Journal of Mathematical Systems, Estimation, and Control
Vol. 4, No. 2, 1994, pp. 1–25

Extended Controller Form and Invariants of Nonlinear Control Systems with a Single Input*

Wei Kang

Abstract

In this paper, a normal form for nonlinear control systems with a single-input by studying its k-jet is found. It is a generalization of linear Brunovsky form to nonlinear control systems. A set of invariants which characterizes the homogeneous parts of a control system is also found. Then the problem of convergence and uniqueness of the normal forms is studied.

Key words: nonlinear systems, extended controller form, invariants

AMS Subject Classifications: 93C10, 93C15

1 Introduction

A fruitful technique for many dynamic systems and control systems consists of transforming them into a simple form. Normal forms for nonlinear dynamic systems have been studied since the end of last century, which was first addressed by H. Poincaré. Since then, Poincaré's technique has been successfully applied to the research in the area of nonlinear vector fields, Hamilton dynamic systems, nonlinear mappings and bifurcation phenomenon (see, for instance, Arnold [1], Siegel [19], Bruno [6] and Baider [2]). In linear control theory, different kinds of normal forms was found, in which the Brunovsky form is useful for the purpose of the design of control laws. During seventies and eighties, the problem of how to transform a nonlinear control system into a Brunovsky form by a change of coordinates and state feedback was studied by many authors (see Krener [16]). Unfortunately, most nonlinear controllable systems do not admit a Brunovsky form and even when one does, the transformation of a system into the Brunovsky form involves solving a system of first order linear partial differential equations which can be numerically quite difficult. Therefore, it

*Received November 3, 1992; received in final form April 19, 1993. Summary appeared in Volume 4, Number 2, 1994.

is natural to ask the following questions: If a nonlinear system does not admit a Brunovsky form, what kind of simple form (normal form) it can have under changes of coordinates and state feedback? Is there a simple numerical method of finding the change of coordinates and state feedback which transforms a given nonlinear system into its normal form? What are the invariants of a nonlinear system under changes of coordinates and feedback? These questions have been partially answered by A. J. Krener and I in our earlier research, yet there are many interesting problems left unsolved (see Kang-Krener [14], Krener [15]).

In this paper, we try to answer the questions mentioned above for nonlinear systems with a single input. The idea is to apply Poincaré's technique to nonlinear control systems. The normal form given in §2 has a linear part in Brunovsky form and some simple nonlinear terms. The transformation which carries a nonlinear system to its normal form can be found by solving a set of algebraic equations, which is numerically much simpler than solving partial differential equations. In fact, a change of coordinates and feedback which transforms the homogeneous part of certain degree of a control system into that of another control system (not necessarily in normal form) can be found by solving a set of algebraic equations, which are similar to the homological equations of Poincaré for nonlinear vector fields. In §3, A set of homogeneous polynomials associated to each system is found, which is proved to be invariant under homogeneous changes of coordinates and feedback. In fact, the dth homogeneous part of one system can be transformed into that of another if and only if they have the same d-invariants. In other words, the d-invariants characterize the homogeneous parts of a control system under the transformations of homogeneous changes of coordinates and feedback. Furthermore, the d-invariants of a system in the normal form turn out to be equal to some partial derivatives of the nonlinear terms in the normal form. Since the normal form of homogeneous parts of all degree is found, it is necessary to study the property of convergence as degree approaches to infinity. This problem is believed to be more difficult than the problem of finding normal forms for homogeneous terms. The similar problem for the nonlinear vector fields was studied for many years and it is not completely solved yet. Some nice theorems by Poincaré, Dulac and Siegel claimed the convergence of the normal forms and the changes of coordinates if the invariants of a vector field, i.e. the eigenvalues of the linearization, satisfy certain conditions. But the eigenvalues (poles) of the linearized control system are not invariant under linear feedback, therefore the nature of the problem is quite different for control systems. The result proved in §4 is that if an analytic nonlinear control system can be linearized to any degree then it can be linearized by an analytic change of coordinates and feedback. For certain kind of three dimensional system, we can also prove the property of convergence. In §5, examples are given which shows that the normal form of certain degree of a fixed control system is not unique under changes of coordinates and feedback of lower degree. This phenomenon is similar to that of the normal forms of nonlinear vector fields. In fact, the normal forms of formal vector fields found by Poincaré are not uniquely determined by the original vector field. The problems of convergence and uniqueness of the normal forms of vector fields has been studied for a long time by many authors (see Van der Meer [20] or Wiggins [21]) and there are still many problems left open.

2 Extended Normal Forms

From Brunovsky [7] and Kailath [13], we know that any controllable linear system can be transformed into a controller form by a linear change of coordinates. If, in addition, we also allow linear changes of coordinates in the input space and linear state feedback, any controllable linear system can be transformed into a Brunovsky form. Therefore, under linear changes of coordinates and linear state feedback, Brunovsky form is the normal form of controllable linear systems. In this paper, we try to extend this result to the following nonlinear control systems in which the control μ enters the dynamics in a linear fashion:

$$\dot{\xi} = f(\xi) + g(\xi)\mu \tag{1}$$

where $\xi \in \Re^n$, $\mu \in \Re^1$, $f(\xi)$ and $g(\xi)$ are n-dimensional vector fields. We assume that $\xi = 0$ is an equilibrium point of (1), i.e

$$f(0) = 0. \tag{2}$$

In this paper, we only consider the control systems with a single-input. In the following, the Taylor expansion of the system (1) is frequently used:

$$\begin{aligned} \dot{\xi} &= F\xi + G\mu + \sum_{k=2}^{\infty} \left(f^{[k]}(\xi) + g^{[k-1]}(\xi)\mu \right) \\ F &= \frac{\partial f}{\partial \xi}(0) \quad G = g(0), \end{aligned} \tag{3}$$

where the upper index means that the entries of the vectors are homogeneous polynomials of degree k. This kind of upper index is also applied to other vector fields and polynomials (e.g. $\alpha^{[2]}, \phi^{[k]}$). As mentioned before, the Brunovsky form is the normal form of linear controllable systems. Therefore, it is the normal form of the linear part of the system (3) if (F, G) is controllable. In Kang-Krener [14], the normal forms of the quadratic part was found. This is the starting point of finding the normal forms for higher degree terms. In the following, we give the normal forms for the homogeneous parts of degree larger than two. To avoid the problem of convergence at this moment, we consider the system (3) as a formal nonlinear control system with a single input, i.e, the summation is a formal sum, it may or may not be convergent. The analytic case will be discussed in §4.

