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# An Invariant Manifold Approach to Nonlinear Feedback Stabilization on Compacta*

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**Abstract**

In this paper, some results of feedback stabilization on compacta for a nonlinear control system are obtained by using an invariant manifold approach. In particular, a class of globally nonminimum phase systems are treated. Issues such as high gain feedback stabilization on compacta vs. peaking phenomenon, and globally exponentially minimum phase vs. globally asymptotically (critically) minimum phase are also discussed.

**Key words**: nonlinear systems, feedback stabilization, high gain control, invariant manifold

**AMS Subject Classifications**: 93C10, 93B52, 93D15, 93A20

## 1 Introduction

In the past few years there have been quite a few papers dealing with global and/or on compacta (or semiglobal, as is used by some authors) feedback stabilization for nonlinear control systems, for example, [4, 17, 19]. For nonlinear systems, global feedback stabilization, as we know, is in general difficult to achieve and a solution normally calls for a full state nonlinear feedback scheme. On the other hand, for many nonlinear control systems, it is possible to find feedback control laws, which only require partial knowledge of the state variables, such that on one hand, they only render the equilibrium point locally, not globally in general, asymptotically stable, and on the other hand, they contain a gain parameter whose value can be set in such a way that any a priori given bounded set of initial conditions can be contained in the domain of attraction. In this paper, we will use the invariant manifold for a singular perturbation model to

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study the problem of nonlinear feedback stabilization on compacta (the precise definition will be recalled a little later), which is different from the approaches in [4, 17].

We consider the following system:

$$\begin{aligned} \dot{x} &= f(x,\xi) \\ \dot{\xi} &= F(x,\xi) + G(x,\xi)u \\ y &= h(\xi) \end{aligned} \tag{1.1}$$

where $y$ is the output of the system and u the input. $x \in R^n, \xi \in R^q$, $y \in R^m$, $u \in R^m$. The mappings $f$, $F$, $G$ and $h$ are smooth and $f(0) = 0, F(0) = 0$, and $h(0) = 0$.

Let us assume that a high gain control law $u = k\alpha(x,\xi)$, where $k$ can be tuned according to initial conditions, solves stabilization on compacta for (1.1). Then, the closed loop system becomes:

$$\begin{aligned} \dot{x} &= f(x,\xi) \\ \dot{\xi} &= F(x,\xi) + G(x,\xi)k\alpha(x,\xi). \end{aligned} \tag{1.2}$$

Let $\epsilon = \frac{1}{k}$, we have

$$\begin{aligned} \dot{x} &= f(x,\xi) \\ \epsilon\dot{\xi} &= \epsilon F(x,\xi) + G(x,\xi)\alpha(x,\xi). \end{aligned} \tag{1.3}$$

It is well known that under some hypotheses, locally the stability of (1.3) is determined by that of the flow on an invariant manifold of (1.3), which makes the analysis much easier. In fact, this is also true for the "on compacta" case. This issue will be discussed in Section 2.

Let us now recall the definition of feedback stabilization on compacta:

**Definition 1.1.** *System* (1.1) *is said to be stabilizable on compacta by smooth feedback control laws, if for every bounded set $P$ of the values of the state variables, there is a smooth feedback law such that the equilibrium point of the corresponding closed-loop system is asymptotically stable and the domain of attraction contains $P$.*

In this paper we also consider the following type stabilization problem: a feedback control law is required only to render the system have a bounded trajectory, the state vector does not necessarily tend to the equilibrium point. A practical footnote for this is that frequently disturbances in the system make it impossible that the state vector decays to the equilibrium. A bounded trajectory is the best one can expect. For this purpose we define

**Definition 1.2.** *System* (1.1) *is said to be bounded-state (BS for short) stabilizable on compacta by smooth feedback control if given a bounded initial data set $P$, there exists a smooth feedback control law such that the state trajectories of the closed loop system are uniformly bounded with respect to $P$.*

The paper is organized as follows. In section 2, we review some results about invariant manifolds for a singular perturbation model and prove several results of stability on an invariant manifold. In section 3, we give our results on nonlinear feedback stabilization on compacta, in particular, we give a result for the case where the zero dynamics is not globally stable. In section 4, we discuss high gain feedback stabilization on compacta vs. peaking phenomenon and globally exponentially minimum phase vs. globally asymptotically (critically) minimum phase.

In this paper the standard notations will be followed. Throughout we use "$|\cdot|$" to denote the (induced) Euclidean norm.

## 2 Preliminaries

Let us consider the following singular perturbation model:

$$\begin{aligned} \dot{x} &= f(x, z, \epsilon) \\ \epsilon \dot{z} &= F(x, z, \epsilon) \quad x \in R^p, \ z \in R^q \end{aligned} \tag{2.1}$$

where $\epsilon$ is a small positive parameter and $f$ and $F$ are $C^3$, $f(0,0,\epsilon) = 0$, $F(0,0,\epsilon) = 0$, $f_x(x,z,\epsilon)$, $f_z(x,z,\epsilon)$, $F_x(x,z,\epsilon)$ and $F_z(x,z,\epsilon)$ are all uniformly bounded for $0 < \epsilon < \epsilon_0$ when $(x,z)$ is in a bounded set, and there exists a solution $z = \phi(x)$ for $F(x,z,0) = 0$, where $\phi$ is $C^3$, $\phi(0) = 0$ and $Dom\ \phi = R^p$. We also assume

**H1:** $\mathrm{Re}\ \sigma\left(\frac{\partial F(x,\phi(x),0)}{\partial z}\right) \subset C^-, \qquad \forall \quad x \in R^p.$

For (2.1), we have the following result:

**Proposition 2.1** *Under the above assumptions, given a $p > 0$, and $B_p = \{x : \ |x| \leq p\}$ there exists an $\epsilon_0 > 0$ such that when $\epsilon \leq \epsilon_0$ there exists an invariant manifold for* (2.1) *described by $z = \phi(x) + \psi(x,\epsilon)$, for $x \in B_p$, $\epsilon < \epsilon_0$, where,*

$$\begin{aligned} &|\psi(x,\epsilon)| < M(\epsilon), M(\epsilon) \to 0 \ \text{as} \ \epsilon \to 0 \qquad \forall \quad x \in B_p. \\ &|\psi(x_1,\epsilon) - \psi(x_2,\epsilon)| < \ell(\epsilon)|x_1 - x_2|, \ell(\epsilon) \to 0 \ \text{as} \ \epsilon \to 0 \qquad \forall \quad x_1, x_2 \in B_p \end{aligned} \tag{2.2}$$

Proposition 2.1 can be found in [20] and a simpler case where $\frac{\partial F(x,\phi(x),0)}{\partial z}$ is assumed to be constant, can be found in [5]. However, in [20] there are some serious errors in the proof in estimating certain state transition matrices. Because the result is also of independent interest, we give a complete proof of Proposition 2.1 in the appendix.

The invariant manifold is actually a center manifold on compacta. Let us change the time scale $\tau = \frac{t}{\epsilon}$ and consider $\epsilon$ as a variable, then (2.1) becomes

$$\begin{aligned} \epsilon' &= 0 \\ x' &= \epsilon f(x,z,\epsilon) \\ z' &= F(x,z,\epsilon). \end{aligned} \tag{2.3}$$

As in the local case, the stability of system (2.1) about the equilibrium point can be determined by that of the flow on the center manifold:

$$w' = \epsilon f(x, \phi(x) + \psi(x,\epsilon), \epsilon). \tag{2.4}$$

**Proposition 2.2** *If the hypotheses in Proposition 2.1 hold and $F(x,z,0)$ is linear as a function of $z$, then the following statements are true: (a) If the flow on the center manifold (2.4) is unstable, then the equilibrium point of (2.1) is also unstable. (b) For any given $p$, if the solution to (2.4) $w(t)$ is uniformly bounded for all $w(0)$ such that $|w(0)| \le p$ and for all sufficiently small $\epsilon$, then for all $x(0)$ with $|x(0)| \le p$ and all $z(0)$ with $|z(0)| \le d$, where $d$ is any given positive number, when $\epsilon$ is sufficiently small,*

$$\begin{aligned} x(\tau) &= w(\tau) + O(e^{-\gamma\tau}) \\ z(\tau) &= \phi(w(\tau),\epsilon) + O(e^{-\gamma\tau}) \end{aligned} \tag{2.5}$$

*or*

$$\begin{aligned} x(t) &= w(t) + O(e^{-\gamma\frac{t}{\epsilon}}) \\ z(t) &= \phi(w(t),\epsilon) + O(e^{-\gamma\frac{t}{\epsilon}}) \end{aligned} \tag{2.6}$$

*where $\gamma > 0$.*

**Remark**: (b) holds even if the equilibrium point of (2.4) is unstable but the system has a bounded attractor contained in the set $|w| \le p$.

