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System Assignment and Pole Placement for Symmetric Realisations*

R.E. Mahony† U. Helmke†

Abstract

In this paper we consider the problem of system assignment for the class of linear output feedback systems having symmetric state space realisations. For such systems the classical pole placement task can be considered as a degenerate case of a more general system assignment problem. It is shown that a symmetric state space realisation can be assigned arbitrary (real) poles via output feedback if and only if there are at least as many system inputs as states. The task of computing feedback gains for system assignment is approached by deriving gradient flows which minimize suitable least squares distance functions on smooth manifolds of output feedback equivalent realisations. These ordinary differential equations provide insight into the complex structure of the systems assignment and pole placement problems. Computing the limiting values of the flows provides a method of determining optimal feedback gains for the system assignment (pole placement) problem even when exact solutions to the problem does not exist. The methods are also generalised to a simultaneous multiple system assignment problem.

Key words: output feedback system assignment, pole placement, symmetric systems

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1 Introduction

A classical problem in systems theory is that of pole placement or eigenvalue assignment of linear systems via constant gain output feedback. This is clearly a difficult task and despite a number of important results, (cf. Byrnes [3] for an excellent survey,) a complete solution giving necessary and sufficient conditions for a solution to exist has not been developed. It has recently been shown that (strictly proper) linear systems with $mp > n$ can be assigned arbitrary poles using real output feedback [13]. Here n denotes the McMillan degree of a system having m inputs and p outputs. Of course if $mp < n$ for a given linear system then generic pole assignment is impossible, even when complex feedback gain is allowed [9]. The case $mp = n$ remains unresolved, though a number of interesting results are available [9, 14, 2]. Present results do not apply to output feedback systems with symmetries or structured feedback systems. More generally, one is also interested in situations where an optimal state feedback gain is sought such that the closed loop response of the system is a best approximation of a desired response, though the exact response may be unobtainable. In such cases one would still hope to find a constructive method to compute the optimal feedback that achieves the best approximation. The problem appears to be too difficult to tackle directly, however, and algorithmic solutions are an attractive alternative.

The techniques applied in this paper are related to recent work in solving linear algebraic problems using dynamical systems. A survey of this field along with applications in linear systems theory is given in the forthcoming monograph [8]. In addition, we mention Brockett [1] who tackles a least squares matching task, motivated by problems in computer vision algorithms, that is related to the system approximation problem we consider though his paper does not consider the effects of feedback. The work also relates to Chu's paper [5] who develops dynamical system methods to solve inverse singular value problems, a topic that is closely related to the pole placement question. The simultaneous multiple system assignment problem we consider is a generalisation of the single system problem and is reminiscent of Chu's approach [4] to simultaneous reduction of several real matrices.

In this paper, we consider a structured class of systems (those with symmetric state space realisations) for which, to our knowledge, no previous pole placement results are available. The assumption of symmetry of the realisation, besides having a natural network theoretic interpretation, simplifies the geometric analysis considerably. It is shown that a symmetric state space realisation can be assigned arbitrary (real) poles via output feedback if and only if there are at least as many system inputs as states. This result is surprising since a naive counting argument (comparing the number of free variables $\frac{1}{2}m(m+1)$ of symmetric output feedback gain to the number of poles n of a symmetric realization having m inputs and n states) would suggest that $\frac{1}{2}m(m+1) \geq n$ is sufficient for pole placement. To investigate the problem further we derive gradient flows of least squares cost criteria (functions of the matrix entries of realisations) defined on smooth manifolds of output feedback equivalent symmetric realisations. Limiting solutions to these flows occur at minima of the cost criteria and relate directly to finding optimal feedback gains for system assignment and pole placement problems. Cost criteria are proposed for solving the tasks of system assignment, pole placement, and simultaneous multiple system assignment. This work is part of an ongoing investigation into the potential of dynamical methods in systems theory. In particular, we mention a companion paper [10] in which a direct numerical scheme is proposed for computing the algorithms developed in the sequel and a discussion of general linear systems in [8, Chapter 5.3].

The paper consists of seven sections including the introduction. In section two we formulate the problems considered and prove two lemmas which provide necessary conditions for generic pole placement and system assignment. In section three we give a development of the geometry of the set of systems that are considered. Section four contains a dynamical systems approach to computing systems assignment problems for the class of symmetric state space realizations while section five contains three corollaries which apply to the pole placement and the simultaneous multiple system assignment problems. Lastly, section six contains a number of numerical investigations and section seven contains some final conclusions.

2 Statement of the Problem

In this section we present a brief review of symmetric systems before giving the precise formulations of the problems that are considered in the sequel and proving a pole placement result for symmetric state space realizations.

A *symmetric transfer function* is a proper rational matrix function $G(s) \in \mathbf{R}^{m\times m}$ such that

$$G(s) = G(s)^T.$$

For any such transfer function there exists a minimal *signature symmetric* realisation

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned}$$

of $G(s)$ such that $(AI_{pq})^T = AI_{pq}$ and $C^T = I_{pq}B$, with $I_{pq} = \mathrm{diag}(I_p, -I_q)$, a diagonal matrix with its first p diagonal entries 1 and the remaining diagonal entries -1. Symmetric transfer functions correspond to linear models of electrical networks constructed from resistors, capacitors and inductors. A signature symmetric realisation is a dynamical model of a suitable electrical network with p capacitors and q inductors.

Static linear symmetric output feedback is introduced to a state space model via a feedback law

$$u = Ky + v, \quad K = K^T,$$

leading to the "closed loop" system

$$\begin{aligned} \dot{x} &= (A + BKC)x + Bv, \\ y &= B^T x. \end{aligned} \tag{2.1}$$

In particular, symmetric output feedback, where $K = K^T \in \mathbf{R}^{m\times m}$, preserves the structure of signature symmetric realisations and is the only output feedback transformation that has this property.

A *symmetric state space* system (also *symmetric realisation*) is a linear dynamical system

$$\dot{x} = Ax + Bu, \quad A = A^T \tag{2.2}$$

$$y = B^T x, \tag{2.3}$$

with $x \in \mathbf{R}^n$, $u, y \in \mathbf{R}^m$, $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$. Thus, symmetric state space systems correspond to RC-networks, constructed entirely of capacitors and resistors. Without loss of generality assume that $m \leq n$, B is full rank and $B^T B = I_m$ the $m \times m$ identity matrix. We use a matrix pair $(A, B) \in S(n) \times O(n, m)$, where $S(n) = \{X \in \mathbf{R}^{n\times n} \mid X = X^T\}$ the set of symmetric $n \times n$ matrices and $O(n, m) = \{Y \in \mathbf{R}^{n\times m} \mid Y^T Y = I_m\}$ to represent a linear system of the form (2.2) and (2.3). The set $O(n, m)$ is the Stiefel manifold (a smooth $nm - \frac{1}{2}m(m+1)$ dimensional submanifold of $\mathbf{R}^{n\times m}$) of $n \times m$ matrices with orthonormal columns [8, pg. 24].

Two symmetric state space systems (A_1, B_1) and (A_2, B_2) are called *output feedback equivalent* if

$$(A_2, B_2) = (\Theta(A_1 + B_1 K B_1^T)\Theta^T, \Theta B_1) \tag{2.4}$$

holds for $\Theta \in O(n) = \{U \in \mathbf{R}^{n\times n} \mid U^T U = I_n\}$ the set of $n \times n$ orthogonal matrices and $K \in S(m)$ the set of symmetric $m \times m$ matrices. Thus the system (A_2, B_2) is obtained from (A_1, B_1) using an orthogonal change of basis $\Theta \in O(n)$ in the state space $\mathbf{R}^n$ and a symmetric feedback transformation $K \in S(m)$. It is easily verified that output feedback equivalence is an equivalence relation (cf. [12, pg. 22]) on the set of symmetric state space systems.

In this paper we consider the following problems for the class of symmetric state space systems.

Problem A Let $(A, B) \in S(n) \times O(n, m)$ be a symmetric state space system and choose $(F, G) \in S(n) \times O(n, m)$ to possesses the desired system structure. Consider the potential

$$\begin{aligned} \Psi \quad &: \quad \mathbf{R}^{n \times n} \times O(n, m) \to \mathbf{R}, \\ \Psi(A, B) \quad &:= \quad ||A - F||^2 + 2||B - G||^2, \end{aligned}$$

where $||X||^2 = \mathrm{tr}(X^T X)$ is the Frobenius (or Euclidean) matrix norm. Find a symmetric state space system $(A_{\min}, B_{\min})$ which minimizes Ψ over the set of all output feedback equivalent systems to (A, B). Equivalently, find a pair of matrices $(\Theta_{\min}, K_{\min}) \in O(n) \times S(m)$ such that

$$\psi(\Theta, K) := ||\Theta(A + BKB^T)\Theta^T - F||^2 + 2||\Theta B - G||^2,$$

is minimized over $O(n) \times S(m)$

Such a formulation is particularly of interest when structural properties of the desired realisations are specified. For example, one may wish to choose the "target system" (F, G) with certain structural zeros. If an exact solution to the system assignment problem exists (i.e. $\Psi(A_{\min}, B_{\min}) = 0$) it is easily seen that $(A_{\min}, B_{\min})$ will have the same structural zeros as (F, G). Of course, one expects that an exact solution to the system assignment problem need not always exist. Indeed, unless the symmetric state space systems considered have at least as many inputs as states the following lemma shows that the problem is generically unsolvable.

Lemma 2.1 *Let n and m be integers, $n \geq m$, and let $(F, G) \in S(n) \times O(n, m)$. Consider matrix pairs $(A, B) \in S(n) \times O(n, m)$.*

a) If $m = n$ then for any matrix pair (A, B) of the above form, there exist matrices $\Theta \in O(n)$ and $K \in S(m)$ such that

$$\Theta(A + BKB^T)\Theta^T = F, \quad \Theta B = G.$$

b) If $m < n$ then the set of $(A, B) \in S(n) \times O(n, m)$ for which an exact solution to the system assignment problem exists is measure zero in $S(n) \times O(n, m)$. (I.e. for almost all systems $(A, B) \in S(n) \times O(n, m)$ no exact solution to the system assignment problem exists.)