Definition 1 *The system (1) is said to be linearly controllable if its linear part (F, G) is controllable as a linear control system.*

In this paper, we always assume that a nonlinear control system or a formal nonlinear control system is linearly controllable. As mentioned above, there exists a linear change of coordinates and a linear state feedback which transform a linearly controllable nonlinear system (or formal system) into a system (or formal system) such that its linear part is in Brunovsky form, i.e,

$$\dot{\xi} = A\xi + B\mu + \sum_{k=2}^{\infty} \left(f^{[k]}(\xi) + g^{[k-1]}(\xi)\mu \right), \tag{4}$$

where the linear part (A, B) is in the following form:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{n\times n} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}. \tag{5}$$

Since the linear part of (4) is already in the normal form, we want to leave it invariant under the change of coordinates and feedback. Therefore, the transformation group used in this paper is the group of all formal transformations such as

$$\begin{aligned} \xi &= x + \sum_{k=2}^{\infty} \phi^{[k]}(x) \\ \nu &= \mu + \sum_{k=2}^{\infty} \alpha^{[k]}(\xi) + \sum_{k=2}^{\infty} \beta^{[k-1]}(\xi)\mu. \end{aligned} \tag{6}$$

The following theorem gives us the normal form of a formal nonlinear control system with a single input. Its proof and the proof of Theorem 2 will be given after the proof of Theorem 3. In the present paper, the results are proved using a more intuitive method, which is different from the proofs in [14].

Theorem 1 *By a formal change of coordinates and feedback (6), the formal control system (4) can be transformed into a system in the following normal form:*

$$
\begin{aligned}
\dot{x}_1 &= x_2 + \sum_{j=3}^{n} P_{1j}(x_1, \cdots, x_j) x_j^2 \\
\dot{x}_2 &= x_3 + \sum_{j=4}^{n} P_{2j}(x_1, \cdots, x_j) x_j^2 \\
&\vdots \\
\dot{x}_i &= x_{i+1} + \sum_{j=i+2}^{n} P_{ij}(x_1, \cdots, x_j) x_j^2 \\
&\vdots \\
\dot{x}_{n-2} &= x_{n-1} + P_{n-2n}(x_1, \cdots, x_n) x_n^2 \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= \nu.
\end{aligned}
\tag{7}
$$

Definition 2 *A control system such as (7) is said to be in extended controller form.*

Remark 1 Here $P_{ij}(x_1, \cdots, x_j)$ is a formal polynomial. It is defined by the formal summation

$$
P_{ij}(x_1, \cdots, x_j) = \sum_{k=0}^{\infty} P_{ij}^{[k]}(x_1, \cdots, x_j). \tag{8}
$$

This normal form is a natural generalization of the quadratic normal form in Kang-Krener [14]. The coefficients λ_{ij} there are replaced by the formal polynomials $P_{ij}(x_1, \cdots, x_j)$.

Remark 2 In the theory of vector field, the first kth terms in the Taylor expansion of a vector field is called a k-jet. Therefore, the extended controller form is the normal form of k-jets, $k \geq 1$, for nonlinear control systems.

To investigate the local property of a nonlinear system, higher degree terms of a system is not important. So, sometimes it is enough to transform only lower degree terms of a system into their normal forms and ignore the higher degree terms. The following theorem shows that homogeneous terms of different degree can be transformed into there normal forms separately by homogeneous transformations of the same degree. A (homogeneous) *d-transformation* is defined by

$$
\begin{aligned}
\xi &= x + \phi^{[d]}(x) \\
\nu &= \mu + \alpha^{[d]}(\xi) + \beta^{[d-1]}(\xi)\mu.
\end{aligned}
\tag{9}
$$

A d-transformation leaves all the homogeneous parts of a nonlinear system of degree less than d invariant. So, given a nonlinear system, we can transform the linear part into Brunovsky form at first; then transform its quadratic part into its extended controller form while leaving the linear Brunovsky form invariant; then transform the cubic part into its extended controller form leaving the linear and quadratic part invariant. Continuing in this way until we obtain the normal form of a desired degree. Therefore, the following theorem implies that if terms with degree greater than or equal to $d+1$ is omitted, every linearly controllable nonlinear system can be transformed into its extended controller form (to degree d) by a transformation contains terms of at most degree d. In the following, We use $O(x,\mu)^{d+1}$ to represent the summation of homogeneous parts of at least degree $d+1$.

Theorem 2 *By a suitable d-transformation (9) the system (4) can be transformed into*

$$\dot{x} = Ax + B\nu + \sum_{k=2}^{d-1} \left(f^{[k]}(x) + g^{[k-1]}(x)\nu \right) + \tilde{f}^{[d]}(x) + O(x,\mu)^{d+1} \tag{10}$$

where $\tilde{f}^{[d]}(x)$ is in the extended controller form, i.e,

$$\tilde{f}_i^{[d]}(x) = \sum_{j=i+2}^{n} P_{ij}^{[d-2]}(x_1, \cdots, x_j)x_j^2. \tag{11}$$

The nonlinear term $\tilde{f}^{[d]}(x)$ is uniquely determined by $f^{[d]}$ and $g^{[d-1]}$.

A homogeneous change of coordinates and homogeneous feedback of degree d is needed to transform the dth homogeneous part of a nonlinear control system into its normal form. But how to find such a change of coordinates and feedback? We can ask this question in a more general way, i.e, if two systems have the same linearization (not necessarily in Brunovsky form) and the same homogeneous terms of degree less than d, if the dth homogeneous part of one system can be transformed into that of another by a d-transformation, how to find the d-transformation? The advantage of Poincaré's technique is that instead of solving a set of off line partial differential equation to find out the control law, which is a standard method for many control problems, we only solve a set of linear algebraic equations to find the desired change of coordinates and feedback.

Theorem 3 *The system*

$$\dot{\xi} = F\xi + G\mu + \sum_{k=2}^{d-1} \left(f^{[k]}(\xi) + g^{[k-1]}(\xi)\mu \right) + f^{[d]}(\xi) + g^{[d-1]}(\xi)\mu + O(\xi,\mu)^{d+1} \tag{12}$$

can be transformed into

$$\dot{x} = Fx + G\nu + \sum_{k=2}^{d-1}\left(f^{[k]}(x) + g^{[k-1]}(x)\nu\right) + \bar{f}^{[d]}(x) + \bar{g}^{[d-1]}(x)\nu + O(x,\nu)^{d+1} \tag{13}$$

by a d-transformation (9) if and only if there is $\phi^{[d]}(x)$, $\alpha^{[d]}(x)$ and $\beta^{[d-1]}(x)$ in (9) which satisfy the following equations

$$\begin{array}{l} \left[Fx, \phi^{[d]}(x)\right] + G\alpha^{[d]}(x) = f^{[d]}(\xi) - \bar{f}^{[d]}(x) \\ \left[G, \phi^{[d]}(x)\right] + G\beta^{[d-1]}(x) = g^{[d-1]}(\xi) - \bar{g}^{[d-1]}(x) \end{array} \tag{14}$$

Remark 3 The Lie bracket of two vector fields $v_1(x)$ and $v_2(x)$ is defined by

$$[v_1(x), v_2(x)] = \frac{\partial v_2}{\partial x}v_1 - \frac{\partial v_1}{\partial x}v_2.$$

If the family of all homogeneous polynomials of degree d is considered as a linear space with a basis

$$\left\{x_1^{i_1}\cdots x_n^{i_n} \quad | \quad i_1 + i_2 + \cdots + i_n = d\right\},$$

then the equations in (14) can be written as a set of linear algebraic equations. Therefore, finding its numerical solution is not difficult.