Proposition 2.2 can be shown by first mimicking the proof of Theorem 2.2 in [5], then using an argument in the proof of Theorem 7.3 in [4].

In preparation for studying nonlinear feedback stabilization on compacta, we also want to give the following two results.

**Proposition 2.3** *If $x = 0$ of $\dot{x} = f(x)$ is globally asymptotically stable, then given a Lipschitz $\psi(x,\epsilon)$ with $|\psi(x,\epsilon)| \le M(\epsilon,p)|x|$ in $\{x : \ |x| \le p\}$, where $\lim_{\epsilon\to 0} M(\epsilon,p) = 0$,*

$$\dot{x} = f(x) + \psi(x,\epsilon)$$

*is BS stable on compacta, i.e. given a bounded set* $\{x : |x| \leq p\}$*, there is* $\epsilon_p > 0$ *such that when* $\epsilon < \epsilon_p$*, the solution to the perturbed system, when initialized in* $\{x : |x| \leq p\}$*, is uniformly bounded.*

**Proof**: Since the solution to

$$\dot{x} = f(x)$$

is globally asymptotically stable, by the converse theorem to asymptotic stability test [8, 19], we can find a differentiable positive definite decrescent Liapunov function $v(x)$ with a negative definite total derivative globally, i.e.

$$\phi_1(|x|) \leq v(x) \leq \phi_2(|x|)$$
$$\dot{v} \leq -\rho(|x|)$$

where $\phi_1,\ \phi_2,\ \rho$ are functions of class $K_\infty$ (see [19]). We say a function $\phi :\ R_{\geq 0} \to R_{\geq 0}$ belongs to class $K_\infty$ if it belongs to class $K$ and $\lim_{r\to\infty} \phi(r) = \infty$.

Now for the initial data set $\{x : |x| \leq p\}$ choose a number $\beta > 0$ such that $\phi_2^{-1}(\beta) > p$, then for all points $\bar{x}$ satisfying the equation $v(\bar{x}) = \beta$, we have

$$p < \phi_2^{-1}(\beta) \leq |\bar{x}| \leq \phi_1^{-1}(\beta)$$

and

$$\dot{v}(\bar{x}) \leq -\rho(\phi_2^{-1}(\beta)).$$

Now denote $x_\psi(t)$ the solution to the perturbed system. for all initial points $|x_0| \leq p$, we have $v(x_0) < \beta$, and if $v(x_\psi(t)) = \beta$ for some $x_0$ and $t$,

$$\begin{aligned} \dot{v} &= \frac{\partial v}{\partial x} f(x_\psi) + \frac{\partial v}{\partial x} \psi(x_\psi, \epsilon) \\ &\leq -\rho(\phi_2^{-1}(\beta)) + K(\phi_1^{-1}(\beta)) M(\epsilon, \phi_1^{-1}(\beta)). \end{aligned} \tag{2.7}$$

When $\epsilon$ is sufficiently small, we would have

$$\dot{v} \leq -\frac{1}{2}\rho(\phi_2^{-1}(\beta)) < 0$$

which implies $v(x_\psi(t)) < \beta$ when $|x_0| \leq p$. Then

$$|x_\psi(t)| < \phi_1^{-1}(\beta).$$

**Proposition 2.4** *If* $x = 0$ *of* $\dot{x} = f(x)$ *is globally exponentially stable, then for all Lipschitz* $\psi(x, \epsilon)$ *with* $|\psi(x, \epsilon)| \leq M(\epsilon, p)|x|$ *for all* $|x| \leq p$*, where* $\lim_{\epsilon\to 0} M(\epsilon, p) = 0$,

$$\dot{x} = f(x) + \psi(x, \epsilon)$$

*is exponentially stable on compacta, i.e. given a bounded set* $\{x : |x| \leq p\}$*, there is* $\epsilon_p > 0$ *such that when* $\epsilon < \epsilon_p$ $x = 0$ *of the perturbed system is also exponentially stable and the domain of attraction contains* $\{x : |x| \leq p\}$*.*

**Proof:** From the previous proposition, we know that for the perturbed system, when initialized in $P = \{x : |x| \le p\}$,

$$|x_\psi(t)| < \phi_1^{-1}(\beta).$$

From the converse of a Liapunov stability theorem for exponential stability [8], we also know that there exists a Liapunov function $v(x)$ for the solution of $\dot{x} = f(x)$ such that the following estimates are valid on $\{x : \ |x| \le \phi_1^{-1}(\beta)\}$:

$$a_1(\beta)|x|^2 \le v(x) \le a_2(\beta)|x|^2$$
$$\dot{v}(x(t)) \le -a_3|x(t)|^2, \ |\frac{\partial v}{\partial x}| \le a_4(\beta)|x|$$

for certain positive constants $a_1, ..., a_4$.

**Remark:** In general, one can not exclude the possibility that the positive constants $a_1, a_2$ and $a_4$ may depend on $\beta$ and $a_1$ may even tend to 0 as $\beta \to \infty$.

Then for the perturbed system, when initialized in $P$, (for simplicity, we use $x$ for $x_\psi$)

$$\begin{aligned}\dot{v} &= \frac{\partial v}{\partial x}(f(x) + \psi(x, \epsilon)) \\ &\le -a_3|x|^2 + a_4 M(\epsilon, \phi_1^{-1}(\beta))|x|^2 \\ &= -(a_3 - a_4 M(\epsilon, \phi_1^{-1}(\beta)))|x|^2.\end{aligned} \tag{2.8}$$

When $\epsilon$ is sufficiently small, we have $a_3 - a_4 M(\epsilon, \phi_1^{-1}(\beta)) > \frac{1}{2}a_3$. Since $|x(x_0, t)| < \phi_1^{-1}(\beta)$ when $x_0 \in P$, then

$$|x(x_0, t)|^2 < \frac{a_2}{a_1}|x_0|^2 e^{-\frac{1}{2}\frac{a_3}{a_2}t}.$$

Therefore, the solution to the perturbed system is exponentially stable and the domain of attraction contains $P$.

As a direct application of the results presented in this section, we study the feedback stabilization on compacta for the following state input control system:

$$\begin{aligned}\dot{x} &= f(x, z, u, \epsilon) \\ \epsilon\dot{z} &= F(x, z, u, \epsilon) \quad x \in R^p, \ z \in R^q, \ u \in R^m.\end{aligned} \tag{2.9}$$

Assume $f$ and $F$ are $C^3$, $f(0, 0, 0, \epsilon) = 0$, $F(0, 0, 0, \epsilon) = 0$. We also assume that when $\epsilon$ is set to zero, one is able to solve the second equation of (2.9) to obtain:

$$z = \phi(x, u)$$

where $\phi$ is $C^3$ and

$$\text{Re } \sigma \left( \frac{\partial F(x, \phi(x,u), 0)}{\partial z} \right) \subset C^-, \qquad \forall \quad x \in R^p \; u \in R^m.$$

Since $z$ are faster time scale transients, our main interest here is the stabilization of $x$ modes. The existence of a slow manifold provides us such possibility. Substituting $\phi(x,u)$ to the first equation of (2.9), we get

$$\dot{x} = f(x, \phi(x,u), u, 0) \tag{2.10}$$

which is usually called "the reduced system" or sometimes the "rigid body model".

It is obvious that a rigid body model is much easier to establish and analyze than a model which involves, say, elastic forces. An interesting question is if a stabilizing control law designed based on the reduced model of a system (2.9) is still valid for the original system. We show that the same control law can still be used to stabilize the flow on a slow manifold of (2.9).

**Proposition 2.5** *Suppose the above hypotheses hold for system* (2.9)*, if there exists a smooth feedback control $u_0(x)$ which exponentially stabilizes the following reduced system globally:*

$$\dot{x} = f(x, \phi(x,u), u, 0)$$

*then $u_0(x)$ also exponentially stabilizes the invariant manifold*

$$z = \phi(x, u_0(x)) + \psi(x, \epsilon)$$

*of* (2.9) *in any bounded region $B_p$ when $\epsilon$ is sufficiently small. If $F(x,z,u,0)$ is linear as a function of $z$, then the overall closed-loop system is also exponentially stable on compacta.*

The proof of this proposition is just an application of Proposition 2.4.

