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A Parametrization of the Minimal Square Spectral Factors of a Nonrational Spectral Density*

Augusto Ferrante

Abstract

Given a nonrational spectral density, minimality for square spectral factors with singularities in both the left and the right half planes is defined. A parametrization of all minimal square spectral factors is then provided in terms of left inner divisors of a certain inner function depending only on the spectral density. Some results on matrix inner functions and minor complements on the Lindquist-Picci stochastic realization theory are also obtained as by-products.

Key words: Spectral factor, parametrization, inner function, stochastic realization

1 Introduction

The *spectral factorization problem*, i.e., the problem of finding minimal spectral factors of a matrix-valued spectral density $\Phi(s)$ is of crucial importance in systems and control theory, circuit theory and prediction theory. In the rational case, parametrizations of minimal *stable* spectral factors have been obtained starting from the classical results of J. C. Willems [13] on the Algebraic Riccati Equation (ARE), see also [4]. More recently, Picci and Pinzoni in [12] extended the positive real lemma to systems with poles in $Re(s) \neq 0$ (poles outside the imaginary axis), and gave a parametrization of all minimal spectral factors with a given structure of poles or zeroes via the solutions of two AREs. For the rational case a complete parametrization of all the spectral factors with minimal McMillan degree of a given spectral density was presented in [3]. In [10] and [11], which are our main references, minimality is defined for analytic and coanalytic spectral factors, and parametrizations of these two classes are given. In this paper we

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deal with the nonrational case, as understanding nonrational models may be important for approximation purposes.

In this work we *define* minimality for general spectral factors and we prove that, under a reasonable condition, there is a one-to-one correspondence between minimal spectral factors and the left inner divisors of a certain maximum inner function depending only on the spectral density. This is in fact the generalization of a result obtained in [3], where the above mentioned correspondence was established for rational spectral densities. Also in the rational case, as well as in the nonrational, this parametrization of minimal spectral factors via left inner divisors of an inner function, holds under a mild condition. This condition, as formulated in [3], does not fit well with the nonrational case as it relies on a minimal realization of a certain spectral factor of $\Phi(s)$. For this reason the condition in this paper is different from the one of [3]. Of course the two conditions are related each other and the connections will be discussed below.

As corollaries we establish several results that appear as "dual" of well-known results of [10] and [11]. Finally, we give a geometric interpretation of these results providing a general geometric setup which encompasses the one given in [10] and [11]. The proof requires a series of preliminary results on matrix inner functions that appear to be of independent interest. The paper is organized as follows. In Section 2 we collect some results of the Lindquist-Picci stochastic realization theory. In Section 3 we define minimality for general spectral factors and prove that, in the rational case, this definition coincides with the standard one based on the McMillan degree. We also show that for analytic and coanalytic spectral factors this definition coincide with the one given in [10] and [11]. In Section 4 we state our main result and prove one half of it. Moreover we compare this result to the similar one obtained in [3]. The rest of the proof of the main result hinges on various lemmas on inner functions which we state and prove in Section 5. In Section 6 we complete the proof of the main result. In Section 7 we prove some complements of the Lindquist-Picci stochastic realization theory.

2 Elements of Lindquist-Picci Stochastic Realization Theory

Let $\{y(t);\ t \in \mathbf{R}\}$ be *real*, m-dimensional, *mean square continuous* vector process with *Gaussian stationary increments*. In this paper we shall always assume that the incremental spectral density $\Phi(s)$ of y is *coercive* i.e. $\exists c > 0$ s.t. $\Phi(i\omega) \geq cI\ \forall \omega \in \mathbf{R}$. This assumption will guarantee that the spectral density at infinity $R := \Phi(i\infty)$ is a positive definite matrix. Although many of the results of this paper might be proved without the coercivity condition we shall assume it both for the sake of simplicity and for a more straight comparison with the results of [3] and [12] where this assumption is made.

We shall denote with $H(dy)$ the Hilbert space $H(dy) = \overline{span}\{y_l(t) - y_l(s);\ t, s \in \mathbf{R},\ l = 1, 2, ...m\}$. If A and B are subspaces of H, $A \vee B$ will denote the closure of $\{a + b : a \in A, b \in B\}$ and $A \oplus B$ will denote the orthogonal direct sum. Finally if $A \supset B$, $C = A \ominus B$ means that $C \oplus B = A$. Defining $H_t^+(dy) := \overline{span}\{y_l(r) - y_l(s);\ r, s \geq t,\ l = 1, 2, ...m\}$ and $H_t^-(dy) := \overline{span}\{y_l(r) - y_l(s);\ r, s \leq t,\ l = 1, 2, ...m\}$ we have:

$$H(dy) = H_t^+(dy) \vee H_t^-(dy) \quad \forall t. \tag{2.1}$$

If $t = 0$ we drop the subscript and write $H^-(dy)$ and $H^+(dy)$.

We shall assume that $\{y(t)\}$ is *purely nondeterministic* (p.n.d.) in the sense that both the remote past and the remote future are trivial i.e. $H_{+\infty}^+(dy) = H_{-\infty}^-(dy) = \{0\}$. We recall now the main results of the Lindquist and Picci geometric theory of Markovian splitting subspaces because it plays an important role in what follows. For detailed references see [10],[9]and [11]. We deal with the Hilbert space $H := H(dy)$. The inner product in H will be denoted by $\langle \cdot, \cdot \rangle$. Let X, A and B subspaces of H. With the symbol $E^X \xi$ we denote orthogonal projection of $\xi \in H$ onto X. Moreover, we shall say that A is orthogonal to B given X (notation $A \perp B | X$) if $\langle a - E^X a, b - E^X b \rangle = 0 \ \ \forall a \in A, b \in B$. Since $y(t)$ is mean square continuous, there exists a strongly continuous group U_t of unitary operators on H acting as shifts: $U_t(y_l(s) - y_l(r)) = (y_l(s + t) - y_l(r + t))$. Finally we shall denote by X^- the closure of $\{U_t X :\ t \leq 0\}$ and by X^+ the closure of $\{U_t X :\ t \geq 0\}$.

Definition 2.1 *A subspace X is said to be a* Markovian splitting subspace *if*

$$H^- \vee X^- \perp H^+ \vee X^+ | X. \tag{2.2}$$

A Markovian splitting subspace X is said to be proper *if both $(H^- \vee X^-)^\perp$ and $(H^+ \vee X^+)^\perp$ are full range. (Orthogonal complements being taken in H). Finally a Markovian splitting subspace X is said to be* minimal *if it does not contain any other Markovian splitting subspace as a proper subspace.*

Relation (2.2) says that a Markovian splitting subspace is a natural state space (possibly infinite dimensional) for dy. It was shown in [11, Theorem 4.1] that there exists a one-one correspondence between Markovian splitting subspaces X and couples $(S, \overline{S})$ of subspaces satisfying:

$$\begin{cases} H^- \subset S \\ H^+ \subset \overline{S} \end{cases} \tag{2.3a}$$

$$\begin{cases} U_t S \subset S & \text{for } t \leq 0 \\ U_t \overline{S} \subset \overline{S} & \text{for } t \geq 0 \end{cases} \tag{2.3b}$$

$$S^{\perp} \oplus (S \cap \overline{S}) \oplus \overline{S}^{\perp} = H. \tag{2.3c}$$

This correspondence is given by:

$$X = S \cap \overline{S}. \tag{2.4}$$

We shall write $X \sim (S, \overline{S})$ and we shall say that $(S, \overline{S})$ is the *scattering pair* of X. It can be shown that X is proper if and only if both $S^{\perp}$ and $\overline{S}^{\perp}$ are full range. The minimality of a Markovian splitting subspace X can be usefully expressed in terms of its scattering pair. In particular it can be shown that if $X \sim (S, \overline{S})$ is a Markovian splitting subspace then, defining $\overline{S}_1 := H^+ \vee S^{\perp}$ and $S_1 := H^- \vee \overline{S}_1^{\perp}$, we have that $X_1 \sim (S_1, \overline{S}_1)$ is a *minimal* Markovian splitting subspace contained in X. It follows that a Markovian splitting subspace $X \sim (S, \overline{S})$ is minimal if and only if the two relations

$$\overline{S} = H^+ \vee S^{\perp}, \tag{2.5a}$$

$$S = H^- \vee \overline{S}^{\perp}, \tag{2.5b}$$

hold true. Define now the subspaces:

$$N^- := H^- \cap (H^+)^{\perp}, \tag{2.6a}$$

$$N^+ := H^+ \cap (H^-)^{\perp}. \tag{2.6b}$$

These spaces have an intuitive interpretation: N^- is the part of the past of dy which is orthogonal to the future, N^+ is the part of the future of dy which is orthogonal to the past. We collect now a result from [11].

Lemma 2.1 *If $X \sim (S, \overline{S})$ is a Markovian splitting subspace such that $S \subset (N^+)^{\perp}$ and $\overline{S} = H^+ \vee S^{\perp}$, then X is minimal.*

We shall prove this lemma in the special case in which S and $\overline{S}$ are finite dimensional spaces, which corresponds to the spectral density $\Phi(s)$ being rational.

Proof (finite dimensional case): In view of equations (2.5) it is sufficient to prove that: $S = H^- \vee \overline{S}^{\perp}$ i.e. that:

$$S = H^- \vee ((H^+)^{\perp} \cap S). \tag{2.7}$$

Let $s \in S \subset (N^+)^{\perp} = (H^+)^{\perp} \vee H^-$. Using the hypotheses of finite dimensionality we can assume that $s = h_1 + h_2$ where $h_1 \in (H^+)^{\perp}$ and $h_2 \in H^- \subset S$. Since s and h_2 belong to S we also have $h_1 = s - h_2 \in S$. Thus $s = h_1 + h_2 \in H^- \vee ((H^+)^{\perp} \cap S)$. Conversely let $h \in H^- \vee ((H^+)^{\perp} \cap S)$.

