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Modelling and Controllability of Plate–Beam Systems*

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Abstract

The purpose of this paper is twofold. First, a distributed parameter model for a dynamic elastic system consisting of a thin plate to which a thin beam is rigidly and orthogonally attached to the edge of the plate shall be developed, assuming that the centerline of the beam is coplanar with the middle plane of the plate (in the equilibrium state). Second, it is proved that the dynamical system obtained is exactly controllable by means of controls applied along an appropriate portion of the edge of the plate that excludes the junction region between the plate and beam .

Key words: boundary controllability, exact controllability, Timoshenko beams, Reissner-Mindlin plates

AMS Subject Classifications: 93C20, 35B45, 73K05, 73K10

1 Introduction

The purpose of this paper is twofold. First, we shall develop a distributed parameter model for a dynamic elastic system consisting of a thin plate to which a thin beam is rigidly and orthogonally attached to the edge of the plate. In this model it is assumed that the centerline of the beam is coplanar with the middle plane of the plate (in the equilibrium state). Second, it will be proved that the dynamical system obtained is exactly controllable by means of controls applied along an appropriate portion of the edge of the plate that excludes the junction region between the plate and beam.

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With regard to modelling of junctions between elastic bodies let us mention in particular the Ciarlet-Dystunder approach that starts from the equations of 3-d elasticity for a body that is "thin" in one or more of its dimensions. Using either asymptotic expansions in the appropriate parameters or working directly from the variational equation for the 3-d body, the junction conditions are obtained in the limit as the small parameters go to zero; see the monographs of Ciarlet [1] and Le Dret [10] where this method is systematically used to obtain junction conditions for a variety of multi-body systems. When this approach is applied to bodies of different dimensions it is usually required that the lower dimensional body be embedded in some manner into the higher dimensional one. Therefore, in the context of a plate-beam configuration in which the centerline of the beam is coplanar with the middle plane of the plate, in the limit problem the junction region appears as a slit in the middle surface of the plate (see Gruais [3]). This geometry presents particular difficulties in the context of exact controllability because of the poor regularity properties of solutions of the uncontrolled problem in the neighborhood of the slit. This lack of regularity does not allow for the application of multiplier methods commonly used to derive the necessary observability estimates, since such techniques can be justified only if it is known *a priori* that the solutions are essentially classical. In contrast, in the framework presented below the junction region appears as a part of the *boundary* of the middle surface of the plate. The junction conditions are obtained from the very natural requirements that the interface between the (three dimensional) plate and the (three dimensional) beam be flat and that the displacements of the plate match those of the beam along that interface, followed by linearizations of the displacements of the plate about its mid-plane and the displacements of the beam about its centerline.

Modelling of an elastic plate-beam junction is discussed in the next section. The conditions arising from the requirement of continuity of displacements at the interface will be referred to as *geometric junction conditions*. These restrictions on the motion at the junction of course have specific implications for the balance laws of linear and angular momentum at the junction. The latter may, and will, be deduced from Hamilton's Principle, once the particular structures of the kinetic and strain energies are specified, and they turn out to be *nonlocal*.

In Sections 3 through 5 we study exact controllability of our plate-beam model. It is proved that such a system is exactly controllable (in an appropriate function space) by means of controls acting in either the geometric or mechanical boundary conditions along a certain portion of the edge of the plate that excludes the junction region. Let us remark that the methods employed below could also be used without essential new difficulties to treat more general configurations such as a plate with several beams attached to the edge of the plate, or even a plate to which is attached a *network* of rigidly joined beams (cf. [6]). We note also that exact controllability of a plate-beam distributed parameter model, but one of a completely different character than that considered below, has been established by Puel and Zuazua [15]. The reader is also referred to the paper [14], which motivated the present study, where exact controllability of a model of an elastic string connected to an elastic membrane (or, more generally, to an n-dimensional body modelled as an n-dimensional wave equation) is proved.

2 Modelling of a Plate-beam Junction

2.1 Geometric junction conditions

We begin by considering a thin plate of uniform thickness h. Points within the plate will be denoted by coordinates (x, y, z) with respect to the natural $\mathbf{i}, \mathbf{j}, \mathbf{k}$ basis for $\mathbb{R}^3$. It is assumed that the plate has a *middle surface* midway between its faces which, when the plate is in equilibrium, occupies a bounded, connected region Ω of the plane $z = 0$. Let $\boldsymbol{\nu}(\mathbf{x})$ denote the unit exterior normal vector to $\Gamma := \partial\Omega$ at $\mathbf{x}$ and $\boldsymbol{\tau}(\mathbf{x})$ be the positively oriented unit tangent vector at $\mathbf{x}$, whenever these vectors exist.

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be a fixed orthonormal basis in $\mathbb{R}^3$, $\mathbf{x}_0 \in \mathbb{R}^3$, and let ω be a bounded, simply connected closed set in $\mathbb{R}^2$ such that $0 \in \text{int}(\omega)$ and which is doubly symmetric with respect to the origin, i.e., $(\eta, \zeta) \in \omega$ implies that $(-\eta, \zeta) \in \omega$ and $(\eta, -\zeta) \in \omega$. The undeformed beam, in its reference configuration, occupies the region

$$\{\mathbf{x}_0 + \xi\mathbf{e}_1 + \eta\mathbf{e}_2 + \zeta\mathbf{e}_3 \,|\, 0 \le \xi \le \ell,\ (\eta, \zeta) \in \omega\}.$$

The *centerline* of the beam is $\mathbf{x}_0 + \xi\mathbf{e}_1$, $0 \le \xi \le \ell$, and the *cross-section* at $\mathbf{x}_0 + \xi\mathbf{e}_1$ is defined to be

$$A = \{\eta\mathbf{e}_2 + \zeta\mathbf{e}_3 \,|\, (\eta, \zeta) \in \omega\}.$$

We impose the following assumptions:
(A1) $\mathbf{x}_0 \in \Gamma$ and $\boldsymbol{\tau}(\mathbf{x}_0)$ exists.
(A2) $\mathbf{e}_1 = \boldsymbol{\nu}(\mathbf{x}_0)$, $\mathbf{e}_2 = \boldsymbol{\tau}(\mathbf{x}_0)$, $\mathbf{e}_3 = \mathbf{k}$.
(A3)

$$\omega = \{(\eta, \zeta) \,|\, |\eta| < \frac{\alpha}{2},\ |\zeta| < \frac{h}{2}\}.$$

(A4) Set

$$J_\alpha = \{\mathbf{x}_0 + \eta\boldsymbol{\tau}(\mathbf{x}_0) \,|\, |\eta| < \frac{\alpha}{2}\}.$$

Then $J_\alpha \subset \Gamma$.

The region $J_\alpha \times (-h/2, h/2)$ is the *junction region*. The above assumptions mean that the junction region is *flat*, that the beam is prismatic and its centerline is orthogonal to Γ at $\mathbf{x}_0$ and lies in the xy-plane. In terms of the $\mathbf{e}_i$ basis, the junction region is given by

$$\mathbf{x}_0 + \eta \mathbf{e}_2 + \zeta \mathbf{e}_3, \quad |\eta| < \frac{\alpha}{2}, \ |\zeta| < \frac{h}{2}$$

and we may think of the beam as being "glued" to the plate along the junction region. The assumption (A3) that the cross section ω of the beam be rectangular is unessential. At the cost of a slightly more complicated formalism, we may just as easily consider beams with doubly symmetric (with respect to the origin in $\mathbb{R}^2$) cross sections.

We denote by $\mathbf{R}(x, y, z)$ the *position vector* to the deformed position of the material particle in the plate which is located at (x, y, z) when the plate is in equilibrium, and by $\mathbf{W}(x, y)$ the *displacement vector* of the material particle which occupies position $(x, y, 0)$ in the mid-plane in equilibrium. Similarly, $\mathbf{r}(\xi, \eta, \zeta)$ will mean the position vector to the deformed position of the particle originally at position $\mathbf{x}_0 + \xi \mathbf{e}_1 + \eta \mathbf{e}_2 + \zeta \mathbf{e}_3$ of the beam in equilibrium, and $\mathbf{w}(\xi)$ the displacement vector of the point $\mathbf{x}_0 + \xi \mathbf{e}_1$ of the centerline. We now proceed along the lines of [7]. Let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ denote the natural basis for $\mathbb{R}^3$. Assume that we may write

$$\begin{aligned} \mathbf{R}(x, y, z) &= \mathbf{R}(x, y, 0) + z\mathbf{R}_z(x, y, 0) + O(z^2) \\ &= x\mathbf{i} + y\mathbf{j} + \mathbf{W}(x, y) + z\mathbf{R}_z(x, y, 0) + O(z^2), \end{aligned}$$

$$\begin{aligned} \mathbf{r}(\xi, \eta, \zeta) &= \mathbf{r}(\xi, 0, 0) + \eta \mathbf{r}_\eta(\xi, 0, 0) + \zeta \mathbf{r}_\zeta(\xi, 0, 0) + O(\eta^2 + \zeta^2) \\ &= \mathbf{x}_0 + \xi \mathbf{e}_1 + \mathbf{w}(\xi) + \eta \mathbf{r}_\eta(\xi, 0, 0) + \zeta \mathbf{r}_\zeta(\xi, 0, 0) + O(\eta^2 + \zeta^2), \end{aligned}$$

where $\mathbf{R}_z(x, y, 0), \mathbf{r}_\eta(\xi, 0, 0), \mathbf{r}_\zeta(\xi, 0, 0)$ satisfy the following assumptions (the subscripts denote differentiation).

(A5) There is an orthogonal matrix $O_1(\xi)$ with $\det O_1(\xi) = 1$ such that

$$\mathbf{r}_\eta(\xi, 0, 0) = O_1(\xi)\mathbf{e}_2, \ \ \mathbf{r}_\zeta(\xi, 0, 0) = O_1(\xi)\mathbf{e}_3.$$

(A6) There is an orthogonal matrix $O_2(x, y)$ with $\det O_2(x, y) = 1$ such that

$$\mathbf{R}_z(x, y, 0) = O_2(x, y)\mathbf{k}.$$

Assumption (A5) means that cross-sections of the beam move rigidly, while (A6) is the hypothesis of Reissner-Mindlin plate theory that filaments of the plate orthogonal to the mid-plane in the equilibrium configuration suffer no strain under deformation.

Since every orthogonal matrix of determinant unity is the exponential of a skew-symmetric matrix, we have

$$O_1 = \exp S_1 \approx I + S_1\,, \quad O_2 = \exp S_2 \approx I + S_2,$$

$$S_1 = \begin{pmatrix} 0 & -\psi_3 & \psi_2 \\ \psi_3 & 0 & -\psi_1 \\ -\psi_2 & \psi_1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & -\phi_1 \\ -\phi_2 & \phi_1 & 0 \end{pmatrix}.$$

It follows that

$$\begin{aligned} \mathbf{R}(x,y,z) &\approx (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) + \mathbf{W}(x,y) + z(\phi_2(x,y)\mathbf{i} - \phi_1(x,y)\mathbf{j})) \\ &= (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) + \mathbf{W}(x,y) + z\boldsymbol{\phi}(x,y), \end{aligned}$$

where $\boldsymbol{\phi} = \phi_2\mathbf{i} - \phi_1\mathbf{j}$, and, similarly,

$$\mathbf{r}(\xi,\eta,\zeta) \approx \mathbf{x}_0 + \xi\mathbf{e}_1 + \eta\mathbf{e}_2 + \zeta\mathbf{e}_3 + \mathbf{w}(\xi) + \eta(\psi_1\mathbf{e}_3 - \psi_3\mathbf{e}_1) + \zeta(\psi_2\mathbf{e}_1 - \psi_1\mathbf{e}_2).$$

Thus the respective *displacement vectors* of the points (x,y,z) and $\mathbf{x}_0 + \xi\mathbf{e}_1 + \eta\mathbf{e}_2 + \zeta\mathbf{e}_3$ within the plate and beam, respectively, are approximately given by

$$\mathbf{W}(x,y) + z\boldsymbol{\phi}(x,y)$$

and

$$\mathbf{w}(\xi) + \eta(\psi_1(\xi)\mathbf{e}_3 - \psi_3(\xi)\mathbf{e}_1) + \zeta(\psi_2(\xi)\mathbf{e}_1 - \psi_1(\xi)\mathbf{e}_2).$$

The "angles" ψ_i, ϕ_i have the following interpretations: ψ_1 is a torsional rotation about the $\mathbf{e}_1$ axis, ψ_2 and ψ_3 are rotations about the $\mathbf{e}_2$ and $\mathbf{e}_3$ axes, respectively, corresponding to the bending of the beam in the $\mathbf{e}_1\mathbf{e}_3$ and $\mathbf{e}_1\mathbf{e}_2$ planes, respectively. Similarly, ϕ_1 and ϕ_2 are the rotation angles about the $\mathbf{i}$ and $\mathbf{j}$ axes, respectively, associated with the bending of the plate.

Remark 2.1 Assumption (A5), which requires that cross-sections of the beam move rigidly, is not entirely consistent with the introduction of torsion since the latter may introduce cross-sectional warping. The inclusion of warping effects would result in the addition of a term $\chi(\eta,\zeta)\psi_1(\xi)\mathbf{e}_1$ to the last expression for the approximate displacements of the beam, where the function $\chi(\eta,\zeta)$ is the related to the *Saint-Venant warping function* of the cross-section; see [5]. However, warping is considered negligible if the beam is thin and the torsional rotation is small, which is the situation considered below.

We now define the *geometric junction conditions.* Consider a point $\mathbf{x}_0 + \eta\boldsymbol{\tau}(\mathbf{x}_0) + z\mathbf{k}$ within the junction region. We require that

$$\begin{aligned} &\mathbf{W}(\mathbf{x}_0 + \eta\boldsymbol{\tau}) + z\boldsymbol{\phi}(\mathbf{x}_0 + \eta\boldsymbol{\tau}) \\ &\qquad = \mathbf{w}(0) + \eta(\psi_1(0)\mathbf{e}_3 - \psi_3(0)\mathbf{e}_1) + z(\psi_2(0)\mathbf{e}_1 - \psi_1(0)\mathbf{e}_2), \\ &\qquad\qquad |\eta| < \frac{\alpha}{2},\ |z| < \frac{h}{2}, \end{aligned} \tag{2.1}$$

where $\boldsymbol{\tau} = \boldsymbol{\tau}(\mathbf{x}_0)$, that is

$$\left.\begin{aligned} \mathbf{W}(\mathbf{x}_0 + \eta\boldsymbol{\tau}) &= \mathbf{w}(0) + \eta(\psi_1(0)\mathbf{e}_3 - \psi_3(0)\mathbf{e}_1), \\ \boldsymbol{\phi}(\mathbf{x}_0 + \eta\boldsymbol{\tau}) &= \psi_2(0)\mathbf{e}_1 - \psi_1(0)\mathbf{e}_2, \quad |\eta| < \tfrac{\alpha}{2}. \end{aligned}\right\} \tag{2.2}$$

Conditions (2.2) are of *rigid type*. They are linearizations of the requirement that the displacements of the plate and beam match throughout the junction region.

Write

$$\mathbf{w} = u\mathbf{e}_1 + v\mathbf{e}_2 + w\mathbf{e}_3, \quad \mathbf{W} = U\mathbf{i} + V\mathbf{j} + W\mathbf{k} := \mathbf{U} + W\mathbf{k}.$$

Then (2.2) may be written

$$\left.\begin{aligned} \boldsymbol{\nu} \cdot \mathbf{U}(\mathbf{x}_0 + \eta\boldsymbol{\tau}) &= u(0) - \eta\psi_3(0), \\ \boldsymbol{\tau} \cdot \mathbf{U}(\mathbf{x}_0 + \eta\boldsymbol{\tau}) &= v(0), \\ W(\mathbf{x}_0 + \eta\boldsymbol{\tau}) &= w(0) + \eta\psi_1(0), \\ \boldsymbol{\phi}(\mathbf{x}_0 + \eta\boldsymbol{\tau}) &= \boldsymbol{\psi}(0), \quad |\eta| < \frac{\alpha}{2}, \end{aligned}\right\} \tag{2.3}$$

where $\boldsymbol{\psi} = \psi_2\mathbf{e}_1 - \psi_1\mathbf{e}_2 = \psi_2\boldsymbol{\nu}(\mathbf{x}_0) - \psi_1\boldsymbol{\tau}(\mathbf{x}_0)$. Note that in (2.3) the quantities $W, \boldsymbol{\phi}, w, \psi_1, \psi_2$ related to transverse motion are not coupled to the quantities $\mathbf{U}, u, v, \psi_3$ related to in-plane motion.

2.2 Dynamic conditions

The geometric junction conditions related to transverse motion are

$$\left.\begin{aligned} W(\mathbf{x}_0 + \eta\boldsymbol{\tau}) &= w(0) + \eta\psi_1(0), \\ \boldsymbol{\phi}(\mathbf{x}_0 + \eta\boldsymbol{\tau}) &= \boldsymbol{\psi}(0), \quad |\eta| < \frac{\alpha}{2}. \end{aligned}\right\} \tag{2.4}$$

The dynamic conditions will be obtained from Hamilton's Principle:

$$\delta \int_0^T [\mathcal{K}(t) - \mathcal{S}(t) + \mathcal{W}(t)]\, dt = 0, \tag{2.5}$$

where $\mathcal{K}$ and $\mathcal{S}$ represent the kinetic and strain energies, respectively, of the plate-beam system, $\mathcal{W}$ is the work done on the system by external forces and δ denotes the first variation with respect to the class of admissible displacements. The latter must obey, in particular, the geometric junction conditions (2.4). They must, in addition, satisfy any other geometric restrictions that are imposed on the motion. Since $W, \boldsymbol{\phi}, w, \psi_1, \psi_2$ are not coupled to the quantities $\mathbf{U}, u, v, \psi_3$ in the geometric junction conditions, they will likewise not be coupled to them in the dynamic conditions, at least under the further geometric restrictions which we may impose below. Therefore, in considering (2.5) we need only take into account those portions of $\mathcal{K}$, $\mathcal{P}$ and $\mathcal{W}$ that are connected to the transverse motion.

We assume that Ω has a Lipschitz continuous boundary $\Gamma = \overline{\Gamma}_N \cup \overline{\Gamma}_D \cup \overline{J}$, where Γ_N, Γ_D and J are relatively open in Γ and mutually disjoint. The part Γ_D corresponds to Dirichlet (geometric) boundary conditions

$$W = 0, \quad \boldsymbol{\phi} = 0 \ \text{ on } \Gamma_D, \tag{2.6}$$

while Γ_N corresponds to Neumann (mechanical) boundary conditions. We further assume that J is connected and that $\overline{J}_\alpha \subset J$. The part $J - J_\alpha$ may be either *free*, or it will be geometrically restricted in the following manner. Since $\partial J_\alpha = \mathbf{x}_0 \pm (\alpha/2)\boldsymbol{\tau}(\mathbf{x}_0)$, the set $J - J_\alpha$ may be uniquely partitioned into two disjoint, connected sets $J_0^\pm$ such that

$$\mathbf{x}_0 \pm \frac{\alpha}{2}\boldsymbol{\tau}(\mathbf{x}_0) \in J_0^\pm .$$

Let σ_i, $i = 0, 1$, be $C_0^\infty(J)$ functions such that $\sigma_i = 1$ on J_α. It is required that

$$\left.\begin{aligned} W &= \sigma_0[w(0) \pm \frac{\alpha}{2}\psi_1(0)], \\ \boldsymbol{\phi} &= \sigma_1 \boldsymbol{\psi}(0), \quad \text{on } J_0^\pm . \end{aligned}\right\} \tag{2.7}$$

These conditions require that the displacement W and rotations $\boldsymbol{\phi}$ along J change smoothly from their values on J_α to zero, and give prescribed profiles to these quantities along $J_0^\pm$. We refer to Puel-Zuazua [14], where a similar boundary condition is introduced in a string-membrane problem in a neighborhood of the point where the string meets the membrane. Note that (2.4) and (2.7) may be combined to read

$$\left.\begin{aligned} W(\mathbf{x}) &= \sigma_0(\mathbf{x})[w(0) + \lambda(\mathbf{x})\psi_1(0)], \\ \boldsymbol{\phi}(\mathbf{x}) &= \sigma_1(\mathbf{x})\boldsymbol{\psi}(0), \quad \mathbf{x} \in J, \end{aligned}\right\} \tag{2.8}$$

where the continuous, piecewise linear function λ is given by

$$\lambda(\mathbf{x}) = \begin{cases} \eta \ \text{ if } \mathbf{x} = \mathbf{x}_0 + \eta\boldsymbol{\tau}(\mathbf{x}_0) \in J_\alpha, \\ \pm\alpha/2 \ \text{ if } \mathbf{x} \in J_0^\pm . \end{cases}$$

Thus the geometric conditions which the admissible displacements must obey are (2.4) and (2.6) when $J - J_\alpha$ is free, or (2.6) and (2.8) otherwise.

We write

$$\mathcal{K} = \mathcal{K}_P + \mathcal{K}_B, \quad \mathcal{S} = \mathcal{S}_P + \mathcal{S}_B, \quad \mathcal{W} = \mathcal{W}_P + \mathcal{W}_B,$$

where $\mathcal{K}_P$ and $\mathcal{K}_B$ are the kinetic energies of the plate and beam, respectively, and similarly for $\mathcal{S}_P, \mathcal{S}_B$ and $\mathcal{W}_P, \mathcal{W}_B$. We assume that the plate and beam are homogeneous and elastically isotropic. Implicit in our assumptions and in the approximations of the last section is that the energy of the plate is that associated with Reissner-Mindlin plate theory, while $\mathcal{K}_B$ and $\mathcal{S}_B$ are given by (see [5] or [7, Appendix])

$$\mathcal{K}_B = \frac{1}{2}\int_0^\ell (\rho_2|\dot{w}|^2 + I_\alpha|\dot{\psi}_2|^2 + I|\dot{\psi}_1|^2)\,d\xi, \tag{2.9}$$

where $\dot{} = \partial/\partial t$, ρ_2 is the mass of the beam per unit of reference length,

$$I_\alpha = \rho_2\frac{\alpha^2}{12},\ \ I_\beta = \rho_2\frac{\beta^2}{12},\ \ I = I_\alpha + I_\beta;$$

and

$$\mathcal{S}_B = \frac{1}{2}\int_0^\ell (GM|\psi_1'|^2 + E_2M_\beta|\psi_2'|^2 + K_2|w' + \psi_2|^2))\,d\xi, \tag{2.10}$$

where E_2 is Young's modulus of the beam, K_2 is its shear modulus in bending in the $\mathbf{e}_1\mathbf{e}_3$ plane, G is its modulus of torsional rigidity and

$$M_\alpha = A\frac{\alpha^2}{12},\ \ M_\beta = A\frac{\beta^2}{12},\ \ M = M_\alpha + M_\beta,$$

$A = \alpha\beta$ denoting the cross-sectional area.