Proof: If $m = n$ then $O(n, m) = O(n)$ and $B^T = B^{-1}$. For any $(A, B) \in S(n) \times O(n)$ choose $(\Theta, K) = (GB^T, G^T FG - B^T AB)$. Thus,

$$\Theta(A + BKB^T)\Theta^T = GB^T ABG^T + GB^T B(G^T FG - B^T AB)B^T BG^T = F$$

and $\Theta B = GB^T B = G$.

To prove part b) observe that since output feedback equivalence is an equivalence relation the set of systems for which the system assignment problem is solvable are exactly those systems which are output feedback equivalent to (F, G). Consider the set

$$\mathcal{F}(F,G) = \{(\Theta(F + BGB^T)\Theta^T, \Theta G) \mid (\Theta, K) \in O(n) \times S(m)\}.$$

We use a result proved independently in Section 3, Lemma 3.1, that $\mathcal{F}(F,G)$ is a smooth submanifold of $S(n) \times O(n,m)$. But $\mathcal{F}(F,G)$ is the image of $O(n) \times S(m)$ via the continuous map $(\Theta, K) \mapsto (\Theta(F + BGB^T)\Theta^T, \Theta G)$ and necessarily has dimension at most $\dim O(n) \times S(m) = \frac{1}{2}n(n-1) + \frac{1}{2}m(m+1)$. The dimension of $S(n) \times O(n,m)$ however is $\frac{1}{2}n(n+1) + (nm - \frac{1}{2}m(m+1))$ [8, pg. 24]. Observe that

$$\dim O(n) \times S(m) - \dim S(n) \times O(n,m) = (n-m)(m+1),$$

which is strictly positive for $0 \leq m < n$. Thus, for $m < n$ the set $\mathcal{F}(F,G)$ is a submanifold of $S(n) \times O(n,m)$ of non-zero co-dimension and has zero measure. ∎

A similar task to Problem A is that of pole placement for symmetric state space realizations. The pole placement task for symmetric systems is; given an arbitrary set of numbers $s_1 \geq \ldots \geq s_n$ in $\mathbf{R}$ and an initial $m \times m$ symmetric transfer function $G(s) = G^T(s)$ with a symmetric realisation, find a symmetric matrix $K \in S(m)$ such that the poles of $G_K(s) = (I_m - G(s)K)^{-1}G(s)$ are exactly $s_1, \ldots, s_n$. Rather than tackle this problem directly we consider the following variant of the problem.

Problem B Let $(A, B) \in S(n) \times O(n,m)$ be a symmetric state space system and let $F \in S(n)$ be a symmetric matrix. Define

$$\begin{aligned} \Phi(A,B) &:= ||A - F||^2, \\ \phi(\Theta, K) &:= ||\Theta(A - BKB^T)\Theta^T - F||^2. \end{aligned}$$

Find a symmetric state space system $(A_{\min}, B_{\min})$ which minimizes Φ over the set of all output feedback equivalent systems to (A, B). Respectively, find a pair of matrices $(\Theta_{\min}, K_{\min}) \in O(n) \times S(m)$ which minimizes ϕ over $O(n) \times S(m)$.

Problem B minimizes a cost criterion that assigns the full eigenstructure of the closed loop system. Two symmetric matrices have the same eigenstructure (up to orthogonal similarity transformation) if and only if they have the same eigenvalues (since any symmetric matrix may be diagonalised via an orthogonal similarity transformation.) Thus, Problem B is equivalent to solving the pole-placement problem for symmetric systems (assigning the eigenvalues of the closed loop system.) The advantage of considering Problem B rather than a standard formulation of the pole-placement problem lies in the smooth nature of the optimization problem obtained.

It is of interest to consider generic conditions on symmetric state space systems for the existence of an exact solution to Problem B (i.e. the existence of $(\Theta_{\min}, K_{\min})$ such that $\phi(\Theta_{\min}, K_{\min}) = 0$). This is exactly the classical pole placement question about which much is known for general linear systems [3, 13]. The following result answers (at least in part) this question for symmetric state space systems. It is interesting to note that the necessary conditions for "generic" pole-placement for symmetric state space systems are much stronger than those for general linear systems.

Lemma 2.2 *Let n and m be integers, $n \geq m$, and let $F \in S(n)$ be a real symmetric matrix. Consider matrix pairs $(A, B) \in S(n) \times O(n, m)$.*

a) If $m = n$ then for any matrix pair (A, B) of the above form, there exist matrices $\Theta \in O(n)$ and $K = K^T \in \mathbf{R}^{m \times m}$ such that

$$\Theta(A + BKB^T)\Theta^T = F. \tag{2.5}$$

b) If $m < n$ then there exists an open set of matrix pairs $(A, B) \in S(n) \times O(n, m)$ of the above form such that eigenstructure assignment (to the matrix F) is impossible.

Proof: To prove part a) observe that since $m = n$ one has $B \in O(n)$. Thus, choosing

$$K = B^T(F - A)B,$$

gives $A + BKB^T = A + BB^T(F - A)BB^T = F$. Thus, the pair $(I_n, B^T(F - A)B) \in O(n) \times S(n)$ solves the eigenstructure assignment problem.

To prove part b) observe that the set of matrix pairs

$$\{(A, B) \mid A = BB^T ABB^T\}$$

is Zariski closed in $S(n) \times O(n, m)$ and consequently of measure zero [11]. Thus, either there exists a matrix pair $(A, B) \in S(n) \times O(n, m)$ and matrices $\Theta \in O(n)$ and $K = K^T \in \mathbf{R}^{m \times m}$ such that (2.5) is satisfied and $A \neq BB^T ABB^T$ or part b) is trivially true. Direct manipulations of (2.5), remembering that $B^T B = I_m$, yield

$$K = B^T(\Theta^T F \Theta - A)B.$$

Substituting this back into (2.5) gives

$$\Theta^T F \Theta = (A - BB^T ABB^T) + BB^T \Theta^T F \Theta BB^T.$$

Observe that

$$\begin{aligned} &\operatorname{tr}\left((A - BB^T ABB^T)^T BB^T \Theta^T F \Theta BB^T\right) \\ &\quad = \operatorname{tr}\left((BB^T(A - BB^T ABB^T)BB^T)\Theta^T F \Theta\right) \\ &\quad = 0, \end{aligned}$$

and taking the squared Frobenius norm of $\Theta^T F \Theta$ gives

$$||F||^2 = ||(A - BB^T ABB^T)||^2 + ||BB^T \Theta^T F \Theta BB^T||^2,$$

where we use the invariance of the Frobenius norm under orthogonal transformations. It follows directly that $||F||^2 \geq ||(A - BB^T ABB^T)||^2$.

Since (A, B) was chosen deliberately such that $A \neq BB^T ABB^T$ one may consider the related matrix pair $(A', B') = (\mu A, B)$, where

$$\mu = \left(\frac{||F||^2 + 1}{||A - BB^T ABB^T||^2} \right)^{\frac{1}{2}}.$$

By construction

$$||(A' - B'B'^T A'B'B'^T)||^2 = ||F||^2 + 1 > ||F||^2$$

and no solution to the eigenstructure assignment problem exists for the system (A', B'). Moreover, the map $(A, B) \mapsto ||(A - BB^T ABB^T)||^2$ is continuous and it follows that there is an open neighbourhood of systems around (A', B') for which the eigenstructure assignment task cannot be solved. ■

Remark 2.1 It follows directly from the proof of Lemma 2.2 that eigenstructure assignment of a symmetric state space system $(A, B) \in S(n) \times O(n, m)$ to an arbitrary closed loop matrix $F \in S(n)$ is possible only if

$$||F||^2 \geq ||A - BB^T ABB^T||^2.$$

In particular, if $m < n$ then the 'pole placement map' [3] is not almost onto for symmetric realisations. Of course if feedback is applied which destroys the symmetry of the realisation then the general linear pole placement results would apply. □

Remark 2.2 One may weaken the hypothesis of Lemma 2.2 considerably to deal with matrix pairs $(A, B) \in S(n) \times \mathbf{R}^{n \times m}$, for which B is not constrained to satisfy $B^T B = I_m$ and for which m may be greater than n. One has that eigenstructure assignment is generically possible if and only if rank$B = n$. The proof is similar to that given above observing that the projection operator BB^T is related to the general projection operator $B(B^T B)^\dagger B^T$, where $\dagger$ represents the pseudo-inverse of a matrix. For example, the feedback matrix yielding exact system assignment for rank$B = n$ is

$$K = (B^T B)^\dagger B^T (F - A) B (B^T B)^\dagger.$$

□

A further problem that is considered is that of simultaneous multiple system assignment. This is a difficult problem about which very little is known presently. Our approach is to consider a generalisation of the cost criterion ψ for a single system.

Problem C For any integer $N \in \mathbf{N}$ let $(A_1, B_1), \ldots, (A_N, B_N)$ and (F_1, G_1), …, (F_N, G_N) be two sets of N symmetric state space systems. Define

$$\psi_N(\Theta, K) := \sum_{i=1}^{N} ||\Theta(A_i + B_i K B_i^T)\Theta^T - F_i||^2 + 2\sum_{i=1}^{N} ||\Theta B_i - G_i||^2.$$

Find a pair of matrices $(\Theta_{\min}, K_{\min}) \in O(n) \times S(m)$ which minimizes ψ_N over $O(n) \times S(m)$.

