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Modeling and Control of a Multiple Component Structure*

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Abstract

In this paper, a mathematical model is presented for a multiple component structure (MCS) composed of two Euler-Bernoulli beams, two distributed masses, and a rotating hub through which a torque control is applied. Imposition of a control of this type results in a bounded control operator. Since the control is bounded, damping in the beam model is required to stabilize the system. The Kelvin-Voigt damping model is assumed. The weak formulation is used to show the model is wellposed. A convergent Galerkin finite element approximation scheme is constructed for the model and is used to compute a sequence of controls which approximate an optimal control for the structure. This control is the solution to the linear quadratic regulator (LQR) problem for the MCS.

Key words: distributed parameter system, multiple component structure, linear quadratic regulator, approximation

AMS Subject Classifications: 35, 65N30, 73D30, 73K12, 93D15

1 Introduction

Throughout the past decade, increased interest in the development of sophisticated space structures such as a permanent space station and space based military structures has stimulated research in the mathematics and engineering communities. To handle these tasks, structures composed of several interconnected components are designed; generally, these are termed multiple component structures (MCSs). Composite materials

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are used in the construction of some components, and hence, these components are more flexible than those constructed of traditional materials. Development of schemes to control vibrations inherent in these structures is crucial to the success of these projects.

The mathematical models for MCSs contain coupled partial and ordinary differential equations. Much of the theory for modeling and control of distributed parameter systems has been developed recently and applied to single component structures, MCSs with only one flexible component, or flexible components and no rigid components [8, 9, 13, 19, 20]. In his dissertation, Bontsema considers stabilization via frequency domain methods of a satellite modeled by two flexible beams connected at a rigid hub. This is one of the few reports concerning a MCS with more than one flexible component and rigid components [7]. MCS models which include both rigid and flexible components have more complex dynamics as evidenced by the equations of elastic deformation and the boundary conditions. The flexible components modify the behavior of the rigid bodies, and vice versa, especially when rotations are considered.

Once a mathematical model is posed, it is necessary to prove the existence of a unique solution depending continuously on initial conditions and other parameters, i.e., the wellposedness of the model. One method to prove these models are wellposed relies on the weak formulation of the problem. The theory is described in several texts [2, 15, 16, 18] though all are inadequate for some second order initial value problems arising from MCS control problems. In their paper [4] (see also [2]), Banks, Ito, and Wang provide a theorem which gives conditions for wellposedness of weak solutions for such models. Once wellposedness of the models is established, other aspects such as stability and control can be addressed.

The abstract LQR problem for MCSs cannot be numerically implemented directly since the mathematical models are infinite dimensional. Consequently, an approximating sequence of finite dimensional LQR problems is constructed. The approximation scheme developed in this paper to obtain the sequence is formulated so that the sequence of optimal controls for the approximating LQR problems converges to the optimal control for the MCS. Fundamental results on solution of the LQR problem can be found in [2, 11, 13].

In this paper, modeling, wellposedness, development of a convergent approximation scheme, and solution of the LQR problem for a MCS are discussed. This MCS is composed of two flexible beams and three rigid bodies. The control for the system is the application of a torque at the hub, resulting in a compact feedback operator. Gibson [11] showed that to obtain uniform exponential stabilizability under such conditions, damping must be included in the model; the Kelvin-Voigt damping model is used for both beams. A mathematical model for the structure written in terms of coupled partial and ordinary differential equations is presented in Section 2. After this model is shown to be wellposed in Section 3, the abstract LQR problem is discussed in Section 4. In Section 5, a Galerkin finite element approximation scheme for the model is given along with a proof that the optimal controls for the resulting approximations converge to the optimal control for the MCS. It is demonstrated in Section 6 that optimal controls can be computed for a MCS with more than one flexible component coupled with rigid bodies. The need for results in this area of MCSs has been recognized and is addressed in the Report on Future Directions in Control [10].

2 A Mathematical Model for the MCS

This section contains a mathematical model for two serially connected Euler-Bernoulli beams with distributed masses at the ends and a rigid rotating hub through which the actuating torque is applied. A distributed mass is one which has length as opposed to the more commonly considered point mass.

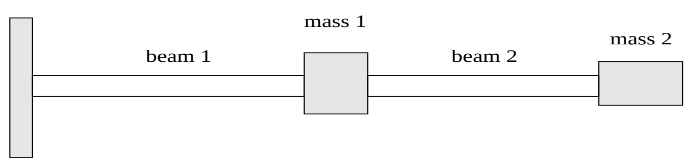

Figure 2.1: MCS

The following notation is used: beam i has length l_i, corresponding Young's modulus E_i, moment of inertia I_i, density ρ_i, and cross sectional area A_i (for brevity, the quantities $\rho_i A_i$ and $E_i I_i$ are denoted as $(\rho A)_i$ and $(EI)_i$ respectively). Additionally, l_{m_i} is the length of mass i, $i = 1, 2$, J_i is moment of inertia for mass i, $i = 0, 1, 2$ where 0 refers to the hub, γ_i is the coefficient of Kelvin-Voigt damping for beam i, $\theta(t)$ is angle of rotation of the hub, and $u(t)$ is the external torque applied to the hub, $u : [0, \infty] \rightarrow \Re$.

To derive the mathematical model corresponding to this MCS, define $v_1(t, x_1)$ to be the vertical displacement of beam one measured from the undeflected position of the beam at time t, position x_1, $0 \leq x_1 \leq l_1$. Define $v_2(t, x_2)$ to be the vertical displacement of beam two measured from the undeflected position of beam two as determined from the deflected position of mass one at time t. Measurement of the displacement of beam two with respect to this coordinate system rather than the equilibrium position in Figure 2.1 results in cantilevered boundary conditions on beam two, simplifying efforts in the approximation scheme. The system of equations describing this MCS contains the standard Euler-Bernoulli beam equations with additional terms due to Kelvin-Voigt damping and hub rotations, and boundary conditions which include the dynamics of the distributed masses and the hub. Unless otherwise noted, $i = 1, 2$.

The equations describing elastic deformations of the beams for $0 \leq t$, $0 \leq x_i \leq l_i$, are

$$(\rho A)_1[\frac{\partial^2 v_1}{\partial t^2}(t,x_1) + x_1\ddot{\theta}(t)] + (\gamma I)_1\frac{\partial^5 v_1}{\partial t \partial x^4}(t,x_1) + (EI)_1\frac{\partial^4 v_1}{\partial x^4}(t,x_1) = 0, \tag{2.1}$$

$$\begin{aligned}(\rho A)_2[&\frac{\partial^2 v_2}{\partial t^2}(t,x_2) + \frac{\partial^2 v_1}{\partial t^2}(t,l_1) + (x_2 + l_{m_1})\frac{\partial^3 v_1}{\partial x \partial t^2}(t,l_1)\\ &+ (x_2 + l_1 + l_{m_1})\ddot{\theta}(t)] + (\gamma I)_2\frac{\partial^5 v_2}{\partial t \partial x^4}(t,x_2) + (EI)_2\frac{\partial^4 v_2}{\partial x^4}(t,x_2) = 0.\end{aligned} \tag{2.2}$$

At the hub there are three boundary conditions: the two cantilevered conditions for beam one, and one for bending moment which states that the external applied torque is equal to the difference between the moment on the hub and the moment at the left end of beam one:

$$v_1(t,0) = 0, \tag{2.3}$$

$$\frac{\partial v_1}{\partial x}(t,0) = 0, \tag{2.4}$$

$$J_0\ddot{\theta}(t) - (\gamma I)_1\frac{\partial^3 v_1}{\partial t \partial x^2}(t,0) - (EI)_1\frac{\partial^2 v_1}{\partial x^2}(t,0) = u(t). \tag{2.5}$$

At the connection between the two beams, i.e., at mass one, there are four boundary conditions: the two cantilevered conditions for beam two and conditions on bending moment and shear force. The boundary condition for bending moment states that the difference of moments on the left end of beam two and the right end of beam one equals the moment generated by the rotation of mass one. The boundary condition for shear force states that the difference of shear forces on the left end of beam one and the right end of beam two equals the force generated by the acceleration of mass one:

$$v_2(t,0) = 0, \tag{2.6}$$

$$\frac{\partial v_2}{\partial x}(t,0) = 0, \tag{2.7}$$

$$\begin{aligned}&(\gamma I)_1[\frac{\partial^3 v_1}{\partial t \partial x^2}(t,l_1) + \frac{l_{m_1}}{2}\frac{\partial^4 v_1}{\partial t \partial x^3}(t,l_1)] + (EI)_1[\frac{\partial^2 v_1}{\partial x^2}(t,l_1) + \frac{l_{m_1}}{2}\frac{\partial^3 v_1}{\partial x^3}(t,l_1)]\\ &- (\gamma I)_2[\frac{\partial^3 v_2}{\partial t \partial x^2}(t,0) - \frac{l_{m_1}}{2}\frac{\partial^4 v_2}{\partial t \partial x^3}(t,0)] - (EI)_2[\frac{\partial^2 v_2}{\partial x^2}(t,0) - \frac{l_{m_1}}{2}\frac{\partial^3 v_2}{\partial x^3}(t,0)]\end{aligned}$$

$$= -J_1[\frac{\partial^3 v_1}{\partial x \partial t^2}(t,l_1) + \ddot{\theta}(t)], \qquad (2.8)$$

$$(\gamma I)_1 \frac{\partial^4 v_1}{\partial t \partial x^3}(t,l_1) + (EI)_1 \frac{\partial^3 v_1}{\partial x^3}(t,l_1) - (\gamma I)_2 \frac{\partial^4 v_2}{\partial t \partial x^3}(t,0) - (EI)_2 \frac{\partial^3 v_2}{\partial x^3}(t,0)$$
$$= m_1[\frac{\partial^2 v_1}{\partial t^2}(t,l_1) + \frac{l_{m_1}}{2}\frac{\partial^3 v_1}{\partial x \partial t^2}(t,l_1) + (l_1 + \frac{l_{m_1}}{2})\ddot{\theta}(t)]. \qquad (2.9)$$

At mass two there are two boundary conditions: one for bending moment and one for shear force which are similar to those described above. The boundary condition for moment states that the moment on the left end of beam two equals the moment generated by the rotation of mass two. The boundary condition for shear force states that the shear force on the left end of beam two equals the force generated by the acceleration of mass two:

$$(\gamma I)_2[\frac{\partial^3 v_2}{\partial t \partial x^2}(t,l_2) + \frac{l_{m_2}}{2}\frac{\partial^4 v_2}{\partial t \partial x^3}(t,l_2)] + (EI)_2[\frac{\partial^2 v_2}{\partial x^2}(t,l_2) + \frac{l_{m_2}}{2}\frac{\partial^3 v_2}{\partial x^3}(t,l_2)]$$
$$= -J_2[\frac{\partial^3 v_2}{\partial x \partial t^2}(t,l_2) + \frac{\partial^3 v_1}{\partial x \partial t^2}(t,l_1) + \ddot{\theta}(t)], \qquad (2.10)$$

$$(\gamma I)_2 \frac{\partial^4 v_2}{\partial t \partial x^3}(t,l_2) + (EI)_2 \frac{\partial^3 v_2}{\partial x^3}(t,l_2) = m_2[\frac{\partial^2 v_2}{\partial t^2}(t,l_2) + \frac{\partial^2 v_1}{\partial t^2}(t,l_1)$$
$$+\frac{l_{m_2}}{2}\frac{\partial^3 v_2}{\partial x \partial t^2}(t,l_2) + (l_{m_1} + \frac{l_{m_2}}{2} + l_2)\frac{\partial^3 v_1}{\partial x \partial t^2}(t,l_1)$$
$$+(l_1 + l_{m_1} + \frac{l_{m_2}}{2} + l_2)\ddot{\theta}(t)]. \qquad (2.11)$$

3 Wellposedness of the Mathematical Model

One way to show the wellposedness of the model in the previous section is to use a weak formulation of the problem based upon sesquilinear forms and Gelfand triples. This framework can be found in [2, 4, 15, 16, 18]. In this section, necessary wellposedness theorems are presented and applied to show that the MCS model from the previous section is wellposed.