The kinetic energy of a Reissner-Mindlin plate is (see, e.g., [8, Chapter 1])

$$\mathcal{K}_P = \frac{1}{2}\int_\Omega \left(\rho_1|\dot{W}|^2 + I_h|\dot{\boldsymbol{\phi}}|^2\right)\,d\Omega, \tag{2.11}$$

where and ρ_1 is the mass density of the plate per unit of reference area and $I_h = \rho_1 h^2/12$. To express the strain energy $\mathcal{S}_P$ we introduce (following Tucsnak [17]) the following notation. Let $\mathcal{A}$ denote the set of two by two symmetric matrices, and $C : \mathcal{A} \mapsto \mathcal{A}$ be the second order tensor defined by

$$C[\varepsilon] = \frac{E_1h^3}{12(1-\mu_1^2)}[\mu_1(\varepsilon_{11} + \varepsilon_{22})\mathcal{I} + (1-\mu_1)\varepsilon],\ \ \forall\varepsilon\in\mathcal{A},$$

where $\mathcal{I}$ is the identity in $\mathcal{A}$, E_1 is Young's modulus of the plate and $\mu_1 \in (0,1)$ is Poisson's ratio. For any function $\mathbf{u} : I\!R^2 \mapsto I\!R^2$ with $\partial u_i/\partial x_j \in L^2(\Omega)$, set

$$\varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^*) = \frac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i).$$

Then $\mathcal{S}_P$ may be expressed as

$$\mathcal{S}_P = \frac{1}{2}\int_\Omega [(C[\varepsilon(\boldsymbol{\phi})], \varepsilon(\boldsymbol{\phi})) + K_1|\nabla W + \boldsymbol{\phi}|^2]\, d\Omega,$$

where for $\varepsilon, \eta \in \mathcal{A}$, $(\eta, \varepsilon) = \sum \eta_{ij}\varepsilon_{ij}$ and K_2 is the shear modulus of the plate.

To obtain the work done on the plate-beam system the applied forces have to be specified. Let $\hat{\mathbf{F}}_P(x, y, z)$ be a distributed body force acting on the plate and $\hat{\mathbf{f}}_P(x, y, z)$ be an edge force distributed along $\Gamma_N \times (-h/2, h/2)$. The work done on the plate by these forces is approximately

$$\int_{-h/2}^{h/2}\int_\Omega \hat{\mathbf{F}}_P \cdot (\mathbf{W} + z\boldsymbol{\phi})\, d\Omega dz + \int_{-h/2}^{h/2}\int_{\Gamma_N} \hat{\mathbf{f}}_P \cdot (\mathbf{W} + z\boldsymbol{\phi})\, d\Gamma dz.$$

Consider the resultant forces

$$\mathbf{F}_P(x, y) = \int_{-h/2}^{h/2} \hat{\mathbf{F}}_P\, dz, \quad \mathbf{f}_P(x, y) = \int_{-h/2}^{h/2} \hat{\mathbf{f}}_P\, dz,$$

and moments

$$\mathbf{M}_P(x, y) = \int_{-h/2}^{h/2} z\hat{\mathbf{F}}_P\, dz, \quad \mathbf{m}_P(x, y) = \int_{-h/2}^{h/2} z\hat{\mathbf{f}}_P\, dz.$$

The portion of the work done on the plate related to its transverse motion may then be written (see, e.g., [8])

$$\mathcal{W}_P = \int_\Omega (F_1 W + \mathbf{M}_1 \cdot \boldsymbol{\phi})\, d\Omega + \int_{\Gamma_N} (f_1 W + \mathbf{m}_1 \cdot \boldsymbol{\phi})\, d\Gamma. \qquad (2.12)$$

where

$$F_1(x, y) = \mathbf{k} \cdot \mathbf{F}_P(x, y), \quad f_1(x, y) = \mathbf{k} \cdot \mathbf{f}_P(x, y),$$
$$\mathbf{M}_1 = (\mathbf{i} \cdot \mathbf{M}_P)\mathbf{i} + (\mathbf{j} \cdot \mathbf{M}_P)\mathbf{j}, \quad \mathbf{m}_1 = (\mathbf{i} \cdot \mathbf{m}_P)\mathbf{i} + (\mathbf{j} \cdot \mathbf{m}_P)\mathbf{j}.$$

Similarly, let $\hat{\mathbf{F}}_B(\xi, \eta, \zeta)$ be a distributed body force acting on the beam and $\hat{\mathbf{f}}_B(\eta, \zeta)$ be a force distributed over the cross-section at $\xi = \ell$. The work done on the beam by these forces is approximately

$$\int_0^\ell \int_\omega \hat{\mathbf{F}}_B \cdot (\mathbf{w}(\xi) + \eta(\psi_1(\xi)\mathbf{e}_3 - \psi_3(\xi)\mathbf{e}_1) + \zeta(\psi_2(\xi)\mathbf{e}_1 - \psi_1(\xi)\mathbf{e}_2))\, d\omega d\ell$$
$$+ \int_\omega \hat{\mathbf{f}}_B \cdot (\mathbf{w}(\ell) + \eta(\psi_1(\ell)\mathbf{e}_3 - \psi_3(\ell)\mathbf{e}_1) + \zeta(\psi_2(\ell)\mathbf{e}_1 - \psi_1(\ell)\mathbf{e}_2))\, d\omega.$$

Consider the resultant forces

$$\mathbf{F}_B(\xi) = \int_\omega \hat{\mathbf{F}}_B \, d\omega, \;\; \mathbf{f}_B = \int_\omega \hat{\mathbf{f}}_B \, d\omega,$$

and set

$$M_{21}(\xi) = \int_\omega (\zeta \mathbf{e}_2 \cdot \hat{\mathbf{F}}_B - \eta \mathbf{e}_3 \cdot \hat{\mathbf{F}}_B) \, d\omega, \;\; M_{22}(\xi) = \int_\omega \zeta \mathbf{e}_1 \cdot \hat{\mathbf{F}}_B \, d\omega,$$

$$m_{21} = \int_\omega (\zeta \mathbf{e}_2 \cdot \hat{\mathbf{f}}_B - \eta \mathbf{e}_3 \cdot \hat{\mathbf{f}}_B) \, d\omega, \;\; m_{22} = \int_\omega \zeta \mathbf{e}_1 \cdot \hat{\mathbf{f}}_B \, d\omega.$$

The quantity M_{21} is a twisting moment around $\mathbf{e}_1$, M_{22} is a bending moment about $\mathbf{e}_2$ and similarly for m_{21} and m_{22}. Then the portion of the work done on the beam related its transverse motion is

$$\mathcal{W}_B = \int_0^\ell [F_2 w + \mathbf{M}_2 \cdot (\psi_1 \mathbf{e}_1 + \psi_2 \mathbf{e}_2)] \, d\xi \\ + f_2 w(\ell) + \mathbf{m}_2 \cdot (\psi_1 \mathbf{e}_1 + \psi_2 \mathbf{e}_2)(\ell), \quad (2.13)$$

where

$$F_2 = \mathbf{k} \cdot \mathbf{F}_B, \;\; f_2 = \mathbf{k} \cdot \mathbf{f}_B,$$
$$\mathbf{M}_2 = M_{21} \mathbf{e}_1 + M_{22} \mathbf{e}_2, \;\; \mathbf{m}_2 = m_{21} \mathbf{e}_1 + m_{22} \mathbf{e}_2.$$

We are now in the position to calculate (2.5), where the variation is taken with respect to displacements satisfying (2.4) and (2.6) if $J - J_\alpha$ is free, or (2.6) and (2.8) otherwise. We discuss only the latter situation, since this is the one which will be considered in subsequent sections. In the calculation we utilize the identity

$$\int_\Omega (C[\varepsilon(\mathbf{u})], \varepsilon(\mathbf{v})) \, d\Omega = -\int_\Omega \mathbf{v} \cdot (\operatorname{div} C[\varepsilon(\mathbf{u})]) \, d\Omega \\ + \int_\Gamma \mathbf{v} \cdot (C[\varepsilon(\mathbf{u})]\boldsymbol{\nu}) \, d\Gamma. \quad (2.14)$$

The following equations of motion and boundary conditions are obtained.

Equations of motion of the plate:

$$\left. \begin{aligned} \rho_1 \ddot{W} - K_1 \operatorname{div}(\nabla W + \boldsymbol{\phi}) &= F_1, \\ I_h \ddot{\boldsymbol{\phi}} - \operatorname{div} C[\varepsilon(\boldsymbol{\phi})] + K_1(\nabla W + \boldsymbol{\phi}) &= \mathbf{M}_1. \end{aligned} \right\} \quad (2.15)$$

Boundary conditions along Γ_N:

$$\left. \begin{aligned} K_1(\nabla W + \boldsymbol{\phi}) \cdot \boldsymbol{\nu} &= f_1, \\ C[\varepsilon(\boldsymbol{\phi})]\boldsymbol{\nu} &= \mathbf{m}_1. \end{aligned} \right\} \quad (2.16)$$

Equations of motion of the beam:

$$\left.\begin{aligned} \rho_2\ddot{w} - K_2(w' + \psi_2)' &= F_2, \\ I_\alpha\ddot{\psi}_2 - E_2M_\beta\psi_2'' + K_2(w' + \psi_2) &= M_{22}, \\ I\ddot{\psi}_1 - GM\psi_1'' &= M_{21}. \end{aligned}\right\} \tag{2.17}$$

Boundary conditions at $\xi = \ell$:

$$\left.\begin{aligned} K_2(w' + \psi_2)(\ell) &= f_2, \\ E_2M_\beta\psi_2'(\ell) &= m_{22}, \\ GM\psi_1'(\ell) &= m_{21}. \end{aligned}\right\} \tag{2.18}$$

In addition, we obtain the variational junction condition

$$\begin{aligned} 0 = &\int_J \left[\hat{\boldsymbol{\phi}} \cdot (C[\varepsilon(\boldsymbol{\phi})]\boldsymbol{\nu}) + K_1\hat{W}(\nabla W + \boldsymbol{\phi}) \cdot \boldsymbol{\nu}\right] d\Gamma \\ &- GM\psi_1'(0)\hat{\psi}_1(0) - E_2M_\beta\psi_2'(0)\hat{\psi}_2(0) - K_2(w' + \psi_2)(0)\hat{w}(0) \end{aligned} \tag{2.19}$$

for all test functions $\hat{W}, \hat{\boldsymbol{\phi}}, \hat{w}, \hat{\boldsymbol{\psi}}$ which satisfy (2.8), where $\hat{\boldsymbol{\psi}} = \hat{\psi}_2\mathbf{e}_1 - \hat{\psi}_1\mathbf{e}_2$. We may deduce from (2.19) the following junction conditions.

Dynamic junction conditions:

$$\left.\begin{aligned} GM\psi_1'(0) &= \int_J \{K_1\lambda\sigma_0\boldsymbol{\nu} \cdot (\nabla W + \boldsymbol{\phi}) \\ &\qquad -\sigma_1\mathbf{e}_2 \cdot (C[\varepsilon(\boldsymbol{\phi})]\boldsymbol{\nu})\}d\Gamma, \\ E_2M_\beta\psi_2'(0) &= \int_J \sigma_1\mathbf{e}_1 \cdot (C[\varepsilon(\boldsymbol{\phi})]\boldsymbol{\nu})\, d\Gamma, \\ K_2(w' + \psi_2)(0) &= K_1\int_J \sigma_0\boldsymbol{\nu} \cdot (\nabla W + \boldsymbol{\phi})\, d\Gamma \quad \text{on } J; \end{aligned}\right\} \tag{2.20}$$

in which it should be recalled that $\mathbf{e}_1 = \boldsymbol{\nu}(\mathbf{x}_0)$, $\mathbf{e}_2 = \boldsymbol{\tau}(\mathbf{x}_0)$.

If desired, one may in the same manner obtain the equations of motion and boundary and junction conditions for the components $\mathbf{U}, u, v, \psi_3$ related to in-plane motion of the plate-beam system.

3 Controllability

In this section exact controllability of the plate-beam system will be studied. Only controllability of the subsystem for $W, \boldsymbol{\phi}, w, \boldsymbol{\psi}$ will be considered;

however, a similar analysis could be carried out on the subsystem for $\mathbf{U}$, u, v and ψ_3. Controls are to be applied through the boundary conditions of the plate only. Two cases will be considered: (i) controls are applied through Dirichlet boundary conditions and there are no mechanical boundary conditions, or (ii) controls are applied through the mechanical boundary conditions on Γ_N. The latter situation is more interesting than the former from a physical point of view, but is technically more complex.

Without loss of generality, we may assume that all distributed forces and moments vanish, as do those at the free end of the beam. (Once the reachable set of the homogeneous problem is identified, that of the inhomogeneous problem is also known.) The system to be considered may then be written

$$\left.\begin{aligned} \rho_1 \ddot{W} - K_1 \operatorname{div}(\nabla W + \boldsymbol{\phi}) = 0,\\ I_h \ddot{\boldsymbol{\phi}} - \operatorname{div} C[\varepsilon(\boldsymbol{\phi})] + K_1(\nabla W + \boldsymbol{\phi}) = 0; \end{aligned}\right\} \tag{3.1}$$

$$\left.\begin{aligned} \rho_2 \ddot{w} - K_2(w' + \psi_2)' = 0,\\ I_\alpha \ddot{\psi}_2 - E_2 M_\beta \psi_2'' + K_2(w' + \psi_2) = 0,\\ I\ddot{\psi}_1 - GM\psi_1'' = 0; \end{aligned}\right\} \tag{3.2}$$

$$K_2(w' + \psi_2)(\ell) = E_2 M_\beta \psi_2'(\ell) = GM\psi_1'(\ell) = 0; \tag{3.3}$$

$$W = 0, \quad \boldsymbol{\phi} = 0 \ \text{ on } \Gamma_0 := \Gamma_D; \tag{3.4}$$

$$W = \sigma_0[w(0) + \lambda\psi_1(0)], \quad \boldsymbol{\phi} = \sigma_1 \boldsymbol{\psi}(0) \ \text{ on } J; \tag{3.5}$$

$$\left.\begin{aligned} GM\psi_1'(0) &= \int_J \{K_1 \lambda \sigma_0 \boldsymbol{\nu} \cdot (\nabla W + \boldsymbol{\phi})\\ &\qquad -\sigma_1 \mathbf{e}_2 \cdot (C[\varepsilon(\boldsymbol{\phi})]\boldsymbol{\nu})\} d\Gamma,\\ E_2 M_\beta \psi_2'(0) &= \int_J \sigma_1 \mathbf{e}_1 \cdot (C[\varepsilon(\boldsymbol{\phi})]\boldsymbol{\nu})\, d\Gamma,\\ K_2(w' + \psi_2)(0) &= K_1 \int_J \sigma_0 \boldsymbol{\nu} \cdot (\nabla W + \boldsymbol{\phi})\, d\Gamma \ \text{ on } J; \end{aligned}\right\} \tag{3.6}$$

When controls are applied in the Dirichlet boundary conditions, the remaining boundary conditions are

$$W = f, \quad \boldsymbol{\phi} = \mathbf{m} \ \text{ on } \Gamma_1 := \Gamma - \overline{\Gamma}_0 - \overline{J}, \tag{3.7}$$

where f and $\mathbf{m} = m_1\mathbf{i} + m_2\mathbf{j}$ are the controls. When the controls are in the mechanical boundary conditions the remaining boundary conditions are

$$K_1(\nabla W + \boldsymbol{\phi}) \cdot \boldsymbol{\nu} = f, \quad C([\varepsilon(\boldsymbol{\phi})]\boldsymbol{\nu} = \mathbf{m} \ \text{ on } \Gamma_1 := \Gamma_N. \tag{3.8}$$

To complete the description of the system, initial conditions are specified:

$$\left.\begin{array}{r} W = W^0, \ \ \dot{W} = W^1, \ \ \boldsymbol{\phi} = \boldsymbol{\phi}^0, \ \ \dot{\boldsymbol{\phi}} = \boldsymbol{\phi}^1, \\ w = w^0, \ \ \dot{w} = w^1, \ \ \psi_i = \psi_i^0, \ \ \dot{\psi}_i = \psi_i^1 \ \text{ at } t = 0, \ i = 1, 2. \end{array}\right\} \tag{3.9}$$

3.1 Description of results

The region Ω is open, bounded and connected with a Lipschitz boundary $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1 \cup \overline{J}$ consisting of a finite number of smooth arcs. The sets Γ_0, Γ_1 and J are relatively open in Γ and mutually disjoint, J is connected and $\overline{\Gamma}_1 \cap \overline{J} = \emptyset$.

For $s \geq 0$ we set

$$\mathcal{H}^s(\Omega) := H^s(\Omega; [\mathbf{i}]) \bigoplus H^s(\Omega; [\mathbf{j}]) \bigoplus H^s(\Omega; [\mathbf{k}]),$$

$$\mathcal{H}^s(0, \ell) = H^s(0, \ell; [\mathbf{e}_1]) \bigoplus H^s(0, \ell; [\mathbf{e}_2]) \bigoplus H^s(0, \ell; [\mathbf{e}_3]),$$

where $[\mathbf{e}]$ denotes the linear span of the vector $\mathbf{e}$. Let

$$\mathbf{\Phi} = \phi_2\mathbf{i} - \phi_1\mathbf{j} + W\mathbf{k} := \boldsymbol{\phi} + W\mathbf{k} \in \mathcal{H}^0(\Omega)$$

and

$$\mathbf{\Psi} = \psi_2\mathbf{e}_1 - \psi_1\mathbf{e}_2 + w\mathbf{e}_3 := \boldsymbol{\psi} + w\mathbf{e}_3 \in \mathcal{H}^0(0, \ell).$$

We define the norms

$$\|\mathbf{\Phi}\|_{\mathcal{H}^0(\Omega)} = \left[\int_\Omega \left(\rho_1|W|^2 + I_h|\boldsymbol{\phi}|^2\right) \, d\Omega\right]^{1/2},$$

$$\|\mathbf{\Psi}\|_{\mathcal{H}^0(0,\ell)} = \left[\int_0^\ell (\rho_2|w|^2 + I_\alpha|\psi_2|^2 + I|\psi_1|^2) \, d\xi\right]^{1/2},$$

and we set

$$H = \mathcal{H}^0(\Omega) \times \mathcal{H}^0(0, \ell)$$

with its natural product topology. For $s > 0$ the norms on $\mathcal{H}^s(\Omega)$ and $\mathcal{H}^s(0, \ell)$ are those induced by the corresponding H^s spaces with their standard norms.

Let γ be a nonempty, relatively open subset of $\Gamma - \overline{J}$, and consider the set

$$\mathcal{H}^1_\gamma(\Omega) = \{\mathbf{\Phi} \in \mathcal{H}^1(\Omega) | \, \mathbf{\Phi} = 0 \ \text{ on } \gamma\}.$$

For $\mathbf{\Phi} \in \mathcal{H}^1_\gamma(\Omega)$ we set

$$\|\mathbf{\Phi}\|_{\mathcal{H}^1_\gamma(\Omega)} = \left\{ \int_\Omega \left[(C[\varepsilon(\boldsymbol{\phi})], \varepsilon(\boldsymbol{\phi})) + K_1 |\nabla W + \boldsymbol{\phi}|^2 \right] \, d\Omega \right\}^{1/2}.$$

According to Korn's Lemma, $\|\cdot\|_{\mathcal{H}^1_\gamma(\Omega)}$ defines a norm on $\mathcal{H}^1_\gamma(\Omega)$ equivalent to that induced by $\mathcal{H}^1(\Omega)$. We further introduce

$$V_\gamma = \{(\mathbf{\Phi}, \mathbf{\Psi}) \,|\, \mathbf{\Phi} \in \mathcal{H}^1_\gamma(\Omega), \quad \mathbf{\Psi} \in \mathcal{H}^1(0,\ell) \text{ and } \mathbf{\Phi} = \sigma \mathbf{\Psi}(0) \text{ on } J\},$$

where σ is a 3 by 3 matrix which with respect to the $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ basis is given by

$$\sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_1 & 0 \\ 0 & -\sigma_0 \lambda & \sigma_0 \end{pmatrix}.$$

Note that the condition $\mathbf{\Phi} = \sigma \mathbf{\Psi}(0)$ just expresses the geometric condition (3.5). We define a norm on V_γ by setting

$$\|(\mathbf{\Phi}, \mathbf{\Psi})\|_{V_\gamma} = \left\{ \|\mathbf{\Phi}\|^2_{\mathcal{H}^1_\gamma(\Omega)} \right. \\ \left. + \left\{ \int_0^\ell (GM|\psi_1'|^2 + E_2 M_\beta |\psi_2'|^2 + K_2 |w' + \psi_2|^2) \, d\xi \right\} \right\}^{1/2}.$$

The Hilbert space V_γ is dense in H with compact injection, so that if H is identified with its dual space we have the compact embeddings $V_\gamma \subset H \subset V_\gamma'$, where V_γ' denotes the dual of V_γ.