Again we can assume that $h = h_1 + h_2$ where $h_1 \in H^- \subset S$ and $h_2 \in S$. This implies that also $h = h_1 + h_2 \in S$. ■

We reformulate now these results in the spectral domain. Let $\Phi(s)$ be the $(m \times m)$ spectral density of dy, and assume that it has rank m almost everywhere on the imaginary axis $\mathbf{I}$. We recall that, by assumption, $\Phi(s)$ is a coercive spectral density i.e. $\exists c > 0$ s.t. $\Phi(i\omega) \geq cI \; \forall \omega \in \mathbf{R}$. It is well-known that $\Phi(s) = \Phi^T(-s)$ and that under the present assumptions on $y(t)$ there exist solutions of the spectral factorization problem:

$$\Phi(s) = W(s)W^T(-s). \tag{2.8}$$

In this paper we shall only consider square solutions of (2.8). Thus, in the following, a *spectral factor* will be a square $(m \times m)$ matrix function which solves (2.8). We shall say that a spectral factor is *analytic* if it has no singularities in the open right half plane. Similarly a spectral factor is *coanalytic* if it has no singularities in the open left half plane. Since dy is p.n.d., equation (2.8) admits analytic and coanalytic solutions whose rows are respectively in the *modified Hardy spaces* $\mathcal{W}_m^2$ and $\overline{\mathcal{W}}_m^2$, see [9]. We recall that:

$$\mathcal{W}_m^2 = \{(1+s)h; h \in H_m^2\} \tag{2.9a}$$

$$\overline{\mathcal{W}}_m^2 = \{(1-s)h; h \in \overline{H}_m^2\} \tag{2.9b}$$

where H_m^2 and $\overline{H}_m^2$ are the usual m-dimensional Hardy spaces on the right and left half planes. In the following we shall assume that elements in these space are row vectors using the same formalism of [10].

It is well-known that to each spectral factor $W(s)$ we can associate a unique m-dimensional Wiener process $du(t)$ in the following way:

$$u(t) - u(s) = \int_{-\infty}^{+\infty} \frac{e^{i\omega t} - e^{i\omega s}}{i\omega} d\hat{u}(i\omega), \tag{2.10a}$$

$$d\hat{u}(i\omega) = W^{-1}(i\omega) d\hat{y}(i\omega), \tag{2.10b}$$

where $d\hat{y}(i\omega)$ is the orthogonal spectral measure of the process $\{dy(t)\}$. More details on this construction can be found in [1], [10], [9] and [11]. It is also well-known [9] that the two relations (2.10a) and (2.10b) generate a one-one correspondence between Wiener processes $\{u(t)\}$ satisfying: $H^-(du) \supset H^-(dy) = H^-$ and analytic spectral factors. Similarly there exists a one-one correspondence between Wiener processes $\overline{u}(t)$ such that: $H^+(d\overline{u}) \supset H^+(dy) = H^+$ and coanalytic spectral factors.

Let $X \sim (S, \overline{S})$ be a Markovian splitting subspace of dy. We know (eq. (2.3a)) that $S \supset H^-$. Then $S = H^-(du)$ where du is a Wiener process related to an analytic spectral factor $W(s)$. Similarly $\overline{S} = H^+(d\overline{u})$

where $d\overline{u}$ is a Wiener process related to a coanalytic spectral factor $\overline{W}(s)$. It can be shown that equation (2.3c) implies that the function $K(s) := \overline{W}^{-1}(s)W(s)$ is *inner* i.e. $K(s) \in H^{\infty}_{m\times m}$ and $K(s)K^T(-s) = K(s)K^*(s) = I$. (If $A(s)$ is a matrix function, we denote by $A^*(s)$ the matrix function $A^T(-s)$). This inner function $K(s)$ is called the *structural function* of the pair $(W(s), \overline{W}(s))$.

We recall [11] that the Markovian splitting subspace X has the two following representations in terms of the two Wiener processes du and $d\overline{u}$:

$$X = \int_{-\infty}^{+\infty} [H^2_m \ominus H^2_m K(s)]d\hat{u}, \tag{2.11a}$$

$$X = \int_{-\infty}^{+\infty} [\overline{H}^2_m \ominus \overline{H}^2_m K^*(s)]d\hat{\overline{u}}, \tag{2.11b}$$

where $H^2_m K(s) := \{h(s)K(s) : \ h(s) \in H^2_m\}$ and $\overline{H}^2_m K^*(s) := \{h(s)K^*(s) : h(s) \in \overline{H}^2_m\}$. In the following, if $K(s)$ is an $m \times m$ inner function, we shall use the shorthand notation $H(K(s))$ instead of $H^2_m \ominus H^2_m K(s)$. Similarly, instead of $\overline{H}^2_m \ominus \overline{H}^2_m K^*(s)$ we shall write $\overline{H}(K^*(s))$. We shall say that X, above defined, is the state space associated to the pair $(W(s), \overline{W}(s))$.

Let $W(s)$ be an analytic spectral factor such that the related Wiener process du has the property:

$$H^-(du) \subset (N^+)^{\perp}. \tag{2.12}$$

Define $S := H^-(du)$ and $\overline{S} := H^+ \vee S^{\perp}$. Moreover let $\overline{W}$ the (coanalytic) spectral factor such that the related Wiener process $d\overline{u}$ satisfy the property: $H^+(d\overline{u}) = \overline{S}$. In view of Lemma 2.1 the pair $(S, \overline{S})$ is the scattering pair of a minimal Markovian splitting subspace X which is the natural state space of $W(s)$. It seems therefore natural to define *minimal* an analytic spectral factor $W(s)$ if the related Wiener process du satisfies equation (2.12). We also define *minimal* a coanalytic spectral factor $\overline{W}$ such that the related Wiener process $d\overline{u}$ satisfy the property

$$H^+(d\overline{u}) \subset (N^-)^{\perp}, \tag{2.13}$$

because we can associate to $\overline{W}$ an analytic spectral factor $W(s)$ such that the state space X corresponding to $(W(s), \overline{W}(s))$ is minimal. Notice that in this way we associate the same state space X to $W(s)$ and to $\overline{W}(s)$.

There are two special Wiener processes $\{u_-(t)\}$ and $\{\overline{u}_+(t)\}$ which have the following properties:

$$H^-(du_-) = H^- \tag{2.14a}$$

$$H^+(d\overline{u}_+) = H^+ \tag{2.14b}$$

These two Wiener processes correspond, via (2.10a) and (2.10b), respectively to the *outer* and *coouter* spectral factors $W_-(s)$ and $\overline{W}_+(s)$. These two spectral factors are unique modulo multiplication on the right by a constant orthogonal matrix. Moreover their inverses are analytic respectively in the right and left half plane. Clearly the Wiener process du_- related to $W_-(s)$ satisfies equation (2.12). Using the previous construction, we can also associate to it a coanalytic spectral factor $\overline{W}_-(s)$ such that the related Wiener process $d\overline{u}_-$ satisfy: $H^+(d\overline{u}_-) = H^+ \vee (H^-(du_-))^\perp = (N^-)^\perp$. The spectral factor $\overline{W}_-(s)$ is clearly minimal and has inverse which is analytic in the open right half plane. Similarly we can associate a minimal analytic spectral factor $W_+(s)$ to $\overline{W}_+(s)$. The spectral factor $W_+(s)$ has inverse which is analytic in the open left half plane and the corresponding Wiener process du_+ satisfy the condition $H^-(du_+) = (N^+)^\perp$ It can be shown [9] that any analytic spectral factor $W(s)$ admits the the outer-inner factorization:

$$W(s) = W_-(s)Q(s) \tag{2.15}$$

where $Q(s)$ is an inner matrix function. Similarly any coanalytic spectral factor admits the coouter-coinner factorization:

$$\overline{W}(s) = \overline{W}_+(s)\overline{Q}(s) \tag{2.16}$$

where $\overline{Q}(s)$ is a *coinner* matrix function i.e. $\overline{Q}^*(s) := \overline{Q}^T(-s)$ is inner. In particular we define $Q_+(s)$ to be the inner matrix function such that: $W_+(s) = W_-(s)Q_+(s)$ and $\overline{Q}_-(s)$ the coinner matrix function such that: $\overline{W}_-(s) = \overline{W}_+(s)\overline{Q}_-(s)$. It can be shown [10] that an analytic spectral factor $W(s)$ is minimal if and only if the inner matrix function $Q(s)$ in the factorization (2.15) is a *left inner divisor* of $Q_+(s)$ i.e. there exists another inner matrix function $V(s)$ such that $Q_+(s) = Q(s)V(s)$. Similarly, a coanalytic spectral factor $\overline{W}(s)$ is minimal if and only if the inner matrix function $\overline{Q}^*(s)$ (where $\overline{Q}(s)$ is the coinner function in the factorization (2.16)) is a *right inner divisor* of $\overline{Q}^*_-(s)$ i.e. there exists another inner matrix function $U(s)$ such that $\overline{Q}^*_-(s) = U(s)\overline{Q}^*(s)$.

In this paper inner and coinner functions play a central role. For this reason we recall some well-known results and set some notations. Let $Q_1(s)$ and $Q_2(s)$ be inner functions: in the set $\mathcal{Q}$ of common left inner divisors of $Q_1(s)$ and $Q_2(s)$ there is an element $Q(s)$ (unique up to multiplication on the right by a constant orthogonal matrix) which is divided on the left side by all elements in $\mathcal{Q}$. This inner $Q(s)$ is called the *greatest common left inner divisor* of $Q_1(s)$ and $Q_2(s)$ and we shall denote this with the formalism: $Q(s) = g.c.l.i.d.(Q_1(s), Q_2(s))$. If $Q(s)$ is the identity (or a constant orthogonal matrix) we shall say that $Q_1(s)$ and $Q_2(s)$ are left coprime and we shall write: $(Q_1(s), Q_2(s))_L = I$. In a symmetric way, we can define the greatest common right inner divisor $Q(s)$ of $Q_1(s)$ and $Q_2(s)$.

We shall denote it with the symbol $Q(s) = g.c.r.i.d.(Q_1(s), Q_2(s))$. If $Q(s)$ is the identity we say that $Q_1(s)$ and $Q_2(s)$ are right coprime and we write: $(Q_1(s), Q_2(s))_R = I$. In the scalar case we shall denote by the symbol $g.c.i.d.(\cdot,\cdot)$ the greatest common inner divisor of two scalar inner function. We shall also drop the subscripts $_L$ or $_R$ for scalar coprime inner function. We are now ready to tie together the geometric definition of minimality and the spectral representation $W(s), \overline{W}(s)$ of the scattering pair $(S, \overline{S})$. Let $(W(s), \overline{W}(s))$ be a couple of analytic and coanalytic minimal spectral factors which admit the factorizations (2.15) and (2.16). The corresponding state space X defined in (2.11) is characterized by three inner matrix functions: the structural function $K(s)$ and the two inner function $Q(s)$ and $\overline{Q}^*(s)$ defined by (2.15) and (2.16). For this reason $(K(s), Q(s), \overline{Q}^*(s))$ is called the *inner triplet* of X, see [11, page 271].