## 3 Nonlinear Feedback Stabilization on Compacta

In this section, we consider the problem of stabilization on compacta for system (1.1) by state feedback control. We will use the normal form proposed by Byrnes and Isidori to develop our results:

$$
\begin{aligned}
\dot{z} &= f_0(z, \xi_1^1, ..., \xi_1^m) \\
\dot{\xi}_1^i &= \xi_2^i \\
&\vdots \\
\dot{\xi}_{r_i}^i &= f_i(z, \xi_1^1, \ldots, \xi_1^m) + \sum_{j=1}^{m} g_j^i(z, \xi^1, \ldots, \xi^m) u_j \\
y_i &= \xi_1^i \qquad\qquad i = 1, \ldots, m
\end{aligned}
\tag{3.1}
$$

where $z \in R^n$, $\xi \in R^q$, $y \in R^m$, $u \in R^m$, $f_i(0) = 0$, $i = 1, \ldots, m$ and $f_0(z, 0) = 0$ implies $z = 0$.

The zero dynamics of (3.1) can be easily characterized by

$$\dot{z} = f_0(z, 0).$$

**Remark:** For the sake of simplicity, we assume in (3.1) that $f_i$ ($i = 1, ..., m$) does not contain the derivatives of the outputs. One may have also noted that in (3.1), besides $z$, only $\xi_1^i$, $i = 1, ..., m$ appear in $f_0$. This is actually a nonpeaking condition. As is widely appreciated, the solution of stabilization on compacta for (1.1) by high gain feedback control laws can be quite subtle for the case of relative degree bigger than 1 due to the so called peaking phenomenon [17]. And some form of a nonpeaking condition typically needs to be assumed. A set of coordinate free conditions are given in [4] so that (1.1) can be transformed into (3.1).

In this section we assume that

**H2:** the matrix $G = (g_j^i(z, \xi^1, \ldots, \xi^m))_{m \times m}$ is nonsingular at every point.

Under the hypothesis, Byrnes and Isidori are able to render the submanifold $M^* = \{\xi^1 = 0, ..., \xi^m = 0\}$ globally invariant by using an appropriate nonlinear state feedback law. If additionally the zero dynamics of the system is globally asymptotically stable, they are able to use a high gain state feedback law to render the system asymptotically stable on compacta. Our method is different. While not necessarily maintaining the manifold $M^*$ invariant (which means less information about the state variables is needed to generate the control law), we are able to use a high gain feedback law to obtain the BS stabilization on compacta for (3.1) in general, and in some cases, even asymptotic stabilization.

Let us denote $p^{r_i}(s) = a_{r_i - 1} s^{r_i - 1} + \cdots + a_0$ ($i = 1, ..., m$) and $p_k^{r_i}(s) = k a_{r_i - 1} s^{r_i - 1} + \cdots + k^{r_i} a_0$.

**Theorem 3.1** *Assume H2 holds for* (3.1). *If the zero dynamics is globally asymptotically stable, then for any Hurwitz polynomials* $s^{r_i} + p_k^{r_i}(s)$ ($i =$

$1, ..., m)$, *the following high gain control law*

$$u_k = G^{-1}\begin{pmatrix} -p_k^{r_1}(\frac{d}{dt})y_1 \\ \vdots \\ -p_k^{r_m}(\frac{d}{dt})y_m \end{pmatrix}$$

*where the gain $k$ can be tuned according to the initial condition set, BS stabilizes system* (3.1) *on compacta.*

**Proof**: Set $\epsilon = \frac{1}{k}$ and

$$\tilde{\xi}_j^i = \epsilon^{j-1}\xi_j^i, \qquad j = 1, ..., r_i, \ i = 1, ..., m$$

when a high gain control law in the theorem is used, the closed loop system becomes

$$\begin{aligned} \dot{z} &= f_0(z, \tilde{\xi}_1^1, ..., \tilde{\xi}_1^m) \\ \epsilon\dot{\tilde{\xi}}_1^i &= \tilde{\xi}_2^i \\ &\vdots \\ \epsilon\dot{\tilde{\xi}}_{r_i}^i &= \epsilon f_i(z, \tilde{\xi}_1^1, ..., \tilde{\xi}_1^m) - p^{r_i}(\tilde{\xi}_{(\cdot)}^i) \quad (i = 1, ..., m) \end{aligned} \tag{3.2}$$

where $p^{r_i}(\xi_{(\cdot)}^i) = a_{r_i-1}\xi_{r_i}^i + ... + a_0\xi_1^i$. By Proposition 2.1. we know there exists a slow manifold for (3.2):

$$\tilde{\xi} = \phi(z, \epsilon)$$

with $|\phi(z, \epsilon)| \to 0$ uniformly as $\epsilon \to 0$. By Proposition 2.2, the stability behavior of system (3.3) is decided by that of the flow on the slow manifold:

$$\begin{aligned} \dot{w} &= f_0(w, \phi_1^1(w, \epsilon), ..., \phi_1^m(w, \epsilon)) \\ &= f_0(w, 0) + \psi(w, \epsilon) \end{aligned} \tag{3.3}$$

where $\psi(w, \epsilon) = f_0(w, \phi_1^1(w, \epsilon), ..., \phi_1^m(w, \epsilon)) - f_0(w, 0)$. since the zero dynamics of system (3.1) is globally asymptotically stable, by Proposition 2.3, (3.3) is BS stable on compacta. Then (3.2) is BS stable on compacta, which implies, of course, the closed-loop system (3.1) is also BS stable on compacta.

If the zero dynamics is globally exponentially stable, instead of being just critically asymptotically stable, we actually exponentially stabilize the system:

**Theorem 3.2** *Suppose H2 holds for* (3.1). *If the zero dynamics is globally exponentially stable, then for any Hurwitz polynomials $s^{r_i} + p_k^{r_i}(s)$ $(i =$*

$1, ..., m$), *the following high gain control law*

$$u_k = G^{-1} \begin{pmatrix} -p_k^{r_1}(\frac{d}{dt})y_1 \\ \vdots \\ -p_k^{r_m}(\frac{d}{dt})y_m \end{pmatrix}$$

*exponentially stabilizes system* (3.1) *on compacta.*

**Proof:** If the zero dynamics is globally exponentially stable, then the flow on the slow manifold governed by (3.3) is exponentially stable on compacta by Proposition 2.4. Therefore the whole system is exponentially stable on compacta by Proposition 2.2.

**Remark:** Once again we want to emphasize that our control law does not cancel the terms $f_i$, $i = 1, ..., m$ in (3.1).

The following result was first proven in [4], which can also be proven by our method.

**Corollary 3.3** *Suppose H2 holds for* (3.1) *and* $f_i = 0$ $i = 1, ..., m$ *at all the points. If the zero dynamics is globally asymptotically stable (uniformly bounded), then for any Hurwitz polynomials* $s^{r_i} + p_k^{r_i}(s)$ $(i = 1, ..., m)$, *the following high gain control law*

$$u_k = G^{-1} \begin{pmatrix} -p_k^{r_1}(\frac{d}{dt})y_1 \\ \vdots \\ -p_k^{r_m}(\frac{d}{dt})y_m \end{pmatrix}$$

*asymptotically (BS) stabilizes system* (3.1) *on compacta.*

**Proof:** In this case, $\phi(z, \epsilon) = 0$ is a slow manifold. The flow on the slow manifold is governed by:

$$\dot{w} = f_0(w, 0).$$

By the hypothesis, it is globally asymptotically stable (uniformly bounded). Then the system is asymptotically (BS) stable on compacta by Proposition 2.2.