Let V, H be complex Hilbert spaces with norms $|\cdot|_V$ and $|\cdot|_H$ respectively; let $\langle\cdot,\cdot\rangle_H$ denote the inner product on H. Assume V is densely and continuously embedded in H, i.e., V is a dense subset of H and a positive constant c exists such that for $\phi \in V$, $|\phi|_H \leq c|\phi|_V$. By applying the Riesz representation theorem, H is identified with H^*, the conjugate dual of H. For each $z \in H$, we can define an element $\psi(z) \in V^*$ by $\psi(z)(\phi) = \langle z, \phi\rangle_H$ for $\phi \in V$. The mapping $\psi : H \to V^*$ is continuous, linear, and one-to-one; it may be shown that $\psi(H)$ is dense in V^* with respect to the V^* norm.

Thus, H is densely and continuously embedded in V^*. This construction is called a Gelfand triple and is denoted

$$V \hookrightarrow H \cong H^* \hookrightarrow V^*$$

with H referred to as the pivot space.

Let σ_1 be a continuous sesquilinear form on V. Since σ_1 is continuous, for each $z \in V$, the mapping $\phi \to \sigma_1(z, \phi)$ is in V^*. Consequently, there exists a continuous linear operator $\mathbf{A} : V \to V^*$ defined by

$$(\mathbf{A}z)(\phi) = \sigma_1(z, \phi) \quad z, \phi \in V.$$

Conversely, given a continuous linear operator $\mathbf{A} \in \mathcal{L}(V, V^*)$, one can define a continuous sesquilinear form on V by $\sigma_1(z, \phi) = (\mathbf{A}z)(\phi)$. Thus there is a one-to-one correspondence between continuous sesquilinear forms on V and operators in $\mathcal{L}(V, V^*)$ which can be written as

$$\sigma_1(z, \phi) = (\mathbf{A}z)(\phi) = \langle \mathbf{A}z, \phi \rangle_{V^*,V}. \tag{3.1}$$

By $\langle \cdot, \cdot \rangle_{V^*,V}$, we refer to the duality pairing on V^*, V, i.e., the extension by continuity of $\langle \cdot, \cdot \rangle_H$ from $H \times V$ to $V^* \times V$. If the restricted domain of $\mathbf{A}$ is defined by

$$\mathcal{D}(\mathbf{A}) = \{z \in V : \mathbf{A}z \in H\}$$

define

$$A = \mathbf{A}|_{\mathcal{D}(\mathbf{A})} \quad \mathcal{D}(A) = \mathcal{D}(\mathbf{A}).$$

Then $A : \mathcal{D}(A) \subset V \subseteq H \to H$, and for $z \in \mathcal{D}(A)$, $\phi \in V$

$$Az(\phi) = \mathbf{A}z(\phi) = \sigma_1(z, \phi) = \langle Az, \phi \rangle_H. \tag{3.2}$$

A sesquilinear form σ_1 on V is *V-elliptic* if there exists a constant $c > 0$ such that for all $\phi \in V$

$$Re\ \sigma_1(\phi, \phi) \geq c|\phi|_V^2$$

and *V-coercive* if there exist constants $c > 0$ and $\lambda \geq 0$ such that for all $\phi \in V$

$$Re\ \sigma_1(\phi, \phi) \geq c|\phi|_V^2 - \lambda|\phi|_H^2.$$

Let σ_2 be a continuous sesquilinear form on V with

$$\sigma_2(z, \phi) = (\mathbf{D}z)(\phi) = \langle \mathbf{D}z, \phi \rangle_{V^*,V}. \tag{3.3}$$

The operator D is defined in a manner similar to that of the operator A, that is, if the restricted domain of $\mathbf{D}$ is defined by

$$\mathcal{D}(\mathbf{D}) = \{\phi \in V : \mathbf{D}\phi \in H\},$$

define

$$D = \mathbf{D}|_{\mathcal{D}(\mathbf{D})} \quad \mathcal{D}(D) = \mathcal{D}(\mathbf{D}).$$

Then $D : \mathcal{D}(D) \subset V \subseteq H \to H$, and for $z \in \mathcal{D}(D)$, $\phi \in V$,

$$Dz(\phi) = \mathbf{D}z(\phi) = \sigma_2(z, \phi) = \langle Dz, \phi\rangle_H. \tag{3.4}$$

A weak formulation of a general second order initial value problem can be written as

$$\begin{aligned} \langle \ddot{z}(t), \phi\rangle_H + \sigma_2(\dot{z}(t), \phi) + \sigma_1(z(t), \phi) = \langle f(t), \phi\rangle_H \quad \text{for all } \phi \in V \\ z(0) = z_0, \quad \dot{z}(0) = z_1. \end{aligned} \tag{3.5}$$

Assuming σ_1 and σ_2 are continuous and V-coercive, the corresponding abstract equation is given by

$$\begin{aligned} \ddot{z}(t) + D\dot{z}(t) + Az(t) = f(t) \quad \text{in } H \\ z(0) = z_0, \quad \dot{z}(0) = z_1 \end{aligned} \tag{3.6}$$

where A and D are given by (3.2) and (3.4) respectively.

To show the wellposedness of a second order system, it is written in first order form and shown to meet the wellposedness conditions for first order systems. If the weak formulation of a generic first order system is written as

$$\begin{aligned} \langle \dot{y}(t), \chi\rangle_{\mathcal{H}} + \sigma(y(t), \chi) = \langle F(t), \chi\rangle_{\mathcal{H}} \quad \text{for all } \chi \in \mathcal{V} \\ y(0) = y_0 \end{aligned} \tag{3.7}$$

where $F : [0, T] \to \mathcal{H}$ and the corresponding abstract form on $\mathcal{H}$ is

$$\begin{aligned} \dot{y}(t) = -\mathcal{A}y(t) + F(t) \\ y(0) = y_0, \end{aligned} \tag{3.8}$$

then the following theorem from [2] can be used to establish the wellposedness of the system.

Theorem 3.1 *If $-\mathcal{A}$ is the infinitesimal generator of an analytic semigroup $\mathcal{S}(t)$ and $F : [0, T] \to \mathcal{H}$ is uniformly Hölder continuous, i.e.,*

$$|F(t) - F(s)|_{\mathcal{H}} \le K|t - s|^{\alpha}, \quad 0 \le s \le t, \ 0 < \alpha \le 1,$$

then for each $y_0 \in \mathcal{H}$, (3.8) has a unique strong solution $y \in C([0, T]; \mathcal{H}) \cap C^1((0, T]; \mathcal{H})$ given by

$$y(t) = \mathcal{S}(t)y_0 + \int_0^t \mathcal{S}(t - s)F(s)ds. \tag{3.9}$$

This solution depends continuously on y_0 and F.

To write the second order system in first order form, define the product spaces $\mathcal{V} = V \times V$ and $\mathcal{H} = V \times H$ with inner products $\langle \cdot, \cdot \rangle_{\mathcal{V}} = \langle \cdot, \cdot \rangle_V + \langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_{\mathcal{H}} = \langle \cdot, \cdot \rangle_V + \langle \cdot, \cdot \rangle_H$ respectively. Define the sesquilinear form σ on $\mathcal{V}$ given by

$$\sigma((\phi, \psi), (\xi, \zeta)) = -\langle \psi, \xi \rangle_V + \sigma_1(\phi, \zeta) + \sigma_2(\psi, \zeta).$$

Then (3.5) can be written as

$$\begin{aligned} \langle \dot{y}(t), \chi \rangle_{\mathcal{H}} + \sigma(y(t), \chi) &= \langle F(t), \chi \rangle_{\mathcal{H}} \quad \text{for all } \chi \in \mathcal{V} \\ y(0) &= y_0 \end{aligned} \tag{3.10}$$

where $y(t) = (z(t), \dot{z}(t)), y_0 = (z_0, z_1), \chi = (\phi, \psi)$ and $F(t) = (0, f(t))$. Using this notation, one can write (3.6) in first order form on $\mathcal{H}$

$$\begin{aligned} \dot{y}(t) &= -\mathcal{A}y(t) + F(t) \\ y(0) &= y_0 \end{aligned} \tag{3.11}$$

where

$$\mathcal{A} = \begin{bmatrix} 0 & -I \\ A & D \end{bmatrix}$$

is defined on the restricted domain

$$\mathcal{D}(\mathcal{A}) = \{(\phi, \psi) \in \mathcal{V} : A\phi + D\psi \in H\} \subseteq \mathcal{V}.$$

If σ is continuous and $\mathcal{V}$-coercive, then $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$ and the operator $-\mathcal{A}$ associated with σ as

$$\sigma(\chi, \eta) = \langle -\mathcal{A}\chi, \eta \rangle_{\mathcal{H}} \quad \chi \in \mathcal{D}(\mathcal{A}), \eta \in \mathcal{V}$$

is the generator of an analytic semigroup $\mathcal{S}(t) \in \mathcal{L}(\mathcal{H}), t \geq 0$. The following theorem gives the wellposedness and form of the solution for (3.6).

Theorem 3.2 *If σ_1 and σ_2 are continuous and V-coercive and $f : [0, T] \rightarrow H$ is uniformly Hölder continuous, then for each $y_0 \in \mathcal{V}$, (3.6) has a unique strong solution $y = (z, \dot{z})$ given by*

$$y(t) = \mathcal{S}(t)y_0 + \int_0^t \mathcal{S}(t - s)F(s)ds \tag{3.12}$$

such that $z \in C([0, T]; V) \cap C^1([0, T]; V)$ and $\dot{z} \in C((0, T); V) \cap C^1((0, T); V)$.

Proof: Since σ_1 and σ_2 are continuous and V-coercive, there exist $c_{\sigma_1} > 0$, $c_{\sigma_2} > 0$, $\lambda_{\sigma_1} \geq 0$, $\lambda_{\sigma_2} \geq 0$, $k_{\sigma_1} > 0$, and $k_{\sigma_2} > 0$ such that

$$\begin{aligned} Re\ \sigma_1(\phi, \phi) &\geq c_{\sigma_1}|\phi|_V^2 - \lambda_{\sigma_1}|\phi|_H^2, \\ Re\ \sigma_2(\phi, \phi) &\geq c_{\sigma_2}|\phi|_V^2 - \lambda_{\sigma_2}|\phi|_H^2, \\ |\sigma_1(\phi, \psi)| &\leq k_{\sigma_1}|\phi|_V|\psi|_V, \\ |\sigma_2(\phi, \psi)| &\leq k_{\sigma_2}|\phi|_V|\psi|_V, \end{aligned}$$

for $\chi = (\phi, \psi) \in \mathcal{V}$. The continuity of σ follows from the continuity of σ_1, σ_2 and the inner product on V. To see that σ is $\mathcal{V}$-coercive, choose $\epsilon > 0$ so that $c_{\sigma_2} - \epsilon > 0$. Then

$$\begin{aligned}
Re\,\sigma(\chi,\chi) &= Re(-\langle\psi,\phi\rangle_V + \sigma_1(\phi,\psi) + \sigma_2(\psi,\psi)) \\
&\geq -|\langle\psi,\phi\rangle_V| - |\sigma_1(\phi,\psi)| + Re\,\sigma_2(\psi,\psi) \\
&\geq -|\psi|_V|\phi|_V - k_{\sigma_1}|\phi|_V|\psi|_V + c_{\sigma_2}|\psi|_V^2 - \lambda_{\sigma_2}|\psi|_H^2 \\
&\geq -\epsilon|\psi|_V^2 - \frac{(1+k_{\sigma_1})^2}{4\epsilon}|\phi|_V^2 + c_{\sigma_2}|\psi|_V^2 - \lambda_{\sigma_2}|\psi|_H^2 \\
&= -(\frac{(1+k_{\sigma_1})^2}{4\epsilon} + 1)|\phi|_V^2 + |\phi|_V^2 + (c_{\sigma_2} - \epsilon)|\psi|_V^2 - \lambda_{\sigma_2}|\psi|_H^2 \\
&\geq c(|\phi|_V^2 + |\psi|_V^2) - \lambda(|\phi|_V^2 + |\psi|_H^2) \\
&= c|\chi|_{\mathcal{V}}^2 - \lambda|\chi|_{\mathcal{H}}^2
\end{aligned}$$

where $c = min(1, c_{\sigma_2} - \epsilon)$, $\lambda = max((\frac{1+k_{\sigma_1})^2}{4\epsilon} + 1, \lambda_{\sigma_2})$. Thus, σ is $\mathcal{V}$-coercive and $-\mathcal{A}$ is the generator of an analytic semigroup $\mathcal{S}(t)$ in $\mathcal{H}$, $t \geq 0$.The result and continuous dependence of y on y_0 and F follow from Theorem 3.1. □

To use this framework to establish the wellposedness of the MCS model, the operators A and D, the sesquilinear forms σ_1 and σ_2, and the spaces H and V must be determined. Gibson and Adamian note that to make the necessary identifications, the operator A must be H-coercive (see [13]). If not, then they explain if A has a finite number of nonpositive eigenvalues, a bounded self-adjoint linear operator A_1 on H can be chosen so that $\tilde{A} = A + A_1$ is H-elliptic. This choice can be made by selecting an operator whose null space is the orthogonal complement in H of the eigenspace of A corresponding to nonpositive eigenvalues. Once A_1 is chosen, V is defined to be the completion of $\mathcal{D}(A)$ with respect to the inner product $\langle z, \tilde{z}\rangle_V = \langle \tilde{A}z, \tilde{z}\rangle_H$ for $z, \tilde{z} \in \mathcal{D}(A)$. Consequently, $V = \mathcal{D}(\tilde{A}^{1/2})$, and $\langle z, \tilde{z}\rangle_V = \langle \tilde{A}^{1/2}z, \tilde{A}^{1/2}\tilde{z}\rangle_H$ for $z, \tilde{z} \in V$. Since V is expressed as the square root of an operator, it cannot always be written explicitly.