The Proof of Theorem 3: Each element in $W^{[d]}$ represents a d-transformation (9), which transforms (12) into

$$\begin{aligned} \dot{x} &= Fx + G\nu + \sum_{k=2}^{d-1}\left(f^{[k]}(x) + g^{[k-1]}(x)\nu\right) \\ &\quad + (f^{[d]}(x) - \left[Fx, \phi^{[d]}(x)\right] - G\alpha^{[d]}(x)) \\ &\quad + \left(g^{[d-1]}(x) - \left[G, \phi^{[d]}(x)\right] - G\beta^{[d-1]}(x)\right)\nu + O(x,\nu)^{d+1}. \end{aligned} \tag{15}$$

It is obvious that equation (15) coincides with (13) up to degree d if and only if the equations (14) are satisfied. □

The following notations will be frequently used in this section. It defines the family of homogeneous parts of control systems and the family of d-transformations as linear spaces.

$$
\begin{aligned}
V^{[d]} &= \left\{ \left(f^{[d]}(\xi), g^{[d-1]}(\xi)\right) \;\middle|\; \begin{array}{l} f^{[d]}(\xi) \textit{ and } g^{[d-1]}(\xi) \textit{ are} \\ \textit{homogeneous vector fields in } (3) \end{array} \right\} \\
W^{[d]} &= \left\{ \left(\phi^{[d]}(\xi), \alpha^{[d]}(\xi), \beta^{[d-1]}(\xi)\right) \;\middle|\; \begin{array}{l} \phi^{[d]}(\xi), \alpha^{[d]}(\xi) \textit{ and } \beta^{[d-1]}(\xi) \textit{ are} \\ \textit{homogeneous vector fields} \\ \textit{and functions in } (9) \end{array} \right\} \\
\tilde{V}^{[d]} &= \left\{ \left(\tilde{f}^{[d]}(x), 0\right) \middle| \textit{ it is in the normal form } (11) \right\}.
\end{aligned}
\tag{16}
$$

It is obvious that $V^{[d]}$, $\tilde{V}^{[d]}$ and $W^{[d]}$ are linear spaces.

Definition 3 *Define a linear map $\mathcal{A}$: $W^{[d]} \longrightarrow V^{[d]}$ by*

$$
\mathcal{A}(\phi^{[d]}, \alpha^{[d]}, \beta^{[d-1]}) = \left(\left[Ax, \phi^{[d]} \right] + B\alpha^{[d]}, \left[B, \phi^{[d]} \right] + B\beta^{[d-1]} \right). \tag{17}
$$

The image of $W^{[d]}$ under $\mathcal{A}$ is denoted by

$$
V_I^{[d]} = \mathcal{A}(W^{[d]}). \tag{18}
$$

Based on Theorem 3 and Definition 3, we can prove Lemma 2 and Lemma 3 in the appendix.

The Proof of Theorem 2: By (ii) of Lemma 3, we know that

$$
\left(f^{[d]}(x), g^{[d-1]}(x)\right) \in \left(\tilde{f}^{[d]}(x), \tilde{g}^{[d-1]}(x)\right) + V_I^{[d]}, \tag{19}
$$

for a unique pair $\left(\tilde{f}^{[d]}(x), \tilde{g}^{[d-1]}(x)\right)$ in normal form (11). Theorem 2 follows (19) and Lemma 2. □

The Proof of Theorem 1: Consider the case $d = 2$. From Theorem 2 or the result in Kang-Krener [14], we can transform (4) into a system whose quadratic part $\left(f^{[2]}(x), g^{[1]}(x)\right)$ is in the quadratic normal form by a quadratic transformation. Then we use a cubic transformation to transform the resulting system into one whose cubic term is in normal form (Theorem 2) and this cubic transformation leaves the quadratic part invariant. Continuing in this way for $d = 4, 5, 6...$, we can find a sequence of d-transformations in the form of (9) for $d \geq 2$. The composition is the formal transformation (6) which transforms (4) into the extended controller form (7). □

3 Invariants Under d-transformations

Suppose two linear control systems are controllable. They can be transformed from one into another by a linear change of coordinates and feedback if and only if they have the same controllability indices. Therefore, controllability indices completely determine a linear system up to the linear change of coordinates and feedback. If a control system is nonlinear, what are the invariants under the homogeneous change of coordinates and feedback (9)? If two systems have the same linear part, we want to find a set of invariants for each homogeneous part of a system so that the degree d homogeneous part of one system can be transformed into that of another by a d-transformation (9) if and only if the two systems have the same set of invariants. We still consider system (3). In Kang-Krener [14], a set of constant numbers was proved to be the invariants of the quadratic part. If the degree $d > 2$, the invariants are not constants any more, in fact, we will find a set of homogeneous polynomials of degree $d-2$ which are invariant under d-transformations of the form (9). Consider an analytic or formal nonlinear control system (3), its linear part is not necessarily in Brunovsky form. If it is linearly controllable, then there exists a constant row vector (H) of dimension n such that

$$HF^iG = \begin{cases} 0 & 0 \le i < n-1, \\ \\ 1 & i = n-1. \end{cases} \tag{20}$$

Let

$$\begin{array}{l} a1^{[d]tr}(\xi) = (d-2)th\ homogeneous\ part\ of \\ HF^{t-1}\left[ad^{r-1}_{F\xi+f^{[d]}(\xi)}\left(G+g^{[d-1]}(\xi)\right), ad^{r-2}_{F\xi+f^{[d]}(\xi)}\left(G+g^{[d-1]}(\xi)\right)\right] \end{array} \tag{21}$$

Definition 4 *The d-invariants are defined to be*

$$a^{[d]tr}(\xi) = a1^{[d]tr}(\xi_1, \cdots, \xi_{n-r+2}, 0, \cdots, 0) \qquad \begin{array}{l} 2 \le r \le n-1 \\ 1 \le t \le n-r. \end{array} \tag{22}$$

So $a^{[d]tr}$ is a homogeneous polynomial of degree $d-2$ with respect to $\xi_1, \cdots, \xi_{n-r+2}$. The following theorem shows that the d-invariants defined above uniquely determines the homogeneous parts under d-transformations. Denote the d-invariants of systems (12) and (13) by

$$\left\{ a^{[d]tr} \left| \begin{array}{l} 2 \le r \le n-1 \\ 1 \le t \le n-r \end{array} \right. \right\}$$

$$\left\{ \bar{a}^{[d]tr} \left| \begin{array}{l} 2 \le r \le n-1 \\ 1 \le t \le n-r \end{array} \right. \right\},$$

respectively.