We denote by $(\mathbf{\Phi}^0, \mathbf{\Psi}^0)$, $(\mathbf{\Phi}^1, \mathbf{\Psi}^1)$ the initial data (3.9), i.e.,

$$\mathbf{\Phi}^0 = \boldsymbol{\phi}^0 + W^0 \mathbf{k}, \quad \mathbf{\Phi}^1 = \boldsymbol{\phi}^1 + W^1 \mathbf{k},$$
$$\mathbf{\Psi}^0 = \boldsymbol{\psi}^0 + w^0 \mathbf{e}_3, \quad \mathbf{\Psi}^1 = \boldsymbol{\psi}^1 + w^1 \mathbf{e}_3.$$

We also set

$$\mathbf{f} = m_1 \mathbf{i} + m_2 \mathbf{j} + f \mathbf{k}$$

and

$$U = L^2(\Gamma_1; [\mathbf{i}]) \bigoplus L^2(\Gamma_1; [\mathbf{j}]) \bigoplus L^2(\Gamma_1; [\mathbf{k}]).$$

The next two theorems concern the well-posedness of the control problem.

Theorem 3.1 *(Well-posedness of Dirichlet control problem.) Let $\gamma = \Gamma - \overline{J}$ and suppose that*

$$(\mathbf{\Phi}^0, \mathbf{\Psi}^0) \in H, \quad (\mathbf{\Phi}^1, \mathbf{\Psi}^1) \in V_\gamma', \quad \mathbf{f} \in L^2(0,T;U).$$

Then (3.1)–(3.7), (3.9) *has a unique solution*

$$(\mathbf{\Phi}, \mathbf{\Psi}) \in C([0,T]; H) \cap C^1([0,T]; V_\gamma').$$

Theorem 3.2 *(Well-posedness of Neumann control problem.) Assume that* $\Gamma_0 \neq \emptyset$ *and that*

$$(\mathbf{\Phi}^0, \mathbf{\Psi}^0) \in H, \ \ (\mathbf{\Phi}^1, \mathbf{\Psi}^1) \in V'_{\Gamma_0}, \ \ \mathbf{f} \in L^2(0, T; U).$$

Then (3.1)–(3.6), (3.8), (3.9) *has a unique solution*

$$(\mathbf{\Phi}, \mathbf{\Psi}) \in C([0, T]; H) \cap C^1([0, T]; V'_{\Gamma_0}).$$

Remark 3.1 The assumption in Theorem 3.2 that $\Gamma_0 \neq \emptyset$ is unessential, but if $\Gamma_0 = \emptyset$ we need to replace the norm on V by $(\|(\mathbf{\Phi}, \mathbf{\Psi})\|_V^2 + \int_\Omega |\mathbf{\Phi}|^2 d\Omega)^{1/2}$.

Suppose that $\mathbf{\Phi}^0 = \mathbf{\Phi}^1 = 0$ and $\mathbf{\Psi}^0 = \mathbf{\Psi}^1 = 0$. For $T > 0$ set

$$R_T = \{((\mathbf{\Phi}(T), \mathbf{\Psi}(T)), (\dot{\mathbf{\Phi}}(T), \dot{\mathbf{\Psi}}(T))) | \, \mathbf{f} \in L^2(0, T; U)\}.$$

In order to say something useful about R_T we shall need to impose geometric restrictions on Γ. We therefore assume that there is a point $\hat{\mathbf{x}}_0 \in I\!R^2$ such that

$$(\mathbf{x} - \hat{\mathbf{x}}_0) \cdot \boldsymbol{\nu} \leq 0 \ \text{ for } \mathbf{x} \in \Gamma_0, \ \ (\mathbf{x} - \hat{\mathbf{x}}_0) \cdot \boldsymbol{\nu} < 0 \ \text{ for } \mathbf{x} \in \overline{J}. \quad (3.10)$$

Set

$$\Gamma_1^+ = \{\mathbf{x} \in \Gamma_1 | \, (\mathbf{x} - \hat{\mathbf{x}}_0) \cdot \boldsymbol{\nu} > 0\}, \ \ \Gamma_1^- = \{\mathbf{x} \in \Gamma_1 | \, (\mathbf{x} - \hat{\mathbf{x}}_0) \cdot \boldsymbol{\nu} \leq 0\}.$$

When the controls act in the Dirichlet boundary conditions, one may assume without loss of generality that $\Gamma_1^+ = \Gamma_1$ or, to say the same thing, that the controls $\mathbf{f}$ are supported in Γ_1^+, which amounts to redefining Γ_0 to be $\Gamma_0 \cup \Gamma_1^-$. However, when the controls act in the Neumann boundary conditions the explicit assumption

$$(\mathbf{x} - \hat{\mathbf{x}}_0) \cdot \boldsymbol{\nu} \geq 0 \ \text{ on } \Gamma_1 \quad (3.11)$$

is needed.

Theorem 3.3 *(Dirichlet boundary control.) Assume that* (3.10) *holds and let* $\gamma = \Gamma - \overline{J}$. *Then there is a time* $T_0 > 0$ *such that* $R_T = H \times V'_\gamma$ *for* $T > T_0$.

Theorem 3.4 *(Neumann boundary control.) Assume that* (3.10) *and* (3.11) *hold, that* $\Gamma_0 \neq \emptyset$ *and that* $\overline{\Gamma}_0$ *and* $\overline{\Gamma}_1$ *either do not intersect or else intersect in a strictly convex corner. Then there is a time* $T_0 > 0$ *such that* $V_{\Gamma_0} \times H \subset R_T$ *for* $T > T_0$.

Remark 3.2 One may eliminate the geometric condition (3.11) in Theorem 3.4 but at the expense of enlarging the control space and working in a weaker state space. For example, if one admits controls in the space

$$(H^1(0,T;U_+))' \bigoplus L^2(0,T;\mathcal{H}^{-1}(\Gamma_1^-)), \tag{3.12}$$

then one may prove that $H \times V'_{\Gamma_0} \subset R_T$ for T large enough, without assumption (3.11). In (3.12),

$$U_+ = L^2(\Gamma_1^+;[\mathbf{i}]) \bigoplus L^2(\Gamma_1^+;[\mathbf{j}]) \bigoplus L^2(\Gamma_1^+;[\mathbf{k}]),$$
$$\mathcal{H}^{-1}(\Gamma_1^-) = H^{-1}(\Gamma_1^-;[\mathbf{i}]) \bigoplus H^{-1}(\Gamma_1^-;[\mathbf{j}]) \bigoplus H^{-1}(\Gamma_1^-;[\mathbf{k}]).$$

The proofs of Theorems 3.1–3.4 are given in Section 5. In the next subsection the observability estimates needed to prove Theorems 3.3 and 3.4 are presented.

3.2 Observability estimates

We consider the problem

$$\left.\begin{aligned} \rho_1 \ddot{Z} - K_1 \operatorname{div}(\nabla Z + \boldsymbol{\varphi}) &= 0, \\ I_h \ddot{\boldsymbol{\varphi}} - \operatorname{div} C[\varepsilon(\boldsymbol{\varphi})] + K_1(\nabla Z + \boldsymbol{\varphi}) &= 0; \end{aligned}\right\} \tag{3.13}$$

$$\left.\begin{aligned} \rho_2 \ddot{z} - K_2(z' + \theta_2)' &= 0, \\ I_\alpha \ddot{\theta}_2 - E_2 M_\beta \theta_2'' + K_2(z' + \theta_2) &= 0, \\ I\ddot{\theta}_1 - GM\theta_1'' &= 0; \end{aligned}\right\} \tag{3.14}$$

$$K_2(z' + \theta_2)(\ell) = E_2 M_\beta \theta_2'(\ell) = GM\theta_1'(\ell) = 0; \tag{3.15}$$

$$Z = 0, \ \ \boldsymbol{\varphi} = 0 \ \text{ on } \Gamma_0; \tag{3.16}$$

$$Z = \sigma_0[z(0) + \lambda\theta_1(0)], \ \ \boldsymbol{\varphi} = \sigma_1 \boldsymbol{\theta}(0) \ \text{ on } J, \tag{3.17}$$

where $\boldsymbol{\theta} = \theta_2 \mathbf{e}_1 - \theta_1 \mathbf{e}_2$;

$$\left.\begin{aligned} GM\theta_1'(0) &= \int_J \{K_1 \lambda \sigma_0 \boldsymbol{\nu} \cdot (\nabla Z + \boldsymbol{\varphi}) \\ &\qquad -\sigma_1 \mathbf{e}_2 \cdot (C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu})\} d\Gamma, \\ E_2 M_\beta \theta_2'(0) &= \int_J \sigma_1 \mathbf{e}_1 \cdot (C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu})\, d\Gamma, \\ K_2(z' + \theta_2)(0) &= K_1 \int_J \sigma_0 \boldsymbol{\nu} \cdot (\nabla Z + \boldsymbol{\varphi})\, d\Gamma \ \text{ on } J; \end{aligned}\right\} \tag{3.18}$$

and either

$$Z = 0, \ \ \boldsymbol{\varphi} = 0 \ \text{ on } \Gamma_1 := \Gamma - \overline{\Gamma}_0 - \overline{J}, \tag{3.19}$$

or

$$K_1(\nabla Z + \boldsymbol{\varphi}) \cdot \boldsymbol{\nu} = 0, \ \ C([\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu} = 0 \ \text{ on } \Gamma_1. \tag{3.20}$$

The initial conditions are

$$\left.\begin{array}{c} Z = Z^0, \ \ \dot{Z} = Z^1, \ \ \boldsymbol{\varphi} = \boldsymbol{\varphi}^0, \ \ \dot{\boldsymbol{\varphi}} = \boldsymbol{\varphi}^1, \\ z = z^0, \ \ \dot{z} = z^1, \ \ \boldsymbol{\theta} = \boldsymbol{\theta}^0, \dot{\boldsymbol{\theta}} = \boldsymbol{\theta}^1 \ \text{ at } t = 0. \end{array} c\right\} \tag{3.21}$$

As in the last section, we define

$$\boldsymbol{\Xi} = \boldsymbol{\varphi} + Z\mathbf{k}, \ \ \boldsymbol{\Theta} = \boldsymbol{\theta} + z\mathbf{e}_3,$$

and $\boldsymbol{\Xi}^0, \boldsymbol{\Xi}^1, \boldsymbol{\Theta}^0, \boldsymbol{\Theta}^1$ with obvious connotations.

The Dirichlet initial-boundary value problem (3.13)–(3.19), (3.21) and the Neumann problem (3.13)–(3.18), (3.20) (3.21) have simple variational formulations. Let

$$\hat{\boldsymbol{\Xi}} = \hat{\boldsymbol{\varphi}} + \hat{Z}\mathbf{k}, \ \ \hat{\boldsymbol{\Theta}} = \hat{\boldsymbol{\theta}} + \hat{z}\mathbf{e}_3,$$

and assume that

$$(\hat{\boldsymbol{\Xi}}, \hat{\boldsymbol{\Theta}}) \in V_\gamma \ \text{ where } \gamma = \begin{cases} \Gamma - \overline{J} \ \text{ for Dirichlet BC on } \Gamma_1, \\ \Gamma_0 \ \text{ for Neumann BC on } \Gamma_1. \end{cases}$$

We multiply the two equations in (3.13) by $\hat{Z}, \hat{\boldsymbol{\varphi}}$, respectively, add the products and integrate over Ω. Similarly, multiply the three equations in (3.14) by $\hat{z}, \hat{\theta}_2$ and $\hat{\theta}_1$, respectively, add the products and integrate over $(0, \ell)$. By adding the integrals over Ω and over $(0, \ell)$ and then carrying out appropriate integrations by parts, utilizing (2.14), we obtain (this is implicit in the manner in which we derived the system in the first place)

$$((\ddot{\boldsymbol{\Xi}}, \ddot{\boldsymbol{\Theta}}), (\hat{\boldsymbol{\Xi}}, \hat{\boldsymbol{\Theta}}))_H + ((\boldsymbol{\Xi}, \boldsymbol{\Theta}), (\hat{\boldsymbol{\Xi}}, \hat{\boldsymbol{\Theta}}))_{V_\gamma} = 0, \ \ \forall (\hat{\boldsymbol{\Xi}}, \hat{\boldsymbol{\Theta}}) \in V_\gamma,$$

or

$$(\ddot{\boldsymbol{\Xi}}, \ddot{\boldsymbol{\Theta}}) + A(\boldsymbol{\Xi}, \boldsymbol{\Theta}) = 0 \ \text{ in } V'_\gamma, \tag{3.22}$$

where A is the Riesz isomorphism of V_γ onto V'_γ. We may now apply standard variational or semigroup theory to conclude that the initial value problem for (3.22) has a unique solution with the following regularity:

$$((\boldsymbol{\Xi}^0, \boldsymbol{\Theta}^0), (\boldsymbol{\Xi}^1, \boldsymbol{\Theta}^1)) \in W \Longrightarrow ((\boldsymbol{\Xi}, \boldsymbol{\Theta}), (\dot{\boldsymbol{\Xi}}, \dot{\boldsymbol{\Theta}})) \in C([0, \infty); W),$$

where W stands for any one of $H \times V'_\gamma$, $V_\gamma \times H$ or $D_A \times V_\gamma$ and where

$$D_A = \{(\hat{\Xi}, \hat{\Theta}) \in V_\gamma \,|\, A(\hat{\Xi}, \hat{\Theta}) \in H\}, \quad \|(\hat{\Xi}, \hat{\Theta})\|_{D_A} = \|A(\hat{\Xi}, \hat{\Theta})\|_H.$$

Moreover, the solution is given by a unitary group in all cases:

$$\|((\Xi(t), \Theta(t)), (\dot{\Xi}(t), \dot{\Theta}(t)))\|_W = \|((\Xi^0, \Theta^0), (\Xi^1, \Theta^1))\|_W, \quad t > 0.$$

Remark 3.3 It may be verified through integrations by parts that

$$\begin{aligned} D_A \supset \{(\Xi, \Theta) |\, &(\Xi, \Theta) \in (\mathcal{H}^2(\Omega) \times \mathcal{H}^2(0, \ell)) \cap V_\gamma, \\ &(\Xi, \Theta) \text{ satisfy the dynamic junction and boundary conditions}\}. \end{aligned}$$

The injection of $\mathcal{H}^2(\Omega) \times \mathcal{H}^2(0, \ell)$ into D_A is continuous and

$$\begin{aligned} (A(\Xi, \Theta), (\hat{\Xi}, \hat{\Theta}))_H = &- \int_\Omega \{[\operatorname{div} C[\varepsilon(\boldsymbol{\varphi})] - K_1(\nabla Z + \boldsymbol{\varphi})] \cdot \hat{\boldsymbol{\varphi}} \\ &+ K_1 \operatorname{div}(\nabla Z + \boldsymbol{\varphi}) \hat{Z}\} \, d\Omega - \int_0^\ell \{GM\theta_1'' \hat{\theta}_1 \\ &- [E_2 M_\beta \theta_2'' - K_2(z' + \theta_2)]\hat{\theta}_2 + K_2(z' + \theta_2)' \hat{z}\} \, d\Omega. \end{aligned}$$

On the other hand, it follows from results of Nicaise [13] that under the stated assumptions on the region Ω, an element $(\Xi, \Theta) \in D_A$ has the following spatial regularity:

$$\Xi \in \mathcal{H}^s(\Omega), \ \forall s < s_0, \ s \le 2, \quad \text{for some } s_0 > 3/2, \tag{3.23}$$

in the case of Dirichlet conditions on Γ_1, and also in the case of Neumann conditions on Γ_1 provided $\overline{\Gamma}_0$ and $\overline{\Gamma}_1$ either have an empty intersection or else intersect in a strictly convex corner. If $\overline{\Gamma}_0$ and $\overline{\Gamma}_1$ meet in a nonconvex corner, (3.23) holds for all $s < s_0$, $s < 3/2$, for some $s_0 > 5/4$. Since $\dim(\Omega) = 2$, it follows from a Sobolev imbedding theorem that

$$\Xi \in C^\alpha(\overline{\Omega}; [\mathbf{i}]) \bigoplus C^\alpha(\overline{\Omega}; [\mathbf{j}]) \bigoplus C^\alpha(\overline{\Omega}; [\mathbf{k}])$$

all $\alpha \in (0, s-1]$ [2, Theorem 1.4.5.2]. With regard to Θ, we have $\Theta \in \mathcal{H}^2(0, \ell)$. The injection of D_A into $\mathcal{H}^s(\Omega) \times \mathcal{H}^2(0, \ell)$ is continuous.

The main results of this section are the following *a priori* estimates.

Proposition 3.1 *Let the hypotheses of Theorem 3.3 hold and suppose that*

$$(\Xi^0, \Theta^0) \in D_A, \quad (\Xi^1, \Theta^1) \in V_\gamma.$$

There is a constant T_0 *such that if* $T > T_0$*, the solution of* (3.13)–(3.19), (3.21) *satisfies*

$$\|((\Xi^0, \Theta^0), (\Xi^1, \Theta^1))\|^2_{V_\gamma \times H} \leq C(T) \int_{\Sigma_1} (\mathbf{r} \cdot \boldsymbol{\nu}) \left[|C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}|^2 + K_1 \left| \frac{\partial Z}{\partial \boldsymbol{\nu}} \right|^2 \right] d\Sigma,$$

where $\mathbf{r} = \mathbf{x} - \hat{\mathbf{x}}_0$ *and* $\Sigma_1 = \Gamma_1 \times (0, T)$*.*

Proposition 3.2 *Let the hypotheses of Theorem 3.4 be satisfied and suppose that*

$$(\Xi^0, \Theta^0) \in D_A, \quad (\Xi^1, \Theta^1) \in V_{\Gamma_0}.$$

There is a constant T_0 *such that if* $T > T_0$*, the solution of* (3.13)–(3.18), (3.20), (3.21) *satisfies*

$$\|((\Xi^0, \Theta^0), (\Xi^1, \Theta^1))\|^2_{V_{\Gamma_0} \times H} \leq C(T) \int_{\Sigma_1} (\mathbf{r} \cdot \boldsymbol{\nu}) |\dot{\Xi}|^2 \, d\Sigma.$$

The following is an immediate consequence of Proposition 3.2, using the idea of "weakening the norm" (see, Lions [12]).

Corollary 3.1 *Let the hypotheses of Theorem 3.4 be satisfied and suppose that*

$$(\Xi^0, \Theta^0) \in V_{\Gamma_0}, \quad (\Xi^1, \Theta^1) \in H.$$

There is a constant T_0 *such that if* $T > T_0$*, the solution of* (3.13)–(3.18), (3.20), (3.21) *satisfies*

$$\|((\Xi^0, \Theta^0), (\Xi^1, \Theta^1))\|^2_{H \times V'_{\Gamma_0}} \leq C(T) \int_{\Sigma_1} (\mathbf{r} \cdot \boldsymbol{\nu}) |\Xi|^2 \, d\Sigma.$$

4 Proofs of Propositions 3.1 and 3.2

Our starting point for the proofs of both propositions is an identity from [4, Lemma 3.3.1] that may be stated as follows.

Lemma 4.1 *Let* $\Xi = \boldsymbol{\varphi} + Z\mathbf{k}$ *and assume that* Ξ *satisfies* (3.23) *for some* $s > 3/2$*. Then*

$$\begin{aligned}
&\int_\Omega \{(\nabla \boldsymbol{\varphi} \mathbf{r}) \cdot [\operatorname{div} C[\varepsilon(\boldsymbol{\varphi})] - K_1(\nabla Z + \boldsymbol{\varphi})] \\
&\qquad + K_1(\mathbf{r} \cdot \nabla Z) \operatorname{div}(\nabla Z + \boldsymbol{\varphi})\} \, d\Omega = K_1 \int_\Omega (\nabla Z + \boldsymbol{\varphi}) \cdot \boldsymbol{\varphi} \, d\Omega \\
&\qquad - \frac{1}{2} \int_\Gamma (\mathbf{r} \cdot \boldsymbol{\nu}) \left\{ (C[\varepsilon(\boldsymbol{\varphi})], \varepsilon(\boldsymbol{\varphi})) + K_1 |\nabla Z + \boldsymbol{\varphi}|^2 \right\} d\Gamma \\
&\qquad + \int_\Gamma \{(\nabla \boldsymbol{\varphi} \mathbf{r}) \cdot (C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}) + K_1(\mathbf{r} \cdot \nabla Z)(\nabla Z + \boldsymbol{\varphi}) \cdot \boldsymbol{\nu}\} \, d\Gamma. \quad (4.1)
\end{aligned}$$

Remark 4.1 If $s \le 2$, the integral on the left may be interpreted (on a term-by-term basis) in the duality between $H_0^{2-s}(\Omega)$ and $H^{s-2}(\Omega)$ since, for $s > 3/2$, $H^{s-1}(\Omega) \subset H^{2-s}(\Omega) = H_0^{2-s}(\Omega)$. All of the other integrals in (4.1) are well-defined. The constant K_1 in (4.1) is arbitrary, but we shall shortly use this identity with $\boldsymbol{\Xi}$ a solution of (3.13).

Later on we shall also need the following identity which, when $K_1 = 0$, is a generalization of Lemma 4.1 from the radial vector field $\mathbf{r}$ to an arbitrary vector field $\mathbf{h}$.

Lemma 4.2 *Let* $\mathbf{h}$ *be any* $W^{1,\infty}$ *vector field in* Ω *and* $\boldsymbol{\varphi} = \varphi_1 \mathbf{i} + \varphi_2 \mathbf{j}$, *where* $\varphi_i \in H^s(\Omega)$ *for some* $s > 3/2$. *Then*

$$\int_\Omega (\nabla \boldsymbol{\varphi} \mathbf{h}) \cdot \operatorname{div} C[\varepsilon(\boldsymbol{\varphi})]\, d\Omega = -\frac{1}{2}\int_\Gamma (\mathbf{h}\cdot\boldsymbol{\nu})(C[\varepsilon(\boldsymbol{\varphi})], \varepsilon(\boldsymbol{\varphi}))\, d\Gamma + \int_\Gamma (\nabla\boldsymbol{\varphi}\mathbf{h}) \cdot C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}\, d\Gamma - \int_\Omega Q(\nabla\boldsymbol{\varphi})\, d\Omega, \quad (4.2)$$

where $Q(\nabla\boldsymbol{\varphi})$ *is a quadratic form in* $\varphi_{i,j} := \partial\varphi_i/\partial x_j$ *given by*

$$\begin{aligned} Q(\nabla\boldsymbol{\varphi}) = \frac{D_1}{2}\{&[(\varphi_{1,1}^2 - \varphi_{2,2}^2) + \frac{1}{2}(1-\mu_1)(\varphi_{2,1}^2 - \varphi_{1,2}^2)](h_{1,1} - h_{2,2}) \\ &+ h_{1,2}[\varphi_{1,1}((1+\mu_1)\varphi_{2,1} + (1-\mu_1)\varphi_{1,2}) + 2\varphi_{2,2}\varphi_{2,1}] \\ &\quad + h_{2,1}[\varphi_{2,2}((1+\mu_1)\varphi_{1,2} + (1-\mu_1)\varphi_{2,1}) + 2\varphi_{1,1}\varphi_{1,2}]\}, \end{aligned}$$

where $D_1 = E_1 h^3/12(1-\mu_1^2)$.