Once H and V are chosen in this way, one can define the total energy space $E = V \times H$ with inner product $\langle z, \tilde{z}\rangle_E = \langle z_1, \tilde{z}_1\rangle_V + \langle z_2, \tilde{z}_2\rangle_H$ for $z = (z_1, z_2), \tilde{z} = (\tilde{z}_1, \tilde{z}_2) \in E$. If $z(t)$ is the solution to the problem under consideration, the kinetic energy for the system is $\frac{1}{2}\langle \dot{z}(t), \dot{z}(t)\rangle_H$ and, in the case that A is H-coercive, $\frac{1}{2}\langle z(t), z(t)\rangle_V$ is the strain energy for the system and $\frac{1}{2}\langle (z(t), \dot{z}(t)), (z(t), \dot{z}(t))\rangle_E$ is the total energy in the system. Although H and V need not be chosen in this fashion to obtain wellposedness results, this choice is attractive because of the physical significance.

After establishing that the abstract system corresponding to $\tilde{A}$ is well-posed, the wellposedness of the original system is obtained by observing that the original system given by A is a bounded perturbation of the wellposed system, i.e., $A = \tilde{A} - A_1$. Applying a theorem from Pazy [17, page 76], one can show that the original problem is wellposed.

Referring to the model for the MCS, the state space H and the state $z(t)$ are chosen to be $H = \Re \times L_2[0,l_1] \times L_2[0,l_2] \times \Re^4$ and $z(t) = (z_1(t), z_2(t), z_3(t), z_4(t), z_5(t), z_6(t), z_7(t))$ in H where

$$\begin{aligned}
z_1(t) &= \theta(t) \\
(z_2(t))(x_1) &= v_1(t,x_1) + x_1\theta(t) \\
(z_3(t))(x_2) &= v_2(t,x_2) + v_1(t,l_1) + (x_2 + l_{m_1})v_{1_x}(t,l_1) \\
&\quad +(x_2 + l_1 + l_{m_1})\theta(t) \\
z_4(t) &= v_{1_x}(t,l_1) + \theta(t) \\
z_5(t) &= v_1(t,l_1) + \frac{l_{m_1}}{2}v_{1_x}(t,l_1) + (l_1 + \frac{l_{m_1}}{2})\theta(t) \\
z_6(t) &= v_{1_x}(t,l_1) + v_{2_x}(t,l_2) + \theta(t) \\
z_7(t) &= v_1(t,l_1) + v_2(t,l_2) + (l_{m_1} + \frac{l_{m_2}}{2} + l_2)v_{1_x}(t,l_1) \\
&\quad +\frac{l_{m_2}}{2}v_{2_x}(t,l_2) + (l_1 + l_{m_1} + \frac{l_{m_2}}{2} + l_2)\theta(t). \qquad (3.13)
\end{aligned}$$

The inner product on H is taken to be

$$\begin{aligned}
\langle z, \tilde{z}\rangle_H &= J_0 z_1\tilde{z}_1 + (\rho A)_1\langle z_2, \tilde{z}_2\rangle_{L_2[0,l_1]} + (\rho A)_2\langle z_3, \tilde{z}_3\rangle_{L_2[0,l_2]} \\
&\quad +J_1 z_4\tilde{z}_4 + m_1 z_5\tilde{z}_5 + J_2 z_6\tilde{z}_6 + m_2 z_7\tilde{z}_7. \qquad (3.14)
\end{aligned}$$

The mathematical model can be written in abstract form

$$\ddot{z}(t) + D\dot{z}(t) + Az(t) = B_0 u(t) \quad \text{in } H$$

where A, D, and B are defined as follows:

$$Az = \begin{bmatrix}
-\frac{(EI)_1}{J_0} z_{2_{xx}}(0) \\
\frac{(EI)_1}{(\rho A)_1} z_{2_{xxxx}}(\cdot) \\
\frac{(EI)_2}{(\rho A)_2} z_{3_{xxxx}}(\cdot) \\
\frac{(EI)_1}{J_1}(z_{2_{xx}}(l_1) + \frac{l_{m_1}}{2} z_{2_{xxx}}(l_1)) - \frac{(EI)_2}{J_1}(z_{3_{xx}}(0) - \frac{l_{m_1}}{2} z_{3_{xxx}}(0)) \\
\frac{1}{m_1}[(EI)_2 z_{3_{xxx}}(0) - (EI)_1 z_{2_{xxx}}(l_1)] \\
\frac{(EI)_2}{J_2}[z_{3_{xx}}(l_2) + \frac{l_{m_2}}{2} z_{3_{xxx}}(l_2)] \\
-\frac{(EI)_2}{m_2} z_{3_{xxx}}(l_2)
\end{bmatrix}$$

$$Dz = \begin{bmatrix} -\frac{(\gamma I)_1}{J_0} z_{2_{xx}}(0) \\ \frac{(\gamma I)_1}{(\rho A)_1} z_{2_{xxxx}}(\cdot) \\ \frac{(\gamma I)_2}{(\rho A)_2} z_{3_{xxxx}}(\cdot) \\ \frac{(\gamma I)_1}{J_1}(z_{2_{xx}}(l_1) + \frac{l_{m_1}}{2} z_{2_{xxx}}(l_1)) - \frac{(\gamma I)_2}{J_1}(z_{3_{xx}}(0) - \frac{l_{m_1}}{2} z_{3_{xxx}}(0)) \\ \frac{1}{m_1}[(\gamma I)_2 z_{3_{xxx}}(0) - (\gamma I)_1 z_{2_{xxx}}(l_1)] \\ \frac{(\gamma I)_2}{J_2}[z_{3_{xx}}(l_2) + \frac{l_{m_2}}{2} z_{3_{xxx}}(l_2)] \\ -\frac{(\gamma I)_2}{m_2} z_{3_{xxx}}(l_2) \end{bmatrix}$$

where

$$\begin{aligned} \mathcal{D}(A) = \{ & z \in H : z_2 \in H^4[0,l_1],\ z_3 \in H^4[0,l_2],\ z_2(0) = 0, \\ & z_2(l_1) = z_5 - \frac{l_{m_1}}{2} z_4,\ z_{2_x}(0) = z_1,\ z_{2_x}(l_1) = z_4 = z_{3_x}(0), \\ & z_3(0) = z_5 + \frac{l_{m_1}}{2} z_4,\ z_3(l_2) = z_7 - \frac{l_{m_2}}{2} z_6,\ z_{3_x}(l_2) = z_6 \}, \end{aligned} \tag{3.15}$$

$\mathcal{D}(D) = \mathcal{D}(A)$, and $B_0 = [1\ 0\ 0\ 0\ 0\ 0\ 0]^T$ is defined on $\Re$.

The rigid body rotation of the hub induces a zero eigenvalue for A, hence A is not H-coercive. By choosing a bounded operator A_1 such that $\langle A_1 z, \tilde{z} \rangle_H = J_0 z_1 \tilde{z}_1$, we define an $\tilde{A} = A + A_1$ which is H-coercive. Then $V = \mathcal{D}(\tilde{A}^{1/2}) = \mathcal{D}(A^{1/2})$ and is contained in the set

$$\begin{aligned} \{ & z \in H : z_2 \in H^2[0,l_1],\ z_3 \in H^2[0,l_2],\ z_2(0) = 0, \\ & z_2(l_1) = z_5 - \frac{l_{m_1}}{2} z_4,\ z_{2_x}(0) = z_1,\ z_{2_x}(l_1) = z_4 = z_{3_x}(0), \\ & z_3(0) = z_5 + \frac{l_{m_1}}{2} z_4,\ z_3(l_2) = z_7 - \frac{l_{m_2}}{2} z_6,\ z_{3_x}(l_2) = z_6 \}. \end{aligned} \tag{3.16}$$

The inner product on V is taken to be

$$\begin{aligned} \langle z, \tilde{z} \rangle_V &= \langle \tilde{A}^{1/2} z, \tilde{A}^{1/2} \tilde{z} \rangle_H \\ &= (EI)_1 \langle z_{2_{xx}}, \tilde{z}_{2_{xx}} \rangle_{L_2[0,l_1]} + (EI)_2 \langle z_{3_{xx}}, \tilde{z}_{3_{xx}} \rangle_{L_2[0,l_2]} \\ &\quad + \langle A_1 z, \tilde{z} \rangle_H. \end{aligned} \tag{3.17}$$

To obtain the wellposedness results, define $\tilde{D} = D + A_1$ where A_1 is defined above. To prove that the problem is wellposed, the sesquilinear forms associated with $\tilde{A}$ and $\tilde{D}$ must be determined. Consider $\langle \tilde{A} z, \Phi \rangle_H$

for $z \in \mathcal{D}(A), \Phi \in V$. Integrating by parts and using the fact that $\Phi \in V$, we obtain the sesquilinear form $\tilde{\sigma_1}$

$$\begin{aligned}
\tilde{\sigma_1}(z,\Phi) &= \langle \tilde{A}z, \Phi\rangle_H \\
&= (EI)_1\langle z_{2_{xx}}, \phi_{2_{xx}}\rangle_{L_2[0,l_1]} + (EI)_2\langle z_{3_{xx}}, \phi_{3_{xx}}\rangle_{L_2[0,l_2]} + J_0 z_1\phi_1 \\
&= \sigma_1(z,\Phi) + \langle A_1 z, \Phi\rangle_H. \qquad (3.18)
\end{aligned}$$

Similarly, the sesquilinear form $\tilde{\sigma_2}$ can be written as

$$\begin{aligned}
\tilde{\sigma_2}(z,\Phi) &= \langle \tilde{D}z, \Phi\rangle_H \\
&= (\gamma I)_1\langle z_{2_{xx}}, \phi_{2_{xx}}\rangle_{L_2[0,l_1]} + (\gamma I)_2\langle z_{3_{xx}}, \phi_{3_{xx}}\rangle_{L_2[0,l_2]} + J_0 z_1\phi_1 \\
&= \sigma_2(z,\Phi) + \langle A_1 z, \Phi\rangle_H. \qquad (3.19)
\end{aligned}$$

A weak formulation of the mathematical model for the MCS is given by

$$\begin{aligned}
\langle \ddot{z}(t), \psi\rangle_H + \sigma_2(\dot{z}(t),\psi) + \sigma_1(z(t),\psi) = \langle B_0 u(t), \psi\rangle_H \quad \text{for all } \psi \in V \\
z(0) = z_0, \;\; \dot{z}(0) = z_1. \qquad (3.20)
\end{aligned}$$

To show the MCS model is wellposed, Theorem 3.2 is applied to the system

$$\begin{aligned}
\langle \ddot{z}(t), \psi\rangle_H + \tilde{\sigma_2}(\dot{z}(t),\psi) + \tilde{\sigma_1}(z(t),\psi) = \langle B_0 u(t), \psi\rangle_H \\
z(0) = z_0, \;\; \dot{z}(0) = z_1. \qquad (3.21)
\end{aligned}$$

Application of the theorem from Pazy shows that the mathematical model for the MCS is wellposed.