The high gain feedback control we use depend on $G$ and is in general nonlinear. If the following additional hypothesis holds for (3.1):

**H3**: the matrix $G$ in H2 depends only on $z$ and either of the following two cases is true: $G(z)$ is diagonal and $\sigma(G(z)) \subset C^-$ or $G(z)$ is diagonal and $\sigma(G(z)) \subset C^+$ $\forall z \in R^n$,

then, we can actually use a *linear* high gain feedback control only depending on $\xi$ to stabilize the system:

**Theorem 3.4** *If H2 and H3 hold, then there exists a linear high gain feedback control in the form of*

$$u_k = s \begin{pmatrix} -p_k^{r_1}(\frac{d}{dt})y_1 \\ \vdots \\ -p_k^{r_m}(\frac{d}{dt})y_m \end{pmatrix} \tag{3.4}$$

*which BS stabilizes system* (3.1) *on compacta, where* $s = 1$ *if* $\sigma(G(z)) \subset C^+$ *and* $s = -1$ *if* $\sigma(G(z)) \subset C^-$.

**Proof:** Set $\epsilon = \frac{1}{k}$ and

$$\tilde{\xi}_j^i = \epsilon^{j-1}\xi_j^i, \qquad j = 1, ..., r_i, \ i = 1, ..., m$$

when the linear high gain control law is plugged in, the closed loop system becomes

$$\begin{aligned} \dot{z} &= f_0(z, \tilde{\xi}_1^1, ..., \tilde{\xi}_1^m) \\ \epsilon\dot{\tilde{\xi}}_1^i &= \tilde{\xi}_2^i \\ &\vdots \\ \epsilon\dot{\tilde{\xi}}_{r_i}^i &= \epsilon f_i(z, \tilde{\xi}_1^1, ..., \tilde{\xi}_1^m) - g_i(z)p^{r_i}(\tilde{\xi}_{(\cdot)}^i) \quad (i = 1, ..., m) \end{aligned} \tag{3.5}$$

where $p^{r_i}(\xi_{(\cdot)}^i) = a_{r_i-1}\xi_{r_i}^i + ... + a_0\xi_1^i$, and without loss of generality, we assume $G(z)$ is diagonal and $\sigma(G(z)) \subset C^+$, so $s = 1$ and $0 < r_p \leq g_i(z) \leq R_p \ \forall |z| \leq 2p$.

In order to make use of the slow manifold results, we need to show that the eigenvalues corresponding to the linearized terms of $\tilde{\xi}$ can be assigned to the left half plane by appropriate choices of $p^{r_i}(s)$ $i = 1, ..., m$. Namely, we need to show that there are $p^{r_i}(s)$ $(1 \leq i \leq m)$ such that $s^{r_i} + g_i(z)p^{r_i}(s)$ are Hurwitz polynomials for each $|z| \leq p$. From the root locus analysis, it is easy to see that when a $p_0^{r_i}(s)$ is Hurwitz, the closed loop loci for $\frac{p_0^{r_i}(s)}{s^{r_i}}$ are all located in the open left half plane when the gain is sufficiently large. Therefore, we can find a positive constant $L$ such that $Lp_0^{r_i}(s)$ satisfies our requirement (because $g_i(z)$ is bounded below from zero). From now on, we can just follow the proof of Theorem 3.1.

By using the linear high gain control (3.4), similarly we have the following corollaries:

**Corollary 3.5** *Suppose H2 and H3 hold for* (3.1). *If the zero dynamics is globally exponentially stable, then there exists a linear high gain control* (3.4) *which exponentially stabilizes system* (3.1) *on compacta.*

**Corollary 3.6** *Suppose H2 and H3 hold for* (3.1) *and* $f_i = 0$ $i = 1, ..., m$ *at all the points. If the zero dynamics is globally asymptotically stable (uniformly bounded), then there exists a linear high gain control* (3.4) *which asymptotically (BS) stabilizes system* (3.1) *on compacta.*

Before presenting a result for the nonminimum phase case, for the sake of simplicity we assume $r_1 = r_2 = \cdots = r_m = r$. If H2 holds, after a feedback transformation (3.1) becomes

$$\begin{aligned} \dot{z} &= f_0(z, \xi_1) \\ \dot{\xi}_1 &= \xi_2 \\ &\vdots \\ \dot{\xi}_r &= u \\ y &= \xi_1 . \end{aligned} \tag{3.6}$$

We also assume

**H4**: In (3.6), $f_0(0, \xi_1) = 0$ and $\frac{\partial f_0(z,\xi_1)}{\partial \xi_1 \partial z}|_{z=0} = 0$ $\forall \xi_1 \in R^m$.

**Remark**: for a general affine nonlinear control system, H4 can be stated in a coordinate free fashion (see [11]).

**Theorem 3.7** *Suppose H4 hold for* (3.6) *and there exists a* $C^r$ *map* $q$: $R^n \to R^m$ *such that*

$$\dot{z} = f_0(z, q(z))$$

*is globally exponentially stable, then there exists high gain control* $u_k(z, \xi)$ *which exponentially stabilize* (3.6) *on compacta. In particular, we can take the control as*

$$u_k(t) = -p_k^r(\frac{d}{dt})(y - \frac{k|z|^2}{1 + k|z|^2} q(z)) + (\frac{d}{dt})^r (\frac{k|z|^2}{1 + k|z|^2} q(z)) \tag{3.7}$$

*where* $s^r + p_k^r(s) = s^r + kb_{r-1}s^{r-1} + \cdots + k^r b_0$, $k > 0$, *is a Hurwitz polynomial.*

**Remark**: Here we do not assume $q(0) = 0$, otherwise it is a trivial result. First we should notice the fact that if $\dot{z} = f(z, q(z))$ is globally exponentially stable, then the zero dynamics of (3.6) must be locally exponentially stable due to H4 (if H4 holds, $q(z)$ can not change the linear part of the zero dynamics equation).

**Proof**: First, suppose $r = 1$.

Let $\tilde{y} = y - \frac{k|z|^2}{1+k|z|^2}q(z)$, then the control law in (3.7) can be expressed as

$$u = -k\tilde{y} + \frac{d}{dt}(\frac{k|z|^2}{1+k|z|^2}q(z)).$$

By setting $\epsilon = \frac{1}{k}$, we can write the closed-loop system of (3.6) as:

$$\begin{aligned} \dot{z} &= f_0(z, \tilde{y} + \frac{|z|^2}{\epsilon + |z|^2}q(z)) \\ \epsilon\dot{\tilde{y}} &= -\tilde{y}. \end{aligned} \tag{3.8}$$

Obviously (3.8) satisfies the assumptions in (2.1) and in Proposition 2.1. Therefore, by Proposition 2.2, the stability of (3.8) is determined by that of the flow on the invariant manifold $\tilde{y} = 0$:

$$\dot{z} = f_0(z, \frac{|z|^2}{\epsilon + |z|^2}q(z)). \tag{3.9}$$

We rewrite (3.9) as:

$$\dot{z} = f_0(z, q(z)) + f_0(z, \frac{|z|^2}{\epsilon + |z|^2}q(z)) - f_0(z, q(z)).$$

Since

$$\dot{z} = f_0(z, q(z))$$

is globally exponentially stable, in order to show (3.9) is exponentially stable on compacta, we only need to show that

$$\psi(z, \epsilon) = f_0(z, \frac{|z|^2}{\epsilon + |z|^2}q(z)) - f_0(z, q(z))$$

satisfies the growth condition in Proposition 2.4.

When H4 is satisfied, it is easy to show that for all $z$ in $P = \{z : |z| \leq p\}$,

$$|f_0(z, \frac{|z|^2}{\epsilon + |z|^2}q(z)) - f_0(z, q(z))| \leq \epsilon M(p)|z|^2 \frac{|q(z)|}{\epsilon + |z|^2} \leq M_q(p)\frac{\epsilon|z|}{\epsilon + |z|^2}|z|.$$

A straight forward computation shows

$$\frac{\epsilon|z|}{\epsilon + |z|^2} \leq \frac{1}{2}\epsilon^{\frac{1}{2}}.$$

Therefore, by Proposition 2.4, $z = 0$ of (3.9) is exponentially stable and the domain of attraction contains $P$. Since $p$ can be any positive value, by Proposition 2.2 we conclude that for $(z, y)$ in any bounded set, the origin of (3.8) of is exponentially stable and the domain of attraction contains the set.