In this setup minimality conditions (2.5) become coprimeness conditions between $K(s)$, $Q(s)$ and $\overline{Q}^*(s)$). Indeed, a state space X (given by (2.11)) is minimal if and only if $Q(s)$ is a left inner divisor of $Q_+(s)$, $\overline{Q}^*(s)$ is a right inner divisor of $\overline{Q}^*_-(s)$ and the two coprimeness conditions

$$(Q(s), K(s))_R = I, \tag{2.17a}$$
$$(\overline{Q}^*(s), K(s))_L = I, \tag{2.17b}$$

hold true.

Definition 2.2 *A couple $(W(s), \overline{W}(s))$ of analytic and coanalytic spectral factors is said to be a* Lindquist Picci pair *(L.P. pair) if the function $\overline{W}^{-1}(s)W(s)$ is inner. A L.P. pair is said to be a* minimal Lindquist Picci pair *(minimal L.P. pair) if the corresponding state space X (eq. (2.11)) is minimal.*

Let $K(s)$ be a $m \times m$ inner matrix function. The *invariant factors* of $K(s)$ are scalar inner functions $k_1(s), k_2(s), ...k_m(s)$defined as follows [10]: set $q_0(s) = 1$ and, for $i = 1, 2, ..m$, define $q_i(s)$ to be the greatest common inner divisor of all $i \times i$ minors of $K(s)$. Then set $k_i(s) := q_i(s)/q_{i-1}(s)$ for $i = 1, 2, ..m$. Two inner functions with the same invariant factors are said to be *quasi-equivalent* [5]. Obviously two scalar inner functions are quasi-equivalent if and only if they coincide up to a constant of absolute value one. It is proved in [8] that if $K_1(s)$ and $K_2(s)$ are structural functions of two minimal L.P. pairs, then $K_1(s)$ and $K_2(s)$ are quasi-equivalent (in particular they have the same determinant). In the set $\mathcal{M}$ of minimal L.P. pairs of dy the two extreme elements $(W_-(s), \overline{W}_-(s))$ and $(W_+(s), \overline{W}_+(s))$ generate respectively, the two structural functions $K_-(s) := \overline{W}_-^{-1}(s)W_-(s)$ and $K_+(s) := \overline{W}_+^{-1}(s)W_+(s)$, which will play a crucial role below.

Let $T(i\omega) \in \mathcal{L}^{\infty}_{m\times m}(\mathbf{I})$. The Hankel operator with symbol T is defined by

$$H_T: \begin{array}{ccc} \overline{H}^2 & \longrightarrow & H^2 \\ h & \longrightarrow & P^{H^2}[T(s)h], \end{array} \tag{2.18}$$

where P^{H^2} is the orthogonal projection onto H^2.

If $Q(s)$ and $K(s)$ are inner matrix functions then the function $T(s) := Q(s)K^*(s)$ is an *all-pass* function i.e. $T(s)T^*(s) = I$. The above factorization of $T(s)$ is said to be *coprime* if $(Q(s), K(s))_R = I$. Similarly the factorization $K^*(s)Q(s)$ is said to be coprime if $(Q(s), K(s))_L = I$. As it is shown in [5] an all-pass function $T(s)$ admits the two coprime factorization $T(s) := Q(s)K^*(s) = K_1^*(s)Q_1(s)$ if and only if it is *strictly noncyclic* i.e. the orthogonal complement in H^2 of the range of the Hankel operator with symbol T is a subspace of full range. Any couple of spectral factors $W_1(s)$ and $W_2(s)$ defines a function $T(s) := W_1^{-1}(s)W_2(s)$ which is clearly an all-pass function. Among these functions there is the one defined by: $T_0(s) := \overline{W}_+^{-1}(s)W_-(s)$ which is the multivariate version of the well-known *phase function* see [10] and [11]. Following these references, we shall say that the process $y(t)$ is *strictly noncyclic* if the phase function $T_0(s)$, uniquely determined by $y(t)$, is strictly noncyclic. As it is shown in [10], a couple of spectral factors $(W(s), \overline{W}(s))$ is a minimal L.P. pair if and only if the phase function $T_0(s)$ admits the coprime factorization $T_0(s) = \overline{Q}(s)K(s)Q^*(s)$ where $(K(s), Q(s), \overline{Q}^*(s))$ is the inner triplet related to $(W(s), \overline{W}(s))$. In section 7 we shall give a dual version of this result and a generalization to spectral factors with no analyticity property.

3 Definition of Minimality and State Space for General Spectral Factors

So far, minimality for spectral factors was defined only in the analytic or coanalytic case. In the rational case minimality is defined in a natural way for any spectral factor, no matter where its poles are. More precisely a rational spectral factor is said to be *minimal* if it has the least possible McMillan degree i.e. the smallest possible dimension of a minimal realization. In general (nonrational case), this definition is not useful (the state space is infinite dimensional), but we can use this idea of minimality and say that a spectral factor is minimal if its natural state space does not contain properly any other state space.

Definition 3.1 *Let $W_1(s)$ be an arbitrary spectral factor . We say that $W_1(s)$ is* minimal *if there exists a minimal L.P. pair $(W(s), \overline{W}(s))$ (def. 2.2) such that:*

$$U(s) := \overline{W}^{-1}(s)W_1(s) \tag{3.1a}$$

$$V(s) := W_1^{-1}(s)W(s) \tag{3.1b}$$

are inner.

A first justification of the previous definition is given by the following proposition.

Proposition 3.1 *In the rational case, Definition 3.1 coincides with the usual one based on the McMillan degree. More precisely, a rational spectral factor $W_1(s)$ is minimal (in accordance to Def. 3.1) if and only if it has the least possible McMillan degree.*

Before proving this proposition we have to set some notation to distinguish the different minimality definitions. We shall simply say that a spectral factor is minimal if it is minimal according to Definition 3.1. We use the word *McMillan minimal* to denote a spectral factor minimal in the sense of the McMillan degree. It is also worth noticing that the definition of minimality for analytic and coanalytic spectral factors given by equations (2.12) and (2.13) coincides with McMillan minimality [11]. We give now a preliminary result which can be found, with a slightly different formulation in [4].

Lemma 3.1 *Let A be a stability matrix and (A, B) be a controllable pair. Moreover let Q_n be the unique symmetric negative definite solution of the ARE*

$$-AQ - QA^T + QBB^TQ = 0. \tag{3.2}$$

Then

$$K(s) = I + B^T(sI - A)^{-1}Q_nB \tag{3.3}$$

is a minimal realization of an inner function. Moreover the set $\mathcal{K}$ of the left inner divisors of $K(s)$ can be parametrized as follows:

$$\mathcal{K} = \{K_i(s) = I + B^T(sI - A)^{-1}Q_iB : Q_i \text{ is a symm. sol. of (3.2)}\}. \tag{3.4}$$

Proof of Proposition 3.1: Let $\Phi(s)$ be rational and $W_1(s)$ be a McMillan minimal spectral factor. Let

$$W_1(s) = H(sI - F)^{-1}G + R^{1/2} \tag{3.5}$$

be a minimal realization of $W_1(s)$. We recall that here $R := \Phi(i\omega)$ is positive definite. Our program is to use the results of [12] to build up a couple of analytic and coanalytic spectral factors, and then to prove that this couple is a minimal L.P. pair.

Consider the following ARE:

$$F^T Q + QF + QGG^T Q = 0. \tag{3.6}$$

It is well-known that in the set symmetric solutions of equation (3.6) there are a maximal and a minimal element Q_+ and Q_-, (in the partial ordering of semidefinite positive matrices). We can associate to this two matrices the two spectral factors

$$\overline{W}(s) := H_+(sI - F_+)^{-1}G + R^{-1/2}, \tag{3.7a}$$

$$W(s) := H_-(sI - F_-)^{-1}G + R^{-1/2}, \tag{3.7b}$$

where $F_+ := F + BB^T Q_+$, $F_- := F + BB^T Q_-$, $H_+ := H + R^{-1/2}G^T Q_+$ and $H_- := H + R^{-1/2}G^T Q_-$.

We have that the spectral factors $\overline{W}(s)$ and $W(s)$ above defined are co-analytic and analytic, respectively, and they are McMillan minimal spectral factors [12]. Moreover it is easy to see that $\overline{W}^{-1}(s)$, $W^{-1}(s)$ and $W_1^{-1}(s)$ have minimal realizations with the same state matrix $\Gamma := F - GR^{-1/2}H$. We shall say that $\overline{W}(s)$, $W(s)$ and $W_1(s)$ have the same *zero matrix*. Again an easy computation shows that

$$\overline{W}^{-1}(s)W_1(s) \quad \text{and} \tag{3.8a}$$

$$W_1^{-1}(s)W(s) \tag{3.8b}$$

are inner functions.

We know that $W(s)$ is McMillan minimal, thus we can associate to $W(s)$ a coanalytic spectral factor $\overline{W}_1(s)$ such that $(W(s), \overline{W}_1(s))$ is a minimal L.P. pair, (see section 2 or [11]) Then, by corollary (11.2) in [11], $\overline{W}_1(s)$ has the same zero matrix of $W(s)$, and then of $W_1(s)$. Thus $\overline{W}_1(s)$ has a minimal realization of the form

$$\overline{W}_1(s) := H_1(sI - F_1)^{-1}G + R^{-1/2}, \tag{3.9}$$

where $F_1 := F + BB^T Q_1$, $H_1 := H + R^{-1/2}G^T Q_1$ and Q_1 is a symmetric solution of (3.6).

But the only coanalytic spectral factor of this form is $\overline{W}(s)$ thus we have $\overline{W}(s) = \overline{W}_1(s)$. Hence $(W(s), \overline{W}(s))$ is a minimal L.P. pair. This together with (3.8) concludes one direction of the proof.

Conversely, let $W_1(s)$ be a minimal spectral factor of a rational spectral density. Then, by definition, there exists a minimal L.P. pair $(W(s), \overline{W}(s))$ such that:

$$W_1(s) = \overline{W}(s)U(s), \tag{3.10a}$$

$$W(s) = W_1(s)V(s), \tag{3.10b}$$

with $U(s)$ and $V(s)$ inner functions. We remark that $W(s)$ and $\overline{W}(s)$ are McMillan minimal and have the same zero matrix and then, if $\overline{W}(s) = C(sI + A^T)^{-1}B + R^{-1/2}$ is a minimal realization of $\overline{W}(s)$, $W(s)$ has the following minimal realization [12]:

$$W(s) = H(sI - F)^{-1}B + R^{1/2} \tag{3.11}$$

where $F = -A^T + BB^TQ_n$, $H = C + R^{1/2}B^TQ_n$ and Q_n is the unique negative definite solution of the ARE (3.2).