3 Geometry of Output Feedback Orbits

It is necessary to briefly review the Riemannian geometry of the spaces on which the optimization problems stated in Section 2 are posed. The reader is referred to Helgason [7] for technical details on Lie-groups and homogeneous spaces and Helmke and Moore [8] for a development of dynamical systems methods for optimization along with applications in linear systems theory. In the sequel we often use the Lie-bracket notation $[X, Y] = XY - YX$, where X and Y are square matrices.

The set $O(n) \times S(m)$ forms a Lie group under the group operation $(\Theta_1, K_1) \cdot (\Theta_2, K_2) = (\Theta_1\Theta_2, K_1 + K_2)$. It is known as the *output feedback group* for symmetric state space systems. The tangent spaces of $O(n) \times S(m)$ are

$$T_{(\Theta,K)}(O(n) \times S(m)) = \{(\Omega\Theta, \Xi) \mid \Omega \in Sk(n), \Xi \in S(m)\},$$

where $Sk(n) = \{\Omega \in \mathbf{R}^{n\times n} \mid \Omega = -\Omega^T\}$ the set of $n \times n$ skew symmetric matrices. The Euclidean inner product on $\mathbf{R}^{n\times n} \times \mathbf{R}^{n\times m}$ is given by

$$\langle (A, B), (X, Y) \rangle = \mathrm{tr}(A^T X) + \mathrm{tr}(B^T Y). \tag{3.1}$$

By restriction, this induces a non-degenerate inner product on the tangent space $T_{(I_n,0)}(O(n) \times S(m)) = Sk(n) \times S(m)$. The Riemannian metric we consider on $O(n) \times S(m)$ is the *right invariant group metric*

$$\langle (\Omega_1\Theta, \Xi_1), (\Omega_2\Theta, \Xi_2) \rangle = 2\mathrm{tr}(\Omega_1^T \Omega_2) + 2\mathrm{tr}(\Xi_1^T \Xi_2).$$

The right invariant group metric is generated by the induced inner product on $T_{(I_n,0)}(O(n) \times S(m))$, mapped to each tangent space by the linearisation of the diffeomorphism $(\theta, k) \mapsto (\theta\Theta, k + K)$ for $(\Theta, K) \in (O(n) \times S(m))$. It is readily verified that this defines a Riemannian metric which corresponds, up to a scaling factor, to the induced Riemannian metric on $O(n) \times S(m)$ considered as a submanifold of $\mathbf{R}^{n\times n} \times \mathbf{R}^{n\times m}$. The scaling factor 2 serves to simplify the algebraic expressions obtained in the sequel.

Let $(A, B) \in S(n) \times O(n, m)$ be a symmetric state space system. The *symmetric output feedback orbit* of (A, B) is the set

$$\mathcal{F}(A, B) = \{(\Theta(A + B^T K B)\Theta^T, \Theta B) \mid \Theta \in O(n), K \in S(m)\}, \tag{3.2}$$

of all symmetric realisations that are output feedback equivalent to (A, B). Observe that no assumption on the controllability of the matrix pair (A, B) is made.

Lemma 3.1 *The symmetric output feedback orbit $\mathcal{F}(A, B)$ is a smooth submanifold of $S(n) \times O(n, m)$ with tangent space at a point (A, B) given by*

$$T_{(A,B)}\mathcal{F}(A, B) = \{([\Omega, A] + B\Xi B^T, \Omega B) \mid \Omega \in Sk(n), \Xi \in S(m)\}. \tag{3.3}$$

Proof: The set $\mathcal{F}(A, B)$ is an orbit of the smooth semi-algebraic group action

$$\begin{aligned} &\beta : (O(n) \times S(m)) \times (S(n) \times O(n, m)) \to (S(n) \times O(n, m)), \\ &\beta((\Theta, K), (A, B)) := (\Theta(A + B^T K B)\Theta^T, \Theta B). \end{aligned} \tag{3.4}$$

It follows that $\mathcal{F}(A, B)$ is a smooth submanifold of $S(n) \times \mathbf{R}^{n\times m}$ [6, Appendix B]. For an arbitrary matrix pair (A, B) the map

$$f(\Theta, K) := (\Theta(A + BKB^T)\Theta^T, \Theta B)$$

is a smooth submersion of $O(n) \times S(m)$ onto $\mathcal{F}(A, B)$ [6, pg. 74]. The tangent space of $\mathcal{F}(A, B)$ at (A, B) is the range of the linearisation of f at $(I_n, 0)$

$$\begin{aligned} T_{(I_n,0)}f \quad : \quad & T_{(I_n,0)}(O(n) \times S(m)) \to T_{(A,B)}\mathcal{F}(A, B) \\ & (\Omega, \Xi) \mapsto ([\Omega, A] + B\Xi B^T, \Omega B), \;\; (\Omega, \Xi) \in Sk(n) \times S(m). \end{aligned}$$

■

The space $\mathcal{F}(A, B)$ is also a Riemannian manifold when equipped with the so-called *normal* metric [8]. Fix $(A, B) \in S(n) \times O(n, m)$ a symmetric state space system and consider the map

$$f(\Theta, K) := (\Theta(A + BKB^T)\Theta^T, \Theta B).$$

The tangent map $T_{(I_n,0)}f$ induces a decomposition

$$T_{(I_n,0)}(O(n) \times S(m)) = \ker T_{(I_n,0)}f \oplus \mathrm{dom}\; T_{(I_n,0)}f,$$

where

$$\ker T_{(I_n,0)} f = \{(\Omega_\perp, \Xi_\perp) \in Sk(n) \times S(m) \mid [A, \Omega_\perp] = B\Xi_\perp B^T, \Omega_\perp B = 0\}$$

is the kernel of $T_{(I_n,0)} f$ and

$$\begin{aligned}\text{dom } T_{(I_n,0)} f &= \{(\Omega^\perp, \Xi^\perp) \in Sk(n) \times S(m) \mid \operatorname{tr}((\Omega^\perp)^T \Omega_\perp) = 0, \\ &\qquad \operatorname{tr}((\Xi^\perp)^T \Xi_\perp) = 0, \text{ for all } (\Omega_\perp, \Xi_\perp) \in \ker T_{(I_n,0)} f\}\end{aligned}$$

is the orthogonal complement of the kernel with respect to the Euclidean inner product (3.1). Formally, the normal Riemannian metric on $\mathcal{F}(A,B)$ is the inner product (3.1) on $T_{(I_n,0)}(O(n) \times S(m))$ restricted to dom $T_{(I_n,0)} f$ and induced on $T_{(A,B)}\mathcal{F}(A,B)$ via the isomorphism

$$T_{(I_n,0)} f^\perp : \text{dom } T_{(I_n,0)} f \to T_{(A,B)}\mathcal{F}(A,B),$$

the restriction of $T_{(I_n,0)} f$ to dom $T_{(I_n,0)} f$. Thus, for two tangent vectors $([\Omega_i, A] + B\Xi_i B^T, \Omega_i B) \in T_{(A,B)}\mathcal{F}(A,B)$, $i = 1, 2$, the normal Riemannian metric is computed as

$$\begin{aligned}&\langle\langle([\Omega_1, A] + B\Xi_1 B^T, \Omega_1 B), ([\Omega_2, A] + B\Xi_2 B^T, \Omega_2 B)\rangle\rangle \\ &\qquad = 2\operatorname{tr}((\Omega_1^\perp)^T \Omega_2^\perp) + 2\operatorname{tr}((\Xi_1^\perp)^T \Xi_2^\perp).\end{aligned}$$

Here $(\Omega_i, \Xi_i) = ((\Omega_i)_\perp, (\Xi_i)_\perp) \oplus (\Omega_i^\perp, \Xi_i^\perp) \in \ker T_{(I_n,0)} f \oplus \text{dom } T_{(I_n,0)} f$ for $i = 1, 2$. It is readily verified that this construction defines a Riemannian metric on $\mathcal{F}(A,B)$.

4 Least Squares System Assignment

In this section we consider Problem A, i.e. the question of computing a symmetric state space linear system in a given orbit $\mathcal{F}(A,B)$ that most closely approximates a given "target" system in a least squares sense. We provide a brief analysis of the cost functions Ψ and ψ which leads to existence results for global minima. Gradient flows of the cost functions are derived and existence results for their solutions are given.

Lemma 4.1 *Let $(F,G), (A,B) \in S(n) \times O(n,m)$ be symmetric state space linear systems.*

a) The function $\psi : O(n) \times S(m) \to \mathbf{R}$,

$$\psi(\Theta, K) := ||\Theta(A + BKB^T)\Theta^T - F||^2 + 2||\Theta B - G||^2,$$

has compact sublevel sets. I.e. the sets

$$\{(\Theta, K) \in O(n) \times S(m) \mid \psi(\Theta, K) \leq \alpha\}$$

for any $\alpha \geq 0$, are compact subsets of $O(n) \times S(m)$.

b) The function $\Psi : \mathcal{F}(A,B) \to \mathbf{R}$,

$$\Psi(A,B) := ||A-F||^2 + 2||B-G||^2,$$

has compact sublevel sets.

