinetic energy) in the energy variables* q *(elongation of the spring) and* p *(momentum of the mass). The constant* $c \geq 0$ *is the damping constant, and* u *is the external force.*

The co-energy variables are $e_1 = kq$ *(spring force) and* $e_2 = \frac{p}{m}$ *(velocity), leading to the dynamics*

$$\begin{aligned} \begin{bmatrix} \dot{e_1} \\ \dot{e_2} \end{bmatrix} &= \begin{bmatrix} k & 0 \\ 0 & 1/m \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u \\ y &= e_2 \end{aligned} \tag{34}$$

Note that

$$\frac{d}{dt}\frac{1}{2}e^TQ^{-1}e = e^TFe + e^TBu = -e^TRe + u^Ty \tag{35}$$

and thus if $Q \geq 0$ then $V(e) = \frac{1}{2}e^TQ^{-1}e$ (the Legendre transform of $H(x) = \frac{1}{2}x^TQx$) is a storage function of (29). $V(e)$ is called the *co-energy* of the system, which is in this linear case equal to the energy ($V(Qx) = H(x)$).

A main advantage of the co-energy variable representation of a port-Hamiltonian system is that additional *constraints* on the system are often expressed as constraints on the co-energy variables (see also Section 6).

The reduction of the port-Hamiltonian system to its controllable and/or observable part takes the following form in the co-energy variable representation. Interestingly enough, the formula's take a somewhat "dual" form to the formula's obtained in the energy variable representation.

Consider the system (29) in co-energy variable representation. Take linear coordinates $e = (e_1, e_2)^T$ such that the upper part of

$$\begin{aligned} \begin{bmatrix} \dot{e_1} \\ \dot{e_2} \end{bmatrix} &= \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \\ y &= \begin{bmatrix} B_1^T & B_2^T \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \end{aligned} \tag{36}$$

is the reachability subspace $\mathcal{R}$. By invariance of $\mathcal{R}$ this implies

$$\begin{aligned} Q_{21}F_{11} + Q_{22}F_{21} &= 0 \\ Q_{21}B_1 + Q_{22}B_2 &= 0 \end{aligned} \tag{37}$$

Hence the dynamics restricted to $\mathcal{R}$ equals

$$\begin{aligned} \dot{e}_1 &= (Q_{11}F_{11} + Q_{12}F_{21})e_1 + (Q_{11}B_1 + Q_{12}B_2)u \\ &= (Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})F_{11}e_1 + (Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})B_1u \\ y &= B_1^T e_1 \end{aligned} \tag{38}$$

which is a port-Hamiltonian system in co-energy variable representation, with energy matrix $\bar{Q} = Q_{11} - Q_{12}Q_{22}^{-1}Q_{21}$, and interconnection/damping matrix F_{11}. Notice that these formula's are dual to the corresponding formula's (16) for the controllable part of the system in energy variable representation, where the resulting interconnection/damping matrix is a Schur complement, while the resulting energy matrix is Q_{11}.

Analogously, take linear coordinates $e = (e_1, e_2)^T$ such that the lower part of (36) equals the unobservability subspace $\mathcal{N}$. This implies

$$\begin{aligned} Q_{11}F_{12} + Q_{12}F_{22} &= 0 \\ B_2 &= 0 \end{aligned} \tag{39}$$

leading to the observable reduced dynamics

$$\begin{aligned} \dot{e}_1 &= (Q_{11}F_{11} + Q_{12}F_{21})e_1 + Q_{11}B_1u \\ &= Q_{11}(F_{11} - F_{12}F_{22}^{-1}F_{21})e_1 + Q_{11}B_1u \\ y &= B_1^T e_1 \end{aligned} \tag{40}$$

Combination of the above leads to a similar Kalman-decomposition as in the energy variable representation.

5 The scattering variable representation

Another useful representation of port-Hamiltonian systems is the scattering representation (see [10], [11]). In this representation the power u^Ty supplied to the system is split into a non-negative term denoting the power due to an "incoming wave" and a non-positive term denoting the power of an "outgoing wave". This is accomplished by the following well-known change of coordinates in the space of input and output variables

$$\begin{aligned} v &= \tfrac{1}{\sqrt{2}}(u + y) \\ z &= \tfrac{1}{\sqrt{2}}(-u + y) \end{aligned} \tag{41}$$

with inverse

$$\begin{aligned} u &= \tfrac{1}{\sqrt{2}}(v - z) \\ y &= \tfrac{1}{\sqrt{2}}(v + z) \end{aligned} \tag{42}$$

The vector v is called the vector of incoming wave variables, and z is the vector of outgoing wave variables. Note that v measures the deviation of the input u from the situation where the system is terminated on a unit resistance, corresponding to $\bar{u} = -y$. Indeed $v = \frac{1}{\sqrt{2}}(u - \bar{u})$.