For the case $r > 1$, we first do a coordinate change:

$$\tilde{\xi}_i = \epsilon^{i-1}\xi_i - (\epsilon\frac{d}{dt})^{i-1}(\frac{|z|^2}{\epsilon + |z|^2}q(z)),\ i = 1, ..., r$$

where $\epsilon = \frac{1}{k}$. Then in new coordinates (3.6) becomes

$$\begin{aligned}\dot{z} &= f_0(z, \tilde{\xi}_1 + \frac{|z|^2}{\epsilon + |z|^2}q(z)) \\ \epsilon\dot{\tilde{\xi}}_1 &= \tilde{\xi}_2 \\ &\vdots \\ \epsilon\dot{\tilde{\xi}}_r &= -b_{r-1}\tilde{\xi}_r - \cdots - b_0\tilde{\xi}_1.\end{aligned} \tag{3.10}$$

Obviously,

$$\tilde{\xi}_i = 0,\ i = 1, ..., r$$

is a slow manifold for (3.10). Therefore the stability of (3.10) is determined by that of the flow on the slow manifold:

$$\dot{z} = f_0(z, \frac{|z|^2}{\epsilon + |z|^2}q(z)).$$

From now on, we can follow the proof for the case $r = 1$.

**Example 3.1** Consider

$$\begin{aligned}\dot{z} &= -z + 3z^2 - yz^2 \\ \dot{y} &= u.\end{aligned} \tag{3.11}$$

If we take $q(z) = 3$, the hypotheses of Theorem 3.4 are satisfied. In fact, we can use

$$u = -k(y - \frac{3kz^2}{1 + kz^2}) - \frac{6k(z^2 - 3z^3 + yz^3)}{(1 + kz^2)^2}$$

to solve feedback stabilization on compacta.

## 4 Feedback Stabilization on Compacta and Peaking Phenomenon

It is widely understood that when using high gain control for the purpose of stabilization on compacta, one has to take into consideration the possible "peaking" phenomenon (see, for example, [17, 18] ) when the relative degree(s) of the system is greater than 1. As far as we are aware, the normal form (3.1) is perhaps the only nonpeaking condition which can be characterized in a coordinate free fashion. In this section our main purpose is to present an interesting example. In preparation for that we first give a fairly straight forward result on high gain feedback stabilization on compacta, without assuming the nonpeaking condition (3.1).

For the sake of simplicity, we only study the case of relative degree 2:

$$\begin{aligned} \dot{z} &= f(z, \xi_1, \xi_2) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u \\ y &= \xi_1 \end{aligned} \tag{4.1}$$

where $z \in R^n$ and, without loss of generality, we assume the system is SISO. We also assume that the zero dynamics

$$\dot{z} = f(z, 0)$$

is globally asymptotically stable.

It is well known that one can not always use the high gain control

$$u = -ak^2\xi_1 - bk\xi_2 \qquad a > 0,\ b > 0 \tag{4.2}$$

to stabilize (4.1) on compacta. A counter example was given in [18].

Now let us plug in (4.2) to (4.1):

$$\begin{aligned} \dot{z} &= f(z, \xi_1, \xi_2) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= -ak^2\xi_1 - bk\xi_2 \\ y &= \xi_1. \end{aligned} \tag{4.3}$$

Set $\epsilon = \frac{1}{k}$ and $\tilde{\xi_1} = k\xi_1$, one obtain

$$\begin{aligned} \dot{z} &= f(z, \epsilon\tilde{\xi_1}, \xi_2) \\ \epsilon\dot{\tilde{\xi_1}} &= \xi_2 \\ \epsilon\dot{\xi}_2 &= -a\tilde{\xi_1} - b\xi_2 \\ y &= \xi_1. \end{aligned} \tag{4.4}$$

By using the slow manifold approach, one can easily show that (4.4) is asymptotically stable on compacta. However, since

$$\xi_1 = \epsilon\tilde{\xi_1}$$

the original closed-loop system is not necessarily asymptotically stable on compacta. As a matter of fact, one can see precisely when the high gain control (4.2) might go wrong: when the closed-loop system initiates at a point with large values both in $\xi_1$ and $z$ or $\xi_2$. For example, in the counter example in [18], it is shown that the set $z\xi_1 > 1$ is invariant.

Let us now consider the following modified high gain control:

$$u = -ak\xi_1 - bk\xi_2 + c\xi_1 \qquad a > 0,\ b > 0,\ c > 0. \tag{4.5}$$

The difference with the high gain control (4.5) is that the associated polynomial only has one root, instead of two as in the case of (4.2), tend to infinity as $k$ tends to infinity.

**Theorem 4.1** *If there exist a $\gamma > 0$ such that*

$$\begin{aligned} \dot{z} &= f(z, \xi_1, -\gamma\xi_1) \\ \dot{\xi}_1 &= -\gamma\xi_1 \end{aligned} \tag{4.6}$$

*is globally asymptotically stable, then (4.1) is asymptotically stabilized on compacta by (4.5) if $\frac{a}{b} = \gamma$ and $c = \gamma^2$.*

**Proof:** Set $\epsilon = \frac{1}{k}$. The slow manifold is defined by

$$\xi_2 = -\gamma\xi_1$$

and the dynamics on the slow manifold is governed by (4.6).

Now let us consider an example:

**Example 4.1.**

$$\begin{aligned} \dot{z} &= -z^{2n+1} + 2(y_1 + y_1^3)z^4 + (y_1 + y_2)^2 z \\ \dot{y}_1 &= y_2 \\ \dot{y}_2 &= u \\ y &= y_1 \end{aligned} \tag{4.7}$$

where $n$ is a nonnegative integer. The zero dynamics of the system obviously is

$$\dot{z} = -z^{2n+1}$$

which is globally asymptotically stable.

We show that for $n = 0, 1$, there is no control law which could stabilize the system globally or on compacta; and for $n \geq 2$, one can use a linear high gain control to stabilize the system on compacta. In other words, we will show that as far as global or on compacta stabilization is concerned, *globally exponentially minimum phase* is not necessarily a stronger hypothesis than *globally asymptotically (critically) minimum phase.*

Let $P = \{(z, y_1) : p = zy_1 > 1\}$. For all $(z_0, y_{10})$ in $P$,
when $n = 0$:

$$\begin{aligned} \dot{p} &= \dot{z}y_1 + \dot{y}_1 z \\ &= -y_1 z + 2(y_1^2 + y_1^4)z^4 + y_1(y_1 + y_2)^2 z + y_2 z \\ &= 2(p^3 - 1)p + p(y_1 + y_2)^2 + (y_1 + y_2)z + 2p^2 z^2 \\ &> (y_1 + y_2)^2 + (y_1 + y_2)z + 2z^2 > 0 \end{aligned} \tag{4.8}$$

when $n = 1$:

$$\begin{aligned} \dot{p} &= \dot{z}y_1 + \dot{y}_1 z \\ &= -y_1 z^3 + 2(y_1^2 + y_1^4)z^4 + y_1(y_1 + y_2)^2 z + y_2 z \\ &= 2(p^3 - \frac{1}{2})p + p(y_1 + y_2)^2 + (y_1 + y_2)z + (2p^2 - p)z^2 \\ &> (y_1 + y_2)^2 + (y_1 + y_2)z + z^2 > 0. \end{aligned} \tag{4.9}$$

Therefore, for the cases $n = 0, 1$, $P$ is invariant forward in time. As a matter of fact, every trajectory initialized in $P$ has a finite escape time.

Now for the cases of $n \geq 2$, in order to show the system is asymptotically stabilizable on compacta by a high gain control of the type (4.5), we show that for (4.7), (4.6) in Theorem 4.1 is globally asymptotically stable for $\gamma = 1$:

$$\begin{aligned} \dot{z} &= -z^{2n+1} + 2(y_1 + y_1^3)z^4 \\ \dot{y}_1 &= -y_1. \end{aligned} \tag{4.10}$$

We first need the following result:

**Lemma 4.2** *Let $a > 0$, $,b > 0$, $p > 0$, $q > 0$ and $x \in R$, $y \in R$. The following inequality:*

$$a|x|^p + b|y|^q - xy \geq 0$$

*with the equality held only at $x = 0$, $y = 0$, holds if*

$$\frac{1}{p} + \frac{1}{q} = 1 \text{ and } (ap)^{\frac{1}{p}}(bq)^{\frac{1}{q}} > 1.$$

This result can be shown by using Young's inequality.