Proof: The triangle inequality yields both

$$||K||^2 = ||BKB^T||^2 \le 2(||A+BKB^T||^2 + ||A||^2)$$

and

$$\begin{aligned} ||A+BKB^T||^2 &= ||\Theta(A+BKB^T)\Theta^T||^2 \\ &\le 2(||\Theta(A+BKB^T)\Theta^T - F||^2 + ||F||^2). \end{aligned}$$

Thus, for $(\Theta, K) \in O(n) \times S(m)$ one has

$$\begin{aligned} ||K||^2 &\le 2(2(||\Theta(A+BKB^T)\Theta^T - F||^2 + ||F||^2) + ||A||^2) \\ &\le 4(||\Theta(A+BKB^T)\Theta^T - F||^2 + 2||\Theta B - G||^2) \\ &\quad +4||F||^2 + 2||A||^2, \\ &= 4\psi(\Theta,K) + 4||F||^2 + 2||A||^2, \end{aligned}$$

where a factor of $8||\Theta B - G||^2$ is added to the middle line to give the correct terms for the cost ψ. Thus, for $(\Theta, K) \in O(n) \times S(m)$, satisfying $\psi(\Theta,K) \le \alpha$, one has

$$\begin{aligned} ||(\Theta,K)||^2 &= ||\Theta||^2 + ||K||^2 \\ &\le \operatorname{tr}(\Theta^T\Theta) + 4\psi(\Theta,K) + 4||F||^2 + 2||A||^2 \\ &\le n + 4\alpha + 4||F||^2 + 2||A||^2, \end{aligned}$$

and the sublevel sets of ψ are bounded. Since ψ is continuous the sublevel sets are closed and compactness follows directly [12, pg. 174]. Part b) follows by observing that $\psi = \Psi \circ f$, where $f(\Theta,K) := (\Theta(A+BKB^T)\Theta^T, \Theta B)$ for given $(A,B) \in \mathcal{F}(A,B)$. Thus, the sublevel sets of Ψ are exactly the images of the corresponding sublevel sets of ψ via the continuous map f. Since continuous images of compact sets are themselves compact [12, pg. 167] the proof is complete. ∎

Corollary 4.1 *Let $(F,G),(A,B) \in S(n) \times O(n,m)$ be symmetric state space linear systems.*

a) There exists a global minimum $(\Theta_{\min}, K_{\min}) \in O(n) \times S(m)$ of ψ,

$$\psi(\Theta_{\min}, K_{\min}) = \inf\{\psi(\Theta,K) \mid (\Theta,K) \in O(n) \times S(m)\}.$$

b) There exists a global minimum $(A_{\min}, B_{\min}) \in \mathcal{F}(A,B)$ *of* Ψ,

$$\Psi(A_{\min}, B_{\min}) = \inf\{\Psi(A,B) \mid (A,B) \in \mathcal{F}(A,B)\}.$$

c) The submanifold $\mathcal{F}(A,B) \subseteq S(n) \times O(n,m)$ *is closed in* $S(n) \times \mathbf{R}^{n\times m}$.

Proof: To prove part a), choose $\alpha \geq 0$ such that the sublevel set $J = (\{(\Theta, K) \mid \psi(\Theta, K) \leq \alpha\}$ is non empty. Then $\psi|_J : J \to [0,\alpha]$ is a continuous map from a compact space into the reals and the minimum value theorem [12, pg. 175] ensures the existence of $(\Theta_{\min}, K_{\min})$. The proof of part b) is analogous to that for part a).

To prove c) Assume that $\mathcal{F}(A,B)$ is not closed. Choose a boundary point $(F,G) \in \overline{\mathcal{F}(A,B)} - \mathcal{F}(A,B)$. By part b) there exists a minimum $(A_{\min}, B_{\min}) \in \mathcal{F}(A,B)$ such that

$$\begin{aligned} \Psi(A_{\min}, B_{\min}) &= \inf\{\Psi(A,B) \mid (A,B) \in \mathcal{F}(A,B)\} \\ &= 0 \end{aligned}$$

since (F,G) is in the closure of $\mathcal{F}(A,B)$. But this implies $||A_{\min} - F||^2 + 2||B_{\min} - G||^2 = 0$ and consequently $(A_{\min}, B_{\min}) = (F,G)$. This contradicts the assumption that $(F,G) \notin \mathcal{F}(A,B)$. ■

Having determined the existence of a solution to the system assignment problem we now consider the problem of computing the global minima of the cost functions Ψ and ψ.

Theorem 4.1 *Let* $(A,B), (F,G) \in S(n) \times O(n,m)$ *be symmetric state space systems. Let*

$$\Psi : \mathcal{F}(A,B) \to \mathbf{R}, \quad \Psi(A,B) := ||A-F||^2 + 2||B-G||^2, \tag{4.1}$$

measure the Euclidean distance between two symmetric realisations. Then

a) The gradient of Ψ *with respect to the normal metric is*

$$\operatorname{grad}\Psi(A,B) = \begin{pmatrix} -[A,([A,F] + BG^T - GB^T)] + BB^T(A-F)BB^T \\ ([A,F] + BG^T - GB^T)B \end{pmatrix} \tag{4.2}$$

b) The critical points of Ψ *are characterised by*

$$\begin{aligned} [A,F] &= GB^T - BG^T, \\ 0 &= B^T(A-F)B. \end{aligned} \tag{4.3}$$

c) Solutions of the gradient flow $(\dot{A}, \dot{B}) = -\mathrm{grad}\Psi(A, B)$,

$$\begin{array}{rcl} \dot{A} & = & [A, ([A, F] + BG^T - GB^T)] - BB^T(A - F)BB^T \\ \dot{B} & = & -([A, F] + BG^T - GB^T)B \end{array} \qquad (4.4)$$

exist for all time $t \geq 0$ *and remain in* $\mathcal{F}(A, B)$.

d) Any solution to (4.4) converges as $t \to \infty$ *to a connected set of matrix pairs* $(A, B) \in \mathcal{F}(A, B)$ *which satisfy (4.3) and lie in a single level set of* Ψ.

Proof: We compute the gradient using the identities[1]

$$[i)] \qquad D\Psi|_{(A,B)}(\eta) = \langle\langle \mathrm{grad}\Psi(A, B), \eta \rangle\rangle,$$
$$\eta = ([\Omega, A] + B\Xi B^T, \Omega B) \in T_{(A,B)}\mathcal{F}(A, B)$$
$$[ii)] \qquad \mathrm{grad}\Psi(A, B) \in T_{(A,B)}\mathcal{F}(A, B),$$

Computing the Fréchet derivative of Ψ in direction $([\Omega, A] + B\Xi B^T, \Omega B)$ gives

$$\begin{aligned} &D\Psi|_{(A,B)}([\Omega, A] + B\Xi B^T, \Omega B) \\ &\quad = 2\mathrm{tr}((A - F)^T([\Omega, A] + B\Xi B^T)) + 4\mathrm{tr}((B - G)^T\Omega B) \qquad (4.5) \\ &\quad = 2\mathrm{tr}((-[A - F, A] + 2B(B - G)^T)\Omega) + 2\mathrm{tr}(B^T(A - F)B\Xi) \\ &\quad = \langle\langle [([A, F] + BG^T - GB^T), A] + BB^T(A - F)BB^T, \\ &\qquad\qquad ([A, F] + BG^T - GB^T)B\big), \\ &\qquad\qquad ([\Omega, A] + B\Xi B^T, \Omega B)\rangle\rangle. \end{aligned}$$

When deriving the above relations it is useful to recall that

$$([\Omega, A] + B\Xi B^T, \Omega B) = ([\Omega^\perp, A] + B\Xi^\perp B^T, \Omega^\perp B)$$

where $(\Omega, \Xi) = (\Omega_\perp, \Xi_\perp) \oplus (\Omega^\perp, \Xi^\perp)$, (cf. the discussion of normal metrics at the end of Section 3). Observing that $([A, F] + BG^T - GB^T) \in Sk(n)$ while $B^T(A - F)B \in S(m)$ completes the proof of part a).

To prove b), observe that the first identity ensures that the Fréchet derivative at a critical point ($\mathrm{grad}\Psi = 0$) is zero in all tangent directions. Setting (4.5) to zero, yields

$$2\mathrm{tr}(([F, A] + GB^T - BG^T)\Omega) + 2\mathrm{tr}(B^T(A - F)B\Xi) = 0$$

for arbitrary $(\Omega, \Xi) \in Sk(n) \times S(m)$ and (A, B) a critical point of Ψ.

[1] $D\Psi|_{(A,B)}(\eta)$ is the Fréchet derivative of Ψ in direction η [8].

For given initial conditions $(A(0), B(0))$ solutions of (4.4) will remain in the sublevel set $\{(A,B) \in \mathcal{F}(A,B) \mid \Psi(A,B) \leq \Psi(A(0), B(0))\}$. Since this set is compact, Lemma 4.1, infinite time existence of the solution follows. This proves c) while d) follows from an application of LaSalle's invariance principle. ■

Remark 4.1 Let $N(s)D(s)^{-1}$ be a coprime factorisation of the symmetric transfer function $G(s) = B^T(sI - A)^{-1}B$. Then the coefficients of the polynomial matrix $N(s) \in \mathbf{R}[s]^{m \times m}$ are invariants of the flow (4.4). In particular the zeros of the system (A, B, B^T) are invariant under the flow (4.4). □

The above theorem provides a method of investigating best approximations to a given "target system" within a symmetric output feedback orbit. However, it does not provide any explicit information on the changing feedback transformations $(\Theta(t), K(t))$. To obtain such information we propose a related flow on the output feedback group $O(n) \times S(m)$. The following result generalises work by Brockett on matching problems [1]. Brockett considers similar cost functions but only allows state space transformations rather than output feedback transformations.