Theorem 4 *The system (12) can be transformed into the system (13) with an error $O(x,\nu)^{d+1}$ by a d-transformation (9) if and only if*

$$a^{[d]tr} = \bar{a}^{[d]tr} \qquad \begin{array}{l} 2 \leq r \leq n-1 \\ 1 \leq t \leq n-r \end{array} \tag{23}$$

Corollary 1 *The system*

$$\dot{\xi} = F\xi + G\nu + f^{[d]}(\xi) + g^{[d-1]}(\xi)\mu + O(\xi,\mu)^{d+1} \tag{24}$$

can be linearized to degree d, or equivalently, there exists a d-transformation (9) which transforms (24) into

$$\dot{x} = Fx + G\nu + O(\xi,\mu)^{d+1}$$

if and only if

$$a^{[d]tr} = 0 \qquad \begin{array}{l} 2 \leq r \leq n-1 \\ 1 \leq t \leq n-r \end{array}.$$

To prove Theorem 4, we first prove the following theorem which reveals a simple relation between the d-invariants of a system in normal form and the coefficients of the nonlinear terms. Like (11), we denote $\tilde{f}^{[d]}(x)$ to be the normal form of degree d, i.e,

$$\tilde{f}_i^{[d]}(x) = \sum_{j=i+2}^{n} P_j^{[d-2]}(x_1,\cdots,x_j)x_j^2. \tag{25}$$

Theorem 5 *There is a natural one-to-one correspondence between the set of all d-invariants and the set of all extended controller forms of degree d. In fact, the d-invariants of the system*

$$\dot{x} = Ax + B\mu + \tilde{f}^{[d]}(x) \tag{26}$$

are

$$a^{[d]tr} = \frac{\partial^2}{\partial x_{n-r+2}^2} P_{t\,n-r+2}^{[d-2]}(x_1,\cdots,x_{n-r+2})x_{n-r+2}^2 \qquad \begin{array}{l} 2 \leq r \leq n-1 \\ 1 \leq t \leq n-r. \end{array} \tag{27}$$

Proof: At first, we prove the following relation for $r \geq 2$:

$$
ad^{r-1}_{A\xi+\tilde{f}^{[d]}(\xi)}(B) = \begin{array}{l} \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \end{array}}\right\} n-r+1 \\ \\ \\ \\ \end{array} \end{array} + \left[\begin{array}{l} \frac{\partial P^{[d-2]}_{1\, n-r+2} x^2_{n-r+2}}{\partial x_{n-r+2}} \\ \vdots \\ \frac{\partial P^{[d-2]}_{n-r\, n-r+2} x^2_{n-r+2}}{\partial x_{n-r+2}} \\ 0 \\ \vdots \\ 0 \end{array}\right]
$$

$$
+h^{[d-1]}_{rn-r+3}(x_1, \cdots, x_{n-r+3})x_{n-r+3} + \cdots + h^{[d-1]}_{rn}(x_1, \cdots, x_n)x_n + O(x)^d, \tag{28}
$$

where $h^{[d-1]}_{ij}(x_1, \cdots, x_j)$ is a vector field depends only on $x_1, \cdots, x_j$. Obviously, for r = 2, the equality (28) is correct because

$$
ad_{A\xi+\tilde{f}^{[d]}(\xi)}(B) = \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{array}\right] + \left[\begin{array}{l} \frac{\partial P^{[d-2]}_{1n} x^2_n}{\partial x_n} \\ \vdots \\ \frac{\partial P^{[d-2]}_{n-2\, n} x^2_n}{\partial x_n} \\ 0 \\ 0 \end{array}\right]. \tag{29}
$$

Suppose that (28) is correct for r, then

$$
\begin{aligned}
&ad^{r}_{A\xi+\tilde{f}^{[d]}(\xi)}(B) \\
&= ad_{A\xi+\tilde{f}^{[d]}(\xi)} \left(\left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \end{array}}\right\} n-r+1 \\ \\ \\ \\ \end{array} + \left[\begin{array}{l} \frac{\partial P^{[d-2]}_{1n-r+2} x^2_{n-r+2}}{\partial x_{n-r+2}} \\ \vdots \\ \frac{\partial P^{[d-2]}_{n-r\, n-r+2} x^2_{n-r+2}}{\partial x_{n-r+2}} \\ 0 \\ \vdots \\ 0 \end{array}\right] \right. \\
&\qquad \left. + \sum_{i=n-r+3}^{n} h^{[d-1]}_{ri}(x_1, \cdots, x_i)x_i + O(x)^d \right)
\end{aligned}
$$

$$
= ad_{A\xi+\tilde{f}^{[d]}(\xi)} \left(\begin{array}{l} \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{array} \right] \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} n-r+1 \\ \\ \\ \\ \end{array} \\ +h^{[d-1]}_{rn-r+2}(x_1,\cdots,x_{n-r+2})x_{n-r+2} \\ +\displaystyle\sum_{i=n-r+3}^{n} h^{[d-1]}_{ri}(x_1,\cdots,x_i)x_i + O(x)^d \end{array} \right), \tag{30}
$$

for some vector $h^{[d-1]}_{rn-r+2}(x_1,\cdots,x_{n-r+2})$. Using the fact

$$
[Ax, h(x_1,x_2,\cdots,x_j)x_j] = \bar{h}(x_1,x_2,\cdots,x_j)x_j + \tilde{h}(x_1,x_2,\cdots,x_{j+1})x_{j+1} \tag{31}
$$

for some $\bar{h}$ and $\tilde{h}$, it is obvious that the right side of (30) equals

$$
\left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{array} \right] \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} n-r \\ \\ \\ \\ \end{array} + \left[\begin{array}{l} \frac{\partial P^{[d-2]}_{1n-r+1}x^2_{n-r+1}}{\partial x_{n-r+1}} \\ \vdots \\ \frac{\partial P^{[d-2]}_{n-r\,n-r+1}x^2_{n-r+1}}{\partial x_{n-r+1}} \\ 0 \\ \vdots \\ 0 \end{array} \right] + \sum_{i=n-r+2}^{n} h^{[d-1]}_{r+1i}(x_1,\cdots,x_i)x_i \tag{32}
$$

plus some higher degree terms. By induction, we proved the equality (28). From (28), it is easy to show that

$$\left[ad^{r-1}_{Ax+f^{[d]}(x)}(B), ad^{r-2}_{Ax+f^{[d]}(x)}(B)\right]$$

$$= \begin{bmatrix} \frac{\partial^2}{\partial x^2_{n-r+2}} P^{[d-2]}_{1n-r+2} x^2_{n-r+2} \\ \vdots \\ \frac{\partial^2}{\partial x_{n-r+2}} P^{[d-2]}_{n-r\,n-r+2} x^2_{n-r+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \sum_{i=n-r+3}^{n} \tilde{h}^{[d-1]}_i x_i + O(x)^d. \tag{33}$$

Therefore

$$a^{[d]tr} = (d-2)th \ homogeneous \ part \ of$$

$$CA^{t-1}\left[ad^{r-1}_{Ax+f^{[d]}(x)}(B), ad^{r-2}_{Ax+f^{[d]}(x)}(B)\right](x_1, \cdots, x_{n-r+2}, 0, \cdots, 0)$$

$$= \frac{\partial^2}{\partial x^2_{n-r+2}}\left(P^{[d-2]}_{t\,n-r+2} x^2_{n-r+2}\right). \tag{34}$$

Theorem 5 is proved. □

The proof of Theorem 4 is based on Theorem 5 and the following lemma.