Now define a Liapunov function:

$$V = \frac{1}{10n - 18}z^{10n-18} + \frac{100}{3}y_1^6 + \frac{100}{9}y_1^{18}$$

then

$$\begin{aligned} \dot{V} &= -z^{12n-18} + 2(y_1 + y_1^3)z^{10n-15} - 200y^6 - 200y^{18} \\ &= -2(P_1 + P_2) \end{aligned}$$

where $P_1 = \frac{1}{4}z^{12n-18} - y_1 z^{10n-15} + 100y_1^6$ and $P_2 = \frac{1}{4}z^{12n-18} - y_1^3 z^{10n-15} + 100y_1^{18}$.

Set $\bar{z} = z^{10n-15}$, one can easily see that $P_1$ satisfies the hypotheses of Lemma 4.2, therefore $P_1 > 0,\ \forall(z, y_1) \neq 0$. Similarly, by setting $\bar{z} = z^{10n-15}$ and $\bar{y}_1 = y_1^3$, one can show $P_2 > 0,\ \forall(z, y_1) \neq 0$. Thus

$$\dot{V} \leq 0$$

and the equality holds only at $z = 0,\ y_1 = 0$. Since $V$ is obviously radially unbounded, the origin of (4.10) is globally asymptotically stable. Therefore, when $n \geq 2$, (4.7) can be asymptotically stabilized on compacta by a high gain control (4.5).

It is not difficult to show that when $n \geq 2$, the system can also be globally stabilized by a full state feedback.

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## Appendix: Proof of Proposition 2.1

**Proof:** Rewrite the system as follows:

$$\begin{aligned} \dot{x} &= f(x, z, \epsilon) \\ \epsilon\dot{z} &= F(x, z, \epsilon) \quad x \in R^p,\ z \in R^q. \end{aligned} \tag{A.1}$$

Make a coordinate change first:

$$z = \phi(x) + y$$

then (A.1) becomes

$$\begin{aligned} \dot{x} &= f(x, \phi(x) + y, \epsilon) \\ \epsilon\dot{y} &= A(x)y + Q(x, y, \epsilon) \end{aligned} \tag{A.2}$$

where

$$A(x) = \frac{\partial F(x, \phi(x), \epsilon)}{\partial z}$$

$$Q(x, y, \epsilon) = F(x, \phi(x) + y, \epsilon) - \frac{\partial F}{\partial z}(x, \phi(x), \epsilon)y - \epsilon\frac{\partial \phi(x)}{\partial x} f(x, \phi(x) + y, \epsilon).$$

Suppose $I_B$ is a $C^\infty$ function with the following properties:

$$I_B(x) = \begin{cases} 1, & x \in B_p; \\ 0, & |x| \geq 2p. \end{cases}$$

Define

$$f_B(x,y) = f(xI_B(x), (\phi(x)+y)I_B(x), \epsilon).$$

Evidently $f_B(x,y)$ agrees with $f(x,y,\epsilon)$ when $x \in B_p$. Now define a collection of mappings on $B_{2p} := \{x : |x| \le 2p\}$:

$$Y_M^\ell = \{Y(x,\epsilon) : |Y(x,\epsilon)| \le M(\epsilon), |Y(x,\epsilon) - Y(\bar{x},\epsilon)| \le \ell(\epsilon)|x-\bar{x}|\}$$

with $\|Y\| = \sup_{x\in B_{2p}} |Y(x,\epsilon)|$. And consider the differential equation:

$$\begin{array}{ll} \dot{x} &= f_B(x, Y(x,\epsilon)) \qquad Y(x,\epsilon) \in Y_M^\ell \\ x(0) &= x_0 \qquad\qquad\qquad\quad x_0 \in B_{2p} \end{array} \tag{A.3}$$

Since $f_B(x,Y(x,\epsilon)) = 0$ when $|x| \ge 2p$, the solution $x_Y(t,x_0,\epsilon)$ to (A.3) will stay in $B_{2p}$, i.e.

$$|x_Y(t,x_0,\epsilon)| \le 2p, \text{ for any } x_0 \in B_{2p},\ Y(x,\epsilon) \in Y_M^\ell,\ -\infty < t < \infty$$

we can formally write:

$$x_Y(t,x_0,\epsilon) = x_0 + \int_0^t f_B(x_Y, Y(x_Y,\epsilon))dt$$

then,

$$\begin{aligned} &|x_{Y_1}(t,x_0^1,\epsilon) - x_{Y_2}(t,x_0^2,\epsilon)| \\ &\le |x_0^1 - x_0^2| + |\int_0^t |f_B(x_{Y_1}, Y_1(x_{Y_1},\epsilon) - f_(x_{Y_2}, Y_2(x_2,\epsilon)|\, d\tau| \\ &\le |x_0^1 - x_0^2| + M_1|\int_0^t (|x_{Y_1} - x_{Y_2}| + |Y_1(x_{Y_1},\epsilon) - Y_2(x_{Y_2},\epsilon)|)\, d\tau| \\ &\le |x_0^1 - x_0^2| + M_1|\int_0^t (|x_{Y_1} - x_{Y_2}| + |Y_1(x_{Y_1},\epsilon) - Y_2(x_{Y_1},\epsilon)| + \\ &+ |Y_2(x_{Y_1},\epsilon) - Y_2(x_{Y_2},\epsilon)|)\, d\tau| \\ &\le |x_0^1 - x_0^2| + M_1|\int_0^t [(1+\ell(\epsilon))|x_{Y_1} - x_{Y_2}| + \|Y_1 - Y_2\|]\, d\tau| \end{aligned} \tag{A.4}$$

where $\|Y\| = \sup_{x\in B_{2p}} |Y(x,\epsilon)|$.

By the Gronwall-Bellman inequality:

$$|x_{Y_1}(t,x_0^1,\epsilon) - x_{Y_2}(t,x_0^2,\epsilon)| \le |x_0^1 - x_0^2|e^{M_1(1+\ell(\epsilon))|t|} + \frac{\|Y_1 - Y_2\|}{1+\ell(\epsilon)}e^{M_1(1+\ell(\epsilon))|t|}$$

Now, we estimate the state transition matrix of:

$$\epsilon\frac{dp}{dt} = A(x_Y(t,x_0,\epsilon))p \tag{A.5}$$

where $A$ is defined in (A.2). Since $B_{2p}$ is closed, then Re $\sigma(A(x)) \leq -2r$, $r > 0$, $\forall\, x \in B_{2p}$, when $\epsilon$ is sufficiently small.

**Lemma A.1** [1] *Consider the system $\epsilon\dot{x} = A(t)x$. If $|A(t)|$ and $|\dot{A}(t)|$ are bounded, and Re $\sigma(A(t)) < -2r$, $r > 0 \quad \forall\, t$, then, when $\epsilon$ is sufficiently small, the state transition matrix is bounded by $|U(t,s)| \leq Ke^{-\frac{r}{\epsilon}(t-s)} \quad t \geq s$.*

So, the state transition matrix of (A.5) is bounded by:

$$|Ux_Y(t,s,\epsilon)| \leq Ke^{-\frac{r}{\epsilon}(t-s)} \quad t \geq s$$

when $\epsilon$ is sufficiently small. Then, the solution of

$$\epsilon\dot{y} = A(x_Y(t,x_0,\epsilon))y + Q(x_Y(t,x_0,\epsilon),y,\epsilon)$$

can be written as

$$y(t) = Ux_Y(t,0,\epsilon)y_0 + \frac{1}{\epsilon}\int_0^t U_{x_Y}(t,s,\epsilon)Q(x_Y(s,x_0,\epsilon),y(s),\epsilon)\,ds.$$

Now define an operator on $Y_M^\ell$:

$$(TY)(x_0) = \frac{1}{\epsilon}\int_{-\infty}^0 U_{x_Y}(0,s,\epsilon)Q(x_Y(s,x_0,\epsilon),Y(x_Y(s,x_0,\epsilon),\epsilon))\,ds.$$

Recall that:

$$Q(x,y,\epsilon) = F(x,\phi(x)+y,\epsilon) - F_z(x,\phi(x),\epsilon)y - \epsilon\phi_x(x)f(x,\phi(x)+y)$$

and $F(x,\phi(x),0) = 0$, then

$$\begin{aligned} |Q(x_Y,Y(x_Y,\epsilon),\epsilon)| &\leq M_2|Y(x_Y,\epsilon)|^2 + \epsilon M_3 \\ &\leq M_2M^2(\epsilon) + \epsilon M_3 \end{aligned} \tag{A.6}$$

then

$$\begin{aligned} |(TY)(x_0)| &\leq \frac{1}{\epsilon}\int_{-\infty}^0 |U_{x_Y}||Q|\,ds \\ &\leq \frac{1}{\epsilon}\int_{-\infty}^0 Ke^{\frac{r}{\epsilon}s}(M_2M^2(\epsilon)+\epsilon M_3)\,ds \\ &= \frac{K}{r}(M_2M^2(\epsilon)+\epsilon M_3) \end{aligned} \tag{A.7}$$

by proper choice of $M(\epsilon)$, we may have $\frac{K}{r}(M_2M(\epsilon) + \frac{\epsilon M_3}{M(\epsilon)}) < 1$, when $\epsilon$ is sufficiently small, i.e.