Theorem 4.2 *Let $(A,B),(F,G) \in S(n) \times O(n,m)$ be symmetric state space linear systems. Define*

$$\psi : O(n) \times S(m) \to \mathbf{R}, \quad \psi(\Theta, K) := ||\Theta(A + BKB^T)\Theta^T - F||^2 + 2||\Theta B - G||^2 \tag{4.6}$$

then:

a) The gradient of ψ with respect to the right invariant group metric is

$$\operatorname{grad}\psi(\Theta, K) = \begin{pmatrix} [\Theta(A + BKB^T)\Theta^T, F]\Theta + (\Theta BG^T - GB^T\Theta^T)\Theta \\ B^T(A + BKB^T - \Theta^T F\Theta)B \end{pmatrix} \tag{4.7}$$

b) The critical points of ψ are characterised by

$$\begin{aligned} [F, \Theta(A + BKB^T)\Theta^T] &= (\Theta BG^T - GB^T\Theta^T), \\ K &= B^T(\Theta^T F\Theta - A)B. \end{aligned} \tag{4.8}$$

c) Solutions of the gradient flow $(\dot{\Theta}, \dot{K}) = -\operatorname{grad}\psi(\Theta, K)$

$$\begin{aligned} \dot{\Theta} &= -[\Theta(A + BKB^T)\Theta^T, F]\Theta - (\Theta BG^T - GB^T\Theta^T)\Theta, \\ \dot{K} &= -B^T(A + BKB^T - \Theta^T F\Theta)B, \end{aligned} \tag{4.9}$$

exist for all time $t \geq 0$ and remain in a bounded subset of $O(n) \times S(m)$. Moreover, as $t \to \infty$ any solution of (4.9) converges to a connected subset of critical points in $O(n) \times S(m)$ which are contained in a single level set of ψ.

d) If $(\Theta(t), K(t))$ is a solution to (4.9) then

$$(\mathcal{A}(t), \mathcal{B}(t)) = (\Theta(t)(A + BK(t)B^T)\Theta(t)^T, \Theta(t)^T B)$$

is a solution of (4.4).

Proof: The computation of the gradient is analogous to that undertaken in the proof of Theorem 4.1 while the characterisation of the critical points follows directly from setting (4.7) to zero. The proof of c) is also analogous to the proof of parts c) and d) in Theorem 4.1.

The linearisation of $f(\Theta, K) := (\Theta(A + BKB^T)\Theta^T, \Theta^T B)$ is readily computed to be

$$T_{(\Theta,K)} f(\Omega\Theta, \Xi) = ([\Omega, \mathcal{A}] + \mathcal{B}\Xi\mathcal{B}^T, \Omega\mathcal{B})$$

where $\mathcal{A} = \Theta(A + BKB^T)\Theta^T$ and $\mathcal{B} = \Theta B$. The image of $(\dot{\Theta}, \dot{K})$ via this linearisation is

$$\begin{aligned} T_{(\Theta,K)} f(\dot{\Theta}, \dot{K}) \quad = \quad & \big([\mathcal{A}, [\mathcal{A}, F] + (\mathcal{B}G^T - G\mathcal{B})] + \mathcal{B}\mathcal{B}^T(\mathcal{A} - F)\mathcal{B}\mathcal{B}^T, \\ & -([\mathcal{A}, F] + \mathcal{B}G^T - G\mathcal{B}^T)\mathcal{B}\big) . \end{aligned}$$

Consequently $(\dot{\mathcal{A}}, \dot{\mathcal{B}}) = -\mathrm{grad}\Psi(\mathcal{A}, \mathcal{B})$. Classical O.D.E. uniqueness results complete the proof. ■

The following lemma provides an alternative approach to determining a bound on the feedback gain $K(t)$. The method of proof for the following result is of interest and the result obtained is somewhat tighter than that obtained in Lemma 4.1.

Lemma 4.2 *Let $(\Theta(t), K(t))$ be a solution of (4.9). Then the bound*

$$||K(t) - K(0)||^2 \leq \frac{1}{2}\psi(T(0), K(0))$$

holds for all time.

Proof: Integrating out (4.9) for initial conditions (Θ_0, K_0) and then taking norms gives the integral bound

$$\begin{aligned} & ||\Theta(t) - \Theta_0||^2 + ||K(t) - K_0||^2 \\ & \quad = || \int_0^t \mathrm{grad}\psi(\Theta(\tau), K(\tau)) d\tau ||^2 \end{aligned}$$

$$\leq \int_0^t ||\mathrm{grad}\psi(\Theta(\tau), K(\tau))||^2 d\tau$$
$$= \frac{1}{2}\int_0^t \langle \mathrm{grad}\psi(\Theta(\tau), K(\tau)), \mathrm{grad}\psi(\Theta(\tau), K(\tau))\rangle d\tau.$$

Also

$$\frac{d}{dt}\psi(\Theta(t), K(t)) = -\langle \mathrm{grad}\psi(\Theta(t), K(t)), \mathrm{grad}\psi(\Theta(t), K(t))\rangle,$$

and thus, integrating between 0 and t and recalling that $0 \leq \psi(\Theta(t), K(t)) \leq \psi(\Theta(0), K(0))$ for all $t \geq 0$ we get

$$\int_0^t \langle \mathrm{grad}\psi(\Theta(\tau), K(\tau)), \mathrm{grad}\psi(\Theta(\tau), K(\tau))\rangle d\tau =$$
$$\psi(\Theta(0), K(0)) - \psi(\Theta(t), K(t)) \leq \psi(\Theta(0), K(0)),$$

and consequently

$$||\Theta(t) - \Theta_0||^2 + ||K(t) - K_0||^2 \leq \frac{1}{2}\psi(\Theta(0), K(0)).$$

The result follows directly. ■

It is advantageous to consider a closely related flow that evolves only on $O(n)$ rather than the full output feedback group $O(n) \times S(m)$. The following development uses similar techniques to those proposed by Chu in [5].

Let $(A, B) \in S(n) \times O(n, m)$ be a given symmetric state space system and define

$$\mathcal{L} = \{BKB^T \mid K \in S(m)\}$$

to be the linear subspace of $S(n)$ corresponding to the range of the linear map $K \mapsto BKB^T$. Similarly, define $\mathcal{L}^\perp$ to be the orthogonal complement of $\mathcal{L}$ with respect to the Euclidean inner product on $\mathbf{R}^{n\times n}$. The projection operators

$$\mathbb{P} : S(n) \to \mathcal{L}, \quad \mathbb{P}(X) := BB^TXBB^T \tag{4.10}$$

and

$$\mathbb{Q} : S(n) \to \mathcal{L}^\perp, \quad \mathbb{Q}(X) := (\mathbb{I} - \mathbb{P})(X) = X - BB^TXBB^T \tag{4.11}$$

are well defined. Here $\mathbb{I}$ represents the identity operator and $B^TB = I_m$ by assumption. The tangent space of $O(n)$ at a point Θ is $T_\Theta O(n) = \{\Omega\Theta \mid \Omega \in Sk(n)\}$ with Riemannian metric $\langle \Omega_1\Theta, \Omega_2\Theta\rangle = 2\mathrm{tr}(\Omega_1^T\Omega_2)$, corresponding to the right invariant group metric on $O(n)$.

Theorem 4.3 *Let* $(A,B),(F,G) \in S(n) \times O(n,m)$ *be symmetric state space systems. Define*

$$\begin{aligned} \psi^* &: O(n) \to \mathbf{R} \\ \psi^* &:= ||\mathbb{Q}(A - \Theta^T F \Theta)||^2 + 2||\Theta B - G||^2 \end{aligned} \tag{4.12}$$

then,

a) The gradient of ψ^* *with respect to the right invariant group metric is*

$$\mathrm{grad}\psi^*(\Theta) = [\Theta\mathbb{Q}(A - \Theta^T F\Theta)\Theta^T, F]\Theta + (\Theta B G^T - G B^T \Theta^T)\Theta.$$

b) The critical points $\Theta \in O(n)$ *of* ψ^* *are characterised by*

$$[F, \Theta\mathbb{Q}(A - \Theta^T F\Theta)\Theta^T] = (\Theta B G^T - G B^T \Theta^T).$$

and correspond exactly to the orthogonal matrix component of the critical points (4.8) of ψ.

c) The negative gradient flow minimizing ψ^* *is*

$$\dot{\Theta} = [F, \Theta\mathbb{Q}(A - \Theta^T F\Theta)\Theta^T]\Theta - (\Theta B G^T - G B^T \Theta^T)\Theta, \quad \Theta(0) = \Theta_0. \tag{4.13}$$

Solutions to this flow exist for all time $t \geq 0$ *and converge as* $t \to \infty$ *to a connected set of critical points contained in a level set of* ψ^*.

Proof: The gradient and the critical point characterisation are proved as for Theorem 4.1. The equivalence of the critical points is easily seen by solving (4.8) for Θ independently of K. Part c) follows from the observation that (4.13) is a gradient flow on a compact manifold. ■

Fixing Θ constant in the second line of (4.9) yields a linear differential equation in K with solution

$$K(t) = e^{-t}(K(0) + B^T(A - \Theta^T F\Theta)B) - B^T(A - \Theta^T F\Theta)B.$$

It follows that $K(t) \to -B^T(A - \Theta^T F\Theta)B$ as $t \to \infty$. Observe that

$$\begin{aligned} \psi^*(\Theta) &= ||\mathbb{Q}(A - \Theta^T F\Theta)||^2 + 2||\Theta B - G||^2 \\ &= ||\Theta(A + B(B^T(A - \Theta^T F\Theta)B)B^T)\Theta^T - F||^2 + 2||\Theta B - G||^2 \\ &= \psi(\Theta, -B^T(A - \Theta^T F\Theta)B). \end{aligned}$$

Recall also that for exact system assignment we have showed that $K = B^T(\Theta F \Theta^T - A)B$, Lemma 2.2. Thus, it is reasonable to consider solutions $\Theta(t)$ of (4.13) along with continuously changing feedback gain

$$K(t) = B^T(\Theta(t)^T F \Theta(t) - A)B, \tag{4.14}$$

as an approach to solving least squares system assignment problems. A numerical scheme based on this approach is presented in [10].

5 Least Squares Pole Placement and Simultaneous System Assignment

Having developed the necessary tools it is a simple matter to derive gradient flow solutions to Problem B and Problem C described in Section 2.