Proof: From Green's formula (2.14) we have

$$\int_\Omega (\nabla\boldsymbol{\varphi}\mathbf{h}) \cdot \operatorname{div} C[\varepsilon(\boldsymbol{\varphi})\, d\Omega = -\int_\Gamma (\mathbf{h}\cdot\boldsymbol{\nu})(C[\varepsilon(\boldsymbol{\varphi})], \varepsilon(\nabla\boldsymbol{\varphi}\mathbf{h}))\, d\Gamma + \int_\Omega (\nabla\boldsymbol{\varphi}\mathbf{h}) \cdot C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}\, d\Gamma,$$

so to prove (4.2) we need to calculate $(C[\varepsilon(\boldsymbol{\varphi})], \varepsilon(\nabla\boldsymbol{\varphi}\mathbf{h}))$. We have

$$\begin{aligned} (C[\varepsilon(\boldsymbol{\varphi})], \varepsilon(\nabla\boldsymbol{\varphi}\mathbf{h})) = D_1\{&\varphi_{1,1}(\nabla\varphi_1\cdot\mathbf{h})_{,1} + \varphi_{2,2}(\nabla\varphi_2\cdot\mathbf{h})_{,2} \\ &+ \mu_1[\varphi_{1,1}(\nabla\varphi_2\cdot\mathbf{h})_{,1} + \varphi_{2,2}(\nabla\varphi_1\cdot\mathbf{h})_{,2}] \\ &+ \frac{1}{2}(1-\mu_1)(\varphi_{1,2}+\varphi_{2,1})[(\nabla\varphi_1\cdot\mathbf{h})_{,2} + (\nabla\varphi_2\cdot\mathbf{h})_{,1}]\}. \end{aligned}$$

A lengthy, but straightforward, calculation shows that the right side of the last equality may be written

$$\frac{D_1}{2}\operatorname{div}\{\mathbf{h}[(\varphi_{1,1}^2+\varphi_{2,2}^2)+2\mu_1\varphi_{1,1}\varphi_{2,2}+\frac{1}{2}(1-\mu_1)(\varphi_{1,2}+\varphi_{2,1})^2]\}+Q(\nabla\boldsymbol{\varphi})$$
$$=\frac{D_1}{2}\operatorname{div}\{\mathbf{h}[(\varepsilon_{11}^2+\varepsilon_{22}^2+2\mu_1\varepsilon_{11}\varepsilon_{22}+2(1-\mu)\varepsilon_{12}^2]\}+Q(\nabla\boldsymbol{\varphi})$$
$$=\frac{1}{2}\operatorname{div}\{\mathbf{h}(C[\varepsilon(\boldsymbol{\varphi})],\varepsilon(\boldsymbol{\varphi}))\}+Q(\nabla\boldsymbol{\varphi}),$$

from which Lemma 4.2 follows.

We need to rewrite the integrals over Γ in (4.1) in forms more suitable for proving Propositions 3.1 and 3.2. We have on Γ

$$(\nabla\boldsymbol{\varphi}\mathbf{r})\cdot(C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu})$$
$$=(\mathbf{r}\cdot\boldsymbol{\nu})\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\nu}}\cdot(C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu})+(\mathbf{r}\cdot\boldsymbol{\tau})\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\tau}}\cdot(C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}) \quad (4.3)$$

and

$$\frac{\partial\boldsymbol{\varphi}}{\partial x_j}=\nu_j\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\nu}}+\tau_j\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\tau}}, \quad (4.4)$$

where $\boldsymbol{\nu}=(\nu_1,\nu_2)$, $\boldsymbol{\tau}=(\tau_1,\tau_2)=(-\nu_2,\nu_1)$. Write $C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}$ in terms of normal and tangential components:

$$C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}=C_{\boldsymbol{\nu}}(\boldsymbol{\varphi})\boldsymbol{\nu}+C_{\boldsymbol{\tau}}(\boldsymbol{\varphi})\boldsymbol{\tau}.$$

It may be verified from (4.4), after some calculation, that

$$C_{\boldsymbol{\nu}}(\boldsymbol{\varphi})=D_1\left[\boldsymbol{\nu}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\nu}}+\mu_1\boldsymbol{\tau}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\tau}}\right], \quad (4.5)$$

$$C_{\boldsymbol{\tau}}(\boldsymbol{\varphi})=\frac{(1-\mu_1)D_1}{2}\left[\boldsymbol{\tau}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\nu}}+\boldsymbol{\nu}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\tau}}\right], \quad (4.6)$$

where $D_1=E_1h^3/12(1-\mu_1^2)$ is the *flexural rigidity* of the plate. Therefore

$$\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\nu}}=\left(\boldsymbol{\nu}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\nu}}\right)\boldsymbol{\nu}+\left(\boldsymbol{\tau}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\nu}}\right)\boldsymbol{\tau}$$
$$=\frac{1}{D_1}\left[C_{\boldsymbol{\nu}}(\boldsymbol{\varphi})\boldsymbol{\nu}+\frac{2}{1-\mu_1}C_{\boldsymbol{\tau}}(\boldsymbol{\varphi})\boldsymbol{\tau}\right]-\mu_1\boldsymbol{\tau}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\tau}}\boldsymbol{\nu}-\boldsymbol{\nu}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\tau}}\boldsymbol{\tau}.$$

Insertion of this expression into (4.3) yields

$$(\nabla\boldsymbol{\varphi}\mathbf{r})\cdot(C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu})=(\mathbf{r}\cdot\boldsymbol{\nu})\left\{\frac{1}{D_1}\left[|C_{\boldsymbol{\nu}}(\boldsymbol{\varphi})|^2+\frac{2}{1-\mu_1}|C_{\boldsymbol{\tau}}(\boldsymbol{\varphi})|^2\right]\right.$$
$$\left.-\left(\mu_1\boldsymbol{\tau}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\tau}}\boldsymbol{\nu}+\boldsymbol{\nu}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\tau}}\boldsymbol{\tau}\right)\cdot(C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu})\right\}+(\mathbf{r}\cdot\boldsymbol{\tau})\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\tau}}\cdot(C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}).$$

Another rather lengthy calculation show that on Γ

$$(C[\varepsilon(\boldsymbol{\varphi})], \varepsilon(\boldsymbol{\varphi})) = \frac{1}{D_1}\left[|C_{\boldsymbol{\nu}}(\boldsymbol{\varphi})|^2 + \frac{2}{1-\mu_1}|C_{\boldsymbol{\tau}}(\boldsymbol{\varphi})|^2\right] + D_1(1-\mu_1^2)\left|\boldsymbol{\tau}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\tau}}\right|^2. \quad (4.7)$$

In addition,

$$(\mathbf{r}\cdot\nabla Z)(\nabla Z+\boldsymbol{\varphi})\cdot\boldsymbol{\nu} = (\mathbf{r}\cdot\boldsymbol{\nu})[|(\nabla Z+\boldsymbol{\varphi})\cdot\boldsymbol{\nu}|^2 - \boldsymbol{\nu}\cdot\boldsymbol{\varphi}(\nabla Z+\boldsymbol{\varphi})\cdot\boldsymbol{\nu}] + (\mathbf{r}\cdot\boldsymbol{\tau})\frac{\partial Z}{\partial\boldsymbol{\tau}}(\nabla Z+\boldsymbol{\varphi})\cdot\boldsymbol{\nu},$$

$$|\nabla Z+\boldsymbol{\varphi}|^2 = |(\nabla Z+\boldsymbol{\varphi})\cdot\boldsymbol{\nu}|^2 + |(\nabla Z+\boldsymbol{\varphi})\cdot\boldsymbol{\tau}|^2.$$

Therefore

$$\begin{aligned}
&-\frac{1}{2}\int_\Gamma(\mathbf{r}\cdot\boldsymbol{\nu})\left\{(C[\varepsilon(\boldsymbol{\varphi})],\varepsilon(\boldsymbol{\varphi})) + K_1|\nabla Z+\boldsymbol{\varphi}|^2\right\}\,d\Gamma \\
&\quad + \int_\Gamma\left\{(\nabla\boldsymbol{\varphi}\mathbf{r})\cdot(C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}) + K_1(\mathbf{r}\cdot\nabla Z)(\nabla Z+\boldsymbol{\varphi})\cdot\boldsymbol{\nu}\right\}\,d\Gamma \\
&\qquad = \frac{1}{2}\int_\Gamma(\mathbf{r}\cdot\boldsymbol{\nu})\left\{\frac{1}{D_1}\left[|C_{\boldsymbol{\nu}}(\boldsymbol{\varphi})|^2 + \frac{2}{1-\mu_1}|C_{\boldsymbol{\tau}}(\boldsymbol{\varphi})|^2\right]\right. \\
&\qquad\qquad \left. + K_1\left|\frac{\partial Z}{\partial\boldsymbol{\nu}} + \boldsymbol{\varphi}\cdot\boldsymbol{\nu}\right|^2\right\}\,d\Gamma \\
&\qquad - \int_\Gamma(\mathbf{r}\cdot\boldsymbol{\nu})\left\{\frac{D_1(1-\mu_1^2)}{2}\left|\boldsymbol{\tau}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\tau}}\right|^2 + \frac{K_1}{2}\left|\frac{\partial Z}{\partial\boldsymbol{\tau}} + \boldsymbol{\varphi}\cdot\boldsymbol{\tau}\right|^2\right. \\
&\qquad\qquad + K_1(\boldsymbol{\varphi}\cdot\boldsymbol{\nu})(\nabla Z+\boldsymbol{\varphi})\cdot\boldsymbol{\nu} \\
&\qquad\qquad \left. + \left(\mu_1\boldsymbol{\tau}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\tau}}\boldsymbol{\nu} + \boldsymbol{\nu}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\tau}}\boldsymbol{\tau}\right)\cdot(C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu})\right\}\,d\Gamma \\
&\qquad + \int_\Gamma(\mathbf{r}\cdot\boldsymbol{\tau})\left\{\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\tau}}\cdot(C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}) + K_1\frac{\partial Z}{\partial\boldsymbol{\tau}}(\nabla Z+\boldsymbol{\varphi})\cdot\boldsymbol{\nu}\right\}\,d\Gamma. \quad (4.8)
\end{aligned}$$

This last expression will eventually be used in the right side of (4.1)

Two other simple identities that will be needed are

$$\begin{aligned}
&\int_\Omega\boldsymbol{\varphi}\cdot[\operatorname{div}C[\varepsilon(\boldsymbol{\varphi})] - K_1(\nabla Z+\boldsymbol{\varphi})]\,d\Gamma \\
&\qquad = -\int_\Omega[(C[\varepsilon(\boldsymbol{\varphi})],\varepsilon(\boldsymbol{\varphi})) + K_1\boldsymbol{\varphi}\cdot(\nabla Z+\boldsymbol{\varphi})]\,d\Gamma \\
&\qquad\qquad + \int_\Gamma\boldsymbol{\varphi}\cdot(C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu})\,d\Gamma, \quad (4.9)
\end{aligned}$$

and

$$\int_\Omega Z\,\mathrm{div}(\nabla Z+\boldsymbol{\varphi})\,d\Omega=-\int_\Omega \nabla Z\cdot(\nabla Z+\boldsymbol{\varphi})\,d\Omega \\ +\int_\Gamma Z(\nabla Z+\boldsymbol{\varphi})\cdot\boldsymbol{\nu}\,d\Gamma. \quad (4.10)$$

Now suppose that $\boldsymbol{\Xi}$ is a solution of (3.13) with regularity

$$\boldsymbol{\Xi}\in C([0,T];\mathcal{H}^s(\Omega))\cap C^1([0,T];\mathcal{H}^{s-1}(\Omega)),\quad s>3/2. \quad (4.11)$$

We integrate the left side of (4.1) with respect to t from 0 to T. Thus

$$\begin{aligned}\int_0^T\int_\Omega(\nabla\boldsymbol{\varphi}\mathbf{r})\cdot\{&\mathrm{div}\,C[\varepsilon(\boldsymbol{\varphi})]-K_1(\nabla Z+\boldsymbol{\varphi})\\ &+K_1(\mathbf{r}\cdot\nabla Z)\,\mathrm{div}(\nabla Z+\boldsymbol{\varphi})\}\;d\Omega dt\\ &=\int_0^T\int_\Omega\left\{I_h(\nabla\boldsymbol{\varphi}\mathbf{r})\cdot\ddot{\boldsymbol{\varphi}}+\rho_1(\mathbf{r}\cdot\nabla Z)\ddot{Z}\right\}\,d\Omega dt\\ &=\alpha_1(t)|_0^T-\int_0^T\int_\Omega\left\{I_h(\nabla\dot{\boldsymbol{\varphi}}\mathbf{r})\cdot\dot{\boldsymbol{\varphi}}+\rho_1(\mathbf{r}\cdot\nabla\dot{Z})\dot{Z}\right\}\,d\Omega dt,\end{aligned}$$

where

$$\alpha_1(t)=\int_\Omega[I_h(\nabla\boldsymbol{\varphi}\mathbf{r})\cdot\dot{\boldsymbol{\varphi}}+\rho_1(\mathbf{r}\cdot\nabla Z)\dot{Z}]\,d\Omega.$$

One has

$$\begin{aligned}\int_\Omega(\nabla\dot{\boldsymbol{\varphi}}\mathbf{r})\cdot\dot{\boldsymbol{\varphi}}\,d\Omega&=\frac12\int_\Omega[\mathrm{div}(\mathbf{r}|\dot{\boldsymbol{\varphi}}|^2)-2|\dot{\boldsymbol{\varphi}}|^2]\,d\Omega\\ &=\frac12\int_\Gamma(\mathbf{r}\cdot\boldsymbol{\nu})|\dot{\boldsymbol{\varphi}}|^2d\Gamma-\int_\Omega|\dot{\boldsymbol{\varphi}}|^2d\Omega\end{aligned}$$

and, similarly,

$$\int_\Omega(\mathbf{r}\cdot\nabla\dot{Z})\dot{Z}\,d\Omega=\frac12\int_\Gamma(\mathbf{r}\cdot\boldsymbol{\nu})|\dot{Z}|^2d\Gamma-\int_\Omega|\dot{Z}|^2d\Omega.$$

Therefore

$$\begin{aligned}\int_Q(\nabla\boldsymbol{\varphi}\mathbf{r})\cdot\{&\mathrm{div}\,C[\varepsilon(\boldsymbol{\varphi})]-K_1(\nabla Z+\boldsymbol{\varphi})\\ &+K_1(\mathbf{r}\cdot\nabla Z)\,\mathrm{div}(\nabla Z+\boldsymbol{\varphi})\}\;dQ\\ &=\alpha_1(t)|_0^T+\int_Q\left[\rho_1|\dot{Z}|^2+I_h|\dot{\boldsymbol{\varphi}}|^2\right]\,dQ\\ &\qquad-\frac12\int_\Sigma(\mathbf{r}\cdot\boldsymbol{\nu})\left[\rho_1|\dot{Z}|^2+I_h|\dot{\boldsymbol{\varphi}}|^2\right]\,d\Sigma,\end{aligned}\quad (4.12)$$

where $Q = \Omega \times (0,T)$ and $\Sigma = \Gamma \times (0,T)$. Substitute (4.12) into the left side of (4.1) and (4.8) into the right side of (4.1) (after integrating (4.1) in t). We obtain

$$\begin{aligned}
\alpha_1(t)|_0^T &+ \int_Q \left[\rho_1|\dot Z|^2 + I_h|\dot{\boldsymbol\varphi}|^2\right] dQ - K_1\int_Q (\nabla Z + \boldsymbol\varphi)\cdot\boldsymbol\varphi\, dQ \\
&= \frac12\int_\Sigma (\mathbf{r}\cdot\boldsymbol\nu)\left[\rho_1|\dot Z|^2 + I_h|\dot{\boldsymbol\varphi}|^2\right] d\Sigma \\
&\quad + \frac12\int_\Sigma (\mathbf{r}\cdot\boldsymbol\nu)\left\{\frac{1}{D_1}\left[|C_{\boldsymbol\nu}(\boldsymbol\varphi)|^2 + \frac{2}{1-\mu_1}|C_{\boldsymbol\tau}(\boldsymbol\varphi)|^2\right]\right. \\
&\qquad\qquad \left. + K_1\left|\frac{\partial Z}{\partial\boldsymbol\nu} + \boldsymbol\varphi\cdot\boldsymbol\nu\right|^2\right\} d\Sigma \\
&\quad - \int_\Sigma (\mathbf{r}\cdot\boldsymbol\nu)\left\{\frac{D_1(1-\mu_1^2)}{2}\left|\boldsymbol\tau\cdot\frac{\partial\boldsymbol\varphi}{\partial\boldsymbol\tau}\right|^2 + \frac{K_1}{2}\left|\frac{\partial Z}{\partial\boldsymbol\tau} + \boldsymbol\varphi\cdot\boldsymbol\tau\right|^2\right. \\
&\qquad\qquad + K_1(\boldsymbol\varphi\cdot\boldsymbol\nu)(\nabla Z + \boldsymbol\varphi)\cdot\boldsymbol\nu \\
&\qquad\qquad \left. + \left(\mu_1\boldsymbol\tau\cdot\frac{\partial\boldsymbol\varphi}{\partial\boldsymbol\tau}\boldsymbol\nu + \boldsymbol\nu\cdot\frac{\partial\boldsymbol\varphi}{\partial\boldsymbol\tau}\boldsymbol\tau\right)\cdot(C[\varepsilon(\boldsymbol\varphi)]\boldsymbol\nu)\right\} d\Sigma \\
&\quad + \int_\Sigma (\mathbf{r}\cdot\boldsymbol\tau)\left\{\frac{\partial\boldsymbol\varphi}{\partial\boldsymbol\tau}\cdot(C[\varepsilon(\boldsymbol\varphi)]\boldsymbol\nu) + K_1\frac{\partial Z}{\partial\boldsymbol\tau}(\nabla Z + \boldsymbol\varphi)\cdot\boldsymbol\nu\right\} d\Sigma. \qquad (4.13)
\end{aligned}$$

One also has

$$\int_Q \boldsymbol\varphi\cdot[\operatorname{div}(C[\varepsilon(\boldsymbol\varphi)]) - K_1(\nabla Z + \boldsymbol\varphi)]\, dQ = \alpha_2(t)|_0^T - I_h\int_Q |\dot{\boldsymbol\varphi}|^2 dQ,$$

$$K_1\int_Q Z\operatorname{div}(\nabla Z + \boldsymbol\varphi)\, dQ = \alpha_3(t)|_0^T - \rho_1\int_Q |\dot Z|^2 dQ,$$

where

$$\alpha_2(t) = I_h\int_\Omega \boldsymbol\varphi\cdot\dot{\boldsymbol\varphi}\, d\Omega,\quad \alpha_3(t) = \rho_1\int_\Omega Z\dot Z\, d\Omega.$$

Use of the last two relations in (4.9) and (4.10) yields

$$\begin{aligned}
\alpha_2(t)|_0^T &+ \int_Q [(C[\varepsilon(\boldsymbol\varphi)], \varepsilon(\boldsymbol\varphi)) - I_h|\dot{\boldsymbol\varphi}|^2]\, dQ \\
&+ K_1\int_Q \boldsymbol\varphi\cdot(\nabla Z + \boldsymbol\varphi)\, dQ = \int_\Sigma \boldsymbol\varphi\cdot(C[\varepsilon(\boldsymbol\varphi)]\boldsymbol\nu)\, d\Sigma, \qquad (4.14)
\end{aligned}$$

and

$$\begin{aligned}
\alpha_3(t)|_0^T &- \rho_1\int_Q |\dot Z|^2 dQ + K_1\int_Q \nabla Z\cdot(\nabla Z + \boldsymbol\varphi)\, dQ \\
&= K_1\int_\Sigma Z(\nabla Z + \boldsymbol\varphi)\cdot\boldsymbol\nu\, d\Sigma, \qquad (4.15)
\end{aligned}$$

respectively.