Lemma 1 *The polynomial $a^{[d]tr}(\xi_1, \cdots, \xi_{n-r+2})$ is invariant under a d-transformation (9).*

Proof: It is enough to show that $a1^{[d]tr}(\xi)$ is invariant. Given any formal vector field

$$V(\xi) = \sum_{i=k_0}^{\infty} V^{[i]}(\xi). \tag{35}$$

By the change of coordinates

$$x = \xi + \phi^{[d]}(\xi), \tag{36}$$

the vector $V(\xi)$ is transformed into

$$\left(I + \frac{\partial \phi^{[d]}(\xi(x))}{\partial \xi}\right) \sum_{i=k_0}^{\infty} V^{[i]}(\xi(x)) = V^{[k_0]}(x) + O(x)^{k_0+1}. \tag{37}$$

So, homogeneous changes of coordinates leave the first nonzero homogeneous part of a formal vector field invariant. Therefore, by a change of coordinates of the form (36), the (d-2)th homogeneous part of

$$\left[ad^{r-1}_{F\xi+f^{[d]}(\xi)}\left(G+g^{[d-1]}(\xi)\right), ad^{r-2}_{F\xi+f^{[d]}(\xi)}\left(G+g^{[d-1]}(\xi)\right)\right] \tag{38}$$

is invariant. This implies that $a1^{[d]tr}$ is invariant under the change of coordinates (36). Now let's consider a state feedback

$$\mu = \nu + \alpha^{[d]}(\xi) + \beta^{[d-1]}(\xi)\nu, \tag{39}$$

the resulting system has the following homogeneous part of degree d:

$$\left(f^{[d]}(\xi) + G\alpha^{[d]}, g^{[d-1]}(\xi) + G\beta^{[d-1]}(\xi)\right). \tag{40}$$

By induction on r, it can be proved that

$$\begin{aligned} & ad^{r-1}_{F\xi+f^{[d]}(\xi)+G\alpha^{[d]}}\left(G+g^{[d-1]}(\xi)+G\beta^{[d-1]}(\xi)\right) \\ &\quad = ad^{r-1}_{F\xi+f^{[d]}(\xi)}\left(G+g^{[d-1]}(\xi)\right) + \sum_{i=0}^{r-1}\lambda_i^{[d-1]}(\xi)F^iG + O(\xi)^d, \end{aligned} \tag{41}$$

where $\lambda_i^{[d-1]}$ are homogeneous polynomials. Denote the Lie bracket between the vectors

$$ad^{r-2}_{F\xi+f^{[d]}+G\alpha^{[d]}}\left(G+g^{[d-1]}+G\beta^{[d-1]}\right)$$

and

$$ad^{r-1}_{F\xi+f^{[d]}+G\alpha^{[d]}}\left(G+g^{[d-1]}+G\beta^{[d-1]}\right)$$

by X, then

$$\begin{aligned} & HF^{t-1}X \\ &\quad = HF^{t-1}\left[ad^{r-2}_{F\xi+f^{[d]}}\left(G+g^{[d-1]}\right), ad^{r-1}_{F\xi+f^{[d]}}\left(G+g^{[d-1]}\right)\right] \\ &\qquad\qquad + HF^{t-1}\sum_{i=0}^{r-1}\bar{\lambda}_i^{[d-2]}F^iG + O(\xi)^{[d-1]} \\ &\quad = a1^{[d]tr}(\xi) + O(\xi)^{d-1}, \end{aligned} \tag{42}$$

for $t \le n-r$. In (42) we used the fact that $HF^jG = 0$ for $j < n-1$. So, it is proved that $a1^{[d]tr}$ are invariant under both homogeneous changes of coordinates and state feedback of degree d. □

The Proof of Theorem 4: Suppose that the system (12) can be transformed into (13) with an error of at least degree $d+1$, by a d-transformation. Then, the two systems have the same d-invariants (Lemma 1). On the other hand, assume that the d-invariants of (12) and (13) are equal to each other. There is a linear change of coordinates and feedback which transforms the linear part of the two systems into Brunovsky form, it can be proved that the resulting systems still have the same d-invariants. So, we can assume that $(F, G) = (A, B)$. Theorem 2 implies that the dth homogeneous part of these two systems can be transformed into two normal forms by a d-transformation, and these two normal forms have the same d-invariants

$$\left\{ a^{[d]tr} \;\middle|\; \begin{array}{c} 2 \leq r \leq n-1 \\ 1 \leq t \leq n-r \end{array} \right\}. \tag{43}$$

By Theorem 5, the dth homogeneous part of these two normal forms must be equal to each other if they have the same d-invariants. Therefore, by a d-transformation, (12) and (13) can be transformed into the same system with an error of at least degree $d+1$. $\square$

4 Some Insight in Convergence and Analyticity

Another problem we are going to discuss in this paper is the following convergence problem. If the nonlinear control system (4) is analytic, i.e, the infinite summations involved are convergent in a region around the origin, is it possible to find a convergent transformation (6) so that the system (4) can be transformed into an extended controller form by it? Under certain special assumptions, this is proved to be true. But in general, the problem is still open.

Theorem 6 *If an analytic system*

$$\dot{x} = Ax + B\mu + \sum_{k=1}^{\infty} \left(f^{[k]}(x) + g^{[k-1]}(x)\mu \right) \tag{44}$$

is linearizable by a formal transformation, then it is linearizable by an analytic transformation.

Proof: System (44) is formally linearizable implies that it is linearizable to any degree d. By the theorem in Krener [15], the distributions D^r, $1 \leq r \leq n-1$ (see (58)), are degree $d-1$ involutive for all d. This implies that D^r is involutive for any $1 \leq r \leq n-1$. Therefore, system (44) is linearizable by an analytic transformation (see Krener [17] or Isidori [12]). $\square$

If a nonlinear analytic system is not linearizable, the convergence problem becomes complicated. The following theorem shows that a special kind of three dimensional system can be transformed into its extended controller form by an analytic transformation.