Define the inner matrix function $K(s)$ by $K(s) := U(s)V(s)$. We easily get $\overline{W}^{-1}(s)W(s) = K(s)$ and computing the calculations yields the minimal realization: $K(s) = I + B^T(sI - A)^{-1}Q_nB$. Hence, by Lemma 3.1 $U(s)$ has the (nonminimal) realization: $U(s) = I + B^T(sI - A)^{-1}Q_1B$, where Q_1 is a symmetric solution of the ARE (3.2). We can now compute $W_1(s) = \overline{W}(s)U(s)$ and this yields $W_1(s) = H(sI - F)^{-1}B + R^{1/2}$ where $F = -A^T + BB^TQ_1$, $H = C + R^{1/2}B^TQ_1$. It is easy to see that this realization of $W_1(s)$ has the same dimension of the minimal realization (3.11) of $W(s)$ which is McMillan minimal. Hence, also $W_1(s)$ is McMillan minimal. ■

We have also to check that for analytic and coanalytic spectral factors this definition is in agreement with the one given in [10] and [11] above recalled in (2.12) and (2.13).

Proposition 3.2 *For analytic and coanalytic spectral factors, Definition 3.1 coincides with the usual one based on (2.12) and (2.13). More precisely, an analytic (coanalytic) spectral factor $W_1(s)$ is minimal (in accordance to Def. 3.1) if and only if equation (2.12) (equation (2.13)) is satisfied.*

Proof: Let $W_1(s)$ be an analytic spectral factor minimal (in the sense of (2.12)). We can associate to $W_1(s)$ a spectral factor $\overline{W}(s)$ such that $(W(s), \overline{W}(s))$ is a minimal L.P. pair. Then, in the notation of Definition 3.1, we can simply take $U(s) = K(s)$ and $V(s) = I$, and this proves that $W_1(s)$ is minimal in the sense of Definition 3.1.

Conversely, let $W_1(s)$ be analytic and minimal (in the sense of definition 3.1). Thus there exists a minimal L.P. pair $(W(s), \overline{W}(s))$ such that equations (3.1) hold. Moreover there exists an inner function $Q_1(s)$ such that $W_1(s) = W_-(s)Q_1(s)$. Let $(K(s), Q(s), \overline{Q}^*(s))$ be the inner triplet related to $(W(s), \overline{W}(s))$, we recall that

$$(K, Q)_R = I \tag{3.12a}$$

$$(K, \overline{Q}^*)_L = I. \tag{3.12b}$$

Using the same notation of Definition 3.1 and taking into account the definition of $Q(s)$ we have $V(s) = W_1^{-1}(s)W(s) = Q_1^*(s)Q(s)$. Thus by

(3.12a) $V(s)$ is a constant orthogonal matrix that we identify with the identity as our spectral factors are defined up to right multiplication by a constant orthogonal matrix. We have proved that $W_1(s) = W(s)$. But $W(s)$ is the analytic element of a minimal L.P. pair and then it is minimal in the sense of (2.12). The proof for coanalytic spectral factors is similar. ∎

Definition 3.2 *Let* $(W(s), \overline{W}(s))$ *be a, nonnecessarily minimal, L.P. pair, and let* du *and* $d\overline{u}$ *be the corresponding Wiener processes. Let* $W_1(s)$ *be a spectral factor such that the matrix functions* $U(s)$ *and* $V(s)$ *defined by (3.1b) and (3.1a) are inner.*

Set:

$$X' := \int_{-\infty}^{+\infty} H(V(s))d\hat{u} \subset S = H^-(du), \tag{3.13a}$$

$$X'' := \int_{-\infty}^{+\infty} \overline{H}(U^*(s))d\hat{\overline{u}} \subset \overline{S} = H^+(d\overline{u}). \tag{3.13b}$$

We define the state space X_1 *of* $W_1(s)$ *by*

$$X_1 := X' \vee X''. \tag{3.14}$$

In Proposition 3.3 below we show that the sum in equation (3.14) is actually orthogonal. In this definition the spectral factor $W_1(s)$ may have singularities in both the left and the right half planes.

Proposition 3.3 *Suppose that* X *is the state space of* $(W(s), \overline{W}(s))$ *[equations (2.11)], where* $(W(s), \overline{W}(s))$ *is as in Definition 3.2. Then relation*

$$X = X_1 = X' \oplus X'' \tag{3.15}$$

holds true. Moreover $X' \subset H^+(du_1)$ *and* $X'' \subset H^-(du_1)$*, where* du_1*is the Wiener process corresponding to* $W_1(s)$*:* $d\hat{u}_1 := W_1^{-1}(s)d\hat{y}$*.*

Proof: Since the structural function $K(s)$ of the pair $(W(s), \overline{W}(s))$ can be factored as: $K(s) = U(s)V(s)$, it is clear that $H_m^2 K(s) = H_m^2 U(s)V(s) \subset H_m^2 V(s)$. Consequently $H(K(s)) \supset H(V(s))$. This inclusion, together with equations (2.11a) and (3.13a), implies that $X \supset X'$. Similarly it can be proved that $X \supset X''$. It can be easily checked that $d\hat{u} = V^*(s)d\hat{u}_1$ and $d\hat{\overline{u}} = U(s)d\hat{u}_1$. We can then write:

$$X' = \int_{-\infty}^{+\infty} H(V(s))V^*(s)d\hat{u}_1 \subset H^+(du_1), \tag{3.16a}$$

$$X'' = \int_{-\infty}^{+\infty} \overline{H}(U^*(s))U(s)d\hat{u}_1, \subset H^-(du_1) \tag{3.16b}$$

where the two inclusions are due to the fact that $H(V(s))V^*(s) \subset \overline{H}_m^2$ and, symmetrically, $\overline{H}(U^*(s))U(s) \subset H_m^2$. Hence $X' \perp X''$. We now prove that $X \ominus X' = X''$. From the definitions of X and X' we have:

$$X \ominus X' = \int_{-\infty}^{+\infty} [H(K(s)) \ominus H(V(s))] d\hat{u}. \tag{3.17}$$

It is easy to see that $H(K(s)) \ominus H(V(s)) = H_m^2 V(s) \ominus H_m^2 K(s)$ Plugging this equation into (3.17) and using the fact that $d\hat{u} = K^*(s) d\hat{\overline{u}}$ we get:

$$X \ominus X' = \int_{-\infty}^{+\infty} [H_m^2 V(s) \ominus H_m^2 K(s)] K^*(s) d\hat{\overline{u}}. \tag{3.18}$$

Denoting by $\langle \cdot, \cdot \rangle$ the scalar product in $\mathcal{L}_m^2(\mathbf{I})$ we get:

$$\begin{aligned}
&[H_m^2 V(s) \ominus H_m^2 K(s)] K^*(s) = \\
&\{h(s)V(s)K^*(s) = h(s)U^*(s) : \\
&h(s) \in H_m^2,\ \langle h(s)V(s), k(s)K(s) \rangle = 0 \ \ \forall k(s) \in H_m^2\} = \\
&\{h(s)U^*(s) \in \overline{H}_m^2 :\ h(s) \in H_m^2\} = \\
&\{j(s) \in \overline{H}_m^2 :\ j(s)U(s) \in H_m^2\} = \overline{H}(U^*(s)).
\end{aligned} \tag{3.19}$$

The assertion now follows by comparing this equation and equation (3.17) to the definition of X''. ■

This proposition gives a system theoretic justification of Definition 3.1. In fact, equation (3.15), using Definition 3.2, implies that a spectral factor is minimal if and only if its state space is so.

Remark 3.1 Let $(W(s), \overline{W}(s))$ be a minimal L.P. pair with state space X, and let $W_1(s)$ be a minimal spectral factor related to $(W(s), \overline{W}(s))$ by equations (3.1). The state space X, which is a space of random variables, is the same for $W(s)$, $\overline{W}(s)$ and $W_1(s)$, [Proposition 3.3], but the Wiener process corresponding to each spectral factor maps X into different function spaces. More precisely, the Wiener process du (corresponding to $W(s)$) establishes a correspondence between the space X and the space of analytic functions $H(K(s))$, [equation (2.11a)]. Similarly the Wiener process $d\overline{u}$ (corresponding to $\overline{W}(s)$) establishes a correspondence between X and the space of coanalytic functions $\overline{H}(K^*(s))$, [equation (2.11b)]. The spectral factor $W_1(s)$ has singularities in both the left and the right half planes, and the corresponding Wiener processes du_1 maps the state space

X into a suitable pair of function spaces ($H(V(s))$ and $\overline{H}(U^*(s))$), the first being a space of analytic functions, the second being a space of coanalytic functions, [equations (3.13)]. It is then natural to embed the function spaces corresponding to $W(s)$ and $\overline{W}(s)$ into the pairs $(H(K(s)), \{0\})$ and $(\{0\}, \overline{H}(K^*(s)))$ respectively.

It is well-known that, in a Hilbert space setting, both in the deterministic and in the stochastic case, the state space, as a function space,is the orthogonal complement of an invariant subspace, see [5], [10] and [11]. For example, to the analytic spectral factor $W(s)$, there corresponds the space of analytic functions $H(K(s))$ which is the orthogonal complement in H^2_m of the invariant subspace $H^2_m K(s)$. Symmetrically, to $\overline{W}(s)$ there corresponds the space of coanalytic functions $\overline{H}(K^*(s))$ which is the orthogonal complement, in $\overline{H}^2_m$, of $\overline{H}^2_m K^*(s)$. In the general case, the state space, as a function space, may be decomposed into the orthogonal complement of an invariant subspace of H^2, due to the analytic part of the spectral factor, and the orthogonal complement of an invariant subspace of $\overline{H}^2$, due to the coanalytic part. In the extreme cases (analytic and coanalytic) one of the two spaces vanishes.