Let $0 < \delta < 1$. Multiply (4.14) and (4.15) by $1-\delta$ and by δ, respectively, and then add the product to (4.13). We obtain the following expression, where $\alpha = \alpha_1 + (1-\delta)\alpha_2 + \delta\alpha_3$:

$$
\begin{aligned}
\alpha(t)|_0^T + \int_Q \{(1-\delta)\rho_1|\dot{Z}|^2 + \delta I_h|\dot{\boldsymbol{\varphi}}|^2 + (1-\delta)(C[\varepsilon(\boldsymbol{\varphi})], \varepsilon(\boldsymbol{\varphi}))& \\
+ \delta K_1(|\nabla Z|^2 - |\boldsymbol{\varphi}|^2)\} dQ& \\
= \frac{1}{2}\int_\Sigma (\mathbf{r}\cdot\boldsymbol{\nu}) \left[\rho_1|\dot{Z}|^2 + I_h|\dot{\boldsymbol{\varphi}}|^2\right] d\Sigma& \\
+ \frac{1}{2}\int_\Sigma (\mathbf{r}\cdot\boldsymbol{\nu}) \left\{ \frac{1}{D_1}\left[|C_{\boldsymbol{\nu}}(\boldsymbol{\varphi})|^2 + \frac{2}{1-\mu_1}|C_{\boldsymbol{\tau}}(\boldsymbol{\varphi})|^2\right]\right.& \\
\left. + K_1\left|\frac{\partial Z}{\partial\boldsymbol{\nu}} + \boldsymbol{\varphi}\cdot\boldsymbol{\nu}\right|^2\right\} d\Sigma& \\
- \int_\Sigma (\mathbf{r}\cdot\boldsymbol{\nu}) \left\{ \frac{D_1(1-\mu_1^2)}{2}\left|\boldsymbol{\tau}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\tau}}\right|^2 + \frac{K_1}{2}\left|\frac{\partial Z}{\partial\boldsymbol{\tau}} + \boldsymbol{\varphi}\cdot\boldsymbol{\tau}\right|^2\right.& \\
+ K_1(\boldsymbol{\varphi}\cdot\boldsymbol{\nu})(\nabla Z + \boldsymbol{\varphi})\cdot\boldsymbol{\nu}& \\
\left. + \left(\mu_1\boldsymbol{\tau}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\tau}}\boldsymbol{\nu} + \boldsymbol{\nu}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\tau}}\boldsymbol{\tau}\right)\cdot(C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu})\right\} d\Sigma& \\
+ \int_\Sigma (\mathbf{r}\cdot\boldsymbol{\tau}) \left\{ \frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\tau}}\cdot(C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}) + K_1\frac{\partial Z}{\partial\boldsymbol{\tau}}(\nabla Z + \boldsymbol{\varphi})\cdot\boldsymbol{\nu}\right\} d\Sigma& \\
+ \int_\Sigma \{\delta K_1 Z(\nabla Z + \boldsymbol{\varphi})\cdot\boldsymbol{\nu} + (1-\delta)\boldsymbol{\varphi}\cdot(C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu})\} d\Sigma& \\
:= \mathcal{I}_{\Sigma_0} + \mathcal{I}_{\Sigma_1} + \mathcal{I}_{\Sigma_J},&
\end{aligned}
$$

where $\mathcal{I}_{\Sigma_0}$, $\mathcal{I}_{\Sigma_1}$ and $\mathcal{I}_{\Sigma_J}$ denote integrals over $\Sigma_0 := \Gamma_0 \times (0,T)$, $\Sigma_1 := \Gamma_1 \times (0,T)$ and $\Sigma_J := J \times (0,T)$, respectively. If $\boldsymbol{\Xi}$ satisfies (3.16), then

$$
\mathcal{I}_{\Sigma_0} = \frac{1}{2}\int_{\Sigma_0} (\mathbf{r}\cdot\boldsymbol{\nu}) \left\{ \frac{1}{D_1}\left[|C_{\boldsymbol{\nu}}(\boldsymbol{\varphi})|^2 + \frac{2}{1-\mu_1}|C_{\boldsymbol{\tau}}(\boldsymbol{\varphi})|^2\right] + K_1\left|\frac{\partial Z}{\partial\boldsymbol{\nu}}\right|^2\right\} d\Sigma.
$$

When $\boldsymbol{\Xi}$ satisfies the *Dirichlet* boundary conditions (3.19) on Σ_1, then

$$
\mathcal{I}_{\Sigma_1} = \frac{1}{2}\int_{\Sigma_1} (\mathbf{r}\cdot\boldsymbol{\nu}) \left\{ \frac{1}{D_1}\left[|C_{\boldsymbol{\nu}}(\boldsymbol{\varphi})|^2 + \frac{2}{1-\mu_1}|C_{\boldsymbol{\tau}}(\boldsymbol{\varphi})|^2\right] + K_1\left|\frac{\partial Z}{\partial\boldsymbol{\nu}}\right|^2\right\} d\Sigma.
$$

while if Ξ satisfies the *Neumann* boundary conditions (3.20) on Σ_1, then

$$\mathcal{I}_{\Sigma_1} = \frac{1}{2}\int_{\Sigma_1} (\mathbf{r}\cdot\boldsymbol{\nu})\left[\rho_1|\dot{Z}|^2 + I_h|\dot{\boldsymbol{\varphi}}|^2\right] d\Sigma$$
$$-\frac{1}{2}\int_{\Sigma_1} (\mathbf{r}\cdot\boldsymbol{\nu})\left\{D_1(1-\mu_1^2)\left|\boldsymbol{\tau}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\tau}}\right|^2 + K_1\left|\frac{\partial Z}{\partial\boldsymbol{\tau}} + \boldsymbol{\varphi}\cdot\boldsymbol{\tau}\right|^2\right\} d\Sigma.$$

As a consequence, we obtain the following estimates.
(1) If Ξ has regularity (4.11) and satisfies (3.13), (3.16) and (3.19), and if the geometric conditions (3.10) are satisfied, then

$$\alpha(t)|_0^T + \int_Q \{(1-\delta)\rho_1|\dot{Z}|^2 + \delta I_h|\dot{\boldsymbol{\varphi}}|^2 + (1-\delta)(C[\varepsilon(\boldsymbol{\varphi})],\varepsilon(\boldsymbol{\varphi}))$$
$$+\,\delta K_1(|\nabla Z|^2 - |\boldsymbol{\varphi}|^2)\}dQ$$
$$\leq \frac{1}{2}\int_{\Sigma_1}(\mathbf{r}\cdot\boldsymbol{\nu})\left\{\frac{1}{D_1}\left[|C_{\boldsymbol{\nu}}(\boldsymbol{\varphi})|^2 + \frac{2}{1-\mu_1}|C_{\boldsymbol{\tau}}(\boldsymbol{\varphi})|^2\right]\right.$$
$$\left. + K_1\left|\frac{\partial Z}{\partial\boldsymbol{\nu}}\right|^2\right\} d\Sigma + \mathcal{I}_{\Sigma_J}. \quad (4.16)$$

(2) If Ξ has regularity (4.11) and satisfies (3.13), (3.16) and (3.20), and if the geometric conditions (3.10) are satisfied, then

$$\alpha(t)|_0^T + \int_Q \{(1-\delta)\rho_1|\dot{Z}|^2 + \delta I_h|\dot{\boldsymbol{\varphi}}|^2 + (1-\delta)(C[\varepsilon(\boldsymbol{\varphi})],\varepsilon(\boldsymbol{\varphi}))$$
$$+\,\delta K_1(|\nabla Z|^2 - |\boldsymbol{\varphi}|^2)\}dQ$$
$$\leq \frac{1}{2}\int_{\Sigma_1}(\mathbf{r}\cdot\boldsymbol{\nu})\left[\rho_1|\dot{Z}|^2 + I_h|\dot{\boldsymbol{\varphi}}|^2\right] d\Sigma$$
$$-\frac{1}{2}\int_{\Sigma_1}(\mathbf{r}\cdot\boldsymbol{\nu})\left\{D_1(1-\mu_1^2)\left|\boldsymbol{\tau}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\tau}}\right|^2\right.$$
$$\left. + K_1\left|\frac{\partial Z}{\partial\boldsymbol{\tau}} + \boldsymbol{\varphi}\cdot\boldsymbol{\tau}\right|^2\right\} d\Sigma + \mathcal{I}_{\Sigma_J}. \quad (4.17)$$

We proceed to estimate further in (4.16) and (4.17). One has

$$|\nabla Z|^2 - |\boldsymbol{\varphi}|^2 \geq \frac{1}{2}|\nabla Z + \boldsymbol{\varphi}|^2 - 2|\boldsymbol{\varphi}|^2$$

and, since $\Gamma_0 \neq \emptyset$, by Korn's Lemma

$$\int_\Omega |\boldsymbol{\varphi}|^2 d\Omega \leq C(\Omega)\int_\Omega (C[\varepsilon(\boldsymbol{\varphi})],\varepsilon(\boldsymbol{\varphi}))\, d\Omega.$$

We choose and fix $\delta < 1$ so that $2\delta K_1 C(\Omega) = (1-\delta)/2$. We then obtain the estimate

$$
\begin{aligned}
\int_Q \{(1-\delta)\rho_1|\dot{Z}|^2 &+ \delta I_h|\dot{\boldsymbol{\varphi}}|^2 + (1-\delta)(C[\varepsilon(\boldsymbol{\varphi})], \varepsilon(\boldsymbol{\varphi})) \\
&+ \delta K_1(|\nabla Z|^2 - |\boldsymbol{\varphi}|^2)\} dQ \\
\geq & c_0 \int_Q \{\rho_1|\dot{Z}|^2 + I_h|\dot{\boldsymbol{\varphi}}|^2 + (C[\varepsilon(\boldsymbol{\varphi})], \varepsilon(\boldsymbol{\varphi})) + K_1|\nabla Z + \boldsymbol{\varphi}|^2\} dQ \\
= & c_0 \int_0^T (\|\dot{\boldsymbol{\Xi}}(t)\|^2_{\mathcal{H}^0(\Omega)} + \|\boldsymbol{\Xi}(t)\|^2_{\mathcal{H}^1_\gamma(\Omega)})\, dt
\end{aligned}
\tag{4.18}
$$

for some constant $c_0 > 0$.

Let us next estimate the integrals over Σ_J. Assume that $\boldsymbol{\Theta} = \boldsymbol{\theta} + z\mathbf{e}_3$ has regularity

$$
\boldsymbol{\Theta} \in C([0,T]; \mathcal{H}^2(0,\ell)) \cap C^1([0,T]; \mathcal{H}^1(0,\ell)),
\tag{4.19}
$$

and that $\boldsymbol{\Xi}, \boldsymbol{\Theta}$ satisfy the geometric junctions conditions (3.17). We then have the following rough estimates:

$$
\begin{aligned}
-\frac{1}{2}\int_{\Sigma_J} (\mathbf{r}\cdot\boldsymbol{\nu}) \left\{ D_1(1-\mu_1^2)\left|\boldsymbol{\tau}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\tau}}\right|^2 + K_1\left|\frac{\partial Z}{\partial\boldsymbol{\tau}} + \boldsymbol{\varphi}\cdot\boldsymbol{\tau}\right|^2 \right\} d\Sigma \\
\leq C\left(\|\sigma_0\|^2_{H^1(J)} + \|\sigma_1\|^2_{H^1(J)}\right) \int_0^T |\boldsymbol{\Theta}(0,t)|^2 dt;
\end{aligned}
$$

$$
\begin{aligned}
&-K_1\int_{\Sigma_J} (\mathbf{r}\cdot\boldsymbol{\nu})(\boldsymbol{\nu}\cdot\boldsymbol{\varphi})(\nabla Z + \boldsymbol{\varphi})\cdot\boldsymbol{\nu}\, d\Gamma \\
&\qquad \leq \epsilon K_1 \int_{\Sigma_J} |\mathbf{r}\cdot\boldsymbol{\nu}||(\nabla Z + \boldsymbol{\varphi})\cdot\boldsymbol{\nu}|^2 d\Sigma + C_\epsilon \int_J \sigma_1^2\, d\Gamma \int_0^T |\boldsymbol{\theta}(0,t)|^2 dt;
\end{aligned}
$$

where $\epsilon > 0$ is arbitrary,

$$
\begin{aligned}
&-\int_{\Sigma_J} (\mathbf{r}\cdot\boldsymbol{\nu})\left(\mu_1\boldsymbol{\tau}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\tau}}\boldsymbol{\nu} + \boldsymbol{\nu}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\tau}}\boldsymbol{\tau}\right)\cdot(C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu})\, d\Sigma \\
&\qquad \leq \epsilon \int_{\Sigma_J} |\mathbf{r}\cdot\boldsymbol{\nu}||C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}|^2 d\Sigma + C_\epsilon \int_J \left|\frac{\partial\sigma_1}{\partial\boldsymbol{\tau}}\right|^2 d\Gamma \int_0^T |\boldsymbol{\theta}(0,t)|^2 dt;
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\Sigma_J} (\mathbf{r}\cdot\boldsymbol{\tau})\left\{\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\tau}}\cdot(C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}) + K_1\frac{\partial Z}{\partial\boldsymbol{\tau}}(\nabla Z + \boldsymbol{\varphi})\cdot\boldsymbol{\nu}\right\} d\Sigma \\
&\qquad \leq \epsilon \int_{\Sigma_J} |\mathbf{r}\cdot\boldsymbol{\nu}|[|C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}|^2 + K_1|(\nabla Z + \boldsymbol{\varphi})\cdot\boldsymbol{\nu}|^2]\, d\Sigma \\
&\qquad\qquad + C_\varepsilon\left(\|\sigma_0\|^2_{H^1(J)} + \|\sigma_1\|^2_{H^1(J)}\right) \int_0^T |\boldsymbol{\Theta}(0,t)|^2 dt;
\end{aligned}
$$

and

$$\int_{\Sigma_J} \{\delta K_1 Z(\nabla Z + \boldsymbol{\varphi}) \cdot \boldsymbol{\nu} + (1-\delta)\boldsymbol{\varphi} \cdot (C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu})\}\, d\Sigma \\ \le \epsilon \int_{\Sigma_J} |\mathbf{r}\cdot\boldsymbol{\nu}| \{K_1 (\nabla Z + \boldsymbol{\varphi})\cdot\boldsymbol{\nu}|^2 + |C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}|^2\} d\Sigma \\ + C_\epsilon \int_J |\sigma_0|^2 d\Gamma \int_0^T |\boldsymbol{\Theta}(0,t)|^2 dt.$$

It follows that

$$\mathcal{I}_{\Sigma_J} \le \frac{1}{2}\int_{\Sigma_J} (\mathbf{r}\cdot\boldsymbol{\nu}) \left[\rho_1 |\dot{Z}|^2 + I_h |\dot{\boldsymbol{\varphi}}|^2\right] d\Sigma \\ + (C_1 - \epsilon)\int_{\Sigma_J} (\mathbf{r}\cdot\boldsymbol{\nu})[|C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}|^2 + K_1|(\nabla Z + \boldsymbol{\varphi})\cdot\boldsymbol{\nu}|^2]\, d\Sigma \\ + C_\epsilon \int_0^T |\boldsymbol{\Theta}(0,t)|^2 dt, \quad (4.20)$$

for a suitable constant C_1, where C_ϵ depends on ϵ, the physical parameters of the plate, on $\sup_J |\mathbf{r}\cdot\boldsymbol{\nu}|^{-1}$ and on the $H^1(J)$ norms of σ_0 and σ_1.

We substitute (4.18) and (4.20) into (4.16) and (4.17) and obtain the following estimates:

$$\alpha(t)|_0^T + c_0 \int_0^T (\|\dot{\boldsymbol{\Xi}}\|^2_{\mathcal{H}^0(\Omega)} + \|\boldsymbol{\Xi}\|^2_{\mathcal{H}^1_\gamma(\Omega)})\, dt \\ \le C_0 \int_{\Sigma_1} (\mathbf{r}\cdot\boldsymbol{\nu}) \left(|C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}|^2 + K_1 \left|\frac{\partial Z}{\partial \boldsymbol{\nu}}\right|^2\right) d\Sigma \\ + \frac{1}{2}\int_{\Sigma_J} (\mathbf{r}\cdot\boldsymbol{\nu}) \left[\rho_1 |\dot{Z}|^2 + I_h |\dot{\boldsymbol{\varphi}}|^2\right] d\Sigma \\ + (C_1 - \epsilon)\int_{\Sigma_J} (\mathbf{r}\cdot\boldsymbol{\nu})[|C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}|^2 + K_1|(\nabla Z + \boldsymbol{\varphi})\cdot\boldsymbol{\nu}|^2]\, d\Sigma \\ + C_\epsilon \int_0^T |\boldsymbol{\Theta}(0,t)|^2 dt \quad (4.21)$$

in the case of Dirichlet boundary conditions (3.19), where $C_0 = 1/D_1(1-\mu_1)$; and

$$\alpha(t)|_0^T + c_0 \int_0^T (\|\dot{\boldsymbol{\Xi}}\|^2_{\mathcal{H}^0(\Omega)} + \|\boldsymbol{\Xi}\|^2_{\mathcal{H}^1_\gamma(\Omega)})\, dt \\ \le \frac{1}{2}\int_{\Sigma_1} (\mathbf{r}\cdot\boldsymbol{\nu}) \left[\rho_1|\dot{Z}|^2 + I_h|\dot{\boldsymbol{\varphi}}|^2\right] d\Sigma \\ - \frac{1}{2}\int_{\Sigma_1} (\mathbf{r}\cdot\boldsymbol{\nu}) \left\{ E_1 \left|\boldsymbol{\tau}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\tau}}\right|^2 + K_1 \left|\frac{\partial Z}{\partial\boldsymbol{\tau}} + \boldsymbol{\varphi}\cdot\boldsymbol{\tau}\right|^2 \right\} d\Sigma \\ + (C_1 - \epsilon)\int_{\Sigma_J} (\mathbf{r}\cdot\boldsymbol{\nu})[|C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}|^2 + K_1|(\nabla Z + \boldsymbol{\varphi})\cdot\boldsymbol{\nu}|^2]\, d\Sigma \\ + \frac{1}{2}\int_{\Sigma_J} (\mathbf{r}\cdot\boldsymbol{\nu}) \left[\rho_1|\dot{Z}|^2 + I_h|\dot{\boldsymbol{\varphi}}|^2\right] d\Sigma + C_\epsilon \int_0^T |\boldsymbol{\Theta}(0,t)|^2 dt, \quad (4.22)$$

in the case of Neumann boundary conditions (3.20).

We next need to obtain an estimate analogous to (4.22) on the time integral of the total beam energy

$$\mathcal{E}_B(t) = \int_0^\ell (\rho_2|\dot{z}|^2 + I_\alpha|\dot{\theta}_2|^2 + I|\dot{\theta}_1|^2)\, d\xi \\ + \int_0^\ell (GM|\theta_1'|^2 + E_2 M_\beta |\theta_2'|^2 + K_2|z' + \theta_2|^2)\, d\xi.$$

Let $\boldsymbol{\Theta}$ be a solution of (3.14) with regularity (4.19) and denote by $E_B(\xi,t)$ the energy density:

$$\mathcal{E}_B(t) = \int_0^\ell E_B(\xi,t)\, d\xi.$$

It follows from [5, Proposition 3.1] that the following identity holds for $\boldsymbol{\Theta}$:

$$\beta(t)|_0^T + \frac{1}{2}\int_0^T \int_0^\ell [\kappa' E_B - 2\kappa\rho_2\dot{z}\dot{\theta}_2 + 2\kappa K_2\theta_2'(z'+\theta_2)]\, d\xi\, dt \\ = \frac{\kappa(\ell)}{2}\int_0^T E_B(\ell,t)\, dt - \frac{\kappa(0)}{2}\int_0^T E_B(0,t)\, dt,$$

where κ is an arbitrary $C^1[0,\ell]$ function and

$$\beta(t) = \int_0^\ell \kappa[I\dot{\theta}_1\theta_1' + \rho_2\dot{z}(z'+\theta_2) + I_\alpha\dot{\theta}_2\theta_2']\, d\xi.$$

We choose κ so that

$$\kappa \le 0, \quad \kappa' > 0, \quad \kappa^2\rho_2 - (\kappa')^2 I_\alpha < 0, \\ \kappa^2 K_2 - (\kappa')^2 E_2 M_\beta < 0, \quad 0 \le \xi \le \ell.$$

For example, one may choose

$$\kappa(\xi) = -\exp(-k\xi), \quad k > \max(\sqrt{\rho_2/I_\alpha}, \sqrt{K_2/E_2M_\beta}).$$

Then the quadratic forms

$$\kappa'\rho_2 q_1^2 \pm 2\kappa\rho_2 q_1 q_2 + \kappa' I_\alpha q_2^2$$

and

$$\kappa' E_2 M_\beta q_1^2 \pm 2\kappa K_2 q_1 q_2 + \kappa' K_2 q_2^2$$

are uniformly positive definite on $0 \le \xi \le \ell$, hence

$$\begin{aligned}
&\kappa' E_B - 2\kappa\rho_2 \dot{z}\dot{\theta}_2 + 2\kappa K_2\theta_2'(z' + \theta_2) \\
&\qquad = \kappa'[I|\dot{\theta}_1|^2 + GM|\theta_1'|^2] + [\kappa'\rho_2|\dot{z}|^2 - 2\kappa\rho_2\dot{z}\dot{\theta}_2 + \kappa' I_\alpha|\dot{\theta}_2|^2] \\
&\qquad\quad + [\kappa' E_2 M_\beta|\theta_2'|^2 + 2\kappa K_2\theta_2'(z' + \theta_2) + \kappa' K_2|z' + \theta_2|^2 > c_0 E_B
\end{aligned}$$

for some $c_0 > 0$ and the following estimate holds for the solution $\mathbf{\Theta}$:

$$\beta(t)|_0^T + c_0 \int_0^T \mathcal{E}_B(t)\, dt \le -\frac{\kappa(0)}{2} \int_0^T E_B(0,t)\, dt. \tag{4.23}$$

We estimate in the right side of (4.23), using the junction conditions (3.18), as follows:

$$\begin{aligned}
&-\frac{\kappa(0)}{2} \int_0^T E_B(0,t)\, dt \\
&\qquad = -\frac{\kappa(0)}{2} \int_0^T (\rho_2|\dot{z}(0,t)|^2 + I_\alpha|\dot{\theta}_2(0,t)|^2 + I|\dot{\theta}_1(0,t)|^2)\, dt \\
&\quad -\frac{\kappa(0)}{2} \int_0^T (GM|\theta_1'(0,t)|^2 + E_2M_\beta|\theta_2'(0,t)|^2 + K_2|(z' + \theta_2)(0,t)|^2)\, dt \\
&\qquad \le C\left(\int_0^T |\dot{\mathbf{\Theta}}(0,t)|^2 dt + \int_J \sigma_0^2\, d\Gamma \int_{\Sigma_J} |\boldsymbol{\nu}\cdot(\nabla Z + \boldsymbol{\varphi})|^2 d\Sigma\right. \\
&\qquad\qquad\qquad\qquad\qquad \left. + \int_J \sigma_1^2\, d\Gamma \int_{\Sigma_J} |C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}|^2 d\Sigma\right).
\end{aligned} \tag{4.24}$$

We then obtain from (4.23) the estimate

$$\begin{aligned}
&\beta(t)|_0^T + c_0\left(\int_0^T \|\dot{\mathbf{\Theta}}\|^2_{\mathcal{H}^0(0,\ell)} dt\right. \\
&\qquad \left. + \int_0^T \int_0^\ell (GM|\theta_1'|^2 + E_2M_\beta|\theta_2'|^2 + K_2|z' + \theta_2|^2)\, d\xi dt\right) \\
&\le C\left(\int_0^T |\dot{\mathbf{\Theta}}(0,t)|^2 dt + \int_{\Sigma_J} (|C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}|^2 + \boldsymbol{\nu}\cdot(\nabla Z + \boldsymbol{\varphi})|^2)\, d\Sigma\right)
\end{aligned} \tag{4.25}$$

where C depends the physical parameters of the beam and on the $L^2(J)$ norms of σ_0 and σ_1.