Theorem 7 *Consider*

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 + f_1(\xi_2) + f_2(\xi_2)\xi_3 + f_3(\xi_1, \xi_2, \xi_3)\xi_3^2 \\
\dot{\xi}_2 &= \xi_3 \\
\dot{\xi}_3 &= \mu.
\end{aligned} \tag{45}$$

Suppose that $f_i(\xi), i = 1, 2, 3$, are analytic. Then, there is an analytic change of coordinates and feedback (6) which transforms the system (45) into an analytic system in its extended controller form.

Proof: Let

$$\tilde{f}_2(\xi_2) = \int_0^{\xi_2} f_2(t)dt. \tag{46}$$

Define

$$\begin{aligned}
x_1 &= \xi_1 - \tilde{f}_2(\xi_2) \\
x_2 &= \xi_2 + f_1(\xi_2) \\
x_3 &= \xi_3 + \frac{\partial f_1(\xi_2)}{\partial \xi_2}\xi_3 \\
\\
\nu &= \dot{x}_3 = \mu + \frac{\partial^2 f_1(\xi_2)}{\partial \xi_2^2}\xi_3^2 + \frac{\partial f_1(\xi_2)}{\partial \xi_2}\xi_3\mu.
\end{aligned} \tag{47}$$

Under the new coordinates we have

$$\begin{aligned}
\dot{x}_1 &= \xi_2 + f_1(\xi_2) + f_2(\xi_2)\xi_3 + f_3(\xi_1, \xi_2, \xi_3)\xi_3^2 - f_2(\xi_2)\xi_3 \\
&= x_2 + f_3(\xi_1, \xi_2, \xi_3)\xi_3^2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= \nu.
\end{aligned} \tag{48}$$

From (47) we have

$$\xi_3 = x_3 + \psi(x)x_3 \tag{49}$$

for some function $\psi(x)$. Substitute this into (48), we obtain

$$\begin{aligned}
\dot{x}_1 &= x_2 + f_3(\xi(x))(1 + \psi(x))^2 x_3^2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= \nu.
\end{aligned} \tag{50}$$

This is a system in the extended controller form and the transformations used are analytic. □

5 Examples

The following example gives us the extended controller form and invariants of linearly controllable systems of dimension three.

Example 1 *The extended controller form of a three dimensional formal control system is*

$$\begin{aligned}
\dot{x}_1 &= x_2 + \sum_{k=2}^{\infty} P_{13}^{[k-2]}(x_1, x_2, x_3) x_3^2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= \mu.
\end{aligned} \tag{51}$$

The d-invariants, by Theorem 5, is simply

$$\frac{\partial^2}{\partial x_3^2}(P_{13}^{[d-2]}(x_1, x_2, x_3) x_3^2).$$

For example, the quadratic invariant is a constant $P_{13}^{[0]}$.

Theorem 1 shows that any formal control system with a single-input can be transformed into the extended controller form (7) if it is linearly controllable. This result is related to the classification of the formal control systems under formal transformations. But it is not the solution of this problem because a formal system, in general, can be transformed into different systems which are in the extended controller form (7), i.e, the normal form given above is not uniquely determined by the original system. In fact, the normal form of degree d of a system is unique under d-transformations (Theorem 2), but it is not unique under d-transformations of lower degrees. This is similar to the theory of vector fields. It is known that the normal forms of formal vector fields found by Poincaré are not uniquely determined by the original vector field. This phenomenon has been studied for many years by several authors, notably by Kummer [18], Bruno [4], Bruno [5], Bruno [6], Van der Meer [20], Baider-Churchill [3] and Baider [2]. Right now, we only know very little about the uniqueness problem of the extended controller form of nonlinear control systems. The following example shows a nonlinear system and its different extended controller form.

Example 2 *Consider*

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 + \xi_3^2 \\
\dot{\xi}_2 &= \xi_3 \\
\dot{\xi}_3 &= \mu.
\end{aligned} \tag{52}$$

This system is already in extended controller form. By the following quadratic change of coordinates and state feedback

$$\begin{aligned} x_1 &= \xi_1 + \xi_1^2 + \frac{4}{3}(\xi_2 + 2\xi_1\xi_2)^2 \\ x_2 &= \xi_2 + 2\xi_1\xi_2 \\ x_3 &= \xi_3 + 2\xi_2(\xi_2 + \xi_3^2) + 2\xi_1\xi_3 \end{aligned} \tag{53}$$

$$\nu = \dot{x}_3 = \mu + 6\xi_2\xi_3 + 4\xi_3^2 + 4\xi_2\xi_3\mu + 2\xi_1\mu,$$

system (52) is transformed into

$$\begin{aligned} \dot{x}_1 &= x_2 + x_3^2 - 2x_1x_3^2 + O(x)^4 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= \nu. \end{aligned} \tag{54}$$

This is also in the extended controller form. Therefore, the normal form of (52) is not unique.

The reason why this system can be transformed into different extended controller forms is because there is nontrivial quadratic transformations which does not change quadratic part but can make changes to the higher degree terms and this change can not be canceled by a d-transformation of degree greater than or equal to three. In general, *the extended controller form of degree d is unique under transformations of degree $\geq d$, but not unique under lower degree transformations.*

6 Conclusions

The results in this paper are the extension of the results in Kang-Krener [14] to higher degree terms. In [14], we find a normal form and a set of quadratic invariants of a nonlinear control system. In this paper, we continue the study by finding the normal forms and invariants of a nonlinear control system with a single-input for each homogeneous part of degree $d > 2$. The problem of approximate linearization of a nonlinear system by dynamic feedback is closely related to the results of the present paper and Charlet [8], [9]. Some results on this problem will be addressed in another paper. The results in this paper are valid around a fixed equilibrium point. The problem of approximate linearization of nonlinear control systems around a manifold is addressed in Hauser [10], [11] and Xu [22].

Appendix

Lemma 2 *System (4) can be transformed into system (10) by a d - transformation if and only if*

$$\left(f^{[d]}(x) - \tilde{f}^{[d]}(x), g^{[d-1]}(x)\right) \in V_I^{[d]}. \tag{55}$$

Proof: It follows Theorem 3 and Definition 3.

Lemma 3 *(i) If $\tilde{f}^{[d]}$ is in the normal form (11), then $(\tilde{f}^{[d]}, 0)$ is not in V_I.*

(ii) $V^{[d]} = V_I^{[d]} \bigoplus \tilde{V}^{[d]}$.

(iii) If

$$\left(\phi^{[d]}, \alpha^{[d]}, \beta^{[d-1]}\right) \in \ker(\mathcal{A}) \tag{56}$$

then $\phi_1^{[d]}, \phi_2^{[d]}, \cdots, \phi_{n-1}^{[d]}$ are functions independent of x_n and $\phi_{i+1}^{[d]} = L_{Ax}\phi_i^{[d]}$ for $1 \le i \le n-1$.