Theorem 3.3 *The sesquilinear forms $\tilde{\sigma_1}$ and $\tilde{\sigma_2}$ are continuous and V-coercive; hence given initial conditions $z_0, z_1 \in V$, (3.21) is wellposed. Since the MCS model is a bounded perturbation of (3.21), it is also wellposed.*

4 The LQR Control Problem

As previously mentioned, to compute a numerical control for an MCS described by an infinite dimensional system, it is necessary to have a scheme to approximate the original system by a sequence of finite dimensional systems. Controls are then computed for this sequence. For the sequence of controls to converge to the control for the infinite dimensional system, the approximation scheme must satisfy certain properties. The abstract LQR problem is presented below and an approximation scheme and precise sense of convergence is discussed in the following section.

The LQR control problem can be formulated as follows. Assume a system of the form

$$\begin{aligned} \dot{y}(t) &= Ay(t) + Bu(t), \quad t > 0 \\ y(0) &= y_0 \\ w(t) &= Cy(t) \end{aligned} \tag{4.1}$$

where $y(t) \in H$, $A : \mathcal{D}(A) \subset H \to H$, $B \in \mathcal{L}(U,H)$, $C \in \mathcal{L}(H,W)$, and the state space H, the control space U, and the observation space W are real Hilbert spaces. Assume A generates a C_0-semigroup, $S(t)$ on H.

The system in (4.1) has an associated cost functional

$$J(y_0, u) = \int_0^\infty (\langle Qy(t), y(t)\rangle_H + \langle Ru(t), u(t)\rangle_U)dt \tag{4.2}$$

where the control weighting, $R \in \mathcal{L}(U)$, is positive definite and self-adjoint. Let the state weighting, $Q \in \mathcal{L}(H)$ be given by $Q = C^*C$. Then the abstract LQR problem on H can be stated as:

$$\min_{u \in L_2((0,\infty),U)} J(y_0, u),$$

subject to $y(t)$ satisfying (4.1).

A function $u \in L_2((0,\infty),U)$ is an admissible control for $y_0 \in H$ if $J(y_0,u)$ is finite.

Consider the system given by (4.1). The pair (A,B) is stabilizable if there exists an operator $K \in \mathcal{L}(H,U)$ such that $(A - BK)$ generates a uniformly exponentially stable C_0-semigroup on H. The pair (A,C) is detectable if there exists an operator $F \in \mathcal{L}(W,H)$ such that $(A - FC)$ generates a uniformly exponentially stable C_0-semigroup on H. It is known if (4.1) is stabilizable and detectable, then there exists a unique control $u_{opt} \in L_2((0,\infty),U)$ such that

$$J(y_0, u_{opt}) = \min_{u \in L_2((0,\infty),U)} J(y_0, u),$$

(see [12]). This control can be written in the feedback form

$$u_{opt}(t) = -R^{-1}B^*\Pi y(t) \tag{4.3}$$

where $\Pi \in \mathcal{L}(H)$, $\Pi : \mathcal{D}(A) \to \mathcal{D}(A^*)$ is the nonnegative self-adjoint solution of the algebraic Riccati equation

$$A^*\Pi + \Pi A - \Pi BR^{-1}B^*\Pi + Q = 0 \tag{4.4}$$

on H. To assure the existence of such an optimal control for (4.1), the following result from [13] can be used.

Theorem 4.1 *Let the operators A, B, Q, and R be as previously defined. The algebraic Riccati equation (4.4) has a unique nonnegative self-adjoint solution if and only if for each $y_0 \in H$ there exists an admissible control $u(t)$. If such a Π exists, the unique control which minimizes $J(y_0, \cdot)$ is given by equation (4.3) and the corresponding optimal trajectory $y_{opt}(t)$ is given by*

$$y_{opt}(t) = T(t)y_0$$

where $T(t)$ is the uniformly exponentially stable C_0-semigroup generated by $A - BR^{-1}B^*\Pi$. Furthermore,

$$J(y_0, u_{opt}) = \min_{u \in L_2((0,\infty),U)} J(y_0, u) = \langle \Pi y_0, y_0 \rangle_H. \tag{4.5}$$

In general, (4.4) is a nonlinear partial functional differential equation for which a direct analytical solution is usually impossible. Therefore, one seeks computational solutions requiring approximation of the entire control problem. The next section addresses issues of developing a convergent approximation method for the MCS control problem.

5 An Approximation Scheme

The objective of an approximation scheme is to produce a sequence of suboptimal controls, $u^N_{opt}(t)$, such that $u^N_{opt}(t)$ converges to $u_{opt}(t)$ in an appropriate sense and if $u^N_{opt}(t)$ is applied to the infinite dimensional system in (4.1), the closed loop system response is nearly optimal for any initial condition. A convergent approximation scheme is one satisfying these two conditions. One way to construct the approximations is to project the infinite dimensional control problem defined on H onto a series of finite dimensional subspaces H^N, $N = 1, 2, \ldots$.

Let P^N denote a sequence of orthogonal projections $P^N : H \to H^N \subseteq H$. Let $A^N \in \mathcal{L}(H^N)$ be the infinitesimal generators of the C_0-semigroups $S^N(t)$ on H^N. Given operators $B^N \in \mathcal{L}(U, H^N)$, and $Q^N \in \mathcal{L}(H^N)$, the associated sequence of finite dimensional LQR problems on H^N is given by

$$\min_{u \in L_2((0,\infty),U)} J^N(y_0^N, u) \ \text{ subject to}$$

$$\begin{aligned} \dot{y}^N(t) &= A^N y(t) + B^N u(t), \quad t > 0 \\ y^N(0) &= y_0^N = P^N y_0 \\ w^N(t) &= C^N y(t) \end{aligned} \tag{5.1}$$

where

$$J^N(y_0^N, u) = \int_0^\infty (\langle Q^N y^N(t), y^N(t) \rangle_H + \langle Ru(t), u(t) \rangle_U) dt. \tag{5.2}$$

By applying the previously cited result of [12] to (5.1), we see that if the pairs (A^N, B^N) and (A^N, C^N) are stabilizable and detectable, then there is a unique optimal control $u^N_{opt}(t) \in L_2((0,\infty;U)$ of the form

$$u^N_{opt}(t) = -R^{-1}B^{N*}\Pi^N y^N_{opt}(t) \tag{5.3}$$

where $\Pi^N \in \mathcal{L}(H^N)$ is the unique nonnegative self-adjoint solution of the algebraic Riccati equation on H^N

$$A^{N*}\Pi^N + \Pi^N A^N - \Pi^N B^N R^{-1} B^{N*}\Pi^N + Q^N = 0. \tag{5.4}$$

The trajectory $y^N_{opt}(t)$ is the corresponding solution of (5.1) with $u = u^N_{opt}$.

To establish convergence of the approximation scheme for the second order system defined on V and H, one might attempt to write the problem as a first order system on the product spaces $\mathcal{V}$ and $\mathcal{H}$ and apply the fundamental convergence theorem from [6]. However, one condition of that theorem is that V be compactly embedded in H. Banks and Ito point out in [3] that for the first order form, the condition that $\mathcal{V}$ is compactly embedded in $\mathcal{H}$ would imply that V is compactly embedded in V which is false for infinite dimensional spaces. Thus, they provide the following theorem which is applicable to second order systems. Note that the notation in the theorem is the same as that used previously in Section 3.

Theorem 5.1 *Suppose $(\mathcal{A},\mathcal{B})$ is stabilizable, $(\mathcal{A},\mathcal{C})$ is detectable, and the following hold:*

(C1) *For each $z \in V$, there exists an element $\tilde{z}^N$ in H^N such that*

$$|z - \tilde{z}^N|_V \le \epsilon(N), \quad \text{where } \epsilon(N) \to 0 \text{ as } N \to \infty,$$

(C2) *The embedding of V into H is compact.*

Let $\mathcal{T}(t)$ be the C_0-semigroup generated by $\mathcal{A} - \mathcal{B}\mathcal{R}^{-1}\mathcal{B}^*\Pi$ and $\mathcal{T}^N(t)$ be the sequence of C_0-semigroups generated by $\mathcal{A}^N - \mathcal{B}^N\mathcal{R}^{-1}\mathcal{B}^{N*}\Pi^N$ where the approximations are obtained by a Galerkin finite element scheme. Then

$$\Pi^N P^N\xi \to \Pi\xi \quad \text{for every } \xi \in \mathcal{H},$$

$$\mathcal{T}^N(t)P^N\xi \to \mathcal{T}(t)\xi \quad \text{for every } \xi \in \mathcal{H},$$

where convergence is uniform in t on bounded subsets of $[0,\infty)$ and

$$|\mathcal{T}(t)| \le M_1 e^{-\omega t} \quad \text{for } t \ge 0.$$

Moreover, $u^N_{opt}(t) \to u_{opt}(t)$ in U and $y^N_{opt}(t) \to y_{opt}(t)$ in $\mathcal{V}$, uniformly on $[0,T]$.

For $N \geq 1$, assume a general Galerkin finite element approximation of the form

$$z^N(t) = \sum_{j=1}^{N} \zeta_j^N(t) e_j^N, \tag{5.5}$$

where $\{e_1^N, e_2^N, \ldots, e_N^N\}$ is a basis for the approximating space $H^N \in V$. If we require that (3.20) hold for $z = z^N$ and $\phi \in H^N$, it is clear that $\zeta^N(t) = [\zeta_1^N(t), \zeta_2^N(t), \ldots, \zeta_N^N(t)]^T$ satisfies a system of the form

$$M^N \ddot{\zeta}^N(t) + D^N \dot{\zeta}^N(t) + K^N \zeta^N(t) = B_0^N u(t) \tag{5.6}$$

where the mass matrix M^N, damping matrix D^N, stiffness matrix K^N, and actuator influence matrix B_0^N are given by

$$\begin{aligned} M_{ij}^N &= \langle e_i^N, e_j^N \rangle_H, \\ D_{ij}^N &= \sigma_2(e_i^N, e_j^N), \\ K_{ij}^N &= \sigma_1(e_i^N, e_j^N), \\ B_{0ij}^N &= \langle e_i^N, B_{0j} \rangle_H. \end{aligned} \tag{5.7}$$

To proceed, equation (5.6) is written in first order form by letting $\eta^N = (\zeta^N, \dot{\zeta}^N)$:

$$\dot{\eta}^N = A^N \eta^N + B^N u, \tag{5.8}$$

where

$$A^N = \begin{bmatrix} 0 & I \\ -(M^N)^{-1} K^N & -(M^N)^{-1} D^N \end{bmatrix},$$

$$B^N = \begin{bmatrix} 0 \\ -(M^N)^{-1} B_0^N \end{bmatrix}.$$

Since $e_i^N \in V$, it is a vector of seven components denoted

$$e_i^N = (e_{i1}^N, e_{i2}^N, e_{i3}^N, e_{i4}^N, e_{i5}^N, e_{i6}^N, e_{i7}^N) \in \Re \times H^2[0, l_1] \times H^2[0, l_2] \times \Re^4.$$

If we refer to (3.16), we see that the following must hold:

$$\begin{gathered} e_{i2}^N(0) = 0,\ e_{i2}^N(l_1) = e_{i5}^N - \tfrac{l_{m_1}}{2} e_{i4}^N,\ e_{i2_x}^N(0) = e_{i1}^N, \\ e_{i2_x}^N(l_1) = e_{i4}^N = e_{i3_x}^N(0),\ e_{i3_x}^N(l_2) = e_{i6}^N,\ e_{i3}^N(0) = e_{i5}^N + \tfrac{l_{m_1}}{2} e_{i4}^N, \\ e_{i3}^N(l_2) = e_{i7}^N - \tfrac{l_{m_2}}{2} e_{i6}^N. \end{gathered} \tag{5.9}$$

Cubic B-splines will be used to represent the displacements of beams one and two. Given an interval $I = [0, L]$ and $h = \frac{L}{N}$, define the uniform partition of I,

$$\Delta^N = \{0, h, 2h, \ldots, Nh = L\}.$$

Define the set

$$S_3^N(I) = \{\phi \in C^2[0,L] \quad : \quad \phi \text{ is a cubic polynomial on } [jh,(j+1)h], j = 0,\ldots,N-1\}.$$

We want to approximate $H^2[0,l_i]$ by $S_3^N(I_i)$ where $I_i = [0,l_i]$. Let $B_i^N(x)$ be the i^{th} cubic B-spline defined as:
$B_i^N(x) =$

$$\begin{cases} \frac{1}{h^3}(x-x_{i-2})^3 & x \in [x_{i-2}, x_{i-1}] \\ 1 + \frac{3}{h}(x-x_{i-1}) + \frac{3}{h^2}(x-x_{i-1})^2 - \frac{3}{h^3}(x-x_{i-1})^3 & x \in [x_{i-1}, x_i] \\ 1 + \frac{3}{h}(x_{i+1}-x) + \frac{3}{h^2}(x_{i+1}-x)^2 - \frac{3}{h^3}(x_{i+1}-x)^3 & x \in [x_i, x_{i+1}] \\ \frac{1}{h^3}(x_{i+2}-x)^3 & x \in [x_{i+1}, x_{i+2}] \\ 0 & \text{otherwise.} \end{cases}$$

Then $B_{-1}^N, B_0^N, \ldots, B_{N+1}^N$ provide an $N+3$ dimensional basis for $S_3^N(I)$.