Multiply (4.25) by ϵ and add the product to (4.21) and to (4.22) to obtain

$$
\begin{aligned}
&[\alpha(t)+\epsilon\beta(t)]_0^T + c_0\int_0^T \|\dot{\boldsymbol{\Xi}}\|^2_{\mathcal{H}^0(\Omega)}dt \\
&\qquad\qquad + c_0\epsilon\int_0^T \|\dot{\boldsymbol{\Theta}}\|^2_{\mathcal{H}^0(0,\ell)}dt + c_0\int_0^T \|\boldsymbol{\Xi}\|^2_{\mathcal{H}^1_\gamma(\Omega)}dt \\
&\qquad + c_0\epsilon\int_0^T\int_0^\ell (GM|\theta_1'|^2 + E_2M_\beta|\theta_2'|^2 + K_2|z'+\theta_2|^2)\,d\xi\,dt \\
&\quad\le C_0\int_{\Sigma_1}(\mathbf{r}\cdot\boldsymbol{\nu})\left(|C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}|^2 + K_1\left|\frac{\partial Z}{\partial\boldsymbol{\nu}}\right|^2\right))\,d\Sigma \\
&\qquad + \frac{1}{2}\int_{\Sigma_J}(\mathbf{r}\cdot\boldsymbol{\nu})\left[\rho_1|\dot{Z}|^2 + I_h|\dot{\boldsymbol{\varphi}}|^2\right]d\Sigma + \epsilon C\int_0^T|\dot{\boldsymbol{\Theta}}(0,t)|^2dt \\
&\qquad + (C_1-\epsilon-\epsilon C)\int_{\Sigma_J}(\mathbf{r}\cdot\boldsymbol{\nu})[|C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}|^2 + K_1|(\nabla Z+\boldsymbol{\varphi})\cdot\boldsymbol{\nu}|^2]\,d\Sigma \\
&\qquad + C_\epsilon\int_0^T|\boldsymbol{\Theta}(0,t)|^2dt
\end{aligned}
$$

in the case of Dirichlet boundary conditions (3.19) on Σ_1; and

$$
\begin{aligned}
&[\alpha(t)+\epsilon\beta(t)]_0^T + c_0\int_0^T \|\dot{\boldsymbol{\Xi}}\|^2_{\mathcal{H}^0(\Omega)}dt \\
&\qquad\qquad + c_0\epsilon\int_0^T \|\dot{\boldsymbol{\Theta}}\|^2_{\mathcal{H}^0(0,\ell)}dt + c_0\int_0^T \|\boldsymbol{\Xi}\|^2_{\mathcal{H}^1_{\Gamma_0}(\Omega)}dt \\
&\qquad + c_0\epsilon\int_0^T\int_0^\ell (GM|\theta_1'|^2 + E_2M_\beta|\theta_2'|^2 + K_2|z'+\theta_2|^2)\,d\xi\,dt \\
&\quad\le \frac{1}{2}\int_{\Sigma_1}(\mathbf{r}\cdot\boldsymbol{\nu})\left[\rho_1|\dot{Z}|^2 + I_h|\dot{\boldsymbol{\varphi}}|^2\right]d\Sigma + C_\epsilon\int_0^T|\boldsymbol{\Theta}(0,t)|^2dt \\
&\qquad - \frac{1}{2}\int_{\Sigma_1}(\mathbf{r}\cdot\boldsymbol{\nu})\left\{E_1\left|\boldsymbol{\tau}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\tau}}\right|^2 + K_1\left|\frac{\partial Z}{\partial\boldsymbol{\tau}}+\boldsymbol{\varphi}\cdot\boldsymbol{\tau}\right|^2\right\}d\Sigma \\
&\qquad + (C_1-\epsilon-\epsilon C)\int_{\Sigma_J}(\mathbf{r}\cdot\boldsymbol{\nu})[|C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}|^2 + K_1|(\nabla Z+\boldsymbol{\varphi})\cdot\boldsymbol{\nu}|^2]\,d\Sigma \\
&\qquad + \frac{1}{2}\int_{\Sigma_J}(\mathbf{r}\cdot\boldsymbol{\nu})\left[\rho_1|\dot{Z}|^2 + I_h|\dot{\boldsymbol{\varphi}}|^2\right]d\Sigma + \epsilon C\int_0^T|\dot{\boldsymbol{\Theta}}(0,t)|^2dt
\end{aligned}
$$

in the case of Neumann boundary conditions (3.20) on Σ_1. Choose $\epsilon > 0$ so that $C_1 - \epsilon(1+C) \geq 0$. Also, it is possible to choose ϵ so small that

$$\frac{1}{2}\int_{\Sigma_J} (\mathbf{r}\cdot\boldsymbol{\nu})\left[\rho_1|\dot{Z}|^2 + I_h|\dot{\boldsymbol{\varphi}}|^2\right] d\Sigma + \epsilon C\int_0^T |\dot{\boldsymbol{\Theta}}(0,t)|^2 dt \leq 0. \quad (4.26)$$

In fact, by utilizing the geometric junction conditions (3.17) one obtains

$$\begin{aligned}
&\int_{\Sigma_J} (\mathbf{r}\cdot\boldsymbol{\nu})\left[\rho_1|\dot{Z}|^2 + I_h|\dot{\boldsymbol{\varphi}}|^2\right] d\Sigma \\
&\qquad = \int_{\Sigma_J} (\mathbf{r}\cdot\boldsymbol{\nu})\{\rho_1\sigma_0^2|\dot{z}(0,t) + \lambda\dot{\theta}_1(0,t)|^2 \\
&\qquad\qquad + I_h\sigma_1^2(|\dot{\theta}_1(0,t)|^2 + |\dot{\theta}_2(0,t)|^2)\}d\Sigma.
\end{aligned}$$

It is easy to check that the quadratic form

$$\rho_1\sigma_0^2|\dot{z}(0,t) + \lambda\dot{\theta}_1(0,t)|^2 + I_h\sigma_1^2(|\dot{\theta}_1(0,t)|^2 + |\dot{\theta}_2(0,t)|^2)$$

is positive semi-definite on J. Since $\mathbf{r}\cdot\boldsymbol{\nu} < 0$ there,

$$\int_{\Sigma_J}\{\cdots\}d\Sigma \leq \int_{\Sigma_{J_\alpha}}\{\cdots\}d\Sigma,$$

where $\Sigma_{J_\alpha} = J_\alpha \times (0,T)$. On J_α we have $\sigma_i(\mathbf{x}) = 1$ and

$$\mathbf{r}\cdot\boldsymbol{\nu} = (\mathbf{x}_0 + \eta\boldsymbol{\tau}(\mathbf{x}_0) - \hat{\mathbf{x}}_0)\cdot\boldsymbol{\nu}(\mathbf{x}_0) = (\mathbf{x}_0 - \hat{\mathbf{x}}_0)\cdot\boldsymbol{\nu}(\mathbf{x}_0) = \text{constant} < 0.$$

Since

$$\int_{J_\alpha} d\Gamma = \alpha, \quad \int_{J_\alpha}\lambda d\Gamma = 0, \quad \int_{J_\alpha}\lambda^2 d\Gamma = \frac{\alpha^3}{12},$$

we have

$$\begin{aligned}
&\int_{\Sigma_J} (\mathbf{r}\cdot\boldsymbol{\nu})\left[\rho_1|\dot{Z}|^2 + I_h|\dot{\boldsymbol{\varphi}}|^2\right] d\Sigma \\
&\qquad \leq (\mathbf{x}_0 - \hat{\mathbf{x}}_0)\cdot\boldsymbol{\nu}(\mathbf{x}_0)\int_0^T \Big\{\rho_1|\dot{z}(0,t)|^2 \\
&\qquad\qquad + \alpha\left(\rho_1\frac{\alpha^2}{12} + I_h\right)|\dot{\theta}_1(0,t)|^2 + \alpha I_h|\dot{\theta}_2(0,t)|^2\Big\} dt,
\end{aligned}$$

from which (4.26) immediately follows.

We have therefore proved the following estimates: if the initial data satisfy

$$((\boldsymbol{\Xi}^0, \boldsymbol{\Theta}^0), (\boldsymbol{\Xi}^1, \boldsymbol{\Theta}^1)) \in D_A \times V_\gamma,$$

where

$$\gamma = \begin{cases} \Gamma - \overline{J} \text{ for Dirichlet BC on } \Gamma_1, \\ \Gamma_0 \text{ for Neumann BC on } \Gamma_1, \end{cases}$$

and if the hypotheses of Theorem 3.3 are satisfied, the following estimate for the solution of (3.13)–(3.19), (3.21) holds for some positive constants c_0, C_0, C_1:

$$\begin{aligned} &[\alpha(t) + \epsilon\beta(t)]_0^T + c_0 \int_0^T \|((\boldsymbol{\Xi}, \boldsymbol{\Theta}), (\dot{\boldsymbol{\Xi}}, \dot{\boldsymbol{\Theta}})\|^2_{V_\gamma \times H} dt \\ &\qquad \le C_0 \int_{\Sigma_1} (\mathbf{r} \cdot \boldsymbol{\nu}) \left(|C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}|^2 + K_1 \left| \frac{\partial Z}{\partial \boldsymbol{\nu}} \right|^2 \right) d\Sigma \\ &\qquad\qquad + C_1 \int_0^T |\boldsymbol{\Theta}(0,t)|^2 dt. \end{aligned}$$

For the solution of (3.13)–(3.18), (3.20), (3.21) the estimate is

$$\begin{aligned} &[\alpha(t) + \epsilon\beta(t)]_0^T + c_0 \int_0^T \|((\boldsymbol{\Xi}, \boldsymbol{\Theta}), (\dot{\boldsymbol{\Xi}}, \dot{\boldsymbol{\Theta}}))\|^2_{V_{\Gamma_0} \times H} dt \\ &\qquad \le \frac{1}{2} \int_{\Sigma_1} (\mathbf{r} \cdot \boldsymbol{\nu}) \left(\rho_1 |\dot{Z}|^2 + I_h |\dot{\boldsymbol{\varphi}}|^2 \right) d\Sigma \\ &\qquad\quad - \frac{1}{2} \int_{\Sigma_1} (\mathbf{r} \cdot \boldsymbol{\nu}) \left\{ E_1 \left| \boldsymbol{\tau} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{\tau}} \right|^2 + K_1 \left| \frac{\partial Z}{\partial \boldsymbol{\tau}} + \boldsymbol{\varphi} \cdot \boldsymbol{\tau} \right|^2 \right\} d\Sigma \\ &\qquad\quad + C_1 \int_0^T |\boldsymbol{\Theta}(0,t)|^2 dt. \end{aligned}$$

Since the solution is given by a unitary group on $V_\gamma \times H$, we have

$$\int_0^T \|((\boldsymbol{\Xi}, \boldsymbol{\Theta}), (\dot{\boldsymbol{\Xi}}, \dot{\boldsymbol{\Theta}}))\|^2_{V_\gamma \times H} dt = T \|((\boldsymbol{\Xi}^0, \boldsymbol{\Theta}^0), (\boldsymbol{\Xi}^1, \boldsymbol{\Theta}^1))\|^2_{V_\gamma \times H}.$$

Moreover, it is easy to see that

$$\begin{aligned} |\alpha(t) + \epsilon\beta(t)| &\le C \|((\boldsymbol{\Xi}, \boldsymbol{\Theta}), (\dot{\boldsymbol{\Xi}}, \dot{\boldsymbol{\Theta}}))\|^2_{V \times H} \\ &= C \|((\boldsymbol{\Xi}^0, \boldsymbol{\Theta}^0), (\boldsymbol{\Xi}^1, \boldsymbol{\Theta}^1))\|^2_{V_\gamma \times H}. \end{aligned}$$

Therefore, the last two estimates may be replaced by

$$\begin{aligned} &(c_0 T - 2C) \|((\boldsymbol{\Xi}^0, \boldsymbol{\Theta}^0), (\boldsymbol{\Xi}^1, \boldsymbol{\Theta}^1))\|^2_{V_\gamma \times H} \\ &\qquad \le C_0 \int_{\Sigma_1} (\mathbf{r} \cdot \boldsymbol{\nu}) \left(|C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}|^2 + K_1 \left| \frac{\partial Z}{\partial \boldsymbol{\nu}} \right|^2 \right) d\Sigma \\ &\qquad\qquad + C_1 \int_0^T |\boldsymbol{\Theta}(0,t)|^2 dt, \qquad (4.27) \end{aligned}$$

and

$$\begin{aligned}(c_0T-2C)&\|((\mathbf{\Xi}^0,\mathbf{\Theta}^0),(\mathbf{\Xi}^1,\mathbf{\Theta}^1))\|^2_{V_{\Gamma_0}\times H}\\ \le&\frac{1}{2}\int_{\Sigma_1}(\mathbf{r}\cdot\boldsymbol{\nu})\left(\rho_1|\dot Z|^2+I_h|\dot{\boldsymbol{\varphi}}|^2\right)d\Sigma\\ &-\frac{1}{2}\int_{\Sigma_1}(\mathbf{r}\cdot\boldsymbol{\nu})\left\{E_1\left|\boldsymbol{\tau}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\tau}}\right|^2+K_1\left|\frac{\partial Z}{\partial\boldsymbol{\tau}}+\boldsymbol{\varphi}\cdot\boldsymbol{\tau}\right|^2\right\}d\Sigma \quad (4.28)\\ &+C_1\int_0^T|\mathbf{\Theta}(0,t)|^2dt,\end{aligned}$$

respectively. Except for the last term on the right side, (4.27) is the estimate of Propositions 3.1, where $T_0=2C/c_0$. Similarly, when $\mathbf{r}\cdot\boldsymbol{\nu}\ge 0$ on Γ_1, the second integral over Σ_1 on the right side of (4.28) may be dropped and we obtain the estimate

$$\begin{aligned}(c_0T-2C)&\|((\mathbf{\Xi}^0,\mathbf{\Theta}^0),(\mathbf{\Xi}^1,\mathbf{\Theta}^1))\|^2_{V_{\Gamma_0}\times H}\\ &\le\frac{1}{2}\int_{\Sigma_1}(\mathbf{r}\cdot\boldsymbol{\nu})\left(\rho_1|\dot Z|^2+I_h|\dot{\boldsymbol{\varphi}}|^2\right)d\Sigma+C_1\int_0^T|\mathbf{\Theta}(0,t)|^2dt. \quad (4.29)\end{aligned}$$

Therefore, to complete the proofs, it suffices to show that for $T>T_0$,

$$\int_0^T|\mathbf{\Theta}(0,t)|^2dt\le C\int_{\Sigma_1}(\mathbf{r}\cdot\boldsymbol{\nu})\left(|C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}|^2+K_1\left|\frac{\partial Z}{\partial\boldsymbol{\nu}}\right|^2\right)d\Sigma \quad (4.30)$$

in the case of (4.27), and

$$\int_0^T|\mathbf{\Theta}(0,t)|^2dt\le\frac{1}{2}\int_{\Sigma_1}(\mathbf{r}\cdot\boldsymbol{\nu})\left(\rho_1|\dot Z|^2+I_h|\dot{\boldsymbol{\varphi}}|^2\right)d\Sigma \quad (4.31)$$

in the case of (4.29). The proofs of both (4.30) and (4.31) follow along standard lines (see Bardos, Lebeau, Rauch [12, Appendix II] and Zuazua [12, Appendix I]), so we shall only sketch the ideas in the case of (4.30).

One first proves that

$$\begin{aligned}\|((\mathbf{\Xi}^0,\mathbf{\Theta}^0),&(\mathbf{\Xi}^1,\mathbf{\Theta}^1))\|^2_{V_\gamma\times H}\\ &\le C(T)\int_{\Sigma_1}(\mathbf{r}\cdot\boldsymbol{\nu})\left(|C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}|^2+K_1\left|\frac{\partial Z}{\partial\boldsymbol{\nu}}\right|^2\right)d\Sigma\\ &\qquad+\|\mathbf{\Xi}\|^2_{L^\infty(0,T;\mathcal{H}^0(\Omega))} \quad (4.32)\end{aligned}$$

for $T>T_0$ by showing that

$$\begin{aligned}\int_0^T|\mathbf{\Theta}(0,t)|^2dt\le C(T)&\int_{\Sigma_1}(\mathbf{r}\cdot\boldsymbol{\nu})\left(|C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}|^2+K_1\left|\frac{\partial Z}{\partial\boldsymbol{\nu}}\right|^2\right)d\Sigma\\ &+\|\mathbf{\Xi}\|^2_{L^\infty(0,T;\mathcal{H}^0(\Omega))}. \quad (4.33)\end{aligned}$$

Estimate (4.33) may be proved by contradiction, using (4.27) and the compactness of the injection (see Simon [16, Section 8])

$$(\mathbf{\Xi}, \mathbf{\Theta}) \in L^\infty(0,T;V_\gamma) \cap W^{1,\infty}(0,T;H) \mapsto L^\infty(0,T;[V_\gamma, H]_\alpha), \quad 0 < \alpha \le 1,$$

where $[\cdot,\cdot]_\alpha$ denotes the interpolation space of order α.

Let $\mathcal{X}$ be the space of all $(\mathbf{\Xi}, \mathbf{\Theta})$ which satisfy the conditions

$$\left.\begin{aligned} &(\mathbf{\Xi}, \mathbf{\Theta}) \in L^\infty(0,T;V_\gamma) \cap W^{1,\infty}(0,T;H),\\ &(\mathbf{\Xi}, \mathbf{\Theta}) \ \text{ satisfy } (3.13)\text{–}(3.19),\\ &C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu} = 0,\ \frac{\partial Z}{\partial \boldsymbol{\nu}} = 0 \ \text{ on } \Sigma_1, \end{aligned}\right\} \tag{4.34}$$

with norm

$$\|(\mathbf{\Xi}, \mathbf{\Theta})\|_{\mathcal{X}} = \|((\mathbf{\Xi}, \mathbf{\Theta}), (\dot{\mathbf{\Xi}}, \dot{\mathbf{\Theta}}))\|_{L^\infty(0,T;V_\gamma\times H)}.$$

We wish to show that $\mathcal{X} = \{0\}$. This is done in two steps.

(1) One first proves that $\mathcal{X}$ is finite dimensional. This is achieved by using (4.32), (4.34) and conservation of total energy to show that if $(\mathbf{\Xi}, \mathbf{\Theta}) \in \mathcal{X}$ then $(\dot{\mathbf{\Xi}}, \dot{\mathbf{\Theta}}) \in \mathcal{X}$ and the map

$$(\mathbf{\Xi}, \mathbf{\Theta}) \mapsto (\dot{\mathbf{\Xi}}, \dot{\mathbf{\Theta}}) : \mathcal{X} \mapsto \mathcal{X} \tag{4.35}$$

is continuous. Since the injection

$$\{(\mathbf{\Xi}, \mathbf{\Theta}) \in \mathcal{X} \,|\, (\dot{\mathbf{\Xi}}, \dot{\mathbf{\Theta}}) \in \mathcal{X}\} \mapsto \mathcal{X}$$

is compact [16], it follows that $\mathcal{X}$ itself is compact and, therefore, finite dimensional.

(2) Next, one shows that $\mathcal{X} = \{0\}$. In fact, since the map (4.35) is continuous, it has an eigenvalue λ (here we need to work in the complexification of $\mathcal{X}$). In particular, $\dot{\mathbf{\Xi}} = \lambda\mathbf{\Xi}$, so that $\mathbf{\Xi} = \boldsymbol{\varphi} + Z\mathbf{k}$ satisfies

$$\left.\begin{aligned} \rho_1\lambda^2 Z - K_1 \operatorname{div}(\nabla Z + \boldsymbol{\varphi}) = 0,\\ \lambda^2 I_h\boldsymbol{\varphi} - \operatorname{div}(C[\varepsilon(\boldsymbol{\varphi})] + K(\nabla Z + \boldsymbol{\varphi}) = 0, \end{aligned}\right\}$$

$$\boldsymbol{\varphi} = 0, \quad Z = 0, \quad C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu} = 0, \quad \frac{\partial Z}{\partial \boldsymbol{\nu}} = 0 \ \text{ on } \Gamma_1.$$

Since this is a second order elliptic system with constant coefficients and with Cauchy data on Γ_1, we may conclude that $\boldsymbol{\varphi} = 0$ and $Z = 0$.

Finally, one may prove (4.30) by contradiction, using the fact that $\mathcal{X} = \{0\}$.