4 Main Result

To state our main result, it is useful to introduce the inner function

$$U_M(s) := \overline{W}_-(s)^{-1} W_+(s). \tag{4.1}$$

This function plays a role similar to that of the multivariate phase function $T_0(s) = \overline{W}_+(s)^{-1} W_-(s)$. In fact both $U_M(s)$ and $T_0(s)$ are all-pass function and carry all the information about singularities and zeroes (i.e. singularity of the inverse) of the spectral density $\Phi(s)$. In contrast to the phase function, however, $U_M(s)$ is analytic on the closed right half plane i.e. it is an inner function, and this will be of fundamental importance below.

If A is a square matrix we shall denote by $|A|$ the determinant of A.

Theorem 4.1 *Let $y(t)$ be a real, m-dimensional, mean square continuous process with stationary increments and let $\Phi(s)$ be spectral density of dy which is assumed to be coercive. Assume that dy is strictly noncyclic and that $|K_+(s)|$ and $|Q_+(s)|$ are coprime:*

$$(|K_+(s)|, |Q_+(s)|) = I. \tag{4.2}$$

Then there is a one to one correspondence between the set $\mathcal{U}$ of left inner divisors of $U_M(s)$ and the set $\mathcal{F}$ of minimal spectral factors of $\Phi(s)$. More precisely, we have:

$$\mathcal{F} = \{W(s) = \overline{W}_-(s)U(s) : U(s) \in \mathcal{U}\}. \tag{4.3}$$

We first prove the inclusion

$$\mathcal{F} \subset \{W(s) = \overline{W}_-(s)U(s) : U(s) \in \mathcal{U}\}. \tag{4.4}$$

Proof of (4.4): By definition there exists a minimal analytic spectral factor $W(s)$ such that the matrix $V(s)$ defined by:

$$V(s) := W_1^{-1}(s)W(s) \tag{4.5}$$

is inner. Every minimal analytic spectral factor $W(s)$ can be written as in (2.15) and $Q(s)$ is a left inner divisor of $Q_+(s)$ [10, prop 7.2]. Substituting (2.15) in (4.5) we get $W_-(s) = W_1(s)V(s)Q^*(s)$ and multiplying this equation on the right side by $Q_+(s) = W_-^{-1}W_+$ we get

$$W_+(s) = W_1(s)V(s)Q^*(s)Q_+(s) = W_1(s)V_1(s), \tag{4.6}$$

where $V_1(s)$, defined by $V_1(s) = Q^*(s)Q_+(s)$, is inner because $Q(s)$ is a left inner divisor of $Q_+(s)$. In a completely symmetric way we can find an inner function $U_1(s)$ such that:

$$W_1(s) = \overline{W}_-(s)U_1(s). \tag{4.7}$$

Comparing equations (4.6) and (4.7) with the definition of $U_M(s)$, it follows immediately that $U_1(s)$ is a left inner divisor of $U_M(s)$. ■

The inclusion

$$\mathcal{F} \supset \{W(s) = \overline{W}_-(s)U(s) : U(s) \in \mathcal{U}\} \tag{4.8}$$

is quite difficult to prove. To this end we shall establish some preliminary results which, however, appear to be of independent interest. Before doing this we collect below some relevant observations.

Remark 4.1 The correspondence (4.3) between minimal spectral factors and left inner divisors of $U_M(s)$ was first established, for rational spectral densities, in [3], where the following theorem was proved.

Theorem 4.2 *Let $\Phi(s)$ be a rational coercive spectral density and let*

$$\overline{W}_-(s) = C(sI + A^T)^{-1}B + D \tag{4.9}$$

be a minimal realization of the spectral factor $\overline{W}_-(s)$, and $\Gamma := -A^T - BD^{-1}C$ be its zero matrix. Assume that A and Γ have nonintersecting spectra:

$$\sigma(A) \cap \sigma(\Gamma) = \emptyset. \tag{4.10}$$

Then there is a one to one correspondence between the set $\mathcal{U}$ of left inner divisors of $U_M(s)$ and the set $\mathcal{F}$ of minimal spectral factors of $\Phi(s)$. More precisely, we have:

$$\mathcal{F} = \{W(s) = \overline{W}_-(s)U(s) : U(s) \in \mathcal{U}\}. \tag{4.11}$$

It is worth noticing that, in the rational case, condition (4.2) has been recently proved to be also necessary, [2]. Precisely, if it fails, there always exist left inner divisors of $U_M(s)$ such that the corresponding spectral factors are not minimal.

Theorem (4.2) is very similar to Theorem (4.1) except for the condition (4.10) which does not apply in the nonrational setup where the realization (4.9) is no longer available. It is also worth noticing that, in the rational case, strict noncyclicity is not a restrictive condition. In fact, in that case, $T_0(s)$ is rational and therefore it is strictly noncyclic.

Although, at the first glance, the two conditions (4.10) and (4.2) seem very different they are strictly connected. In fact, in the next proposition we see that (4.2), in the rational case coincides exactly with (4.10).

Proposition 4.1 *In the rational case the two conditions (4.2) and (4.10) coincide.*

Proof: Let (4.9) be a minimal realization of $\overline{W}_-(s)$. Then we know from [3] that the inner functions $K_-(s)$ and $Q_+(s)$ have the following minimal realizations: $K_-(s) = I + B^T(sI - A)^{-1}Q_M B$ and $Q_+(s) = I - R^{-1/2}H(sI - \Gamma)^{-1}P_M H^T R^{-1/2}$, where Q_M is the unique symmetric negative definite solution of the ARE:

$$-AQ - QA^T + QBB^TQ = 0. \tag{4.12}$$

P_M the unique symmetric positive definite solution of the ARE:

$$\Gamma P + P\Gamma^T + PH^TR^{-1}HP = 0 \tag{4.13}$$

and $H := R^{1/2}B^TQ_M + C$. The determinants of $K_-(s)$ and of $Q_+(s)$ are clearly finite Blaschke products and we have

$$\{Poles(|K_-(s)|)\} \subset \{Poles(K_-(s))\} \subset \{\sigma(A)\}, \tag{4.14}$$

similarly

$$\{Poles(|Q_+(s)|)\} \subset \{Poles(Q_+(s))\} \subset \{\sigma(\Gamma)\}. \tag{4.15}$$

And then, taking into account the well-known fact that $|K_-(s)| = |K_+(s)|$ [10], condition (4.10) implies condition (4.2).

Conversely we have that condition (4.2) implies equation (4.3). Of course this fact still holds when specialized to the case of rational spectral density, therefore, in particular in the rational case, we have that condition (4.2) implies equation (4.11). But we have already noticed that condition (4.10) is equivalent to equation (4.11). ∎

Remark 4.2 In the proof of the first part of the previous theorem we have not used condition (4.2). Hence, clearly, the inclusion (4.4) holds true also if condition (4.2) fails. Conversely, there are examples in which condition (4.2) fails and there exist left inner divisors of $U_M(s)$ producing spectral factors that are not minimal. Consider the following case taken from [3]:

$$\Phi(s) = \begin{bmatrix} \frac{s^2-1}{s^2-4} & 0 \\ 0 & \frac{s^2-4}{s^2-1} \end{bmatrix} \qquad \overline{W}_-(s) = \begin{bmatrix} \frac{s+1}{s-2} & 0 \\ 0 & \frac{s+2}{s-1} \end{bmatrix} \tag{4.16}$$

and

$$U_M(s) = \frac{(s-2)(s-1)}{(s+2)(s+1)} I_2 \qquad U_1(s) = \frac{1}{s+1} \begin{bmatrix} s & 1 \\ 1 & s \end{bmatrix}. \tag{4.17}$$

$U_1(s)$ is a left inner divisor of $U_M(s)$ but

$$W_1(s) = \overline{W}_-(s) U_1(s) = \begin{bmatrix} \frac{s}{s-2} & \frac{1}{s-2} \\ \frac{s+2}{s^2-1} & \frac{s(s+2)}{s^2-1} \end{bmatrix} \tag{4.18}$$

is not minimal. Thus condition (4.2) may perhaps be weakened but certainly not left out to prove the second part of Theorem 4.1.

Remark 4.3 In the scalar case ($m = 1$) condition (4.2) is automatically satisfied. In fact, in this situation $|K_+(s)| = K_+(s)$ and $|Q_+(s)| = Q_+(s)$ and it is well-known [10] that $K_+(s)$ and $Q_+(s)$ are coprime.

5 Lemmas on Inner Functions

Lemma 5.1 *Let $u(s)$, $q(s)$ and $k(s)$be scalar inner functions and suppose that $u(s)$ is coprime with both $q(s)$ and $k(s)$. Then $u(s)$ is coprime with the product $q(s)k(s)$.*

Proof: It is well-known that two inner function $j(s)$ and $v(s)$ are coprime if and only if $j(s)H^2 \vee v(s)H^2 = H^2$. Therefore we have:

$$q(s)H^2 \vee u(s)H^2 = H^2 \tag{5.1a}$$

$$k(s)H^2 \vee u(s)H^2 = H^2. \tag{5.1b}$$

From (5.1a)it readily follows that

$$q(s)k(s)H^2 \vee u(s)k(s)H^2 = k(s)H^2. \tag{5.2}$$

Plugging (5.2) into (5.1b) we get

$$q(s)k(s)H^2 \vee u(s)k(s)H^2 \vee u(s)H^2 = H^2, \tag{5.3}$$

where we have used the fact that $(q(s)k(s)H^2 \vee u(s)k(s)H^2) \vee u(s)H^2 = q(s)k(s)H^2 \vee (u(s)k(s)H^2 \vee u(s)H^2)$. Since $k(s)H^2 \subset H^2$, and consequently $u(s)k(s)H^2 \vee u(s)H^2 = u(s)H^2$, (5.3) gives:

$$q(s)k(s)H^2 \vee u(s)H^2 = H^2, \tag{5.4}$$

and this yields the assertion. ∎

Corollary 5.1 *Let $u(s)$, $q(s)$ and $k(s)$be scalar inner functions and suppose that $u(s)$ is an inner divisor of $q(s)k(s)$. Then $u(s)$ can be factored as $u(s) = u_q(s)u_k(s)$ where $u_q(s)$ is an inner divisor of $q(s)$ and $u_k(s)$ is an inner divisor of $k(s)$.*