Let $\{a_i^N\}_{i=-1}^{N+1}$ denote the $N+3$ cubic B-splines on the interval $[0,l_1]$ and $\{b_i^N\}_{i=-1}^{N+1}$ denote the $N+3$ cubic B-splines on the interval $[0,l_2]$, i.e., $\{a_i^N\}$ and $\{b_i^N\}$ will be used to approximate the displacements of beams one and two, respectively. It was noted in Section 2 that the coordinate system for the MCS was chosen so that the essential boundary conditions are zero for both beams. To accommodate these boundary conditions, the set of splines used for the first beam and their derivatives must be zero at $x_1 = 0$ and the set of splines used for the second beam and their derivatives must be zero at $x_2 = 0$. For each of $\{a_i^N\}$ and $\{b_i^N\}$, if we take the last N splines and an appropriate linear combination of the first three splines, we obtain a set of $N+1$ splines which satisfies the essential boundary conditions. Thus, the elements of the basis functions corresponding to the displacements of beams one and two are

$$\alpha_i^N = \begin{cases} a_0^N - 2a_{-1}^N - 2a_1^N & i = 1 \\ a_i^N & i = 2,\ldots,N+1 \end{cases}$$

$$\beta_i^N = \begin{cases} b_0^N - 2b_{-1}^N - 2b_1^N & i = 1 \\ b_i^N & i = 2,\ldots,N+1 \end{cases}.$$

If we denote the space spanned by α_i^N as H_1^N and the space spanned by β_i^N as H_2^N, we can write $H_1^N = S_3^N(I_1) \cap H^2[0,l_1]$ with the essential boundary conditions for beam one, and $H_2^N = S_3^N(I_2) \cap H^2[0,l_2]$ with the essential boundary conditions for beam two.

To approximate V, $2N+3$ total basis functions $\{e_1^N, ..., e_{2N+3}^N\}$ are chosen as follows:

$$e_1^N(x) = \begin{bmatrix} 1 \\ x_1 \\ x_2 + l_1 + l_{m_1} \\ 1 \\ l_1 + \frac{l_{m_1}}{2} \\ 1 \\ l_1 + l_{m_1} + l_2 + \frac{l_{m_2}}{2} \end{bmatrix}$$

$$e_i^N(x) = \begin{bmatrix} 0 \\ \alpha_{i-1}^N \\ \alpha_{i-1}^N(l_1) + (x_2 + l_{m_1})(\alpha_{i-1}^N)_x(l_1) \\ (\alpha_{i-1}^N)_x(l_1) \\ \alpha_{i-1}^N(l_1) + \frac{l_{m_1}}{2}(\alpha_{i-1}^N)_x(l_1) \\ (\alpha_{i-1}^N)_x(l_1) \\ \alpha_{i-1}^N(l_1) + (l_{m_1} + l_2 + \frac{l_{m_2}}{2})(\alpha_{i-1}^N)_x(l_1) \end{bmatrix}$$

$$e_j^N(x) = \begin{bmatrix} 0 \\ 0 \\ \beta_{j-N-1}^N \\ 0 \\ 0 \\ (\beta_{j-N-1}^N)_x(l_2) \\ \beta_{j-N-1}^N(l_2) + \frac{l_{m_2}}{2}(\beta_{j-N-1}^N)_x(l_2) \end{bmatrix}$$

where $i = 2, \ldots, N+2$, and $j = N+3, \ldots, 2N+3$. Thus, the approximation space is $H^N = \Re \times H_1^N \times H_2^N \times \Re^4$.

To complete the formulation of the sequence of finite dimensional LQR problems, Q^N and R must be specified. We take $Q = I$ which corresponds to observing all states. In the context of this problem, this choice corresponds to the state weighting term $\langle Qz, z \rangle_V$ being twice the total energy in the structure plus the square of the rigid body rotation [13]. The Grammian matrix $\tilde{W}^N$ is the matrix representation of Q on H^N where

$$\tilde{W}^N = \begin{bmatrix} J_0 & 0 & 0 \\ 0 & K^N & 0 \\ 0 & 0 & M^N \end{bmatrix}.$$

For this MCS, the control input space, U, is $\Re$. Since there is only one control input to the problem, the control weighting, R, is taken to be 1.

To obtain the sequence of controls solving the LQR problem, a system of Matlab subroutines can be used. To generate the system matrices, the L_2 inner products of the basis functions and their second derivatives must be computed to determine the elements of M^N, K^N and D^N. These matrices are used to form the matrices A^N, B^N, Q^N, which along with R are used by the LQR subroutine in the Matlab Control Toolbox to compute the optimal control for the finite dimensional system. This control can be determined from the gain matrix, $K = R^{-1}B^{N*}\Pi^N$, which is the output of the LQR routine.

The optimal control for the abstract LQR problem has the feedback form

$$u(t) = -\langle f, z(t)\rangle_V - \langle g, \dot{z}(t)\rangle_H \tag{5.10}$$

where $z(t)$ is the state given in (3.13) and

$$f = (f_1, f_2, f_3, f_4, f_5, f_6, f_7) \in V,$$

$$g = (g_1, g_2, g_3, g_4, g_5, g_6, g_7) \in H.$$

By expanding (5.10) in terms of the inner products on V and H, we obtain the control law

$$\begin{aligned}
u(t) \quad = \quad & -J_0k_1\theta(t) - (EI)_1\int_0^{l_1} k_2(x_1)v_{1_{xx}}(t,x_1)dx_1 \\
& -(EI)_2\int_0^{l_2} k_3(x_2)v_{2_{xx}}(t,x_2)dx_2 - J_0k_4\dot{\theta}(t) \\
& -(\rho A)_1\int_0^{l_1} k_5(x_1)(\dot{v}_1(t,x_1) + x_1\dot{\theta}(t))dx_1 \\
& -(\rho A)_2\int_0^{l_2} k_6(x_2)(\dot{v}_2(t,x_2) + \dot{v}_1(t,l_1) + (x_2 + l_{m_1})\dot{v}_{1_x}(t,l_1) \\
& \qquad +(x_2 + l_1 + l_{m_1})\dot{\theta}(t))dx_2 \\
& -J_1k_7(\dot{v}_{1_x}(t,l_1) + \dot{\theta}(t)) - m_1k_8(\dot{v}_1(t,l_1) + \frac{l_{m_1}}{2}\dot{v}_{1_x}(t,l_1) \\
& \qquad +(l_1 + \frac{l_{m_1}}{2})\dot{\theta}(t)) \\
& -J_2k_9(\dot{v}_{2_x}(t,l_2) + \dot{v}_{1_x}(t,l_1) + \dot{\theta}(t)) - m_2k_{10}(\dot{v}_2(t,l_2) + \dot{v}_1(t,l_1) \\
& \qquad +\frac{l_{m_2}}{2}\dot{v}_{2_x}(t,l_2) + (l_{m_1} + \frac{l_{m_2}}{2} + l_2)\dot{v}_{1_x}(t,l_1) \\
& \qquad +(l_1 + l_{m_1} + \frac{l_{m_2}}{2} + l_2)\dot{\theta}(t)),
\end{aligned} \tag{5.11}$$

where

$$k_1 = f_1, \quad k_2(x_1) = f_{2_{xx}}(x_1), \quad k_3(x_2) = f_{3_{xx}}(x_2),$$

$$k_4 = g_1, \quad k_5(x_1) = g_2(x_1), \quad k_6(x_2) = g_3(x_2),$$

$$k_7 = g_4, \quad k_8 = g_5, \quad k_9 = g_6, \quad k_{10} = g_7. \tag{5.12}$$

The functions k_2 and k_5 are the bending and velocity gains corresponding to beam one, and k_3 and k_6 are the bending and velocity gains corresponding to beam two. The discrete gains, k_1, k_4, k_7, k_8, k_9 and k_{10} are the gains corresponding to hub position, hub velocity, velocity of the slope of mass one, velocity of mass one, velocity of the slope of mass two, and velocity of mass two, respectively.

The feedback form of the optimal control for the approximating sequence of LQR problems has the same form, i.e.

$$u^N(t) = -\langle f^N, z^N(t)\rangle_V - \langle g^N, \dot{z}^N(t)\rangle_H \tag{5.13}$$

with $f^N \to f$ in V, and $g^N \to g$ in H. If we let

$$\tilde{K}^N = R^{-1} B^{N*} \Pi^N (M^N)^{-1} = [\tilde{k}_1^N, \ldots, \tilde{k}_{4N+6}^N],$$

the approximate gains can be written as

$$k_1^N = \tilde{k}_1^N (e_1^N)_1, \quad k_2^N = \sum_{i=2}^{N+2} \tilde{k}_i^N (D^2 e_i^N)_2, \quad k_3^N = \sum_{i=N+3}^{2N+3} \tilde{k}_i^N (D^2 e_i^N)_3,$$

$$k_4^N = \tilde{k}_{2N+4}^N (e_i^N)_1, \quad k_5^N = \sum_{i=1}^{N+2} \tilde{k}_{i+2N+3}^N (e_i^N)_2, \quad k_6^N = \sum_{i=1}^{2N+3} \tilde{k}_{i+2N+3}^N (e_i^N)_3,$$

$$k_7^N = \sum_{i=1}^{N+2} \tilde{k}_{i+2N+3}^N (e_i^N)_4, \quad k_8^N = \sum_{i=1}^{N+2} \tilde{k}_{i+2N+3}^N (e_i^N)_5,$$

$$k_9^N = \sum_{i=1}^{2N+3} \tilde{k}_{i+2N+3}^N (e_i^N)_6, \quad k_{10}^N = \sum_{i=1}^{2N+3} \tilde{k}_{i+2N+3}^N (e_i^N)_7, \tag{5.14}$$

where $(e_i^N)_j$ denotes the j^{th} entry in the basis function e_i^N defined previously.

To show that this approximation scheme is convergent, the system described by equations (3.5) and (3.6) must be shown to satisfy the conditions of Theorem 5.1.

Lemma 5.1 *The pair $(\mathcal{A}, \mathcal{B})$ is stabilizable.*

Proof: We need to apply the following theorem from [2]:

> Given a sesquilinear form σ_1 which is V-elliptic, continuous, and symmetric, and a sesquilinear form σ_2 which is H-elliptic, continuous and symmetric, then $\mathcal{A}$ (see (3.11)) is the infinitesimal generator of an exponentially stable C_0-semigroup in $\mathcal{H} = V \times H$.