5 Proofs of Theorems 3.1–3.4

5.1 Proofs of Theorems 3.1 and 3.2

Theorem 3.2 will be proved first. The system (3.1)–(3.6), (3.8) may be written as a variational equation by forming

$$\begin{aligned}0 = \int_\Omega \{[\rho_1 \ddot{W} - K_1 \operatorname{div}(\nabla W + \boldsymbol{\phi})]\hat{W} \\ + [I_h \ddot{\boldsymbol{\phi}} - \operatorname{div} C[\varepsilon(\boldsymbol{\phi})] + K_1(\nabla W + \boldsymbol{\phi})] \cdot \hat{\boldsymbol{\phi}}\} d\Omega \\ + \int_0^\ell \{(\rho_2 \ddot{w} - K_2(w' + \psi_2)')\hat{w} + (I\ddot{\psi}_1 - GM\psi_1'')\hat{\psi}_1 \\ + (I_\alpha \ddot{\psi}_2 - E_2 M_\beta \psi_2'' + K_2(w' + \psi_2))\hat{\psi}_2\} d\xi. \quad (5.1)\end{aligned}$$

Set

$$\hat{\boldsymbol{\Phi}} = \hat{\boldsymbol{\phi}} + \hat{W}\mathbf{k}, \quad \hat{\boldsymbol{\Psi}} = \hat{\psi}_2 \mathbf{e}_1 - \hat{\psi}_1 \mathbf{e}_2 + \hat{w}\mathbf{e}_3 := \hat{\boldsymbol{\psi}} + \hat{w}\mathbf{e}_3,$$

and suppose that $(\hat{\boldsymbol{\Phi}}, \hat{\boldsymbol{\Psi}}) \in V_{\Gamma_0} := V$. Upon carrying out integrations by parts in (5.1) one obtains with the aid of (2.14)

$$((\ddot{\boldsymbol{\Phi}}, \ddot{\boldsymbol{\Psi}}), (\hat{\boldsymbol{\Phi}}, \hat{\boldsymbol{\Psi}}))_H + ((\boldsymbol{\Phi}, \boldsymbol{\Psi}), (\hat{\boldsymbol{\Phi}}, \hat{\boldsymbol{\Psi}}))_V = \int_{\Gamma_1} [\mathbf{m} \cdot \hat{\boldsymbol{\phi}} + f\hat{W}]\, d\Gamma, \quad (5.2)$$

where $\boldsymbol{\Phi} = \boldsymbol{\phi} + W\mathbf{k}$, $\boldsymbol{\Psi} = \psi_2 \mathbf{e}_1 - \psi_1 \mathbf{e}_2 + w\mathbf{e}_3$. We have

$$\left| \int_{\Gamma_1} [\mathbf{m} \cdot \hat{\boldsymbol{\phi}} + f\hat{W}]\, d\Gamma \right| \le \|\mathbf{f}\|_U \|(\hat{\boldsymbol{\Phi}}, \hat{\boldsymbol{\Psi}})\|_V,$$

where $\mathbf{f} = \mathbf{m} + f\mathbf{k}$, so there is an operator $B \in \mathcal{L}(U, V')$ such that

$$\int_{\Gamma_1} [\mathbf{m} \cdot \hat{\boldsymbol{\phi}} + f\hat{W}]\, d\Gamma = \langle B\mathbf{f}, (\hat{\boldsymbol{\Phi}}, \hat{\boldsymbol{\Psi}}) \rangle_V, \quad (5.3)$$

$\langle \cdot, \cdot \rangle_V$ denoting the pairing in the duality between V and V'. Therefore (5.2) may be written

$$(\ddot{\boldsymbol{\Phi}}, \ddot{\boldsymbol{\Psi}}) + A(\boldsymbol{\Phi}, \boldsymbol{\Psi}) = B\mathbf{f}$$

or, alternately, as the first order system

$$\dot{X} = \mathcal{A}X + \mathcal{B}\mathbf{f}, \quad (5.4)$$

where

$$X = \begin{pmatrix} (\boldsymbol{\Phi}, \boldsymbol{\Psi}) \\ (\dot{\boldsymbol{\Phi}}, \dot{\boldsymbol{\Psi}}) \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}, \quad \mathcal{B}\mathbf{f} = \begin{pmatrix} 0 \\ B\mathbf{f} \end{pmatrix}.$$

It is well known that $\mathcal{A}$, as an unbounded operator in $H \times V$ with domain $V \times H$, generates a C_0-unitary group on $H \times V'$. The conclusion of Theorem 3.2 follows immediately, since $\mathcal{B}\mathbf{f} \in L^2(0,T;H \times V')$.

To obtain a variational equation for the Dirichlet control problem (3.1)–(3.7) we again begin with (5.1) where we now assume that $(\hat{\mathbf{\Phi}}, \hat{\mathbf{\Psi}}) \in D_A$. Since $(\hat{\mathbf{\Phi}}, \hat{\mathbf{\Psi}}) = 0$ on $\Gamma - \overline{J}$ we have

$$\begin{aligned}
&\int_\Omega \{-K_1 \operatorname{div}(\nabla W + \boldsymbol{\phi})\hat{W} - [\operatorname{div} C[\varepsilon(\boldsymbol{\phi})] - K_1(\nabla W + \boldsymbol{\phi})] \cdot \hat{\boldsymbol{\phi}}\}\, d\Omega \\
&\quad = \int_\Omega \{-K_1 \operatorname{div}(\nabla \hat{W} + \hat{\boldsymbol{\phi}})W - [\operatorname{div} C[\varepsilon(\hat{\boldsymbol{\phi}})] - K_1(\nabla \hat{W} + \hat{\boldsymbol{\phi}})] \cdot \boldsymbol{\phi}\}\, d\Omega \\
&\qquad\qquad + \int_J \{-K_1 \boldsymbol{\nu} \cdot (\nabla W + \boldsymbol{\phi})\hat{W} + K_1 \boldsymbol{\nu} \cdot (\nabla \hat{W} + \hat{\boldsymbol{\phi}})W \\
&\qquad\qquad\qquad - \hat{\boldsymbol{\phi}} \cdot (C[\varepsilon(\boldsymbol{\phi})]\boldsymbol{\nu}) + \boldsymbol{\phi} \cdot (C[\varepsilon(\hat{\boldsymbol{\phi}})]\boldsymbol{\nu})\}\, d\Gamma \\
&\qquad\qquad\qquad\qquad + \int_{\Gamma_1} [\mathbf{m} \cdot (C[\varepsilon(\hat{\boldsymbol{\phi}})]\boldsymbol{\nu}) + K_1 f \frac{\partial \hat{W}}{\partial \boldsymbol{\nu}}]\, d\Gamma.
\end{aligned}$$

Also

$$\begin{aligned}
&\int_0^\ell \{-K_2(w' + \psi_2)')\hat{w} - GM\psi_1''\hat{\psi}_1 - [E_2 M_\beta \psi_2'' - K_2(w' + \psi_2)]\hat{\psi}_2\} d\xi \\
&= \int_0^\ell \{-K_2(\hat{w}' + \hat{\psi}_2)')w - GM\hat{\psi}_1''\psi_1 - [E_2 M_\beta \hat{\psi}_2'' - K_2(\hat{w}' + \hat{\psi}_2)]\psi_2\} d\xi \\
&\quad + K_2(w' + \psi_2)(0)\hat{w}(0) + GM\psi_1'(0)\hat{\psi}_1(0) + E_2 M_\beta \psi_2'(0)\hat{\psi}_2(0) \\
&\qquad - K_2(\hat{w}' + \hat{\psi}_2)(0)w(0) - GM\hat{\psi}_1'(0)\psi_1(0) - E_2 M_\beta \hat{\psi}_2'(0)\psi_2(0).
\end{aligned}$$

Since $(\mathbf{\Phi}, \mathbf{\Psi})$ and $(\hat{\mathbf{\Phi}}, \hat{\mathbf{\Psi}})$ each satisfy the geometric and dynamic junction conditions, it follows that the sum of the integrals over J with the boundary terms at zero in the last two equations vanishes and therefore (5.1) may be written (see Remark 3.3)

$$\begin{aligned}
&((\ddot{\mathbf{\Phi}}, \ddot{\mathbf{\Psi}}), (\hat{\mathbf{\Phi}}, \hat{\mathbf{\Psi}}))_H + ((\mathbf{\Phi}, \mathbf{\Psi}), A(\hat{\mathbf{\Phi}}, \hat{\mathbf{\Psi}}))_H \\
&\qquad = \int_{\Gamma_1} [\mathbf{m} \cdot (C[\varepsilon(\hat{\boldsymbol{\phi}})]\boldsymbol{\nu}) + K_1 f \frac{\partial \hat{W}}{\partial \boldsymbol{\nu}}]\, d\Gamma. \qquad (5.5)
\end{aligned}$$

Write $(\hat{\mathbf{\Phi}}, \hat{\mathbf{\Psi}}) = A^{-1}(\tilde{\mathbf{\Phi}}, \tilde{\mathbf{\Psi}})$ where $(\tilde{\mathbf{\Phi}}, \tilde{\mathbf{\Psi}}) \in H$. Then (5.5) takes the form

$$\begin{aligned}
&((\ddot{\mathbf{\Phi}}, \ddot{\mathbf{\Psi}}), A^{-1}(\tilde{\mathbf{\Phi}}, \tilde{\mathbf{\Psi}}))_H + ((\mathbf{\Phi}, \mathbf{\Psi}), (\tilde{\mathbf{\Phi}}, \tilde{\mathbf{\Psi}}))_H \\
&\qquad = \int_{\Gamma_1} [\mathbf{m} \cdot (C[\varepsilon(\hat{\boldsymbol{\phi}})]\boldsymbol{\nu}) + K_1 f \frac{\partial \hat{W}}{\partial \boldsymbol{\nu}}]\, d\Gamma. \qquad (5.6)
\end{aligned}$$

One has, for $s > 3/2$,

$$\left| \int_{\Gamma_1} [\mathbf{m} \cdot (C[\varepsilon(\hat{\boldsymbol{\phi}})]\boldsymbol{\nu}) + K_1 f \frac{\partial \hat{W}}{\partial \boldsymbol{\nu}}] \, d\Gamma \right| \le C \|\mathbf{f}\|_U \|\hat{\boldsymbol{\Phi}}\|_{\mathcal{H}^s(\Omega)}$$
$$\le C \|\mathbf{f}\|_U \|(\hat{\boldsymbol{\Phi}}, \hat{\boldsymbol{\Psi}})\|_{D_A}$$
$$= C \|\mathbf{f}\|_U \|(\tilde{\boldsymbol{\Phi}}, \tilde{\boldsymbol{\Psi}})\|_H.$$

Thus

$$\int_{\Gamma_1} [\mathbf{m} \cdot (C[\varepsilon(\hat{\boldsymbol{\phi}})]\boldsymbol{\nu}) + K_1 f \frac{\partial \hat{W}}{\partial \boldsymbol{\nu}}] \, d\Gamma$$
$$= (B\mathbf{f}, (\tilde{\boldsymbol{\Phi}}, \tilde{\boldsymbol{\Psi}}))_H, \ \ (\hat{\boldsymbol{\Phi}}, \hat{\boldsymbol{\Psi}}) = A^{-1}(\tilde{\boldsymbol{\Phi}}, \tilde{\boldsymbol{\Psi}}), \tag{5.7}$$

for some $B \in \mathcal{L}(U, H)$, so that (5.6) may be written

$$((\ddot{\boldsymbol{\Phi}}, \ddot{\boldsymbol{\Psi}}), A^{-1}(\tilde{\boldsymbol{\Phi}}, \tilde{\boldsymbol{\Psi}}))_H + ((\boldsymbol{\Phi}, \boldsymbol{\Psi}), (\tilde{\boldsymbol{\Phi}}, \tilde{\boldsymbol{\Psi}}))_H = (B\mathbf{f}, (\tilde{\boldsymbol{\Phi}}, \tilde{\boldsymbol{\Psi}}))_H,$$
$$\forall (\tilde{\boldsymbol{\Phi}}, \tilde{\boldsymbol{\Psi}}) \in H.$$

This is the same as the equation

$$(\ddot{\boldsymbol{\Phi}}, \ddot{\boldsymbol{\Psi}}) + A(\boldsymbol{\Phi}, \boldsymbol{\Psi}) = AB\mathbf{f} \ \text{ in } D'_A, \tag{5.8}$$

where A is an isomorphism of H onto D'_A (it is the extension to H by continuity of the Riesz isomorphism of V_γ onto V'_γ through the formula

$$\langle Au, v \rangle_{V_\gamma} = (u, Av)_H, \ \ \forall u \in V, \ v \in D_A).$$

Equation (5.8) may be written as (5.4) with

$$\mathcal{B}\mathbf{f} = \begin{pmatrix} 0 \\ AB\mathbf{f} \end{pmatrix}, \ \ \mathcal{B} \in \mathcal{L}(U, V'_\gamma \times D'_A).$$

The operator $\mathcal{A}$, as an unbounded operator in $V'_\gamma \times D'_A$ with domain $H \times V'_\gamma$, generates a C_0-group of unitary operators. Therefore, for initial data

$$(\boldsymbol{\Phi}^0, \boldsymbol{\Psi}^0) \in V'_\gamma, \ \ (\boldsymbol{\Phi}^1, \boldsymbol{\Psi}^1) \in D'_A,$$

the initial value problem for (5.8) has a unique solution with

$$(\boldsymbol{\Phi}, \boldsymbol{\Psi}) \in C([0,T], V'_\gamma) \times C^1([0,T], D'_A).$$

We need to show that if the initial data satisfy the stronger regularity assumptions of Theorem 3.1, then the solution has the regularity stated in that theorem. This result cannot be obtained from abstract semigroup theory but rather is a consequence of the following regularity estimate for solutions of the homogeneous system (3.13)–(3.19).

Lemma 5.1 *Let* $(\mathbf{\Xi}, \mathbf{\Theta})$ *be the solution of* (3.13)–(3.19) *with initial data*

$$(\mathbf{\Xi}^0, \mathbf{\Theta}^0) \in V_\gamma, \ \ (\mathbf{\Xi}^1, \mathbf{\Theta}^1) \in H,$$

where $\gamma = \overline{J}$. *Then*

$$\int_{\Sigma_1} \left[|C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}|^2 + K_1 \left(\frac{\partial Z}{\partial \boldsymbol{\nu}}\right)^2 \right] d\Sigma \\ \leq C_0(T+1)\|((\mathbf{\Xi}^0, \mathbf{\Theta}^0), (\mathbf{\Xi}^1, \mathbf{\Theta}^1))\|^2_{V_\gamma \times H}. \quad (5.9)$$

The proof is deferred until the end of this section.

To complete the proof of Theorem 3.1, we utilize the idea of transposition. Set $V := V_\gamma$. The solution of (5.8) having initial data

$$X^0 = \begin{pmatrix} (\mathbf{\Xi}^0, \mathbf{\Theta}^0) \\ (\mathbf{\Xi}^1, \mathbf{\Theta}^1) \end{pmatrix}$$

is given by

$$X(t) = \exp(t\mathcal{A})X^0 + \int_0^t \exp((t-s)\mathcal{A})\mathcal{B}\mathbf{f}(s)\,ds, \ \ 0 \leq t \leq T,$$

where $\exp(t\mathcal{A})$ is the unitary group on $V' \times D'_A$ generated by $\mathcal{A}$. Suppose that $X^0 \in H \times V'$, let $Y^0 \in V \times D_A$ and $\mathcal{B}' \in \mathcal{L}(V \times D_A; U)$ be the dual of $\mathcal{B}$, defined by

$$\langle \mathcal{B}\mathbf{f}, Y^0 \rangle_{V \times D_A} = (\mathbf{f}, \mathcal{B}'Y^0)_U, \ \ \forall \mathbf{f} \in U, \ Y^0 \in V \times D_A.$$

Let $\tau \in (0, T]$ be fixed. We have

$$\langle X(\tau), Y^0 \rangle_{V \times D_A} = \langle X^0, \exp(\tau\mathcal{A}')Y^0 \rangle_{H \times V} \\ + \int_0^\tau (\mathbf{f}(s), \mathcal{B}' \exp((\tau - s)\mathcal{A}')Y^0)_U\, ds. \quad (5.10)$$

Here $\mathcal{A}'$ is the dual of $\mathcal{A}$, defined by

$$\langle \mathcal{A}X^0, Y^0 \rangle_{V \times D_A} = \langle X^0, \mathcal{A}'Y^0 \rangle_{H \times V}, \ \ \forall X^0 \in H \times V', \ Y^0 \in V \times D_A.$$

One has

$$\mathcal{A}' = \begin{pmatrix} 0 & -A \\ I & 0 \end{pmatrix}, \ \ D(\mathcal{A}') = V \times D_A.$$

As is well-known, $\mathcal{A}'$ generates a unitary group $\exp(t\mathcal{A}')$ on $H \times V$ and $\exp(t\mathcal{A}')$ is the dual of the restriction of $\exp(t\mathcal{A})$ to $H \times V'$. Therefore (5.10) is the same as

$$\langle X(\tau), Y^0 \rangle_{V \times D_A} = \langle X^0, Y(\tau) \rangle_{H \times V} + \int_0^\tau (\mathbf{f}(s), \mathcal{B}'Y(s))_U\, ds, \quad (5.11)$$

where $Y(t) := \exp((\tau - t)\mathcal{A}')Y^0$, $0 \le t \le \tau$, satisfies

$$\dot{Y} = -\mathcal{A}'Y, \ \ 0 \le t \le \tau, \ \ Y(\tau) = Y^0. \tag{5.12}$$

If we write

$$Y = ((\mathbf{\Xi}_1, \mathbf{\Theta}_1), (\mathbf{\Xi}, \mathbf{\Theta})), \ \ Y^0 = ((\mathbf{\Xi}^1, \mathbf{\Theta}^1), (\mathbf{\Xi}^0, \mathbf{\Theta}^0)),$$

(5.12) signifies that

$$\left.\begin{array}{c}(\ddot{\mathbf{\Xi}}, \ddot{\mathbf{\Theta}}) + A(\mathbf{\Xi}, \mathbf{\Theta}) = 0, \ \ (\mathbf{\Xi}_1, \mathbf{\Theta}_1) = -(\dot{\mathbf{\Xi}}, \dot{\mathbf{\Theta}}), \\ (\mathbf{\Xi}(\tau), \mathbf{\Theta}(\tau)) = (\mathbf{\Xi}^0, \mathbf{\Theta}^0), \ \ (\dot{\mathbf{\Xi}}(\tau), \dot{\mathbf{\Theta}}(\tau)) = -(\mathbf{\Xi}^1, \mathbf{\Theta}^1).\end{array}\right\} \tag{5.13}$$

In addition,

$$\langle \mathcal{B}\mathbf{f}, Y\rangle_{V\times D_A} = \langle AB\mathbf{f}, (\mathbf{\Xi}, \mathbf{\Theta})\rangle_{D_A} = (B\mathbf{f}, A(\mathbf{\Xi}, \mathbf{\Theta}))_H.$$

From (5.7) we have

$$(B\mathbf{f}, A(\mathbf{\Xi}, \mathbf{\Theta}))_H = \int_{\Gamma_1} [\mathbf{m}\cdot (C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}) + Kf\frac{\partial Z}{\partial \boldsymbol{\nu}}]\, d\Gamma,$$

where $\mathbf{\Xi} = \boldsymbol{\varphi} + Z\mathbf{k}$. It follows that

$$\mathcal{B}'Y = C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu} + K_1\frac{\partial Z}{\partial \boldsymbol{\nu}}\mathbf{k}\Big|_{\Sigma_1}.$$

We insert this expression into (5.11) to obtain the estimate

$$\begin{aligned}|\langle X(\tau), Y^0\rangle_{V\times D_A}| &\le \|X^0\|_{H\times V'}\|Y^0\|_{H\times V} \\ &\quad + \|\mathbf{f}\|_{L^2(0,T;U)}\|C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu} + K_1\frac{\partial Z}{\partial \boldsymbol{\nu}}\mathbf{k}\|_{L^2(0,T;U)} \\ &\le C_0(T+1)\{\|X^0\|_{H\times V'} + \|\mathbf{f}\|_{L^2(0,T;U)}\}\|Y^0\|_{H\times V}, \ \ 0 \le \tau \le T,\end{aligned}$$

in view of Lemma 5.1. It follows that $X \in L^\infty(0,T; H\times V')$. One may pass from L^∞ to C by a standard argument.

Remark 5.1 That $X \in C([0,T]; H\times V')$ also follows directly from a "lifting theorem" of Lasiecka and Triggiani [9] once Lemma 5.1 is established.

Proof of Lemma 5.1: It suffices to prove (5.9) for initial data in $D_A \times V$. We first use a trick from [14]. Let $\zeta \in C^\infty(\overline{\Omega})$ such that $\zeta = 1$ is a neighborhood of Γ_1 and $\zeta = 0$ in a neighborhood of J, and set $\tilde{\mathbf{\Xi}} = \zeta\mathbf{\Xi}$.