Proof: Let u_q be defined as follows:

$$u_q(s) := g.c.i.d.(u(s), q(s)) \tag{5.5}$$

and let $u_1(s)$ and $q_1(s)$ be the two inner functions defined by:

$$u_1(s) := u(s)u_q^*(s), \tag{5.6a}$$

$$q_1(s) := q(s)u_q^*(s). \tag{5.6b}$$

Obviously $u_1(s)$ and $q_1(s)$ are coprime. Now define u_k by:

$$u_k(s) := g.c.i.d.(u_1(s), k(s)). \tag{5.7}$$

Again let u_2 and k_2 be the two coprime inner functions defined by:

$$u_2(s) := u_1(s)u_k^*(s), \tag{5.8a}$$

$$k_2(s) := k(s)u_k^*(s). \tag{5.8b}$$

We then have

$$u(s) = u_q(s)u_k(s)u_2(s). \tag{5.9}$$

In view of the coprimeness between $q_1(s)$ and $u_1(s)$ we also have

$$(u_2(s), q_1(s)) = 1. \tag{5.10}$$

By hypothesis, $u(s)$ divides $q(s)k(s)$ i.e. that there exists an inner function $j(s)$ such that $u(s)j(s) = q(s)k(s)$. Substituting (5.9), (5.8b) and (5.6b) in this equation we get:

$$u_2(s)j(s) = q_1(s)k_2(s). \tag{5.11}$$

We can now apply Lemma 5.1 to (5.11) concluding that $u_2(s)$ is a constant of absolute value one. The conclusion then follows from (5.9). ∎

Lemma 5.2 *Let $T(s) \in \mathcal{L}^{\infty}_{m\times m}$ be an all-pass matrix. Then $T(s)$ admits a coprime inner-coinner factorization:*

$$T(s) = U(s)J^*(s) \quad U(s), J(s) \text{ inner and } \left(U(s), J(s)\right)_R = I \tag{5.12}$$

if and only if it admits a coprime coinner-inner factorization:

$$T(s) = K^*(s)V(s) \quad V(s), K(s) \text{inner and } \left(V(s), K(s)\right)_L = I. \tag{5.13}$$

Moreover these two factorizations are unique up to constant orthogonal matrices and the two relations $|U(s)| = |V(s)|$ and $|J(s)| = |K(s)|$ hold.

Proof: Suppose that $T(s)$ admits the coprime factorization (5.12). Then, as it is shown in [5], the closure of the range of the Hankel operator with symbol T is:

$$\overline{Range}(H_T) = H^2_m \ominus H^2_m U(s). \tag{5.14}$$

It follows that $T(s)$ is strictly noncyclic, and hence it admits the coprime factorization (5.13). The converse is similar. From (5.13) it follows that the kernel of the Hankel operator with symbol T is $KerH_T = \overline{H}^2_m V^*(s)$ and then, in view of Beurling-Lax Theorem [5], [6], $V(s)$ is unique up to multiplication on the left side by constant orthogonal matrices. In the same way, from equation (5.14) it follows that the factorization (5.12) is essentially unique. Finally suppose that $T(s)$ admits the two coprime factorizations (5.12) and (5.13). We can then write also $K(s)U(s) = V(s)J(s)$ with $\left(U(s), J(s)\right)_R = I$ and $\left(V(s), K(s)\right)_L = I$. As it is shown in [8, Lemma 3] this is the same as saying that $U(s)$ and $V(s)$ are quasi-equivalent. Hence we have $|U(s)| = |V(s)|$. The condition $|J(s)| = |K(s)|$ is now obvious. ∎

Lemma 5.3 *Let $J(s), K(s), U(s), V(s)$ be four inner matrix functions such that: $J(s) = U(s)K^*(s)V(s)$ and $K(s)$ is right coprime with $U(s)$ and left coprime with $V(s)$. Then $K(s)$ is a constant orthogonal matrix.*

Proof: We have that $K^*(s)V(s) = U^*(s)J(s)$ where the first factorization is coprime, but the second need not be. Let then $U_d(s)$ be the greatest common left inner divisor of $U(s)$ and $J(s)$, and $U_1(s)$ and $J_1(s)$ the two inner matrix functions (left inner coprime) such that: $U(s) = U_d(s)U_1(s)$ and $J(s) = U_d(s)J_1(s)$. With this notation, we have the two coprime factorizations:

$$K^*(s)V(s) = U_1^*(s)J_1(s). \tag{5.15}$$

As the coprime coinner-inner factorization is unique up to constant orthogonal matrices (Lemma 5.2) we can assume $U_1(s) = K(s)$. But $U_1(s)$ is a right inner divisor of $U(s)$ and hence, in view of right coprimeness between $U(s)$ and $K(s)$, $U_1(s)$ (and then also $K(s)$) must be a constant orthogonal matrix. ∎

Corollary 5.2 *Let $J(s), K(s), U(s), V_1(s), Q(s)$ be five inner matrix functions such that $J(s)Q^*(s) = U(s)K^*(s)V_1(s)$ are two coprime factorization of the same all pass matrix. Then $U(s)$ is a left inner divisor of $J(s)$.*

Proof: It is easy to see that the function $U(s)K^*(s)V_1(s)Q(s)$ is inner. We have that $U(s)$ and $K(s)$ are right coprime and then, in view of previous lemma, $K(s)$ is a left inner divisor of $V_1(s)Q(s)$. The conclusion is now clear because $K^*(s)V_1(s)Q(s)$ is inner. ■

Lemma 5.4 *Let $U(s)$ be an $(m \times m)$ inner matrix function. Then $U(s)$ admits the inner factorization $U(s) = Q(s)K(s)$ if and only if the scalar inner function $|U(s)|$ admits the inner factorization $|U(s)| = q(s)k(s)$ with $|Q(s)| = q(s)$ and $|K(s)| = k(s)$.*

Proof: If $U(s) = Q(s)K(s)$, it is obvious that $|U(s)| = |Q(s)||K(s)| = q(s)k(s)$. Conversely, let $|U(s)| = q(s)k(s)$. $U(s)$ is quasi-equivalent to a diagonal matrix

$$\Delta(s) = diag(\delta_1(s), \delta_2(s), \ldots \delta_m(s)), \tag{5.16}$$

where $\delta_i(s)$ are the invariant factors of $U(s)$ [5]. We have

$$|U(s)| = |\Delta(s)| = \delta_1(s)\delta_2(s)\ldots\delta_m(s) = q(s)k(s). \tag{5.17}$$

It now follows from Corollary 5.1 that each $\delta_i(s)$ admits the inner factorization $\delta_i(s) = q_i(s)k_i(s)$ where $q_i(s)$ and $k_i(s)$ are inner divisors respectively of $q(s)$ and $k(s)$. Define two diagonal inner matrices

$$\Delta_q(s) = diag(q_1(s), q_2(s), \ldots q_m(s)), \tag{5.18a}$$
$$\Delta_k(s) = diag(k_1(s), k_2(s), \ldots k_m(s)). \tag{5.18b}$$

It is clear that $|\Delta_q(s)| = q(s)$, $|\Delta_k(s)| = k(s)$ and

$$\Delta(s) = \Delta_q(s)\Delta_k(s). \tag{5.19}$$

Quasi equivalence between $U(s)$ and $\Delta(s) = \Delta_q(s)\Delta_k(s)$ implies that there exist two matrix functions $A(s), B(s) \in H^\infty_{m\times m}$ such that:

$$A(s)U(s) = \Delta_q(s)\Delta_k(s)B(s), \tag{5.20a}$$
$$(A(s), \Delta_q(s)\Delta_k(s))_L = I, \qquad (U(s), B(s))_R = I. \tag{5.20b}$$

Let us now define the inner matrix $U_k(s)$ as the greatest common right inner divisor of $U(s)$ and $\Delta_k(s)B(s)$. Then there exist a matrix function $U_q(s)$ (inner) and $C(s)$ (in H^∞) such that $U(s) = U_q(s)U_k(s)$ and $\Delta_k(s)B(s) = C(s)U_k(s)$. Hence:

$$A(s)Uq(s) = \Delta_q(s)C(s), \quad (A(s), \Delta_q(s))_L = I, \quad (U_q(s), C(s))_R = I. \tag{5.21}$$

The last equation with the coprimeness conditions implies that $U_q(s)$ is quasi-equivalent to $\Delta_q(s)$ [8], and hence $|U_q(s)| = |\Delta_q(s)| = q(s)$. Obviously, we also have $|U_k(s)| = k(s)$.

■

6 Completion of Proof of Theorem 4.1

We now prove the second part of Theorem 4.1 using the previous results.

Proof of inclusion (4.8): Let $U_1(s)$ be a left inner divisor of $U_M(s)$ and let $W_1 := \overline{W}_-(s)U_1(s)$. Then $|U_1(s)|$ is an inner divisor of $|U_M(s)| = |\overline{Q}_-^*(s)||K_+(s)|$. In view of Corollary 5.1, $|U_1(s)| = q_1(s)k_1(s)$ where $q_1(s)$ divides $|\overline{Q}_-^*(s)|$ and $k_1(s)$ divides $|K_+(s)|$. We are now under the hypotheses of Lemma 5.4. We then know that $U_1(s)$ admits the inner factorization

$$U_1(s) = U_q(s)U_k(s), \tag{6.1}$$

where $|U_q(s)| = q_1(s)$ and $|U_k(s)| = k_1(s)$. Thus we have the coprimeness:

$$(|U_k(s)|, |Q_+(s)| = |\overline{Q}_-^*(s)|) = I. \tag{6.2}$$

The next step is to show that $U_q(s)$ is a left inner divisor of $\overline{Q}_-^*(s)$. As $U_1(s)$ is a left inner divisor of $U_M(s)$, there exists an inner matrix function $V_1(s)$ such that: $U_M(s) = \overline{Q}_-^*(s)K^+(s) = U_1(s)V_1(s)$. Multiplying this equation on the left side by $U_q^*(s)$ we get:

$$U_q^*(s)\overline{Q}_-^*(s)K_+(s) = U_k(s)V_1(s). \tag{6.3}$$

Let us now define the inner matrix $U_d(s)$ as the greatest common left inner divisor of $U_q(s)$ and $\overline{Q}_-^*(s)$, and $Q_r(s)$ and $U_r(s)$ the two inner matrix functions (left inner coprime) such that $U_q(s) = U_d(s)U_r(s)$ and $\overline{Q}_-^*(s) = U_d(s)Q_r(s)$. With this notation the all-pass function $U_q^*(s)\overline{Q}_-^*(s)$ admits the coprime coinner-inner factorization: $U_q^*(s)\overline{Q}_-^*(s) = U_r^*(s)Q_r(s)$. It then admits also the corresponding inner-coinner coprime factorization (Lemma 5.2)

$$U_q^*(s)\overline{Q}_-^*(s) = Q_l(s)U_l^*(s). \tag{6.4}$$

Plugging this equation into (6.3) yields:

$$Q_l(s)U_l^*(s)K_+(s) = U_k(s)V_1(s). \tag{6.5}$$

From Lemma 5.2, it follows that $|U_l(s)| = |U_r(s)|$. It is now apparent that $|U_l(s)|$ is coprime with $|K_+(s)|$. *A fortiori* $U_l(s)$ and $K_+(s)$ must be right coprime. Moreover $U_k(s)V_1(s)$ is inner. We can then apply Lemma 5.3 to equation (6.5), concluding that $U_l(s)$ is a constant orthogonal matrix which is the same as saying that

$$U_q(s) \text{ is a left inner divisor of } \overline{Q}^*_-(s). \tag{6.6}$$

In a completely symmetric way, we can prove that $V_1(s)$ admits the inner factorization $V_1(s) = V_k(s)V_q(s)$ where $V_q(s)$ is a right inner divisor of Q_+ and $|V_k|$ divides $|K_+(s)| = |K_-(s)|$, and then the coprimeness:

$$(|V_k|, |Q_+(s)| = |Q_-(s)|) = I \tag{6.7}$$

holds.