Recall from the wellposedness theorem that $\tilde{\sigma_1}$ and $\tilde{\sigma_2}$ are continuous and V-elliptic; clearly, they are also symmetric. By the embedding structure of V and H, $\tilde{\sigma_2}$ is also H-elliptic. Thus, $\tilde{\mathcal{A}}$ is the infinitesimal generator of an exponentially stable C_0-semigroup on $\mathcal{H}$ where

$$\tilde{\mathcal{A}} = \begin{bmatrix} 0 & -I \\ \tilde{A} & \tilde{D} \end{bmatrix} = \begin{bmatrix} 0 & -I \\ A & D \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ A_1 & A_1 \end{bmatrix}.$$

Since $A_1 z = z_1$ for $z \in \mathcal{D}(A)$, $A_1 = B_0$. If we let $K = [0, I]$, then $\tilde{\mathcal{A}} = \mathcal{A} - \mathcal{B}K$ and the pair $(\mathcal{A}, \mathcal{B})$ is stabilizable. □

Lemma 5.2 *The pair $(\mathcal{A}, \mathcal{C})$ is detectable.*

Proof: Since $\mathcal{C}$ is the identity, the result follows. □

Lemma 5.3 *The approximation scheme satisfies* (C1).

Proof: The crux of this proof relies on showing (C1) holds for the beam components. Let I_i^N denote the cubic B-spline interpolant operator for beam i, i.e., $I_i^N \phi$ is the unique element in H^N that agrees with $\phi \in V$ at $x = \frac{i}{N}$, $i = 0, 1, \ldots, N$. Estimates for cubic spline interpolants can be found in Appendix A.4 and Section III.2.2 of [5]. Let P^N denote the orthogonal projection of V onto H^N and P_1^N and P_2^N denote the parts of P^N corresponding to beams one and two respectively. By (III.2.7) of [5], we have for $z \in V$,

$$|P_i^N z - z|_V \leq |I_i^N z - z|_V \leq \frac{k_i}{N^2} |z|_V.$$

Thus

$$|P^N z - z|_V \leq |P_1^N z - z|_V + |P_2^N z - z|_V \leq \frac{k_1 + k_2}{N^2} |z|_V \to 0 \text{ as } N \to \infty.$$

Letting $\tilde{z}^N = P^N z$ yields (C1). □

Lemma 5.4 *V is compactly embedded in H.*

Proof: The Rellich-Kondrachov Theorem in [1] gives the compact embedding of $H^2[0, l_i]$ into $L_2[0, l_i]$. Since $\Re^n$ is compactly embedded in $\Re^n$, the lemma is true. □

Theorem 5.2 *The approximation scheme described in this section is convergent.*

Proof: Since the approximation scheme satisfies Lemmas 5.1, 5.2, 5.3, 5.4, we have convergence by Theorem 5.1. □

6 Numerical Results

This section contains numerical results obtained by implementing the approximation scheme developed in the previous section. An example was constructed in which an aluminum beam and a steel beam were chosen for beams one and two respectively. The specific parameter values used for both beams and rigid bodies are listed in the following tables. The beam dimensions were chosen to satisfy the basic assumptions of the Euler-Bernoulli model.

Table 6.1: Rigid Body Parameters

parameter	Hub	Mass 1	Mass 2
J	$1\,kg\,m^2$	$1\,kg\,m^2$	$1\,kg\,m^2$
m		$1\,kg$	$1\,kg$
l_m		$.25\,m$	$.25\,m$

Using the above parameters, the optimal controls for the finite dimensional LQR problems were computed using the Matlab Control Toolbox on a Macintosh IIsi. Plots of the resulting functional gains for the finite dimensional approximating systems obtained for $N = 8, 12$, and 16 can be found in Figures 6.1 and 6.2. The discrete gains can be found in Table 6.3.

The beams are the only components of the structure contributing to strain (refer to (5.11), (5.12), (5.14)). Thus, in Figure 6.1, the first two meters of the plot is the bending gain for the first beam (k_2^N) and the last three meters is the bending gain for the second (k_3^N). The gain for beam one is nearly identical to that obtained for a single Euler-Bernoulli beam with a torque applied at the hub and a tip mass at the right end, using the above parameters and omitting mass length. One might expect this as this MCS could be interpreted as a perturbation of the single beam problem. The largest value of the gain (in magnitude) occurs at the hub, where one would expect that the greatest strain exists. The gains near the centers of both beams are zero indicating those are the points of least strain.

Figure 6.2 shows velocity gains for both beams and both masses since all components have associated velocity gains. The dotted boxes denote the masses with the gain shown as a point in the center. As expected, the gain at the hub is zero which reflects the fact that beam one is fixed at the hub. The velocity gains appear continuous over the entire structure, i.e., if

Table 6.2: Beam Parameters

parameter	Beam 1: Aluminum	Beam 2: Steel
L	$2\,m$	$3\,m$
w	$2.54\,cm$	$3.81\,cm$
h	$0.3175\,cm$	$.15875\,cm$
A	$8.0645 \times 10^{-5}\,m^2$	$6.0483 \times 10^{-5}\,m^2$
ρ	$2700\,\frac{kg}{m^3}$	$7800\,\frac{kg}{m^3}$
ρA	$.21774\,\frac{kg}{m}$	$.47177\,\frac{kg}{m}$
E	$7.0 \times 10^{10}\,\frac{N}{m^2}$	$2.0 \times 10^{11}\,\frac{N}{m^2}$
I	$6.7746 \times 10^{-11}\,m^4$	$1.27 \times 10^{-11}\,m^4$
EI	$4.74222\,Nm^2$	$2.54\,Nm^2$
γI	$.01\,\frac{kg}{m\,sec}$	$.01\,\frac{kg}{m\,sec}$

the gains for the structure were smoothly connected, the gains for masses one and two would lie on that graph.

We want to emphasize that in the functional gain plots, gains for the finite dimensional approximating systems corresponding to values of $N = 8$, 12, *and* 16 are plotted, though often only one graph is evident since the gains converge rapidly. In general, the difference in the gain values at the nodes for $N = 8$ and 16 is on the order of 10^{-5} or 10^{-6}. This convergence is also shown in the table of discrete gains below. The eigenvalues also showed rapid convergence, though they are not presented here.

To verify the performance of the control scheme for the MCS, open and closed loop simulations were performed for three initial conditions using the LSIM routine of the Matlab Control Toolbox: one for an initial rotation of the hub, one for an initial displacement of beam one, and one for an initial displacement of beam two. Given a time array, LSIM computes the time

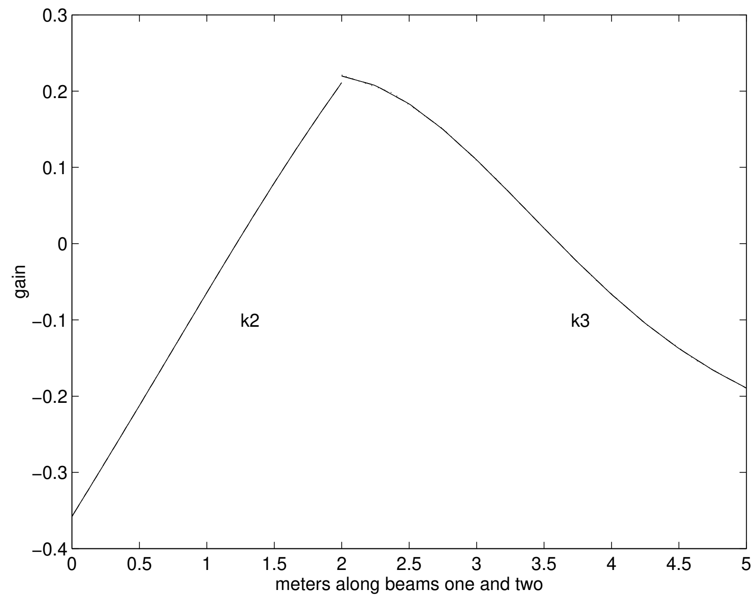

Figure 6.1: Bending Gains, N = 8, 12, 16

response, w, to the input time history, u, for a given system of the form

$$\dot{y} = Ay + Bu$$
$$w = Cy + Du.$$

The open loop simulations were performed by sending the matrices A, B, and C—where C is the matrix representation of identity—to the LSIM routine with D and u equal to zero. To observe the effect of the LQR control, the simulations were performed on the closed loop system with zero input. For all simulations, $N = 8$. The following figures show the results of the simulations. The beams are represented by lines and the masses by '+'. For these components, the vertical axis represents linear displacement, i.e., meters from equilibrium. The hub is represented by 'o'; in this case, the vertical axis represents radian displacement from equilibrium.

To see the effect of the control scheme under initial rotation of the hub, the initial condition was chosen in which $\theta(0) = \frac{\pi}{18}$, and all other states were equal to zero. Initial velocities of all states were chosen to be zero. The behavior of the open loop system over 15 seconds is shown in Figure 6.3. As expected, with zero initial velocities and no control applied to the system, the structure remained in the initial configuration.

The behavior of the closed loop system is shown in Figures 6.4 and 6.5. The initial deflection of the structure is shown by the top line in Figure 6.4.

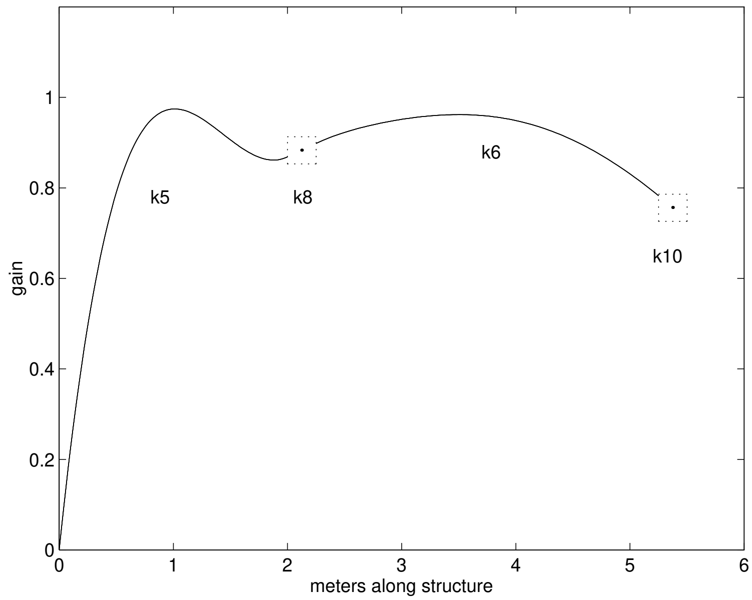

Figure 6.2: Velocity Gains, N = 8, 12, 16

The plots represent the motion of the structure from 0 to 33 seconds with each plot given at three second intervals. As time proceeds, the control rotates the structure toward equilibrium. The last four plots show the beginning of convergence to equilibrium. To view the subsequent behavior more closely, the scale for the vertical axis is magnified in Figure 6.5 and the motion of the structure is plotted from 36 to 66 seconds. The structure has rotated past equilibrium and the control is bringing it back to zero. A plot of the torque which was applied to control the structure in the closed loop simulation is shown in Figure 6.6.

To see the effect of the control scheme on an initial displacement of beam one, the initial condition $v_1(0, x_1) = \frac{1}{72}x_1^3$ was specified. All other states (including velocities) were zero. This particular state was chosen to support the small angle assumption of the Euler-Bernoulli beam model. The behavior of the open loop system over 30 seconds is shown in Figure 6.7. The initial configuration is depicted by a dashed-dotted line. Subsequently, the structure flexes concavely and convexly, achieving no equilibrium.

In contrast, as shown by Figures 6.8, 6.9, and 6.10, the closed loop system is driven to rest by the controller. In Figure 6.8, the initial deflection of the structure is represented by the dashed line. The second plot, corresponding to three seconds shows that the control initially acts by straightening beam one. The third plot, corresponding to six seconds,

Table 6.3: Discrete Gains

	N = 8	N = 12	N = 16
k_1^N	.999999996	.999999992	.999999941
k_4^N	2.4514488	2.4514491	2.4514492
k_7^N	.1190124	.1190145	.1190148
k_8^N	.8833131	.8833187	.8833197
k_9^N	-.2070311	-.2070289	-.2070286
k_{10}^N	.7566707	.7566699	.7566697

shows that beam one is slightly higher than in the previous plot, but the structure is generally rotated toward equilibrium. In Figure 6.9, the motion of the structure from 9 to 33 seconds is shown. The structure is essentially straight as in the first simulation, and is gradually rotated toward equilibrium and slightly past. To see the behavior in subsequent times, the scale of the vertical axis is magnified in Figure 6.10 and the rotation of the structure toward equilibrium is shown. A plot of the controlling torque for the closed loop simulation is shown in Figure 6.11.