For any $\boldsymbol{\varphi} = \varphi_1 \mathbf{i} + \varphi_2 \mathbf{j}$ with $\varphi_i \in H^2(\Omega)$ we have

$$\begin{aligned}
&\operatorname{div} C[\varepsilon(\zeta\boldsymbol{\varphi})] \\
&= D_1 \begin{pmatrix} \dfrac{\partial}{\partial x_1}(\varepsilon_{11}(\zeta\boldsymbol{\varphi}) + \mu_1\varepsilon_{22}(\zeta\boldsymbol{\varphi})) + \dfrac{\partial}{\partial x_2}(1-\mu_1)\varepsilon_{12}(\zeta\boldsymbol{\varphi}) \\ \dfrac{\partial}{\partial x_1}(1-\mu_1)\varepsilon_{12}(\zeta\boldsymbol{\varphi}) + \dfrac{\partial}{\partial x_2}(\varepsilon_{22}(\zeta\boldsymbol{\varphi}) + \mu_1\varepsilon_{11}(\zeta\boldsymbol{\varphi})) \end{pmatrix} \\
&= \zeta \operatorname{div} C[\varepsilon(\boldsymbol{\varphi})] + 2C[\varepsilon(\boldsymbol{\varphi})]\nabla\zeta + D_1 \frac{1-3\mu_1}{2} \begin{pmatrix} \varphi_{2,2} & -\varphi_{2,1} \\ -\varphi_{1,2} & \varphi_{1,1} \end{pmatrix} \nabla\zeta \\
&\qquad + D_1 \begin{pmatrix} \zeta_{,11} + \dfrac{1-\mu_1}{2}\zeta_{,22} & \dfrac{1+\mu_1}{2}\zeta_{,12} \\ \dfrac{1+\mu_1}{2}\zeta_{,12} & \zeta_{,22} + \dfrac{1-\mu_1}{2}\zeta_{,11} \end{pmatrix} \boldsymbol{\varphi}.
\end{aligned}$$

It follows that $\tilde{\boldsymbol{\Xi}}$ satisfies the equations

$$\left.\begin{aligned}
\rho_1 \ddot{\tilde{Z}} - K_1 \operatorname{div}(\nabla\tilde{Z} + \tilde{\boldsymbol{\varphi}}) &= \tilde{F}, \\
I_h \ddot{\tilde{\boldsymbol{\varphi}}} - \operatorname{div} C[\varepsilon(\tilde{\boldsymbol{\varphi}})] + K_1(\nabla\tilde{Z} + \tilde{\boldsymbol{\varphi}}) &= \tilde{\mathbf{M}},
\end{aligned}\right\} \tag{5.14}$$

where

$$\tilde{F} = -K_1 \nabla\zeta \cdot (\nabla Z + \boldsymbol{\varphi})$$

and

$$\begin{aligned}
\tilde{\mathbf{M}} = &-2C[\varepsilon(\boldsymbol{\varphi})]\nabla\zeta - D_1 \frac{1-3\mu_1}{2} \begin{pmatrix} \varphi_{2,2} & -\varphi_{2,1} \\ -\varphi_{1,2} & \varphi_{1,1} \end{pmatrix} \nabla\zeta \\
&- D_1 \begin{pmatrix} \zeta_{,11} + \dfrac{1-\mu_1}{2}\zeta_{,22} & \dfrac{1+\mu_1}{2}\zeta_{,12} \\ \dfrac{1+\mu_1}{2}\zeta_{,12} & \zeta_{,22} + \dfrac{1-\mu_1}{2}\zeta_{,11} \end{pmatrix} \boldsymbol{\varphi} + K_1(Z\nabla\zeta + \zeta\boldsymbol{\varphi});
\end{aligned}$$

the boundary conditions

$$\tilde{Z} = 0, \quad \tilde{\boldsymbol{\varphi}} = 0 \quad \text{on } \Sigma; \tag{5.15}$$

and the initial conditions

$$\tilde{\boldsymbol{\Xi}}(0) = \zeta\boldsymbol{\Xi}^0, \quad \dot{\tilde{\boldsymbol{\Xi}}}(0) = \zeta\boldsymbol{\Xi}^1.$$

For a solution of (5.14), (5.15) with initial data $(\tilde{\boldsymbol{\Xi}}(0), \dot{\tilde{\boldsymbol{\Xi}}}(0)) \in \mathcal{H}^1_\Gamma(\Omega) \times \mathcal{H}^0(\Omega) := \mathcal{H}$ we have the energy estimate

$$\|(\tilde{\boldsymbol{\Xi}}(t), \dot{\tilde{\boldsymbol{\Xi}}}(t))\|^2_{\mathcal{H}} \le \|(\tilde{\boldsymbol{\Xi}}(0), \dot{\tilde{\boldsymbol{\Xi}}}(0))\|^2_{\mathcal{H}} + C\int_0^t \|\tilde{\mathbf{F}}(t)\|^2_{\mathcal{H}^0(\Omega)} dt,$$

where $\tilde{\mathbf{F}} = \tilde{\mathbf{M}} + \tilde{F}\mathbf{k}$. We shall show that

$$\int_\Sigma \left[|C[\varepsilon(\tilde{\boldsymbol{\varphi}})]\boldsymbol{\nu}|^2 + K_1 \left(\frac{\partial \tilde{Z}}{\partial \boldsymbol{\nu}} \right)^2 \right] d\Sigma$$
$$\leq C \left[\int_0^T \|\tilde{\mathbf{F}}(t)\|^2_{\mathcal{H}^0(\Omega)} dt + (T+1) \|(\tilde{\boldsymbol{\Xi}}, \dot{\tilde{\boldsymbol{\Xi}}})\|^2_{L^\infty(0,T;\mathcal{H})} \right]. \quad (5.16)$$

Since

$$\|(\tilde{\boldsymbol{\Xi}}, \dot{\tilde{\boldsymbol{\Xi}}})\|^2_{L^\infty(0,T;\mathcal{H})} \leq \|(\tilde{\boldsymbol{\Xi}}(0), \dot{\tilde{\boldsymbol{\Xi}}}(0))\|^2_{\mathcal{H}} + C \int_0^T \|\tilde{\mathbf{F}}(t)\|^2_{\mathcal{H}^0(\Omega)} dt$$
$$\leq C \left(\|(\boldsymbol{\Xi}^0, \boldsymbol{\Xi}^1)\|^2_{\mathcal{H}^1_\gamma(\Omega)\times\mathcal{H}^0(\Omega)} + \int_0^T \|\tilde{\mathbf{F}}(t)\|^2_{\mathcal{H}^0(\Omega)} dt \right),$$

$$\int_0^T \|\tilde{\mathbf{F}}(t)\|^2_{\mathcal{H}^0(\Omega)} dt \leq C \int_0^T \|\boldsymbol{\Xi}(t)\|^2_{\mathcal{H}^1_\gamma(\Omega)} dt$$
$$\leq C \int_0^T \|((\boldsymbol{\Xi}(t), \boldsymbol{\Theta}(t)), (\dot{\boldsymbol{\Xi}}(t), \dot{\boldsymbol{\Theta}}(t)))\|^2_{V_\gamma \times H} dt$$
$$= CT \|((\boldsymbol{\Xi}^0, \boldsymbol{\Theta}^0), (\boldsymbol{\Xi}^1, \boldsymbol{\Theta}^1))\|^2_{V_\gamma \times H},$$

and since $\tilde{\boldsymbol{\Xi}} = \boldsymbol{\Xi}$ on Σ_1, Lemma 5.1 follows from (5.16).

The idea leading to (5.16) is standard (cf. [12], for example). One multiplies (5.14) by $\mathbf{h} \cdot \nabla \tilde{Z}$ and $(\nabla \tilde{\boldsymbol{\varphi}})\mathbf{h}$ respectively, where $\mathbf{h}$ is a $W^{1,\infty}$ vector field in Ω such that $\mathbf{h} = \boldsymbol{\nu}$ on Γ, adds the products and integrates the sum over $\Omega \times (0,T)$. One thereby obtains

$$\int_0^T \int_\Omega \{ [\rho_1 \ddot{\tilde{Z}} - K_1 \operatorname{div}(\nabla \tilde{Z} + \tilde{\boldsymbol{\varphi}})] \mathbf{h} \cdot \nabla \tilde{Z}$$
$$+ [I_h \ddot{\tilde{\boldsymbol{\varphi}}} - \operatorname{div} C[\varepsilon(\tilde{\boldsymbol{\varphi}})] + K_1(\nabla \tilde{Z} + \tilde{\boldsymbol{\varphi}})] \cdot ((\nabla \tilde{\boldsymbol{\varphi}})\mathbf{h}) \} \, d\Omega dt$$
$$= \int_0^T \int_\Omega [(\mathbf{h} \cdot \nabla \tilde{Z})\tilde{F} + ((\nabla \tilde{\boldsymbol{\varphi}})\mathbf{h})\tilde{\mathbf{M}}] \, d\Omega dt. \quad (5.17)$$

One has

$$\int_0^T \int_\Omega (\mathbf{h} \cdot \nabla \tilde{Z}) \ddot{\tilde{Z}} \, d\Omega dt = \int_\Omega (\mathbf{h} \cdot \nabla \tilde{Z}) \dot{\tilde{Z}} \, d\Omega |_0^T + \frac{1}{2} \int_0^T \int_\Omega (\operatorname{div} \mathbf{h}) |\dot{\tilde{Z}}|^2 d\Omega dt,$$

$$\int_0^T \int_\Omega (\mathbf{h}\cdot\nabla\tilde{Z})\,\mathrm{div}(\nabla\tilde{Z}+\tilde{\boldsymbol{\varphi}})\,d\Omega dt$$

$$= \int_0^T \int_\Omega \{\mathrm{div}[(\mathbf{h}\cdot\nabla\tilde{Z})\nabla\tilde{Z}] - (\nabla\mathbf{h}\nabla\tilde{Z})\cdot\nabla\tilde{Z}$$

$$-\frac{1}{2}\,\mathrm{div}(\mathbf{h}|\nabla\tilde{Z}|)^2 + \frac{1}{2}(\mathrm{div}\,\mathbf{h})|\nabla\tilde{Z}|^2 + (\mathrm{div}\,\tilde{\boldsymbol{\varphi}})(\mathbf{h}\cdot\nabla\tilde{Z})\}d\Omega dt$$

$$= \int_0^T \int_\Omega \{\frac{1}{2}(\mathrm{div}\,\mathbf{h})|\nabla\tilde{Z}|^2 - (\nabla h\nabla\tilde{Z})\cdot\nabla\tilde{Z}$$

$$+ (\mathrm{div}\,\tilde{\boldsymbol{\varphi}})(\mathbf{h}\cdot\nabla\tilde{Z})\}d\Omega dt + \frac{1}{2}\int_\Sigma \left(\frac{\partial\tilde{Z}}{\partial\boldsymbol{\nu}}\right)^2 d\Sigma,$$

$$\int_0^T \int_\Omega \ddot{\tilde{\boldsymbol{\varphi}}}\cdot(\nabla\tilde{\boldsymbol{\varphi}}\mathbf{h})d\Omega dt = \int_\Omega \dot{\tilde{\boldsymbol{\varphi}}}\cdot(\nabla\tilde{\boldsymbol{\varphi}}\mathbf{h})d\Omega|_0^T + \frac{1}{2}\int_0^T \int_\Omega (\mathrm{div}\,\mathbf{h})|\dot{\tilde{\boldsymbol{\varphi}}}|^2 d\Omega dt.$$

Also, from Lemma 4.2 we have

$$\int_0^T \int_\Omega (\nabla\tilde{\boldsymbol{\varphi}}\mathbf{h})\cdot\mathrm{div}\,C[\varepsilon(\tilde{\boldsymbol{\varphi}})]\,d\Omega dt = -\frac{1}{2}\int_\Sigma (C[\varepsilon(\tilde{\boldsymbol{\varphi}})],\varepsilon(\tilde{\boldsymbol{\varphi}}))\,d\Sigma$$

$$+ \int_\Sigma (\nabla\tilde{\boldsymbol{\varphi}}\mathbf{h})\cdot(C[\varepsilon(\tilde{\boldsymbol{\varphi}})]\boldsymbol{\nu})\,d\Sigma - \int_0^T \int_\Omega Q(\nabla\tilde{\boldsymbol{\varphi}})\,d\Omega dt.$$

When the last four formulas are inserted into (5.17) the result is

$$\int_\Omega \{\rho_1\dot{\tilde{Z}}(\mathbf{h}\cdot\nabla\tilde{Z}) + I_h\dot{\tilde{\boldsymbol{\varphi}}}\cdot(\nabla\tilde{\boldsymbol{\varphi}}\mathbf{h})\}d\Omega\Big|_0^T$$

$$+ \frac{1}{2}\int_0^T \int_\Omega (\mathrm{div}\,\mathbf{h})\{\rho_1|\dot{\tilde{Z}}|^2 + I_h|\dot{\tilde{\boldsymbol{\varphi}}}|^2\}d\Omega dt$$

$$+ \int_0^T \int_\Omega \{Q(\nabla\tilde{\boldsymbol{\varphi}}) + K_1[-\frac{1}{2}(\mathrm{div}\,\mathbf{h})|\nabla\tilde{Z}|^2 + (\nabla\mathbf{h}\nabla\tilde{Z})\cdot\nabla\tilde{Z}$$

$$-(\mathrm{div}\,\tilde{\boldsymbol{\varphi}})(\mathbf{h}\cdot\nabla\tilde{Z}) + (\nabla\tilde{Z}+\tilde{\boldsymbol{\varphi}})\cdot((\nabla\tilde{\boldsymbol{\varphi}})\mathbf{h})]\}d\Omega dt$$

$$-\int_\Sigma \left\{\frac{K_1}{2}\left(\frac{\partial\tilde{Z}}{\partial\boldsymbol{\nu}}\right)^2 + (\nabla\tilde{\boldsymbol{\varphi}}\mathbf{h})\cdot(C[\varepsilon(\tilde{\boldsymbol{\varphi}})]\boldsymbol{\nu}) - \frac{1}{2}(C[\varepsilon(\tilde{\boldsymbol{\varphi}})],\varepsilon(\tilde{\boldsymbol{\varphi}}))\right\}d\Sigma$$

$$= \int_0^T \int_\Omega [\tilde{F}(\mathbf{h}\cdot\nabla\tilde{Z}) + \tilde{\mathbf{M}}\cdot(\nabla\tilde{\boldsymbol{\varphi}}\mathbf{h})]\,d\Omega dt.$$

On Σ on has (see (4.5)–(4.7))

$$\nabla\tilde{\boldsymbol{\varphi}}\mathbf{h} = \frac{\partial\tilde{\boldsymbol{\varphi}}}{\partial\boldsymbol{\nu}},\quad C[\varepsilon(\tilde{\boldsymbol{\varphi}})]\boldsymbol{\nu} = D_1\left[\left(\boldsymbol{\nu}\cdot\frac{\partial\tilde{\boldsymbol{\varphi}}}{\partial\boldsymbol{\nu}}\right)\boldsymbol{\nu} + \frac{1-\mu_1}{2}\left(\boldsymbol{\tau}\cdot\frac{\partial\tilde{\boldsymbol{\varphi}}}{\partial\boldsymbol{\nu}}\right)\boldsymbol{\tau}\right],$$

$$
\begin{aligned}
(C[\varepsilon(\tilde{\boldsymbol{\varphi}})],\varepsilon(\tilde{\boldsymbol{\varphi}})) &= \frac{1}{D_1}\left[|C_{\boldsymbol{\nu}}(\tilde{\boldsymbol{\varphi}})|^2+\frac{2}{1-\mu_1}|C_{\boldsymbol{\tau}}(\tilde{\boldsymbol{\varphi}})|^2\right]\\
&= D_1\left[\left(\boldsymbol{\nu}\cdot\frac{\partial\tilde{\boldsymbol{\varphi}}}{\partial\boldsymbol{\nu}}\right)\boldsymbol{\nu}+\frac{1-\mu_1}{2}\left(\boldsymbol{\tau}\cdot\frac{\partial\tilde{\boldsymbol{\varphi}}}{\partial\boldsymbol{\nu}}\right)\boldsymbol{\tau}\right].
\end{aligned}
$$

Therefore, on Σ

$$
\begin{aligned}
&(\nabla\tilde{\boldsymbol{\varphi}}\mathbf{h})\cdot(C[\varepsilon(\tilde{\boldsymbol{\varphi}})]\boldsymbol{\nu})-\frac{1}{2}(C[\varepsilon(\tilde{\boldsymbol{\varphi}})],\varepsilon(\tilde{\boldsymbol{\varphi}}))\\
&\qquad\qquad= \frac{D_1}{2}\left[\left(\boldsymbol{\nu}\cdot\frac{\partial\tilde{\boldsymbol{\varphi}}}{\partial\boldsymbol{\nu}}\right)^2+\frac{1-\mu_1}{2}\left(\boldsymbol{\tau}\cdot\frac{\partial\tilde{\boldsymbol{\varphi}}}{\partial\boldsymbol{\nu}}\right)^2\right]\\
&\qquad\qquad\geq \frac{1}{2D_1}|C[\varepsilon(\tilde{\boldsymbol{\varphi}})]\boldsymbol{\nu}|^2.
\end{aligned}
$$

In addition,

$$
\begin{aligned}
&\left|\int_0^T\int_\Omega\{\tilde{F}(\mathbf{h}\cdot\nabla\tilde{Z})+\tilde{\mathbf{M}}\cdot(\nabla\tilde{\boldsymbol{\varphi}}\mathbf{h})\}\,d\Omega dt\right|\\
&\qquad\qquad\leq C\int_0^T(\|\tilde{\mathbf{F}}\|^2_{\mathcal{H}^0(\Omega)}+\|\tilde{\boldsymbol{\Xi}}\|^2_{\mathcal{H}^1_\Gamma(\Omega)})\,dt,\\
&\int_\Omega\{\rho_1\dot{\tilde{Z}}(\mathbf{h}\cdot\nabla\tilde{Z})+I_h\dot{\tilde{\boldsymbol{\varphi}}}\cdot(\nabla\tilde{\boldsymbol{\varphi}}\mathbf{h})\}d\Omega\Big|_0^T\leq C\|(\tilde{\boldsymbol{\Xi}},\dot{\tilde{\boldsymbol{\Xi}}})\|^2_{L^\infty(0,T;\mathcal{H})},\\
&\left|\int_0^T\int_\Omega(\operatorname{div}\mathbf{h})\{\rho_1|\dot{\tilde{Z}}|^2+I_h|\dot{\tilde{\boldsymbol{\varphi}}}|^2\}d\Omega dt\right|\leq\|\mathbf{h}\|_{W^{1,\infty}(\Omega)}\int_0^T\|\dot{\tilde{\boldsymbol{\Xi}}}\|^2_{\mathcal{H}^0(\Omega)}dt,\\
&\left|\int_0^T\int_\Omega\{Q(\nabla\tilde{\boldsymbol{\varphi}})+K_1[-\frac{1}{2}(\operatorname{div}\mathbf{h})|\nabla\tilde{Z}|^2+(\nabla\mathbf{h}\nabla\tilde{Z})\cdot\nabla\tilde{Z}\right.\\
&\qquad\left.-(\operatorname{div}\tilde{\boldsymbol{\varphi}})(\mathbf{h}\cdot\nabla\tilde{Z})+(\nabla\tilde{Z}+\tilde{\boldsymbol{\varphi}})\cdot((\nabla\tilde{\boldsymbol{\varphi}})\mathbf{h})]\}d\Omega dt\right|\\
&\qquad\qquad\leq C\int_0^T\|\tilde{\boldsymbol{\Xi}}\|^2_{\mathcal{H}^1_\Gamma(\Omega)}\,dt.
\end{aligned}
$$

It follows from the above estimates that

$$
\begin{aligned}
&\int_\Sigma\left[|C[\varepsilon(\tilde{\boldsymbol{\varphi}})]\boldsymbol{\nu}|^2+K_1\left(\frac{\partial\tilde{Z}}{\partial\boldsymbol{\nu}}\right)^2\right]d\Sigma\\
&\quad\leq C\left[\int_0^T\left(\|\tilde{\mathbf{F}}(t)\|^2_{\mathcal{H}^0(\Omega)}+\|(\tilde{\boldsymbol{\Xi}},\dot{\tilde{\boldsymbol{\Xi}}})\|^2_{\mathcal{H}}\right)dt+\|(\tilde{\boldsymbol{\Xi}},\dot{\tilde{\boldsymbol{\Xi}}})\|^2_{L^\infty(0,T;\mathcal{H})}\right]\\
&\qquad\leq C\left[\int_0^T\|\tilde{\mathbf{F}}(t)\|^2_{\mathcal{H}^0(\Omega)}dt+(T+1)\|(\tilde{\boldsymbol{\Xi}},\dot{\tilde{\boldsymbol{\Xi}}})\|^2_{L^\infty(0,T;\mathcal{H})}\right].
\end{aligned}
$$

5.2 Proofs of Theorems 3.3 and 3.4

We begin with the proof of Theorem 3.3. Consider the control-to-state map

$$S_T\mathbf{f} = \int_0^T \exp((T-s)\mathcal{A})\mathcal{B}\mathbf{f}(s)\,ds.$$

We have already proved that $S_T \in \mathcal{L}(L^2(0,T;U), H\times V'_\gamma)$ for each $T>0$. Since

$$\langle S_T\mathbf{f}, Y^0\rangle_{H\times V_\gamma} = \int_0^T (\mathbf{f}(s), \mathcal{B}'Y(s))_U\,ds, \tag{5.18}$$

the dual map $S'_T : H\times V_\gamma \mapsto L^2(0,T;U)$ is given by $S'_T Y^0 = \mathcal{B}'Y(\cdot)$, where $Y = ((\boldsymbol{\Xi}_1, \boldsymbol{\Theta}_1), (\boldsymbol{\Xi}, \boldsymbol{\Theta}))$ satisfies (5.13) with $\tau = T$. Therefore $\mathrm{Range}(S_T) = H\times V'_\gamma$ is equivalent to

$$\|\mathcal{B}'Y(\cdot)\|_{L^2(0,TU)} \ge c_0\|Y^0\|_{H\times V_\gamma}, \;\; \forall Y^0 \in H\times V_\gamma, \tag{5.19}$$

for some $c_0>0$. From the proof of Theorem 3.1, it is seen that (5.19) is equivalent to showing that

$$\int_{\Sigma_1} \{|C[\varepsilon(\boldsymbol{\varphi})]\boldsymbol{\nu}|^2 + K_1\left|\frac{\partial Z}{\partial \boldsymbol{\nu}}\right|^2\}d\Sigma \ge c_0\|((\boldsymbol{\Xi}^0, \boldsymbol{\Theta}^0), (\boldsymbol{\Xi}^1, \boldsymbol{\Theta}^1))\|^2_{V_\gamma\times H}.$$

The last estimate follows from Proposition 3.1 for T large enough.

The proof of Theorem 3.4 is similar. In this case one again has (5.18), where now $\mathcal{B}'$ is given by (see (5.3)) $\mathcal{B}'Y = \boldsymbol{\Xi}|_{\Sigma_1} = \boldsymbol{\varphi} + Z\mathbf{k}|_{\Sigma_1}$. However, (5.19) will not hold in general, so to prove Theorem 3.4 we use the *Hilbert Uniqueness Method* introduced by Lions [11]. One defines a *norm*

$$\|Y^0\|_{\mathcal{F}} = \|\mathcal{B}'Y(\cdot)\|_{L^2(0,T;U)}, \;\; Y^0 \in H\times V_{\Gamma_0},$$

and a space $\mathcal{F}$ which is the completion of $H\times V_{\Gamma_0}$ in the $\mathcal{F}$ norm. Then S'_T is an isometry of $\mathcal{F}$ onto $L^2(0,T;U)$. Since

$$\langle S_T S'_T Z^0, Y^0\rangle_{\mathcal{F}} = (Z^0, Y^0)_{\mathcal{F}},$$

$S_T S'_T$ is the Riesz isomorphism of $\mathcal{F}$ onto $\mathcal{F}'$, the dual space of $\mathcal{F}$. Therefore $R_T = \mathcal{F}'$ for $T>0$. On the other hand, according to Corollary 3.1 one has $\mathcal{F}\subset V'_{\Gamma_0}\times H$ for T large enough, and therefore $V_{\Gamma_0}\times H \subset \mathcal{F}'$ for T large enough. If $X^0 \in R_T$ and $Y^0 = (S_T S'_T)^{-1}X^0$, then $\mathbf{f} = S'_T Y^0$ is the control of minimum norm in $L^2(0,T;U)$ for which $S_T\mathbf{f} = X^0$.

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