Corollary 5.1 Pole Placement *Let $(A, B) \in S(n) \times O(n, m)$ be a symmetric state space system and let $F \in S(n)$ be a given symmetric matrix. Define*

$$\begin{aligned} \Phi : \mathcal{F}(A,B) \to \mathbf{R}, &\quad (A,B) \mapsto ||A - F||^2, \\ \phi : O(n) \times S(m) \to \mathbf{R}, &\quad (\Theta, K) \mapsto ||\Theta(A + BKB^T)\Theta^T - F||^2, \end{aligned}$$

then

a) The gradient of Φ and ϕ with respect to the normal and the right invariant group metric respectively are

$$\mathrm{grad}\Phi(A,B) = \begin{pmatrix} -[A,[A,F]] + BB^T(A-F)BB^T \\ [A,F]B \end{pmatrix}, \tag{5.1}$$

and

$$\mathrm{grad}\phi(\Theta, K) = \begin{pmatrix} [\Theta(A + BKB^T)\Theta^T, F]\Theta \\ B^T(A + BKB^T - \Theta^T F \Theta)B \end{pmatrix}. \tag{5.2}$$

b) The critical points of Φ and ϕ are characterised by

$$\begin{aligned} [A,F] &= 0 \\ B^T(A-F)B &= 0, \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} [\Theta(A + BKB^T)\Theta^T, F] &= 0 \\ B^T(\Theta^T F \Theta - A)B &= K, \end{aligned} \tag{5.4}$$

respectively.

c) Solutions of the gradient flows $(\dot{A}, \dot{B}) = -\mathrm{grad}\Phi(A,B)$

$$\begin{aligned} \dot{A} &= [A,[A,F]] - BB^T(A-F)BB^T \\ \dot{B} &= -[A,F]B \end{aligned} \tag{5.5}$$

exist for all time $t \geq 0$ and remain in $\mathcal{F}(A,B)$. Moreover, any solution of (5.5) converges as $t \to \infty$ to a connected set of matrix pairs $(A,B) \in \mathcal{F}(A,B)$ which satisfy (5.3) and lie in a single level set of Φ.

d) *Solutions of the gradient flow* $(\dot{\Theta}, \dot{K}) = -\mathrm{grad}\phi(\Theta, K)$

$$\begin{aligned} \dot{\Theta} &= -[\Theta(A + BKB^T)\Theta^T, F]\Theta \\ \dot{K} &= -B^T(A + BKB^T - \Theta^T F\Theta)B \end{aligned} \tag{5.6}$$

exist for all time $t \geq 0$ *and remain in a bounded subset of* $O(n) \times S(m)$. *Moreover, as* $t \to \infty$ *any solution of (5.6) converges to a connected subset of critical points in* $O(n) \times S(m)$ *which are contained in a single level set of* ψ.

e) *If* $(\Theta(t), K(t))$ *is a solution to (5.6) then*

$$\big(\Theta(t)(A + BK(t)B^T)\Theta^T(t), \Theta^T(t)B\big)$$

is a solution of (5.5).

Proof: Consider the symmetric state space system $(A, B) \in S(n) \times O(n, m)$ and the matrix pair $(F, G_0) \in S(n) \times \mathbf{R}^{n\times m}$ where G_0 is the $n \times m$ zero matrix. Observe that $\Psi(A, B) = \Phi(A, B) + 2||B||^2$ and similarly $\psi(\Theta, K) = \phi(\Theta, K) + 2||B||^2$, Ψ and ψ are given by (4.1) and (4.6) respectively. Since the norm $||B||^2$ is constant on $\mathcal{F}(A, B)$ the structure of the above optimization problems is exactly that considered in Theorem 4.1 and Theorem 4.2. The results follow as direct corollaries. ■

Similar to the discussion at the end of Section 4 the pole placement problem can be solved by a gradient flow evolving on the orthogonal group $O(n)$ alone.

Corollary 5.2 *Let* $(A, B) \in S(n) \times O(n, m)$ *be a symmetric state space system and let* $F \in S(n)$ *be a symmetric matrix. Define*

$$\begin{aligned} \varphi^* &: O(n) \to \mathbf{R} \\ \varphi^* &:= ||\mathbb{Q}(A - \Theta^T F\Theta)||^2 \end{aligned}$$

where $\mathbb{Q}(X) = (\mathbb{I} - \mathbb{P})(X) = X - BB^TXBB^T$ *(4.11). Then,*

a) *The gradient of* φ^* *with respect to the right invariant group metric is*

$$\mathrm{grad}\varphi^*(\Theta) = [\Theta\mathbb{Q}(A - \Theta^T F\Theta)\Theta^T, F]\Theta.$$

b) *The critical points* $\Theta \in O(n)$ *of* φ^* *are characterised by*

$$[F, \Theta\mathbb{Q}(A - \Theta^T F\Theta)\Theta^T] = 0.$$

and correspond exactly to the orthogonal matrix component of the critical points (5.4) of φ.

c) The negative gradient flow minimizing φ^* is

$$\dot{\Theta} = [F, \Theta \mathbb{Q}(A - \Theta^T F \Theta)\Theta^T]\Theta, \quad \Theta(0) = \Theta_0. \tag{5.7}$$

Solutions to this flow exist for all time $t \geq 0$ and converge as $t \to \infty$ to a connected set of critical points contained in a level set of φ^*.

Proof: Consider the matrix pair $(F, G_0) \in S(n) \times \mathbf{R}^{n \times m}$ where G_0 is the $n \times m$ zero matrix. It is easily verified that $\psi^*(\Theta) = \varphi^*(\Theta) + 2||B||^2$ where ψ^* is given by (4.12). The corollary follows as a direct consequence of Theorem 4.3. ■

Simultaneous system assignment is known to be a hard problem which generically does not have an exact solution. The best that can be hoped for is an approximate solution provided by a suitable numerical technique. The following discussion is a direct generalisation of the development given in Section 4. The generalisation is similar to that employed by Chu in [4] when considering the simultaneous reduction of real matrices.

For any integer $N \in \mathbf{N}$ let $(A_1, B_1), \ldots, (A_N, B_N) \in S(n) \times O(n, m)$ be given symmetric state space systems. The output feedback orbit for the multiple system case is

$$\begin{aligned} &\mathcal{F}((A_1, B_1), \ldots, (A_N, B_N)) := \\ &\quad \{(\Theta(A_1 + B_1 K B_1^T)\Theta^T, \Theta B_1), \ldots, (\Theta(A_N + B_N K B_N^T)\Theta^T, \Theta B_N) \mid \\ &\quad\quad \Theta \in O(n), K \in S(m)\}. \end{aligned}$$

An analogous argument to Lemma 3.1 shows that $\mathcal{F}((A_1, B_1), \ldots, (A_N, B_N))$ is a smooth manifold. Moreover, the tangent space is given by

$$\begin{aligned} &T_{((A_1, B_1), \ldots, (A_N, B_N))} \mathcal{F}((A_1, B_1), \ldots, (A_N, B_N)) \\ &\quad \{([\Omega, A_1] + B_1 \Xi B_1^T, \Omega B_1), \ldots, ([\Omega, A_N] + B_N \Xi B_N^T, \Omega B_N) \mid \\ &\quad\quad \Omega \in Sk(n), \Xi \in S(m)\}. \end{aligned}$$

Indeed, $\mathcal{F}((A_1, B_1), \ldots, (A_N, B_N))$ is a Riemannian manifold when equipped with the normal metric, defined analogously to the normal metric on $\mathcal{F}(A, B)$.

Corollary 5.3 *For any integer $N \in \mathbf{N}$ let $(A_1, B_1), \ldots, (A_N, B_N)$ and (F_1, G_1), …, (F_N, G_N) be two sets of N symmetric state space systems. Define*

$$\begin{aligned} \Psi_N &= \mathcal{F}((A_1, B_1), \ldots, (A_N, B_N)) \to \mathbf{R} \\ \Psi_N((A_1, B_1), \ldots, (A_N, B_N)) &:= \sum_{i=1}^{N} \left(||A_i - F_i||^2 + 2||B_i - G_i||^2\right) \end{aligned}$$

and

$$\begin{aligned}\psi_N &= O(n)\times S(m)\to \mathbf{R}\\ \psi_N(\Theta,K) &:= \sum_{i=1}^{N}\left(||\Theta(A_i+B_iKB_i^T)\Theta^T-F_i||^2+2||\Theta B_i-G_i||^2\right).\end{aligned}$$

Then,

a) *The negative gradient flows of Ψ_N and ψ_N with respect to the normal and the right invariant group metric are*

$$\begin{aligned}\dot{A}_i &= [A_i,\sum_{j=1}^{N}\left([A_j,F_j]+B_jG_j^T-G_jB_j^T\right)]\\ &\quad -\sum_{j=1}^{N}B_iB_j^T(A_j-F_j)B_j^TB_i,\\ \dot{B}_i &= -\sum_{j=1}^{N}([A_j,F_j]+B_jG_j^T-G_jB_j^T)B_i,\end{aligned}\tag{5.8}$$

for $i=1,\dots N$, and

$$\begin{aligned}\dot{\Theta} &= \sum_{j=1}^{N}\left([A_j,F_j]+B_jG_j^T-G_jB_j^T\right)\Theta\\ \dot{K} &= -\sum_{j=1}^{N}B_j^T(A_j+B_jKB_j^T-\Theta F_j\Theta^T)B_j,\end{aligned}\tag{5.9}$$

respectively.

b) *The critical points of Ψ_N and ψ_N are characterised by*

$$\begin{aligned}\sum_{j=1}^{N}[A_j,F_j] &= \sum_{j=1}^{N}\left(G_jB_j^T-B_jG_j^T\right)\\ \sum_{j=1}^{N}B_j^T(A_j-F_j)B_j &= 0,\end{aligned}\tag{5.10}$$

and

$$\begin{aligned}\sum_{j=1}^{N}[\Theta(A_j+B_jKB_j^T)\Theta^T,F_j] &= \sum_{j=1}^{N}\left(\Theta B_jG_j^T-G_jB_j^T\Theta^T\right)\\ K &= \sum_{j=1}^{N}B_j^T(\Theta F_j\Theta^T-A_j)B_j,\end{aligned}\tag{5.11}$$

respectively.