$$\frac{K}{r}(M_2M^2(\epsilon)+\epsilon M_3) < M(\epsilon).$$

Now:

$$\begin{aligned}
&|(TY_1)(x_0^1) - (TY_2)(x_0^2)| \\
&\leq \frac{1}{\epsilon}\int_{-\infty}^{0} [(|U_{x_{Y_1}}(0,s,\epsilon) - U_{x_{Y_2}}(0,s,\epsilon)|)|Q(x_{Y_1}, Y_1(x_{Y_1},\epsilon),\epsilon)| + \\
&+ |U_{x_{Y_2}}||Q(x_{Y_1}) - Q(x_{Y_2})|]ds.
\end{aligned} \tag{A.8}$$

We first need to estimate $|U_{x_{Y_1}}(t,s,\epsilon) - U_{x_{Y_2}}(t,s,\epsilon)|$ $(s \leq t)$ when $\epsilon$ is sufficiently small. Since

$$\begin{aligned}
&\epsilon\frac{\partial}{\partial t}(U_{x_{Y_1}}(t,s,\epsilon) - U_{x_{Y_2}}(t,s,\epsilon)) \\
&\qquad = A(x_{Y_1}(t))U_{x_{Y_1}}(t,s,\epsilon) - A(x_{Y_2}(t))U_{x_{Y_2}}(t,s,\epsilon) \\
&\qquad = A(x_{Y_1}(t))(U_{x_{Y_1}}(t,s,\epsilon) - U_{x_{Y_2}}(t,s,\epsilon)) + \\
&\qquad + (A(x_{Y_1}(t)) - A(x_{Y_1}(t)))U_{x_{Y_2}}(t,s,\epsilon)
\end{aligned} \tag{A.9}$$

$$\begin{aligned}
&|U_{x_{Y_1}}(t,s,\epsilon) - U_{x_{Y_2}}(t,s,\epsilon)| \\
&= \frac{1}{\epsilon}|\int_s^t U_{x_{Y_1}}(t,\tau,\epsilon)(A(x_{Y_1}(\tau)) - A(x_{Y_1}(\tau)))U_{x_{Y_2}}(\tau,s,\epsilon)d\tau| \\
&\leq \frac{1}{\epsilon}\int_s^t Ke^{-\frac{r}{\epsilon}(t-\tau)}M_4|x_{Y_2}(\tau) - x_{Y_1}(\tau)|Ke^{-\frac{r}{\epsilon}(\tau-s)}d\tau \\
&\leq \frac{K^2M_4}{\epsilon}\int_s^t e^{-\frac{r}{\epsilon}(t-s)}(|x_0^1 - x_0^2| + \frac{\|Y_1 - Y_2\|}{1+\ell(\epsilon)})e^{M_1(1+\ell(\epsilon))|\tau|}d\tau \\
&\leq \frac{K^2M_4}{\epsilon M_1}e^{-\frac{r}{\epsilon}(t-s) - M_1(1+\ell)s}(e^{M_1(1+\ell)(t-s)} - 1)(|x_0^1 - x_0^2| + \|Y_1 - Y_2\|) \\
&\leq \frac{K^2M_4(1+\ell(\epsilon))}{\epsilon}e^{-\frac{r-2\epsilon M_1(1+\ell(\epsilon))}{\epsilon}(t-s)}(t-s)(|x_0^1 - x_0^2| + \|Y_1 - Y_2\|) \\
&\leq \frac{2K^2M_4}{r - 2\epsilon M_1(1+\ell(\epsilon))}e^{-\frac{r-2\epsilon M_1(1+\ell(\epsilon))}{2\epsilon}(t-s)}(|x_0^1 - x_0^2| + \|Y_1 - Y_2\|)
\end{aligned}$$

Here we assume $r - 2\epsilon M_1(1+\ell(\epsilon)) > 0$. Then

$$\begin{aligned}
&\frac{1}{\epsilon}\int_{-\infty}^{0} |U_{x_{Y_1}} - U_{x_{Y_2}}||Q(x_{Y_1}, Y_1(x_{Y_1}), \epsilon)|\,ds \\
&\le \frac{1}{\epsilon}\int_{-\infty}^{0} \frac{2K^2M_4}{r-2\epsilon M_1(1+\ell(\epsilon))} e^{\frac{r-2\epsilon M_1(1+\ell(\epsilon))}{2\epsilon}s}(|x_0^1 - x_0^2| + \|Y_1 - Y_2\|)|Q|\,ds \\
&\le \frac{2K^2M_4}{\epsilon(r-2\epsilon M_1(1+\ell(\epsilon)))}(M_2M^2(\epsilon)+\epsilon M_s)(|x_0^1 - x_0^2| + \\
&+ \|Y_1 - Y_2\|)\int_{-\infty}^{0} e^{\frac{r-2\epsilon M_1(1+\ell(\epsilon))}{2\epsilon}s}ds \\
&= \frac{4K^2M_4}{(r-2\epsilon M_1(1+\ell(\epsilon)))^2}(M_2M^2(\epsilon)+\epsilon M_s)(|x_0^1 - x_0^2| + \|Y_1 - Y_2\|)
\end{aligned} \tag{A.10}$$

and since (see [20])

$$\begin{aligned}
&|Q(x_{Y_1}, Y_1, \epsilon) - Q(x_{Y_2}, Y_2, \epsilon)| \\
&\quad\le (M_5M(\epsilon)+\epsilon M_6)[(1+\ell(\epsilon))|x_{Y_1} - x_{Y_2}| + \|Y_1 - Y_2\|] \\
&\quad\le (M_5M(\epsilon)+\epsilon M_6)[2|x_0^1 - x_0^2|e^{-M_1(1+\ell(\epsilon))s} + \\
&\quad+ 2\|Y_1 - Y_2\|e^{-M_1(1+\ell(\epsilon))s}
\end{aligned} \tag{A.11}$$

We have,

$$\begin{aligned}
&\frac{1}{\epsilon}\int_{-\infty}^{0} |U_{x_{Y_2}}||Q(x_{Y_1}Y_1, \epsilon) - Q(x_{Y_2}, Y_2, \epsilon)|\,ds \\
&\quad\le \frac{2}{\epsilon}K(M_5M(\epsilon)+\epsilon M_6)\int_{-\infty}^{0} e^{(\frac{r}{\epsilon} - M_1(1+\ell(\epsilon)))s}[|x_0^1 - x_0^2| + \|Y_1 - Y_2\|]\,ds \\
&\quad= \frac{2K(M_5M(\epsilon)+\epsilon M_6)}{r-\epsilon M_1(1+\ell(\epsilon))}[|x_0^1 - x_0^2| + \|Y_1 - Y_2\|]
\end{aligned} \tag{A.12}$$

then,

$$\begin{aligned}
|(TY_1)(x_0^1) - (TY_2)(x_0^2)| \le &\left[4K^2M_4\frac{(M_2M^2(\epsilon)+\epsilon M_s)}{(r-2\epsilon M_1(1+\ell(\epsilon)))^2} + \right.\\
&\left.+ \frac{2K(M_5M(\epsilon)+\epsilon M_6)}{r-\epsilon M_1(1+\ell(\epsilon))}\right]\cdot \\
&\cdot(|x_0^1 - x_0^2| + \|Y_1 - Y_2\|)
\end{aligned} \tag{A.13}$$