The basic relation between power variables u, y and wave variables v, z is expressed as

$$u^Ty = \frac{1}{2} \parallel v \parallel^2 - \frac{1}{2} \parallel z \parallel^2 \tag{43}$$

Expressing the power variables u, y into the wave variables v, z by (42) into the equation (1) for a port-Hamiltonian system leads to

$$\begin{aligned} \dot{x} &= (J - R - BB^T)Qx + \sqrt{2}Bv \\ z &= \sqrt{2}B^TQx - v \end{aligned} \tag{44}$$

which is called the scattering representation of the port-Hamiltonian system (1). Note that the term $BB^T \geq 0$ can be regarded as a virtual additional resistive term (corresponding to unit resistances attached to the ports of the systems).

Because of (43), the basic dissipation inequality (2) of any port-Hamiltonian system takes the following form for the scattering representation

$$\frac{d}{dt}\frac{1}{2}x^TQx \;=\; \frac{1}{2}\parallel v\parallel^2 - \frac{1}{2}\parallel z\parallel^2 - x^TQRQx \tag{45}$$

If $Q \geq 0$ then the scattering representation is under minimality conditions *asymptotically stable* (see e.g. [11]).

Proposition 5.1. *Consider the scattering representation (44) with* $Q > 0$. *Assume the pair* $(A = JQ, C = \begin{pmatrix} B^TQ \\ RQ \end{pmatrix})$ *is detectable. Then (44) for* $v = 0$ *is asymptotically stable.*

Proof. The right-hand side of (45) for $v = 0$ equals $-x^TQBB^TQx - x^TQRQx$. Defining the Lyapunov function $V(x) = \frac{1}{2}x^TQx$ it follows that $\{x \mid \dot{V}(x) = 0\} = \{x \mid B^TQx = 0,\ RQx = 0\}$. Detectability thus implies by LaSalle's Invariance principle asymptotic stability. □

The controllability and observability structure of the scattering representation is identical to the power variable representation. This follows from

Proposition 5.2. *The reachability subspace* $\mathcal{R}_s$ *of the scattering representation is equal to the reachability subspace* $\mathcal{R}$ *of (1). The unobservability subspace* $\mathcal{N}_s$ *of the scattering representation is equal to the unobservability subspace* $\mathcal{N}$ *of (1).*

Proof. $\mathcal{R}_s = im[\ \sqrt{2}B \mid (J-R-BB^T)Q\sqrt{2}B \mid \ldots] = im[\ B \mid (J-R)QB \mid \ \ldots\ ] = \mathcal{R}$. Similarly

$$\mathcal{N}_s = ker\begin{bmatrix} \sqrt{2}B^TQ \\ \sqrt{2}B^TQ(J-R-BB^T)Q \\ \vdots \end{bmatrix} = ker\begin{bmatrix} B^TQ \\ B^TQ(J-R)Q \\ \vdots \end{bmatrix} = \mathcal{N}.$$

□

Hence, the Kalman-decomposition of the scattering representation is identical to the Kalman-decomposition of (1).

From (45) it follows that for $v = 0$

$$\tfrac{1}{2}\int_0^T \parallel z(t)\parallel^2 dt = \tfrac{1}{2}x^T(0)Qx(0) - \tfrac{1}{2}x^T(T)Qx(T) \;-\; \int_0^T x^T(t)QRQx(t)dt \tag{46}$$

for all $T \geq 0$. Hence if $Q \geq 0$ we conclude that $\frac{1}{2}\int_0^\infty \parallel z(t)\parallel^2 dt$ exists for all $x(0)$, and is equal to $\frac{1}{2}x^T(0)Mx(0)$, with the *observability Gramian* $M \geq 0$ being the solution to (see [11])

$$Q(J-R-BB^T)^TM + M(J-R-BB^T)Q = -2QBB^TQ \tag{47}$$

Furthermore $kerM = \mathcal{N}$. Note that

$$\frac{1}{2}x^T(0)Mx(0) \;=\; \frac{1}{2}\int_0^\infty \parallel z(t)\parallel^2 dt \tag{48}$$

and thus equals the outgoing energy of the system (for the incoming wave v equal to zero). From (46) if follows that $M \leq Q$.