To see the effect of the control scheme on an initial displacement of beam two, the initial condition $v_2(0, x_2) = \frac{1}{90}x_2^3$ was chosen. Again, all other states including velocities were equal to zero. The behavior for the open loop system is illustrated in Figure 6.12. As in the previous simulation, the initial configuration of the structure is shown by a dashed-dotted line. Subsequent plots over 30 seconds show the oscillation of the structure.

The behavior of the closed loop system is shown in Figures 6.13, 6.14, and 6.15. In Figure 6.13, the initial deflection of the structure is given by the dashed line. The second plot, corresponding to three seconds and given by the upper curve, shows the control initially flexing the entire structure concavely. The third plot, corresponding to six seconds, shows the same shape but rotation of the structure toward equilibrium. The fourth plot, representing the structure at nine seconds, shows the structure adopting a convex shape. These four plots show a whip-like action of the beam. In Figure 6.14, the motion of the structure from 12 to 24 seconds is shown. The scale of the vertical axis is magnified slightly to show the decay of

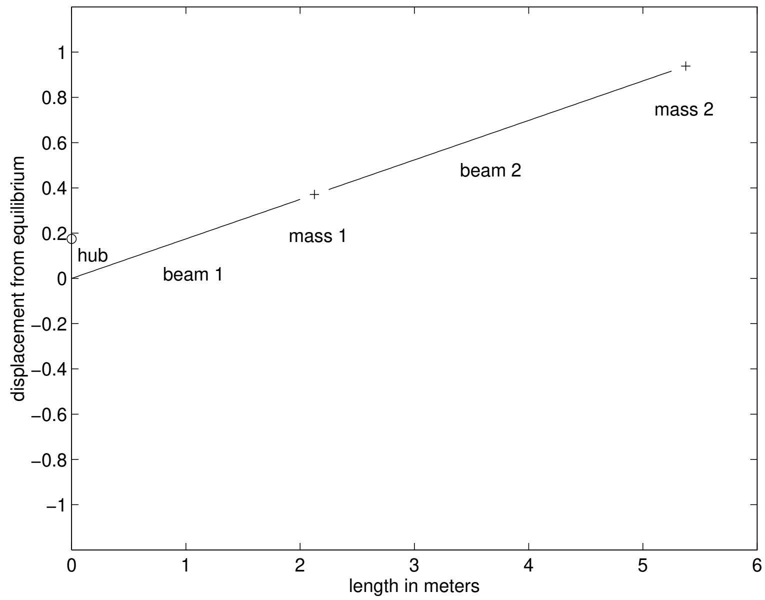

Figure 6.3: Open Loop Simulation, Hub Rotation

the whipping behavior. The structure is straightened and rotated toward equilibrium and slightly past. Figure 6.15 shows the structure rotated back toward, and achieving equilibrium. A plot of the controlling torque for the third closed loop simulation is shown in Figure 6.16.