We can now define a couple $(W(s), \overline{W}(s))$ of analytic and coanalytic spectral factors in the following way:

$$W(s) := W_+(s)V_q^*(s) = W_-(s)Q_+(s)V_q^*(s) \tag{6.8a}$$

$$\overline{W}(s) := \overline{W}_-(s)U_q(s) = \overline{W}_+(s)\overline{Q}_-(s)U_q(s). \tag{6.8b}$$

The spectral factor $W(s)$ is minimal and analytic because $V_q(s)$ is a right inner divisor of $Q_+(s)$. [10, Prop. 7.2]. In the same way $\overline{W}(s)$ is a minimal coanalytic spectral factor because $U_q(s)$ is a left inner divisor of $\overline{Q}^*_-(s)$

We now prove that the couple $(W(s), \overline{W}(s))$ is a minimal L.P. pair. To this end we define the inner function $K(s)$ by $K(s) := \overline{W}^{-1}(s)W(s) = U_k(s)V_k(s)$. It is clear that $K(s)$ is a structural function and that the two relations of coprimeness:

$$(K(s), U_q^*(s)\overline{Q}^*_-(s))_L = I \tag{6.9a}$$

$$(K(s), Q_+(s)V_q^*(s))_R = I \tag{6.9b}$$

follow respectively from (6.2) and (6.7).

It is now easy to show that $W_1(s) := \overline{W}_-(s)U_1(s)$ is minimal in fact $W_1^{-1}(s)W(s) = V_k(s)$ and $\overline{W}^{-1}(s)W_1(s) = U_k(s)$ are inner and, as we noticed above, $(W(s), \overline{W}(s))$ is a minimal L.P. pair: Therefore $W_1(s)$ is minimal by definition. ■

7 Some Ancillary Results

Let $W_1(s)$ be a minimal spectral factor of $\Phi(s)$. We have proved that $W_1(s)$ is uniquely determined by a left inner divisor $U_1(s)$ of $U_M(s)$ such that $W_1(s) = \overline{W}_-(s)U_1(s)$ holds. The inner matrix function $U_1(s)$ is in one-to-one correspondence with another inner $V_1(s)$ such that $U_1(s)V_1(s) = U_M(s)$. Moreover $U_1(s)$ and $V_1(s)$ admit the two inner factorizations $U_1(s) = U_q(s)U_k(s)$ and $V_1(s) = V_k(s)V_q(s)$ and the Lindquist-Picci pair associated to $W_1(s)$ is given by (6.8a) and (6.8b).

Lemma 7.1 *The inner matrix function $U_q(s)$ defined by (6.1) is the greatest common left inner divisor of $\overline{Q}^*_-(s)$ and $U_1(s)$. Analogously, $V_q(s)$ is the greatest common right inner divisor of $Q_+(s)$ and $V_1(s)$.*

Proof: The inner function $U_q(s)$ is by definition a left inner divisor of $U_1(s)$. As we showed before (eq. (6.6)) $U_q(s)$ is also a left inner divisor of $\overline{Q}^*_-(s)$. Define the coinner function $\overline{Q}(s)$ by:

$$\overline{Q}^*_-(s) = U_q(s)\overline{Q}^*(s). \tag{7.1}$$

It remains to prove that $\overline{Q}^*(s)$ and $U_k(s)$ are left coprime: it is clear that $|\overline{Q}^*(s)|$ divides $|\overline{Q}^*_-(s)|$ and that $|U_k(s)|$ divides $|K_-(s)|$ which is coprime with $|\overline{Q}^*_-(s)|$. Thus $|\overline{Q}^*(s)|$ and $|U_k(s)|$ are coprime and, *a fortiori*, $\overline{Q}^*(s)$ and $U_k(s)$ are left coprime. Similarly, it can be shown that $V_q(s) = g.c.r.i.d.(Q_+(s), V_1(s))$ defining $Q(s)$ as the inner function such that

$$Q_+(s) = Q(s)V_q(s). \tag{7.2}$$

■

We have already noticed that $K(s) = U_k(s)V_k(s)$ is the structural function of the couple $(W(s), \overline{W}(s))$, moreover, from (7.2), (7.1), (6.8a) and (6.8b), it follows that $Q(s)$ and $\overline{Q}(s)$ are respectively the inner and the coinner factor in (2.15) and (2.16). Thus we can associate to $U_1(s)$ the inner triplet $(Q(s), K(s), \overline{Q}^*(s))$ related to the state space X of all minimal spectral factors $\widetilde{W}(s)$ ($W_1(s)$ is one of these) such that $\overline{W}^{-1}(s)\widetilde{W}(s)$ and $\widetilde{W}^{-1}(s)W(s)$ are inner. Conversely any inner triplet $(Q(s), K(s), \overline{Q}^*(s))$ with an inner factorization of the structural function $K(s)$ defines via a straightforward calculation the inner function $U_1(s)$. We have then proved the following proposition.

Proposition 7.1 *Suppose that $\Phi(s)$ is strictly noncyclic. Then under condition (4.2) there exists a one-to-one correspondence between minimal spectral factors of $\Phi(s)$ (or equivalently left inner divisors of $U_M(s)$) and inner quadruples $(Q(s), U_k(s), V_k(s), \overline{Q}^*(s))$ where $Q(s)$ and $\overline{Q}^*(s)$ are the inner and coinner factor of a minimal L.P. pair and $U_k(s)V_k(s)$ is an inner factorization of the structural function $K(s)$ of the same minimal L.P. pair.*

We have shown that $U_1(s)$ admits the inner factorization (6.1). In a completely dual way it can be shown that $U_1(s)$ can be also factored as

$$U_1(s) = \widetilde{U}_k(s)\widetilde{U}_q(s), \tag{7.3}$$

where $\widetilde{U}_k(s)$ is a left inner divisor of $K_-(s)$ and $|\widetilde{U}_q(s)|$ divides $|Q_+(s)|$. Similarly the inner function $V_1(s)$ may be factored as

$$V_1(s) = \widetilde{V}_q(s)\widetilde{V}_k(s), \tag{7.4}$$

where $|\widetilde{V}_q(s)|$ divides $|Q_+(s)|$ and $\widetilde{V}_k(s)$ is a right inner divisor of $K_+(s)$. Define another pair, $(\widetilde{W}_-(s), \widetilde{W}_+(s))$, of minimal spectral factors:

$$\widetilde{W}_-(s) := \overline{W}_-(s)\widetilde{U}_k(s), \tag{7.5a}$$

$$\widetilde{W}_+(s) := W_+(s)\widetilde{V}_k^*(s). \tag{7.5b}$$

We show now that the inverse of the spectral factor $\widetilde{W}_-(s)$ is analytic in the open right half plane. In fact we have: $\widetilde{W}_-^{-1}(s) = \widetilde{U}_k^*(s)\overline{W}_-^{-1}(s) = \widetilde{U}_k^*(s)K_-(s)W_-^{-1}(s)$ and this conclude the proof because $\widetilde{U}_k(s)$ is a left inner divisor of $K_-(s)$ and $W_-(s)$ is analytic with its inverse in the open right half plane. In the same way it can be proved that the inverse of the spectral factor $\widetilde{W}_+(s)$ is analytic in the open left half plane.

Equations (7.3) and (7.4) relate each minimal spectral factor $W_1(s)$ (or equivalently each left inner divisor of $U_M(s)$) to a couple $(\widetilde{W}_-(s), \widetilde{W}_+(s))$ defined by (7.5a) and (7.5b), via two inner matrix functions. More precisely we have that the two matrix functions: $\widetilde{W}_-^{-1}(s)\overline{W}_1(s) = \widetilde{U}_q$ and $W_1^{-1}\widetilde{W}_+ = \widetilde{V}_q$ are inner. We can now associate to each minimal spectral factor $W_1(s)$ the matrix function $\widetilde{Q}(s)$ (which is obviously inner) defined by

$$\widetilde{Q}(s) := \widetilde{W}_-^{-1}(s)\widetilde{W}_+(s) = \widetilde{U}_q(s)\widetilde{V}_q(s). \tag{7.6}$$

We shall say that $\widetilde{Q}(s)$ is the *zero function* of the spectral factor $W_1(s)$.

The following lemma is the dual version of Theorem 7.5 in [10].