Furthermore it follows from (45) that

$$\tfrac{1}{2}\int_{-T}^{0}\|v(t)\|^2\,dt = \tfrac{1}{2}\int_{-T}^{0}\|z(t)\|^2\,dt + \int_{-T}^{0} x^T(t)QRQx(t)dt + \tfrac{1}{2}x^T(0)Qx(0) - \tfrac{1}{2}x^T(-T)Qx(-T) \tag{49}$$

for all $T \geq 0$. Hence if $Q \geq 0$ we conclude that $\frac{1}{2}\int_{-\infty}^0 \parallel v(t)\parallel^2 dt$ exists for all $x(0)$, and $\inf_v \frac{1}{2}\int_{-\infty}^0 \parallel v(t)\parallel^2 dt$ is equal to $\frac{1}{2}x^T(0)W^{-1}x(0)$, with the *controllability Gramian* $W \geq 0$ being the solution to (see [11])

$$(J-R-BB^T)QW + WQ(J-R-BB^T)^T = -2BB^T \tag{50}$$

Moreover $imW = \mathcal{R}$. Note also that

$$\frac{1}{2}x^T(0)W^{-1}x(0) \ = \ \inf_v \frac{1}{2}\int_{-\infty}^{0} \parallel v(t) \parallel^2 dt \tag{51}$$

and thus equals the incoming energy of the system (for the outgoing wave z equal to zero). From (49) it follows that $W^{-1} \geq Q$.

Combining the obtained inequalities yields

$$M \leq Q \leq W^{-1} \tag{52}$$

Now bringing the scattering system (44) into a balanced form where $W = M$ (see [7], [12]) and computing the square roots of the eigenvalues of MW which are equal to the Hankel singular values (see [4]) provides us the information about the number of state components of the system to be reduced. These state components require large amount of the incoming energy to be reached and give small amount of the outgoing energy to be observed. Therefore they are less important from the energy point of view and can be removed from the system (see also [1]).

6 Reduction of port-Hamiltonian systems in general case

For a general port-Hamiltonian system in energy (1) or co-energy (29) coordinates with no uncontrollable/unobservable but with "hardly" controllable/observable states we may apply balancing as explained in section 5 and use one of the following structure-preserving reduction techniques.

6.1 Effort-constraint reduction

Consider a full-order port-Hamiltonian system (1). To make the system uniformly asymptotically stable we bring the system (1) into the scattering coordinates (44) applying the coordinate transformation (41). Now we balance the system (44) but in co-energy coordinates (with another change of coordinates (28)), obtaining the following balanced representation of our system

$$\begin{aligned} \dot{e} &= Q(J-R-BB^T)e + \sqrt{2}QBv \\ z &= \sqrt{2}B^T e - v \end{aligned} \tag{53}$$

where the lower part of the state vector $e = (e_1, e_2)^T$ is the most difficult to reach and observe.

Consider the system (44) again, but now in the coordinates where the system (53) is balanced

$$\begin{aligned} \dot{x} &= (J-R-BB^T)e + \sqrt{2}Bv \\ z &= \sqrt{2}B^T e - v \end{aligned} \tag{54}$$

A natural choice for the reduced model would be a model which contains only the e_1 dynamics since the lower part of the state vector e_2 is much less relevant from the energy point of view

$$e_2 = Q_{21}x_1 + Q_{22}x_2 \approx 0 \tag{55}$$

Therefore the reduced system takes the following form

$$\begin{aligned} \dot{x}_1 &= (J_{11}-R_{11}-B_1B_1^T)e_1 + \sqrt{2}B_1 v \\ &= (J_{11}-R_{11}-B_1B_1^T)(Q_{11}x_1+Q_{12}x_2) + \sqrt{2}B_1 v \\ z &= \sqrt{2}B_1^T e_1 - v = \sqrt{2}B_1^T(Q_{11}x_1 + Q_{12}x_2) - v \end{aligned} \tag{56}$$

After substituting $x_2 \approx -Q_{22}^{-1}Q_{21}x_1$ from (55) into (56), assuming that Q_{22}^{-1} exists, the reduced system will take the final form in energy coordinates

$$\begin{aligned} \dot{x}_1 &= (J_{11}-R_{11}-B_1B_1^T)(Q_{11}-Q_{12}Q_{22}^{-1}Q_{21})x_1+\sqrt{2}B_1 v \\ z &= \sqrt{2}B_1^T(Q_{11}-Q_{12}Q_{22}^{-1}Q_{21})x_1 - v \end{aligned} \tag{57}$$

which is again a port-Hamiltonian system.

6.2 Flow-constraint reduction

Another structure-preserving way of model reduction of port-Hamiltonian systems is the so-called *Flow-constraint Method*, when after scattering change of coordinates (41) we balance the system (44) and approximate the lower part of the state vector, but in energy coordinates, *plus* its dynamics. Using the notation $F := J - R - BB^T$ we obtain

$$\begin{aligned} &x_2 \approx 0 \\ &\dot{x}_2 = (F_{21}Q_{11} + F_{22}Q_{21})x_1 + \sqrt{2}B_2 v \approx 0 \end{aligned} \tag{58}$$

with the reduced port-Hamiltonian system of the form

$$\begin{aligned} \dot{x}_1 &= (F_{11}Q_{11} + F_{12}Q_{21})x_1 + \sqrt{2}B_1 v \\ z &= \sqrt{2}(B_1^T Q_{11} + B_2^T Q_{21})x_1 - v \end{aligned} \tag{59}$$