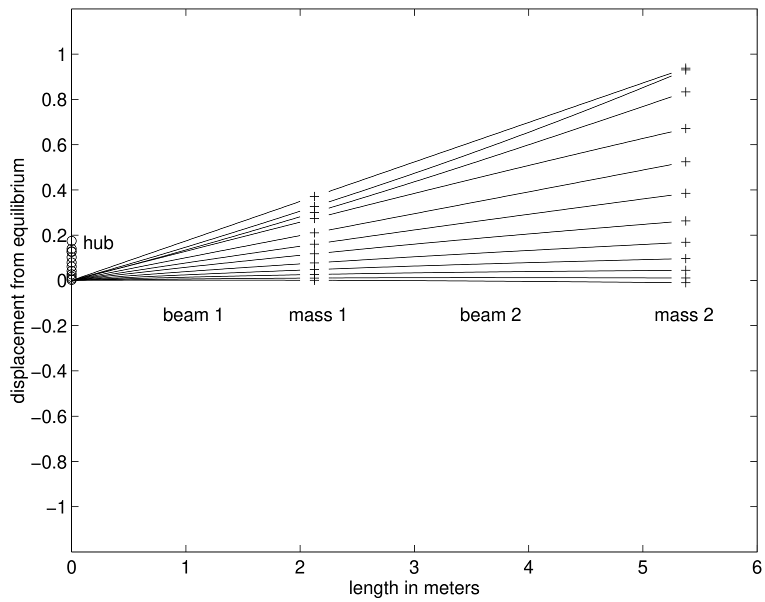

Figure 6.4: Closed Loop Simulation, Hub Rotation, 0 to 33 Seconds

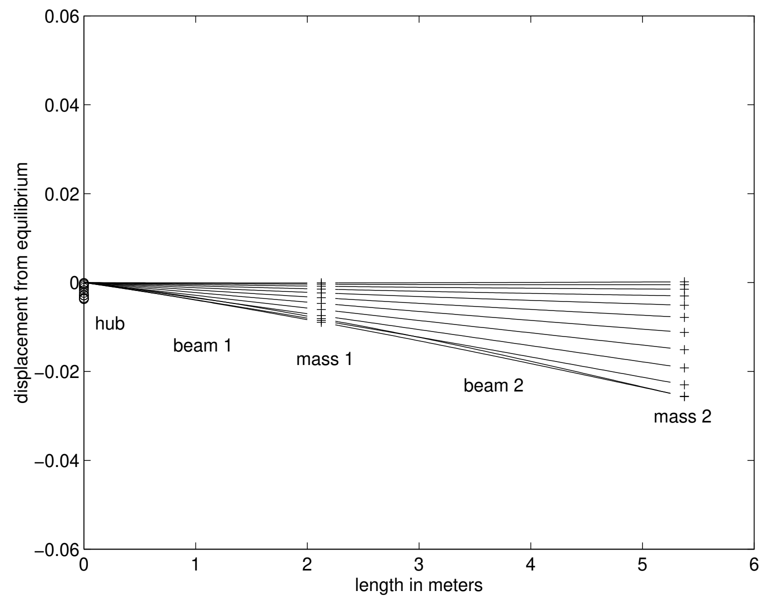

Figure 6.5: Closed Loop Simulation, Hub Rotation, 36 to 66 Seconds

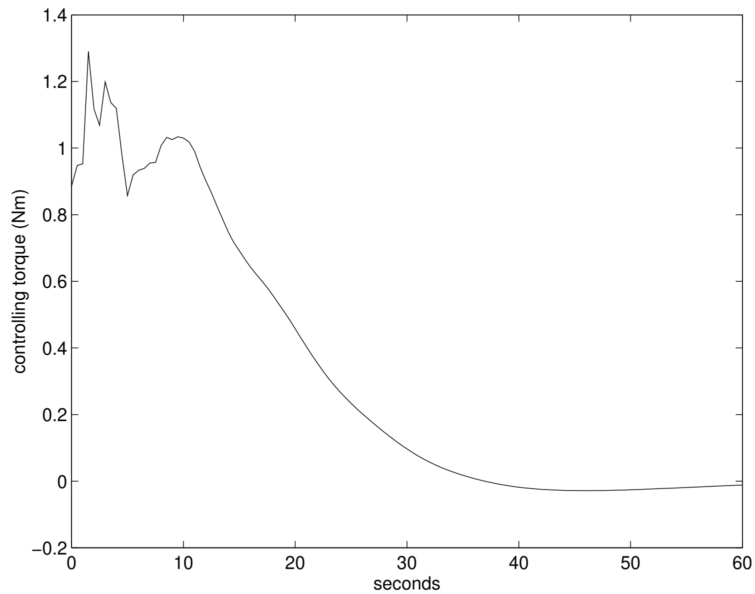

Figure 6.6: Controlling Torque, Simulation 1

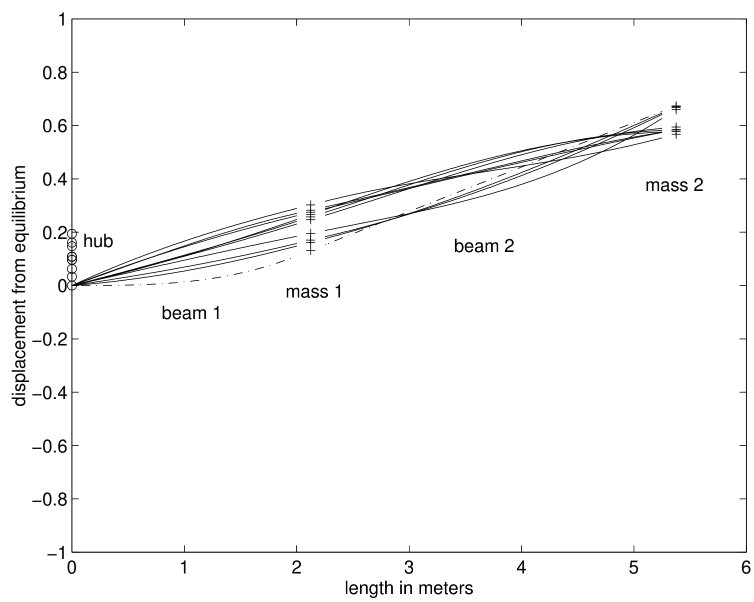

Figure 6.7: Open Loop Simulation, Deflection of Beam 1, 0 to 30 Seconds

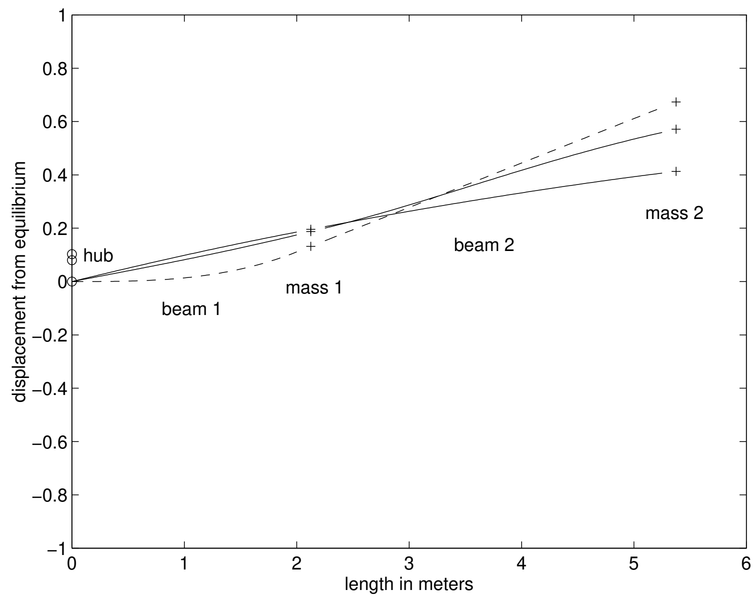

Figure 6.8: Closed Loop Simulation, Deflection of Beam 1, 0 to 6 Seconds

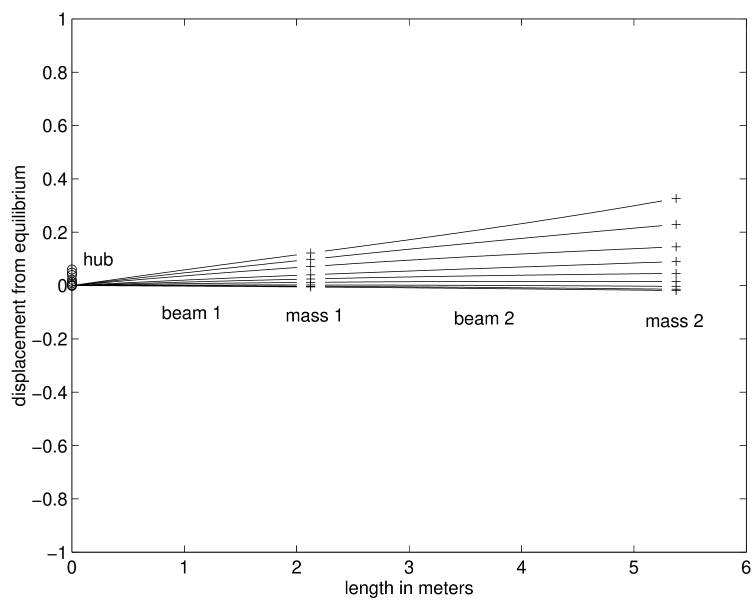

Figure 6.9: Closed Loop Simulation, Deflection of Beam 1, 9 to 33 Seconds

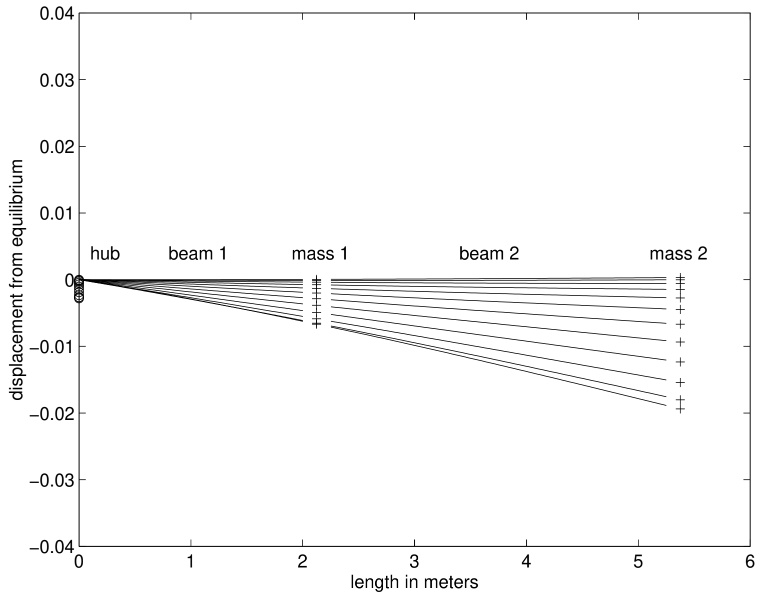

Figure 6.10: Closed Loop Simulation, Deflection of Beam 1, 36 to 69 Seconds

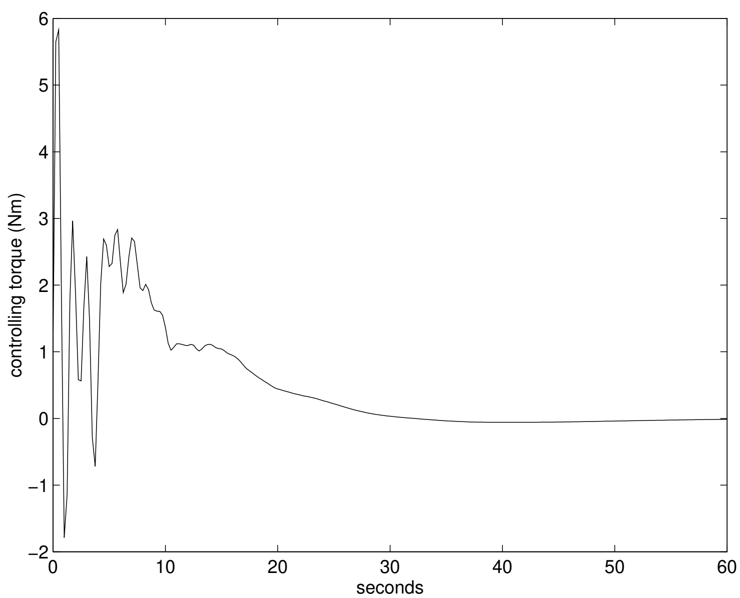

Figure 6.11: Controlling Torque, Simulation 2

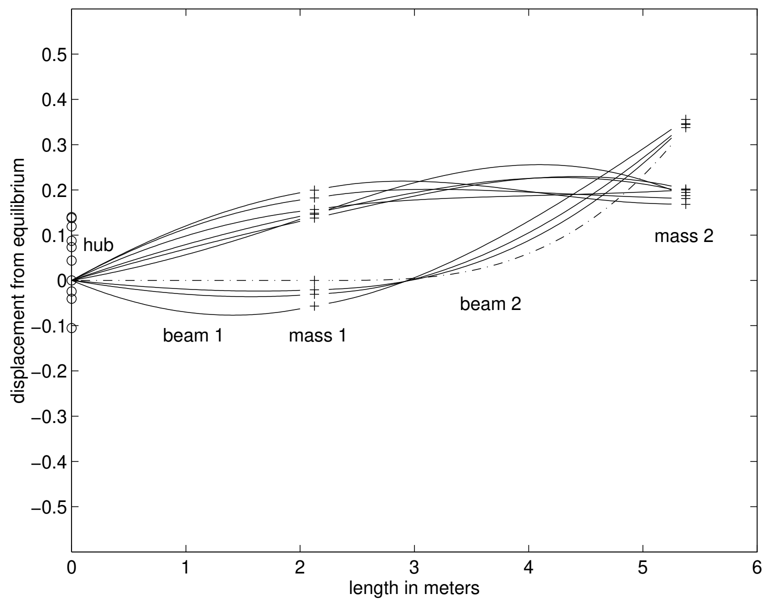

Figure 6.12: Open Loop Simulation, Deflection of Beam 2, 0 to 30 Seconds

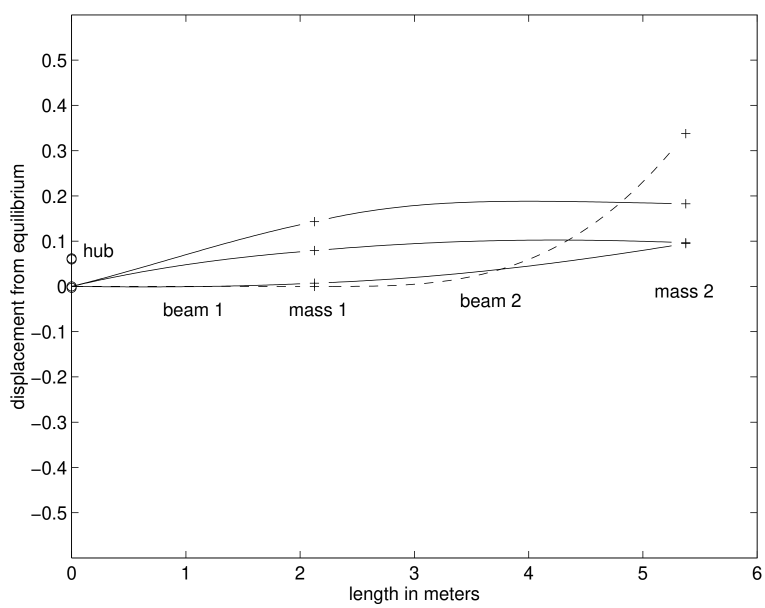

Figure 6.13: Closed Loop Simulation, Deflection of Beam 2, 0 to 9 Seconds

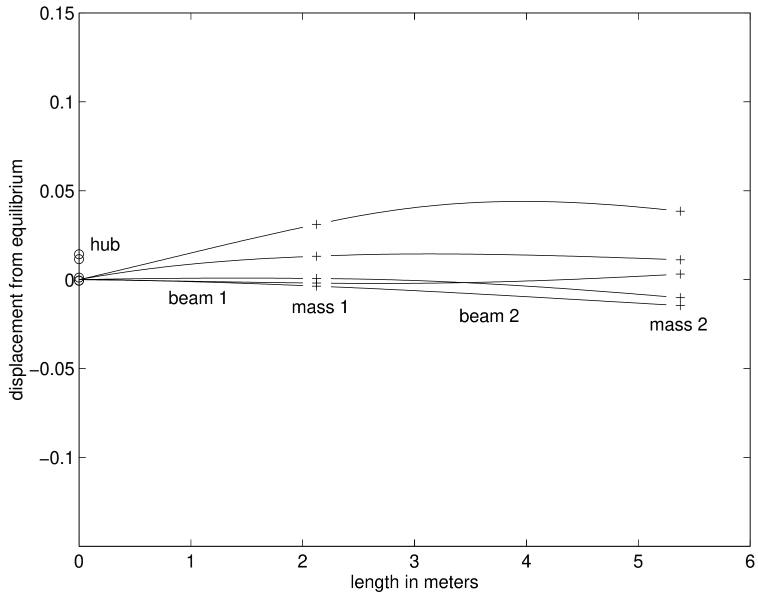

Figure 6.14: Closed Loop Simulation, Deflection of Beam 2, 12 to 24 Seconds

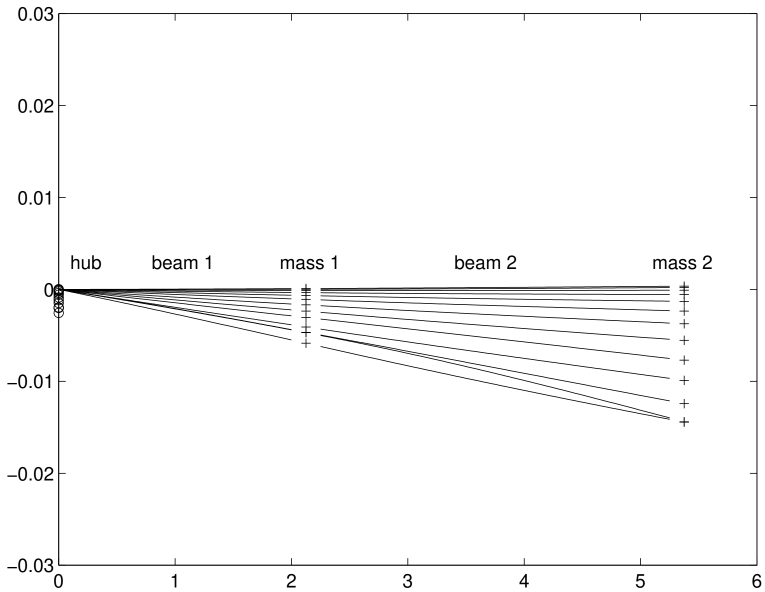

Figure 6.15: Closed Loop Simulation, Deflection of Beam 2, 27 to 63 Seconds

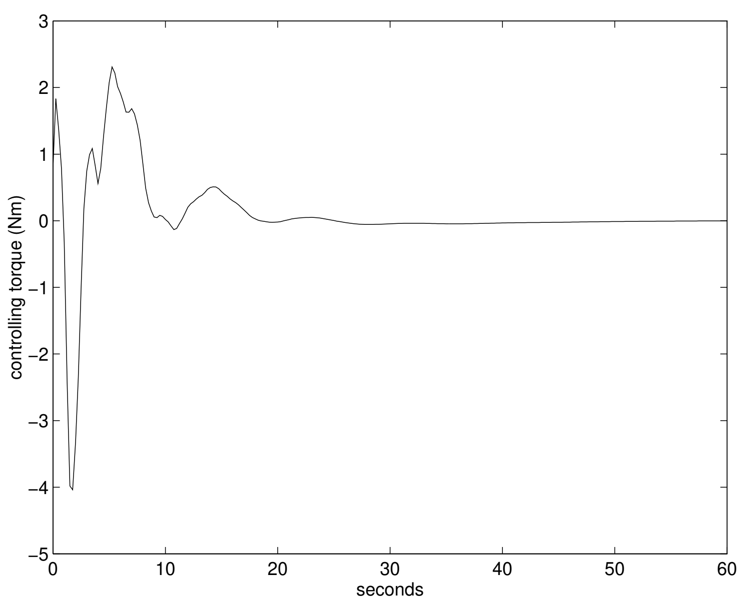

Figure 6.16: Controlling Torque, Simulation 3

7 Conclusions

In conclusion, the approximation scheme developed in Section 5 yields gains and eigenvalues for which the convergence is clear. In most cases, convergence is so rapid that it is not evident that the graphs of functional gains corresponding to values for N of 8, 12, and 16 are distinct. The simulations in Section 6 verify the performance of the computed control for three initial conditions. Since computations of optimal controls solving the LQR problem have not been previously reported for a MCS having more than one flexible component, these results show the viability of the method described in this paper. Other numerical examples have been computed for this structure in which beam and rigid body parameters are varied. A complete description of those results can be found in [14].

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