Proof: The proof of (i). We prove it by contradiction. Suppose $(\tilde{f}^{[d]}, 0)$ is in V_I. Lemma 2 implies that

$$\dot{x} = Ax + \tilde{f}^{[d]}(x) + B\mu \tag{57}$$

is linearizable to degree d. By the theorem in Krener [15], we know that the distributions

$$D^r = C^\infty span\left\{ad^{l-1}_{Ax+\tilde{f}^{[d]}(x)}(B);\ \ 1 \le l \le r\right\}, \quad 1 \le r \le n-1 \tag{58}$$

are degree $d-1$ involutive, i.e,

$$\left[ad^{i-1}_{Ax+\tilde{f}^{[d]}(x)}(B), ad^{j-1}_{Ax+\tilde{f}^{[d]}(x)}(B)\right] = \sum_{k=1}^{max\{i,j\}} \lambda_k ad^{k-1}_{Ax+\tilde{f}^{[d]}(x)}(B) + O(x)^{d-1} \tag{59}$$

for any $i, j \le n-1$. Since

$$\tilde{f}^{[d]} = \begin{bmatrix} \sum_{j=3}^{n} P_{1j}^{[d-2]}(x_1, \cdots, x_j)x_j^2 \\ \sum_{j=4}^{n} P_{2j}^{[d-2]}(x_1, \cdots, x_j)x_j^2 \\ \vdots \\ P_{n-2\,n}^{[d-2]}(x_1, \cdots, x_n)x_n^2 \\ 0 \\ 0 \end{bmatrix}, \tag{60}$$

we assume that the largest j such that $P_{ij}^{[d-2]} \ne 0$ is j_0, and the largest i

such that $P_{ij_0}^{[d-2]} \neq 0$ is i_0. Then we have

$$\tilde{f}^{[d]} = \begin{bmatrix} \sum_{j=3}^{j_0} P_{1j}^{[d-2]}(x_1, \cdots, x_j) x_j^2 \\ \vdots \\ \sum_{j=i_0+2}^{j_0-1} P_{i_0 j}^{[d-2]}(x_1, \cdots, x_j) x_j^2 + P_{i_0 j_0}^{[d-2]} x_{j_0}^2 \\ \sum_{j=i_0+3}^{j_0-1} P_{i_0+1\, j}^{[d-2]}(x_1, \cdots, x_j) x_j^2 \\ \vdots \\ 0 \end{bmatrix}. \tag{61}$$

So

$$\begin{aligned} X_r &= ad_{Ax+\tilde{f}^{[d]}(x)}^{r-1}(B) \\ &= (-1)^{r-1} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} 0 \\ \vdots \\ 1 \end{matrix}}\right\} n-r+1 \\ \\ \\ \\ \end{matrix} \qquad if\ 1 \leq r \leq n - j_0 + 1 \end{aligned} \tag{62}$$

and

$$\begin{aligned} & X_{n-j_0+2} = ad_{Ax+\tilde{f}^{[d]}(x)}^{n-j_0}(B) \\ & = (-1)^{n-j_0+1} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} 0 \\ \vdots \\ 1 \end{matrix}}\right\} j_0 - 1 \\ \\ \\ \\ \end{matrix} - \begin{bmatrix} * \\ \vdots \\ * \\ \frac{\partial P_{i_0 j_0}^{[d-2]} x_{j_0}^2}{\partial x_{j_0}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} * \\ \vdots \\ * \\ \frac{\partial P}{\partial x} \end{matrix}}\right\} i_0 \\ \\ \\ \\ \end{matrix} . \end{aligned} \tag{63}$$

This implies

$$[X_{n-j_0+1}, X_{n-j_0+2}] = (-1)^{n-j_0+1} \begin{bmatrix} * \\ \vdots \\ * \\ \frac{\partial^2 P^{[d-2]}_{i_0 j_0} x_{j_0}^2}{\partial x_{j_0}^2} \\ \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} i_0 \\ \\ \\ \\ \\ \end{array} . \tag{64}$$

Since $\frac{\partial^2 P^{[d-2]}_{i_0 j_0} x_{j_0}^2}{\partial x_{j_0}^2} \neq 0$ and $j_0 - 1 > i_0$, we know that

$$[X_{n-j_0+1}, X_{n-j_0+2}] \neq \sum_{i=1}^{n-j_0+2} \lambda_i X_i + O(x)^{d-1}. \tag{65}$$

This is equivalent to say that D^{n-j_0+2} is not degree $d-1$ involutive. It is a contradiction.

The proof of (ii) and (iii).

$$V_I^{[d]} \bigcap \tilde{V}^{[d]} = \emptyset. \tag{66}$$

So, the summation in the lemma must be a direct sum. We need to show that

$$\dim(V^{[d]}) = \dim(V^{[d]}) + \dim(\tilde{V}^{[d]}). \tag{67}$$

It is obvious that

$$\begin{aligned} \dim(V^{[d]}) &= n \begin{pmatrix} n+d-1 \\ d \end{pmatrix} + n \begin{pmatrix} n+d-2 \\ d-1 \end{pmatrix} \\ \dim(\tilde{V}^{[d]}) &= (n-2) \begin{pmatrix} n+d-3 \\ d-2 \end{pmatrix} \\ &\quad + (n-3) \begin{pmatrix} n+d-4 \\ d-2 \end{pmatrix} + \cdots + \begin{pmatrix} d \\ d-2 \end{pmatrix}. \end{aligned} \tag{68}$$

and

$$\dim(V_I^{[d]}) = \dim(W^{[d]}) - \dim(\ker(\mathcal{A})) = n \begin{pmatrix} n+d-1 \\ d \end{pmatrix}$$

$$+\begin{pmatrix} n+d-1 \\ d \end{pmatrix} + \begin{pmatrix} n+d-2 \\ d-1 \end{pmatrix} - \dim(\ker(\mathcal{A})). \tag{69}$$

By the definition of $\mathcal{A}$, we have

$$\mathcal{A}(\phi^{[d]}, \alpha^{[d]}, \beta^{[d-1]}) = \left(\begin{bmatrix} L_{Ax}\phi_1^{[d]} - \phi_2^{[d]} \\ L_{Ax}\phi_2^{[d]} - \phi_3^{[d]} \\ \vdots \\ L_{Ax}\phi_{n-1}^{[d]} - \phi_n^{[d]} \\ L_{Ax}\phi_n^{[d]} + \alpha^{[d]} \end{bmatrix}, \begin{bmatrix} \frac{\partial \phi_1^{[d]}}{\partial x_n} \\ \frac{\partial \phi_2^{[d]}}{\partial x_n} \\ \vdots \\ \frac{\partial \phi_{n-1}^{[d]}}{\partial x_n} \\ \frac{\partial \phi_n^{[d]}}{\partial x_n} + \beta^{[d-1]} \end{bmatrix} \right), \tag{70}$$

where

$$L_{Ax}\phi = \frac{\partial \phi}{\partial x_1}x_2 + \frac{\partial \phi}{\partial x_2}x_3 + \cdots + \frac{\partial \phi}{\partial x_{n-1}}x_n. \tag{71}$$

The equations (70) and (71) implies that the result in (iii) is correct.