From (58) it immediately follows that $Q_{21}x_1 \approx -F_{22}^{-1}F_{21}Q_{11}x_1 - \sqrt{2}F_{22}^{-1}B_2 v$, assuming that F_{22}^{-1} exists. Substituting in (59) yields

$$\begin{aligned} \dot{x}_1 &= (F_{11} - F_{12}F_{22}^{-1}F_{21})Q_{11}x_1 + \sqrt{2}(B_1 - F_{12}F_{22}^{-1}B_2)v \\ z &= \sqrt{2}(B_1^T - B_2^T F_{22}^{-1}F_{21})Q_{11}x_1 - (2B_2^T F_{22}^{-1}B_2 + 1)v \end{aligned} \tag{60}$$

which is if $(F_{12}F_{22}^{-1})^T = F_{22}^{-1}F_{21}$ again a reduced system in a port-Hamiltonian form.

Remark 6.1. *The reduced-order port-Hamiltonian systems (57) and (60) are automatically passive since the preservation of the port-Hamiltonian structure implies the preservation of the passivity property (see [10]).*

Remark 6.2. *Though the Flow-constraint method is similar to the well-known Truncation ($x_2 \approx 0$, see e.g. [1]) and less-known Singular Perturbation method ($\dot{x}_2 \approx 0$, see [3], [6]), it is different from these reduction methods since it is easy to show that neither of them preserves the port-Hamiltonian structure.*

7 Conclusions

We have shown in section 3 that a full-order uncontrollable/unobservable port-Hamiltonian system can be reduced to a controllable/observable system, which is again port-Hamiltonian, by exploiting the invariance of the reachability/unobservability subspaces of the original systems. We discussed energy and co-energy variable representations of port-Hamiltonian systems in section 4 illustrated by the example of electrical networks where the energy variables are charges and fluxes while the co-energy variables are voltages and currents.

The scattering representation of port-Hamiltonian systems is discussed in section 5 showing that in the scattering coordinates a stable port-Hamiltonian system becomes asymptotically stable, provided an observability condition is satisfied, so that standard balancing can be applied to the port-Hamiltonian system. Balancing is discussed in section 5. The reduction methods Effort-constraint and Flow-constraint are introduced in Section 6 and applied to a general port-Hamiltonian full-order system showing that the proposed approximations preserve the port-Hamiltonian structure for the reduced-order systems as well as the passivity property. Effort- and Flow- constraint methods motivate to investigate further important issues about the error bounds between full-order and reduced-order systems which is the subject for ongoing research.

Bibliography

[1] A. C. Antoulas, *Approximation of Large-Scale Dynamical Systems*, SIAM, 2005.

[2] G. Escobar, A. J. van der Schaft and R. Ortega, *A Hamiltonian viewpoint in the modelling of switching power converters*, Automatica, Special Issue on Hybrid Systems, 35 (1999), pp. 445–452.

[3] K. Fernando, and H. Nicholson, *Singular perturbational model reduction of balanced systems*, IEEE Transactions on Automatic Control, 27 (1982), pp. 466–468.

[4] K. Glover, *All optimal Hankel-norm approximations of linear multivariable systems and their L^∞-error bounds*, International Journal of Control, 39 (1984), pp. 1115–1193.

[5] G. Golo, *Interconnection Structures in Port-Based Modelling: Tools for Analysis and Simulation*, Ph.D. thesis, University of Twente, 2002.

[6] M. Green and D. J. N. Limebeer, *Linear Robust Control*, Prentice-Hall, 1995.

[7] B. C. Moore, *Principal component analysis in linear systems*, IEEE Transactions on Automatic Control, AC-26 (1981), pp. 17–32.

[8] R. Pasumarthy, *On Analysis and Control of Interconnected Finite- and Infinite-dimensional Physical Systems*, Ph.D. thesis, University of Twente, 2006.

[9] J. W. Polderman and J. C. Willems, *Introduction to Mathematical Systems Theory*, Springer-Verlag, 1998.

[10] A. J. van der Schaft, *L2-Gain and Passivity Techniques in Nonlinear Control*, Springer-Verlag, 2000.

[11] A. J. van der Schaft, *On balancing of passive systems*, Proceedings of the European Control Conference 2007, Kos, Greece (2007), pp. 4173–4178.

[12] J. M. A. Scherpen, *Balancing for nonlinear systems*, Ph.D. thesis, University of Twente, 1994.

[13] K. Zhou with J. Doyle and K. Glover, *Robust and Optimal Control*, Prentice-Hall, 1996.

[14] J. C. Willems, *Dissipative dynamical systems, Part I: General Theory*, Archive for Rational Mechanics and Analysis, 45 (1972), pp. 321–351.