Lemma 7.2 *Suppose that $\Phi(s)$ is strictly noncyclic and that (4.2) holds. Let $\widetilde{Q}_1(s)$ and $\widetilde{Q}_2(s)$ the zero functions of two minimal spectral factors. Then $\widetilde{Q}_1(s)$ and $\widetilde{Q}_2(s)$ are quasi-equivalent. In particular each $\widetilde{Q}(s)$ is quasi-equivalent to $Q_+(s)$ and to $\overline{Q}_-^*(s)$.*

Proof: On the one hand we have:

$$\overline{W}_-^{-1}(s)\widetilde{W}_+(s) = \overline{W}_-^{-1}(s)\overline{W}_+^{-1}(s)\overline{W}_+(s)\widetilde{W}_+(s) = \overline{Q}_-^*(s)K_+(s)\widetilde{V}_k^*(s) \tag{7.7}$$

On the other hand:

$$\overline{W}_-^{-1}(s)\widetilde{W}_+(s) = \overline{W}_-^{-1}(s)\widetilde{W}_-(s)\widetilde{W}_-^{-1}(s)\widetilde{W}_+(s) = \widetilde{U}_k(s)\widetilde{Q}(s) \tag{7.8}$$

But we have already proved that $\widetilde{V}_k$ is a right inner divisor of $K_+(s)$, hence the matrix function defined by:

$$\widetilde{K}_+(s) := K_+(s)\widetilde{V}_k^*(s). \tag{7.9}$$

is inner. Comparing these three equations we get:

$$\overline{Q}_-^*(s)\widetilde{K}_+(s) = \widetilde{U}_k(s)\widetilde{Q}(s). \tag{7.10}$$

As we have noted before $\widetilde{U}_k(s)$ is a left inner divisor of $K_-(s)$ and then $|\widetilde{U}_k(s)|$ is coprime with $|\overline{Q}^*_-(s)|$. *A fortiori*, $\widetilde{U}_k(s)$ and $\overline{Q}^*_-(s)$ are left inner coprime. On the other side $\widetilde{K}_+(s)$ is a right inner divisor of $K_+(s)$ and then $|\widetilde{K}_+(s)|$ is coprime with $|\widetilde{U}_q(s)|$ and $|\widetilde{V}_q(s)|$. In view of Lemma 5.1 we can conclude that $|\widetilde{K}_+(s)|$ is coprime also with the product $|\widetilde{U}_q(s)||\widetilde{V}_q(s)| = \widetilde{Q}(s)$. In particular, this implies that $\widetilde{K}_+(s)$ and $\widetilde{Q}(s)$ are right coprime. We can now apply Lemma 3 in [8] and conclude that $\overline{Q}^*_-(s)$ and $\widetilde{Q}(s)$ are quasi-equivalent. The proof now follows from the symmetry and transitivity of the quasi-equivalence relation [5]. ■

As straightforward consequences of this lemma we have the following corollaries:

Corollary 7.1 *Suppose that $\Phi(s)$ is scalar and strictly noncyclic. Then all minimal spectral factors have the same zero function.*

Notice that in the scalar case condition (4.2) is automatically guaranteed.

Corollary 7.2 *To each minimal spectral factor $W_1(s)$ we can associate a zero function $\widetilde{Q}(s)$ and two inner matrix functions $\widetilde{K}_+(s)$ defined by (7.9) and $\widetilde{K}_-(s)$ defined by:*

$$\widetilde{K}_-(s) := \widetilde{U}_k^*(s) K_-(s). \tag{7.11}$$

Among these inner functions the two following relations of coprimeness hold:

$$(\widetilde{K}_+(s), \widetilde{Q}(s))_R = I \quad \text{and} \quad (\widetilde{K}_-(s), \widetilde{Q}(s))_L = I. \tag{7.12}$$

Thus, if $W_1(s)$ is minimal, the phase function $T_0(s) = \overline{W}_+^{-1}(s) W_-(s)$ admits the coprime factorization:

$$T_0(s) = \widetilde{K}_+(s)\widetilde{Q}^*(s)\widetilde{K}_-(s). \tag{7.13}$$

Notice that this factorization is dual respect to the one exhibited in [10]. The following proposition, which is a dual version of Proposition 7.1, summarizes the previous results.

Proposition 7.2 *Suppose that $\Phi(s)$ is strictly noncyclic. Then, under condition (4.2), there exists a one-to-one correspondence between minimal spectral factors of $\Phi(s)$ (or equivalently left inner divisors of $U_M(s)$) and inner quadruples $(\widetilde{K}_-(s), \widetilde{U}_q(s), \widetilde{V}_q(s), \widetilde{K}_+(s))$ such that, defining $\widetilde{Q}(s) := \widetilde{U}_q(s)\widetilde{V}_q(s)$, the phase function $T_0(s)$ admits the doubly coprime factorization $T_0(s) = \widetilde{K}_+(s)\widetilde{Q}^*(s)\widetilde{K}_-(s)$ and there exists a couple of minimal spectral factors $(\widetilde{W}_-(s), \widetilde{W}_+(s))$ whose inverse are analytic respectively in the right and in the left open half plane, such that $\widetilde{Q}(s) = \widetilde{W}_-^{-1}(s)\widetilde{W}_+(s)$.*

Proof: Corollary 7.2 proves one direction. Conversely we have the two coprime factorization of $T_0(s)$: $T_0(s) = \widetilde{K}_+(s)\widetilde{Q}^*(s)\widetilde{K}_-(s) = K_+(s)Q_+^*(s)$. We can apply Corollary 5.2 and conclude that $\widetilde{K}_+(s)$ is a left inner divisor of $K_+(s)$. Symmetrically $\widetilde{K}_-(s)$ is a right inner divisor of $K_-(s)$. This consideration proves that the function $U_1(s)$ defined by $U_1(s) := K_-(s)\widetilde{K}_-^*(s)\widetilde{U}_q(s)$ is an inner function and it is an inner divisor of $U_M(s)$. ■

The set $\mathcal{U}$ of the left inner divisors of $U_M(s)$ is endowed with a partial order which is naturally induced by the division relation. More precisely, denoting with the symbol $\prec$ this partial order, $U_1(s) \prec U_2(s)$ means that $U_1(s)$ is a left inner divisor of $U_2(s)$. (Here $U_1(s), U_2(s) \in \mathcal{U}$). It is well-known that this partial ordering induces in $\mathcal{U}$ a lattice structure with a maximal element $U_M(s)$ and a minimal one which is obviously the identity. This structure is clearly inherited by the set $\mathcal{F}$ of the minimal spectral factors of $\Phi(s)$. In this sense we can say that $\overline{W}_-(s)$ is the minimal element of $\mathcal{F}$ and $W_+(s)$ is the maximal one. This lattice structure can be visualized in the following commutative diagram which, in view of propositions 7.1 and 7.2, holds true for every element $W_1(s)$ in $\mathcal{F}$:

$$
\begin{array}{ccccc}
\overline{W}_-(s) & \xrightarrow{\;U_q(s)\;} & \overline{W}(s) & \xrightarrow{\;\overline{Q}^*(s)\;} & \overline{W}_+(s) \\
\Big\downarrow{\scriptstyle \widetilde{U}_k(s)} & & \Big\downarrow{\scriptstyle U_k(s)} & & \Big\downarrow{\scriptstyle \widetilde{K}_+(s)} \\
\widetilde{W}_-(s) & \xrightarrow{\;\widetilde{U}_q(s)\;} & W_1(s) & \xrightarrow{\;\widetilde{V}_q(s)\;} & \widetilde{W}_+(s) \\
\Big\downarrow{\scriptstyle \widetilde{K}_-(s)} & & \Big\downarrow{\scriptstyle V_k(s)} & & \Big\downarrow{\scriptstyle \widetilde{V}_k(s)} \\
W_-(s) & \xrightarrow{\;Q(s)\;} & W(s) & \xrightarrow{\;V_q(s)\;} & W_+(s)
\end{array}
\tag{7.14}
$$

In this diagram the notation $W_1(s) \xrightarrow{U_2(s)} W_3(s)$ means that $U_2(s)$ is an inner function such that $U_2(s) = W_1^{-1}(s)W_3(s)$ and it is inner. Hence, in the partial order of the set $\mathcal{F}$, $W_3(s)$ is greater then $W_1(s)$.

7.1 A geometric parametrization

In this section we give a parametrization of the subspaces generated by the past and the future of the Wiener processes corresponding to minimal spectral factors.

Proposition 7.3 *Suppose that dy is strictly noncyclic and that the coprimeness relation (4.2) holds. Then the following three conditions are equivalent:*

(i) The spectral factor $W_1(s)$ is minimal.

(ii) The Wiener process du_1 corresponding to $W_1(s)$ satisfies the condition
$$N^- \subset H^-(du_1) \subset (N^+)^\perp. \tag{7.15}$$

(iii) The Wiener process du_1 corresponding to $W_1(s)$ satisfies the condition
$$N^+ \subset H^+(du_1) \subset (N^-)^\perp. \tag{7.16}$$

That condition (7.15) should be related to minimality in the general spectral factor case was suggested to us by Prof. A. Lindquist [7].

Proof: (i)⇒(ii): If $W_1(s)$ is minimal, then, in view of Theorem 4.1, we have that $U_1(s) := \overline{W}_-^{-1}(s)W_1(s)$ and $V_1(s) := W_1^{-1}(s)W_+(s)$ are inner and consequently, as it is shown in [10, Lemma 6.1], we have the relation $H^-(d\overline{u}_-) \subset H^-(du_1) \subset H^-(du_+)$, which corresponds exactly to (7.15). (ii)⇒(i): Using again [10, Lemma 6.1], relation (7.15) implies that $U_1(s) := \overline{W}_-^{-1}(s)W_1(s)$ and $V_1(s) := W_1^{-1}(s)W_+(s)$ are inner. Hence $U_1(s)$ is a left inner divisor of $U_M(s)$, and consequently $W_1(s) = \overline{W}_-(s)U_1(s)$ is minimal. Equivalence (ii)⇔(iii) is trivial. ■

This proposition gives two geometric versions of the diagram (7.14). In particular, as we saw in the proof, the subspaces generated by the past of the Wiener processes corresponding to the spectral factors in diagram (7.14) satisfy the following relations:

$$\begin{array}{ccccc} H^-(d\overline{u}_-) & \subset & H^-(d\overline{u}) & \subset & H^-(d\overline{u}_+) \\ \cap & & \cap & & \cap \\ H^-(d\tilde{u}_-) & \subset & H^-(du_1) & \subset & H^-(d\tilde{u}_+) \\ \cap & & \cap & & \cap \\ H^-(du_-) & \subset & H^-(du) & \subset & H^-(du_+) \end{array} \tag{7.17}$$

Hence, defining $\mathcal{S}_-$ as the set of the subspaces generated by the past of Wiener processes corresponding to minimal spectral factors, we have that

$H^-(du_+)$ and $H^-(d\bar{u}_-)$ are, respectively, the largest and the least element of $\mathcal{S}_-$ with respect to the partial order induced by inclusion. Inverse inclusions clearly hold for the subspaces $H^+(du)$ generated by the future of the Wiener processes.

More research is needed to understand the complete geometric picture underlying the Wiener processes corresponding to these general spectral factors.

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Dipartimento di elettronica e informatica, Universitá di Padova, via Gradenigo 6/A, 35131 Padova, Italy

Communicated by Anders Lindquist