c) *Solutions of the gradient flow (5.8) exist for all time* $t \geq 0$ *and remain in* $\mathcal{F}((A_1, B_1), \ldots, (A_N, B_N))$*. Moreover, any solution of (5.8) converges as* $t \to \infty$ *to a connected set of matrix pairs* $((\mathcal{A}_1, \mathcal{B}_1), \ldots, (\mathcal{A}_N, \mathcal{B}_N)) \in \mathcal{F}((A_1, B_1), \ldots, (A_N, B_N))$ *which satisfy (5.10) and lie in a single level set of* Ψ_N*.*

d) *Solutions of the gradient flow (5.9) exist for all time* $t \geq 0$ *and remain in a bounded subset of* $O(n) \times S(m)$*. Moreover, as* $t \to \infty$ *any solution of (5.9) converges to a connected stubset of critical points in* $O(n) \times S(m)$ *which are contained in a single level set of* ψ_N*.*

e) *If* $(\Theta(t), K(t))$ *is a solution to (5.6) then*

$$(\mathcal{A}_i(t), \mathcal{B}_i(t)) = (\Theta(A_i + B_i K B_i^T)\Theta^T, \Theta B_i),$$

for $i = 1, \ldots, N$*, is a solution of (5.8).*

Proof: Observe that the potentials Ψ_N and ψ_N are linear sums of potentials of the form Ψ and ψ considered in Theorem 4.2 and Theorem 4.1. The proof is then a simple generalisation of the arguments employed in the proofs of these theorems. ∎

6 Simulations

A number of simulations studies have been completed to investigate the properties of the gradient flows presented and obtain general information about the system assignment and pole placement problems. Indeed, computing the gradient flows (4.4) and (5.1) has already improved our knowledge of the problems since it was the non-convergence of our original simulations that that lead us to further investigate conditions on the existence of exact solutions to the pole placement and system assignment problems, and eventually to lemmas 2.1 and 2.2.

To compute the limiting values of the ordinary differential equations considered we used the MATLAB function ODE45. This function integrates ordinary differential equations using the Runge-Kutter-Fehlberg method with an automatic step size selection. Numerical integration is undertaken using fourth order approximations of the integrand while the accuracy of each iteration over the step length is checked against a fifth order approximation. At each step of the interpolation the step length is reduced until the error between the fourth and fifth order approximation of the integral is less than a pre-specified constant $E > 0$. In the simulations undertaken the error bound was set to $E = 1 \times 10^{-7}$, this allowed for reasonable accuracy without excessive computational cost.

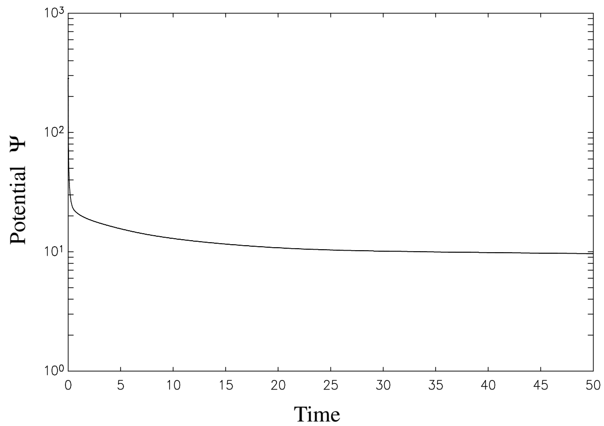

Figure 1: Plot of $(\Psi(A(t),B(t)))$ verses t for $(A(t),B(t))$ a typical solution to (4.4).

Due to Lemma 2.1 we do not expect to see convergence of the solution of (4.4) to an exact solution of the System Assignment problem for arbitrary initial condition. The typical behaviour of solutions to (4.4) is shown in Figure 1, where the potential, $(\Psi(A(t),B(t)))$, for $(A(t),B(t))$ a solution to (4.4), is plotted verses time. The potential is plotted on $\log_{10}$ scaled axis for all the plots presented to display the linear convergence of Figures 2 and 3 better. The initial conditions $(A_0,B_0) \in S(5) \times O(5,4)$ and the target system $(F,G) \in S(5) \times O(5,4)$ are randomly generated apart from symmetry and orthogonality requirements. The state dimension, $n = 5$, and the input and output dimension, $m = 4$, are arbitrarily chosen. Similar behaviour is obtained for all simulations for any choice of n and m for which $m < n$. In Figure 1, observe that the potential converges to a non-zero constant $\lim_{t\to\infty} \Psi(A(t),B(t)) = 9.3$. For the limiting value of the solution to be an exact solution to the system assignment problem we would require $\lim_{t\to\infty} \Psi(A(t),B(t)) = 0$.

In contrast, Lemma 2.2 ensures only that the pole placement task is not solvable on some open set of symmetric state space systems but leaves open the question of whether other open sets of systems exists for which the pole placement problem is solvable. Simulations show that the pole placement problem is indeed solvable for some open sets of symmetric state space systems. Figure 2 shows a plot of the potential $\Phi(A(t),B(t))$ (cf. Corollary 5.1) verses time for $(A(t),B(t))$ a solution to (5.5). The initial conditions and target matrix here are the initial conditions (A_0,B_0) and the state matrix F, from (F,G), used to generate Figure 1. The plot clearly shows that the potential converges exponentially (linearly in the $\log_{10}$ scaled verses unscaled) to zero. Consequently, the solution $(A(t),B(t))$ converges to an

Simulation	$\Phi(A(40), B(40))$
1	2.63×10^{-10}
2	2.09×10^{-9}
3	5.65×10^{-9}
5	3.35×10^{-10}
6	3.16×10^{-11}
7	1.62×10^{-11}
8	1.05×10^{-10}
9	3.68×10^{-10}
10	1.20×10^{-8}
11	2.72×10^{-8}

Table 1: Potentials $\Phi(A_i(40), B_i(40))$ for experiments $i = 1, \ldots, 10$ where $A_i(t), B_i(t))$ is a solution to (4.4) with initial conditions $(A_i(0), B_i(0)) = (A_0 + N_i, U_i B_0) \in S(n) \times O(n, m)$. Here $N_i = N_i^T$ is a randomly generated symmetric matrix with $||N_i|| \leq 0.25$ and $U_i \in O(n)$ is an randomly generated orthogonal matrix with $||U_i - I_n|| \leq 0.25$.

exact solution the pole placement problem, $\lim_{t\to\infty} A(t) = F$. Comparing Figures 1 and 2 and recalling that they were generated using the same initial conditions, we have explicit evidence that the system assignment problem is strictly more difficult than the pole placement problem.

Next we may ask does the particular initial condition (A_0, B_0) lie in an open set of initial conditions for which the pole placement problem can be exactly solved. A series of ten simulations was completed, integrating (5.5) for initial conditions (A_i, B_i) close to (A_0, B_0), $||A_0 - A_i|| + ||B_0 - B_i|| \leq 0.5$. Each integration was carried out over a time interval of forty seconds and the final potential $\Phi(A(40), B(40))$ for each simulation is given in table 6. The plot of $\log(\Phi)$ verses time for each simulation was qualitatively equivalent to Figure 2. It is our conclusion from this that the pole placement problem could be exactly solved for all initial conditions in a neighbourhood of $A_0, B_0)$.

Remark 6.1 It may appear reasonable that the pole placement problem could be solved for any initial condition with initial state matrix A_0 close to the desired structure F. Indeed one would expect that the potential of a solution to (5.5) equipped with such initial conditions would converge exponentially fast to zero. In fact simulations have shown this to be false.

Let $C \in O(n, n-m)$ be a matrix orthogonal to B, (i.e. $\mathrm{tr}(B^T C) = 0$). Observe that a solution to the pole placement problem requires $\Theta^T F \Theta -$

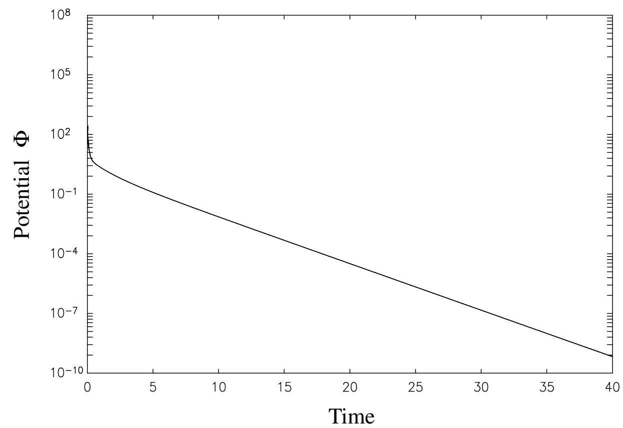

Figure 2: Plot of $(\Phi(A(t),B(t)))$ verses t for $(A(t),B(t))$ a solution to (5.5) with initial conditions (A_0,B_0) for which the solution $(A(t),B(t))$ converges to a global minimum of Φ.

$A=BKB^T$ and thus

$$\Theta^T F\Theta C - AC = 0 \Longrightarrow F\Theta C - \Theta AC = 0.$$

Since A and C are specified by the initial condition (the span of C is the important object) then we see that $\Theta \in \mathbf{R}^{n\times n}$ must lie in the linear subspace defined by the kernel of the linear map $\Theta \mapsto F\Theta C - \Theta AC$. Of course Θ must also lie in the set of orthogonal matrices and the intersection of the kernel of $\Theta \mapsto F\Theta C - \Theta AC$ with the orthogonal matrices provides an exact criterion for the existence of a solution to the pole placement problem.