By the definition of L_{Ax}, we know that if $\phi(x) = \phi(x_1, \cdots, x_j)$ and $\dfrac{\partial \phi}{\partial x_j} \neq 0$, then $L_{Ax}\phi$ depend on x_{j+1}, i.e, $\dfrac{\partial L_{Ax}\phi}{\partial x_{j+1}} \neq 0$. Assume that $\left(\phi^{[d]}, \alpha^{[d]}, \beta^{[d-1]}\right) \in \ker(\mathcal{A})$. If $\phi_1^{[d]}(x) = \phi_1^{[d]}(x_1, \cdots, x_j)$ and if $\dfrac{\partial \phi_1^{[d]}}{\partial x_j} \neq 0$, by the result in (iii) and the property of L_{Ax} we know that $\phi_{n-j+1}^{[d]} = L_{Ax}^{n-j}\phi_1^{[d]}$ and

$$\frac{\partial \phi_{n-j+1}^{[d]}}{\partial x_n} \neq 0. \tag{72}$$

Because only $\phi_n^{[d]}(x)$ can be a function depending on x_n (part (iii)), we have

$$n - j + 1 = n, \tag{73}$$

i.e, $j = 1$. Therefore

$$\phi_1^{[d]} = \lambda x_1^d. \tag{74}$$

So,

$$\dim(\ker(\mathcal{A})) = 1. \tag{75}$$

From (68), (69) and (75), we have

$$\begin{aligned} &\dim(V^{[d]}) - \dim(V_I^{[d]}) = \\ &\quad (n-1)\begin{pmatrix} n+d-2 \\ d-1 \end{pmatrix} - \begin{pmatrix} n+d-1 \\ d \end{pmatrix} + 1. \end{aligned} \tag{76}$$

By using

$$\begin{pmatrix} n+1 \\ k+1 \end{pmatrix} = \begin{pmatrix} n \\ k \end{pmatrix} + \begin{pmatrix} n-1 \\ k \end{pmatrix} + \cdots + \begin{pmatrix} k \\ k \end{pmatrix} \tag{77}$$

we have

$$\begin{aligned} &\dim(V^{[d]}) - \dim(V_I^{[d]}) \\ &\quad = (n-1)\begin{pmatrix} n+d-2 \\ d-1 \end{pmatrix} - \sum_{k=2}^{n} \begin{pmatrix} n+d-k \\ d-1 \end{pmatrix}. \end{aligned} \tag{78}$$

By using (77) again, this equals

$$(n-2)\begin{pmatrix} n+d-3 \\ d-2 \end{pmatrix} + (n-3)\begin{pmatrix} n+d-4 \\ d-3 \end{pmatrix} + \cdots + \begin{pmatrix} d \\ d-2 \end{pmatrix}. \tag{79}$$

This is the dimension of $\tilde{V}^{[d]}$. Therefore (67) is correct and (ii) of Lemma 3 is proved. □

Acknowledgment: The author would like to thank Professor Arthur J. Krener for his creative advice, constant support and friendly encouragement.

References

[1] V. I. Arnold. *Geometrical Methods in the Theory of Ordinary Differential Equations.* New York: Springer-Verlag, 1983.

[2] A. Baider. Unique normal forms for vector fields and hamiltonians, *J. of Differential Equations* **78** (1989), 33-52.

[3] A. Baider and R. C. Churchill. Uniqueness and non-uniqueness of normal forms for vector fields, *Proc. Roy. Soc. Edinburgh Sect. A* **108** (1988), 27-33.

[4] A. Bruno. Analytical form of differential equations, *Trans. Moscow Math. Soc.* **25** (1971), 131-288.

[5] A. Bruno. Analytical form of differential equations, *Trans. Moscow Math. Soc.* **26** (1972), 199-238.

[6] A. D. Bruno. *Local Methods in Nonlinear Differential Equations, Part I. The Local Method of Nonlinear Analysis of Differential Equations, Part II, The Sets of Analyticity of a Normalizing Transformation.* New York: Springer-Verlag, 1989.

[7] P. Brunovsky. A classification of linear controllable systems, *Kybernetika*(Praha) **3** (1970), 173-187.

[8] B. Charlet, J. Lévine and R. Marino. On dynamic feedback linearization, *Systems and Control Letters* **13** (1989), 143-152.

[9] B. Charlet, J. Lévine and R. Marino. Dynamic feedback linearization with applications to aircraft control, *Proceedings, IEEE Conf. on Decision and Control,* Austin, Texas, pp. 701-705, 1988.

[10] J. Hauser. Nonlinear control via uniform system approximation, *Syst. and Cont. Letters* **17** (1991), 145-154.

[11] J. Hauser. Extended uniform system approximation for multivariable nonlinear system, *Proc. ECC-91,* Hermès, Grenoble, 1991.

[12] A. Isidori. *Nonlinear Control Systems,* Second Edition. New York: Springer-Verlag, 1989.

[13] T. Kailath. *Linear Systems.* Englewood Cliffs: Prentice Hall, 1980.

[14] W. Kang and A. J. Krener. Extended quadratic controller normal form and dynamic feedback linearization of nonlinear systems, *SIAM J. Contr. Opt.* **30(6)** (1992), 1319-1337.

[15] A. J. Krener. Approximate linearization by state feedback and coordinate change, *Systems and Control Letters* **5** (1984), pp. 181-185.

[16] A. J. Krener. Normal forms for linear and nonlinear systems, in *Differential Geometry, the Interface Between Pure and Applied Mathematics* (M. Luksik, C. Martin, and W. Shadwick, eds.), Contemporary Mathematics v. 68, American Mathematical Society, Providence, 157-189, 1986.

[17] A. J. Krener, S. Karahan, M. Hubbard, and R. Frezza. Higher order linear approximations to nonlinear control systems, *Proceedings, IEEE Conf. On Decision and Control,* Los Angeles, pp. 519-523, 1987.

[18] M. Kummer. How to avoid "secular" terms in classical and quantum mechanics, *Nuovo Cimento* **B** (1971), 123-148.

[19] C. L. Siegel. On the integrals of canonical systems, *Ann. Math.* **42** (1941), 806-822; (1988), 1237-1239.

[20] J. C. Van Der Meer. *The Hamiltonian Hopf Bifurcation, Lecture Notes in Mathematics,* Vol. 1160. New York: Springer-Verlag: 1985.

[21] S. Wiggins. *Introduction to Applied Nonlinear Dynamical Systems and Chaos.* New York: Springer-Verlag, 1990.

[22] Z. Xu and J. Hauser. High order approximate feedback linearization about a manifold, *Proc. IFAC-NOLCOS 2,* Bordeaux, 1992.

Department of Systems Science and Mathematics, Washington University, St. Louis, MO 63130

Communicated by Alberto Isidori