Set $Y_1 = Y_2$, by proper choice of $\ell(\epsilon)$, we have,

$$|(TY_1)(x_0^1) - (TY_2)(x_0^2)| \le \ell(\epsilon)|x_0^1 - x_0^2|.$$

So, $(TY)(x_0) \in Y_M^\ell$. Set $x_0^1 = x_0^2$ and

$$B(\epsilon) = \left[4K^2 M_4 \frac{(M_2 M^2(\epsilon) + \epsilon M_s)}{(r - 2\epsilon M_1(1+\ell(\epsilon)))^2} + \frac{2K(M_5 M(\epsilon) + \epsilon M_6)}{r - \epsilon M_1(1+\ell(\epsilon))}\right]$$

then

$$|(TY_1)(x_0) - (TY_2)(x_0)| \le B(\epsilon)\|Y_1 - Y_2\| \qquad \forall \quad x_0 \in B_{2p}.$$

So,

$$\|(TY_1)(x_0) - (TY_2)(x_0)\| \le B(\epsilon)\|Y_1 - Y_2\|.$$

When $\epsilon$ is sufficiently small, $T$ is a contraction mapping. Therefore there exists a $Y_e(x_0) \in Y_M^\ell$, such that

$$Y_e(x_0) = \frac{1}{\epsilon}\int_{-\infty}^{0} U_{x_{Y_e}}(0,s,\epsilon) Q[x_{Y_e}(s,x_0,\epsilon), Y_e(x_{Y_e}(s,x_0,\epsilon)),\epsilon]\, ds.$$

Now we show $y = Y_e(x)$ ($|x| \le 2p$) is an invariant manifold for the following system:

$$\begin{aligned} \dot{x} &= f_B(x,y) \\ \epsilon\dot{y} &= A(x)y + Q(x,y,\epsilon). \end{aligned} \tag{A.14}$$

Namely we show that for (A.14), if $x(0) = x_0,\ y(0) = Y_e(x_0)$, then $y(t) = Y_e(x(t))$.

Defining $x_y(t) = x(t)$, we can formally write $y(t)$ as

$$y(t) = Ux_y(t,0,\epsilon)y(0) + \frac{1}{\epsilon}\int_0^t U_{x_y}(t,s,\epsilon)Q(x_y(s,x_0,\epsilon), y(s,x_0,\epsilon),\epsilon)\, ds$$

i.e.,

$$\begin{aligned} y(t) &= Ux_y(t,0,\epsilon)\frac{1}{\epsilon}\int_{-\infty}^{0} U_{x_y}(0,s,\epsilon)Q(x_y(s,x_0,\epsilon), y(s),\epsilon)\, ds + \\ &\quad + \frac{1}{\epsilon}\int_0^t U_{x_y}(t,s,\epsilon)Q(x_y(s,x_0,\epsilon), y(s),\epsilon)\, ds \\ &= \frac{1}{\epsilon}\int_{-\infty}^{t} U_{x_y}(t,s,\epsilon)Q(x_y(s,x_0,\epsilon), y(s,x_0,\epsilon),\epsilon)\, ds \\ &= \frac{1}{\epsilon}\int_{-\infty}^{0} U_{x_y}(t,t+\tau,\epsilon)Q(x_y(t+\tau,x_0,\epsilon), y(t+\tau,x_0,\epsilon),\epsilon)\, d\tau \\ &= \frac{1}{\epsilon}\int_{-\infty}^{0} U_{x_y}^t(0,\tau,\epsilon)Q(x_y(\tau,x(t),\epsilon), y(\tau,x(t),\epsilon),\epsilon)\, d\tau \end{aligned} \tag{A.15}$$

where $U^t_{x_y}$ is the state transition matrix satisfying

$$\frac{\partial U^t_{x_y}(\sigma,\tau,\epsilon)}{\partial\sigma} = A(x_y(\sigma, x_y(t)))U^t_{x_y}(\sigma,\tau,\epsilon).$$

Therefore $y = Y_e(x)$ ($|x| \leq 2p$) is an invariant manifold for (A.14). Since $f_B(x, y)$ agrees with $f(x, y, \epsilon)$ for all $x \in B_p$, it is easy to see that $y = Y_e(x)$ ($x \in B_p$) is an invariant manifold for (A.2). Therefore $z = \phi(x) + Y_e(x)$ ($x \in B_p$) is an invariant manifold for (A.1).

Now consider the recursion formula

$$Y_{n+1}(x_0) = \frac{1}{\epsilon}\int_{-\infty}^{0} U_{x_{Y_n}}(0, s, \epsilon)Q[x_{Y_n}(s, x_0, \epsilon), Y_n(x_{Y_n}(s, x_0, \epsilon)), \epsilon]\, ds$$

where $n = 1, 2, 3, \ldots$, $Y_1(x)$ is an arbitrary $C^1$ mapping in $Y^\ell_M$ with $Y_1(0) = 0$.

Differentiating the recursion formula we have

$$\frac{\partial Y_{n+1}(x)}{\partial x} = \frac{1}{\epsilon}\int_{-\infty}^{0} \frac{\partial}{\partial x}(U_{x_{Y_n}} Q(X_{Y_n}, Y_n, \epsilon))\, ds.$$

By our assumption, $F$, $f$, $\phi$ are $C^3$, which implies $\frac{\partial^2 Y_n(x)}{\partial x_i \partial x_j}$ ($\forall i\ \forall j$) are uniformly bounded in $B_p$. Then, $\frac{\partial Y_n(x)}{\partial x}$ is uniformly continuous in $B_p$, which implies $\{\frac{\partial Y_n(x)}{\partial x}\}$ contains a uniformly convergent subsequence. Therefore, $Y_e(x)$ is also $C^1$ with $Y_e(0) = 0$.

Let $\psi(x, \epsilon) = Y_e(x)$, our theorem is thus proved.

## References


[1] D. Aeyels. Stabilization of a class of nonlinear systems by a smooth feedback control, *Systems and Control Letters* **5** (1985) 289–294.

[2] R. Brockett. *Finite Dimensional Linear Systems.* New York: John Wiley and Sons, 1970.

[3] C. I. Byrnes and A. Isidori. New results and examples in nonlinear feedback stabilization, *System Control Lett.* **12** (1989), 437-442.

[4] C. I. Byrnes and A Isidori. Asymptotic stabilization of minimum phase nonlinear systems, *IEEE Trans. Aut. Control* **36** (1991), 1122-1137.

[5] J. Carr. *Applications of Centre Manifold Theory.* New York: Springer-Verlag, 1981.

[6] J. L. Daleckii and M. G. Krein. *Stability of Solutions of Differential Equations in Banach Space.* Providence: American Mathematical Society, 1974.

[7] N. Fenichel. Geometric singular perturbation theory for ordinary differential equations, *J. Differential Equations* **31** (1979), 53–98.

[8] W. Hahn. *Stability of Motion.* Berlin: Springer-Verlag, 1967.

[9] E. Hille. *Lectures on Ordinary Differential Equations.* London: Addison-Wesley, 1969.

[10] F. Hoppensteadt. Properties of solutions of ordinary differential equations with small parameters, *Comm. Pure Appl. Math.* **24** (1971), 807–840.

[11] X. Hu. Asymptotic tracking with internal stability in the large for nonlinear systems, in *Proceedings of the First European Control Conference* (1991), Grenoble, France, pp. 2058-2063.

[12] X. Hu. Some results in nonlinear output regulation and feedback stabilization, *Int. J. Control* **59** (1994), 1085-1093.

[13] X. Hu. Nonlinear feedback stabilization on compacta and invariant manifolds, in *Proceedings of the Second European Control Conference* (1993), Groningen, the Netherlands, pp. 53-57.

[14] P. Kokotovic. Applications of singular perturbation techniques to control problems, *SIAM Review* **26** (1984), 501-550.

[15] P. Kokotovic and H. Sussmann. A positive real condition for global stabilization of nonlinear systems, *System Control Lett.* **13** (1989), 125-133.

[16] R. Marino. High gain feedback in nonlinear control systems, *System Control Lett.* **4** (1985), 1369-1385.

[17] H. J. Sussmann and P. V. Kokotovic. The peaking phenomenon and the global stabilization of nonlinear systems, *IEEE Trans. Aut. Control* **36** (1991), 424-440.

[18] H. J. Sussmann. Limitations on the stabilizability of globally minimum phase system, *IEEE Trans. Aut. Control* **32** (1990), 117-119.

[19] E.D. Sontag. Smooth stabilization implies coprime factorization, *IEEE Trans. Aut. Control* **31** (1989), 435-443.

[20] K. V. Zadiraka. On a nonlocal integral manifold of a nonregularly perturbed differential system, *American Mathematical Society Translations, Series* **289** (1970), 29–50.

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