The difficulty for initial conditions where $||A_0 - F||$ is small is related to the fact that the solution to the pole placement problem for initial conditions $(A_0,B_0)=(F,B_0)$, (i.e. the state matrix already has the desired structure,) is given by the matrix pair $(I_n,0)\in O(n)\times S(m)$ in the output feedback group. The matrix I_n lies at an extremity of $O(n)$ in $\mathbf{R}^{n\times n}$ and it is reasonable that small perturbations of (A_0,B_0) may shift the kernel of the linear map $\Theta \mapsto F\Theta C - \Theta A_0 C$ such that it no longer intersects with $O(n)$. $\square$

An advantage mentioned in Section 4 in computing the limiting solution of (5.7) (Figure 3) compared to computing the full gradient flow (5.5) (Figure 2) is the associated drop in order of the O.D.E. that must be solved. Interestingly, it appears that the solutions of the projected flow (5.7) will also converge more quickly than those of (5.5). Figure 3 shows the potential $\varphi^*(\Theta(t))$ (cf. Corollary 5.2) verses time for $\Theta(t)$ a solution to (5.7). The

Simulation	λ	λ^*	λ^*/λ
1	2.05	53	25.85
2	1.73	43.5	25.14
3	2.03	27.75	13.66
4	0.52	20	38.46
5	1.6	44	27.5

Table 2: Linear rate of convergence for the solution of (5.5), given by λ, and (5.7) given by λ^*. The final column shows the ratio between the rates of convergence for the two differential equations.

initial conditions for this simulation were $\Theta_0 = I_n$ while the specified symmetric state space system used for computing the norm φ^* was (A_0, B_0) the initial conditions for Figures 1 and 2. Observe that from time $t = 1.2$ to $t = 2$, Figure 3 displays unexpected behaviour which we have interpreted to be numerical error. The presence of this error is not surprising since the potential (and consequently the gradient) is of order E^2, where E is the error bound chosen for the ODE45 routine in MATLAB. The relationship of presence of numerical error to order of the potential being approximately E^2 has been double checked by adjusting the error bound E for a number of early simulations.

The exponential (linear) convergence rates of the solution to (5.7) and the solution to (5.5) are computed by reading off the slope of the linear section of plots 2 and 3. For the example shown in Figures 2 and 3 convergence of the solutions is characterised by

$$\begin{aligned} \Phi(A(t), B(t)) &= e^{-\lambda t}, & \lambda &\approx 2.05 \\ \varphi^*(\Theta(t)) &= e^{-\lambda^* t}, & \lambda^* &\approx 53 \end{aligned}$$

where $(A(t), B(t))$ is a solution to (5.5) and $\Theta(t)$ is a solution to (5.7). Five separate experiments were completed in which the two flows were computed for randomly generated target matrices and initial conditions with $n = 5$ and $m = 4$. The linear convergence rates computed from these five experiments are given in Table 6. We deduce that solutions of (5.7) converge around twenty times faster than solutions to (5.5) when the systems considered have five states and four inputs and outputs. A brief study of the behaviour of systems with other numbers of states and inputs indicate that the ratio between convergence rates is of order ten or higher.

In the system assignment problem Lemma 2.1 ensures that an exact solution to the system assignment problem does not generically exist. The

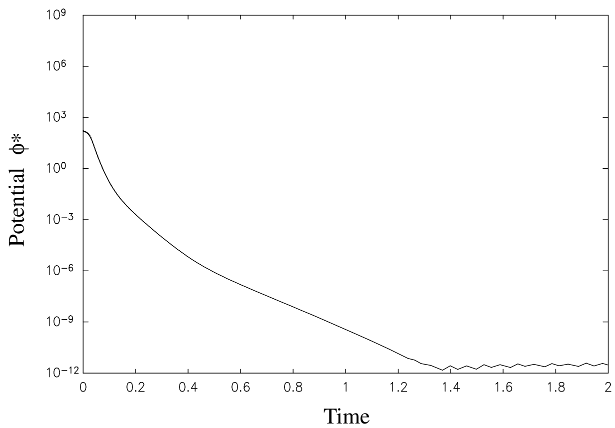

Figure 3: Plot of $(\varphi^*(\Theta(t)))$ verses t for $\Theta(t)$ a solution to (5.5) with initial conditions $\Theta(0) = I_n$ the identity matrix. The potential $\varphi^*(\Theta) := ||(A_0 - \Theta^T F\Theta) - B_0 B_0^T (A_0 - \Theta^T F\Theta) B_0 B_0^T||^2$ is computed with respect to the initial conditions (A_0, B_0) used in Figures 1 and 2.

gradient flow (4.4), however, will certainly converge to a connected set of local minima of the potential Psi, Theorem 4.1. An important question to consider is what structure the critical level set associated with the local minima of Ψ may have. In particular, one may ask is the level set a single point or is it a submanifold (at least locally) of $\mathcal{F}(A, B)$.

Remark 6.2 Observe that critical level sets of Ψ are given by two algebraic conditions $||\text{grad}\Psi(A, B)|| = 0$ and $\Psi(A, B) = \Psi_0$, for some fixed Ψ_0, thus they are algebraic varieties of the closed submanifold $\mathcal{F}(A, B) \subset \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times m}$. It follows, apart from a set of measure zero in $\mathcal{F}(A, B)$ (singularities of the algebraic conditions), that the critical sets will locally have submanifold structure in $\mathcal{F}(A, B)$. □

Rather than consider the computationally huge task of mapping out the local minima of Ψ by integrating out (4.4) for many different initial conditions in $\mathcal{F}(A, B)$, we have tried to obtain some qualitative information (in the vicinity of a given local minima) without incurring the same computational cost. By choosing any initial condition and integrating (4.4) for a suitable time interval an estimate of a local minima (A^∞, B^∞) is obtained. If this point is an isolated minima then it should be locally attractive. By choosing a number of initial conditions (A_i, B_i) in the vicinity of (A^∞, B^∞) and integrating (4.4) a second time we obtain new estimates of local minima (A_i^∞, B_i^∞). If (A^∞, B^∞) approximates an isolated local minima then

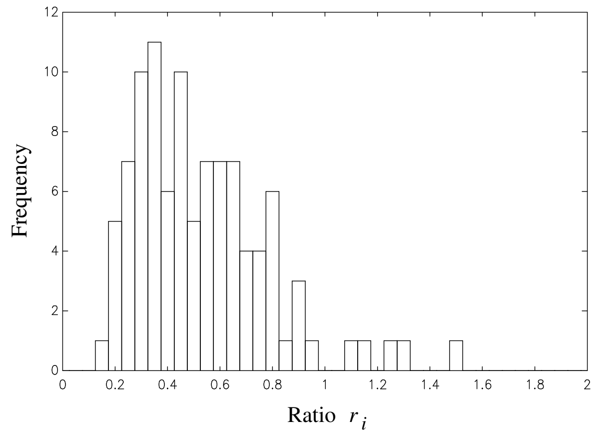

Figure 4: Plot of frequency distribution of r_i given by (6.1) computed for the limiting values of 100 simulations with initial conditions close to (A^∞, B^∞).

the ratio

$$r_i = \frac{||(A_i^\infty, B_i^\infty) - (A^\infty, B^\infty)||}{||(A_i, B_i) - (A^\infty, B^\infty)||} \tag{6.1}$$

should be approximately zero. If (A^∞, B^∞) is not isolated then we expect the ratio r_i to be significantly non-zero. Of course r_i should be less than one on average since we are dealing with a convergent flow. The difficulty in such an approach is deciding on suitable time intervals for the various integrations. The first time interval was determined by repeatedly integrating over longer and longer time intervals (for the same initial conditions) until the norm difference between the final values was less than 1×10^{-8}. An initial time interval of two hundred seconds was found to be suitable. Each subsequent simulation was integrated over a time interval of fifty seconds. The results of one hundred measurements of the ratio r_i for a given estimated local minima (A^∞, B^∞) are plotted as a frequency plot, Figure 4. The frequency divisions for this plot are 0.05, thus in the one hundred experiments undertaken eleven experiments yielded an estimate of r_i between 0.325 and 0.375. It is obvious from Figure 4 that the probability of r_i being zero is small and we conclude that the critical sublevel sets of Ψ have a local submanifold structure. In particular, the local minima of Ψ are not isolated.

7 Conclusion

In this paper we have considered the problems of system assignment (Problem A) and pole placement (Problem B) on the set of symmetric linear state space systems. The pole placement problem has been extensively studied for general linear systems (cf. the survey [3] and [13]), however, little has been done for classes of structured linear systems. A major contribution of this paper is the observation that the additional structure inherent in symmetric linear systems forces the solution to the "classical" pole placement question to be considerably different to that expected based on intuition obtained for the general linear case. In particular, generic pole placement can not be achieved unless the system considered has as many inputs (and outputs) as states.

To compute feedback gains which assign poles as close as possible to desired poles (in a least squares sense) we propose a number of ordinary differential equations. By computing the limiting solution to these equations for arbitrary initial conditions an estimate of the best feedback gain is obtained. A careful study of the properties of the solutions of the gradient flows proposed also provides considerable knowledge of the pole placement problem itself.

Computational methods for determining pole placement feedback gains based on the differential equations proposed in above are discussed in [10]. The methods developed in [10] are based on computing solutions to (5.7) (which appears to converge around twenty times faster than (5.6), cf. Section 6). An additional advantage is such an approach is that the algorithms developed inherit the numerical stability of the gradient flows. This is important since the pole placement task is an inverse eigenvalue problem and such problems tend to be ill conditioned for classical numerical algorithms.

We intend to continue studying the applications of similar techniques to understanding issues in linear systems theory. In particular, we believe that analogous techniques to those developed in this paper will be useful for studying other classes of structured linear systems.

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R. MAHONY AND U. HELMKE

Department of Systems Engineering, Research School of Information Science and Engineering, Australian National University, A.C.T., 0200, Australia

Department of Mathematics, University of Regensburg, 8400 Regensburg, F.R.G.

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