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General Closed Loop Optimal Solutions for Linear Dynamic Systems with Linear Constraints and Functional*

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Abstract

This paper gives a general solution for the problem of deriving closed loop optimal controller for linear dynamical systems with linear constraints and linear functional. The controller is also given for the feasibility problem. Necessary and sufficient condition for the existence of such controller are given. The analysis is completed illustrating some interesting special cases.

Key words: dynamical optimization, constrained systems, positive systems, polyhedral cones

AMS Subject Classifications: 49B99 93C55

1 Introduction

A rather extensive literature has accumulated on input constrained dynamical systems. We cite as examples [1], [2], [3]. Here we use a dual conical condition, given in [4], to investigate the linear dynamical optimization problem with state constraints. The other basic ingredient of our approach is the study of the evolution, backward in time, of the set of admissible states (that is, initial states for which a feasible solution exists). The present results were essentially already given in earlier drafts of this paper (a technical report [5] - and a conference extended abstract [6]). The new presentation is more streamlined and emphasizes the interpretation of our results in terms of feedback solution of linear dynamic optimum problems.

An approach similar to the present one was also exploited in [7] to introduce a generalization of positive systems.

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Some authors (see [8] and [9]) solved some specialized problems in linearly state-control constrained systems with a similar recursion, using at each step a projection algorithm (e.g. in [8] an algorithm is used, which is a slight modification of the well known Fourier-Motzkin elimination method [10]). By contrast in our general setting projections are not used.

We give, under absolutely general conditions, the expression of the closed loop solution for both the feasibility and the optimization problem. This latter seems to have been never examined before in the literature. At the same time we give necessary and sufficient conditions for the existence of corresponding feedback controller.

Besides the theoretical interest, attached with the generality and completeness of the results, our approach has the potential for applications beyond the toy level, despite the complexity problems related to the dual conical optimality condition [4] and [11]). Moreover it has the additional advantage that, for a fixed system, once one computes off-line the structure of the controller (similarly to the Riccati equation in the quadratic optimal regulator) the recomputation of solutions of feasibility or optimality problem defined by any determination of the bounds is very fast.

It might be noted that the computations for positive systems are instead straightforward [7], thanks to the results in [4].

A completely different approach, which leads outside the natural polyhedral setting, is given in [12] and is based on a smoothing of the problem. In this way, standard techniques of dynamical optimization can be applied.

2 Problem Formulation and Recalls

Consider a discrete time linear system described by the input-state recursive equations

$$x(t+1) = A(t)x(t) + B(t)u(t) \tag{2.1}$$

with input (or control) $u(t) \in R^p$ and state $x(t) \in R^n$. Let t_i and t_f (also denoted by T) be, respectively, an initial and final time with $t_f > t_i$. Moreover let the initial state $x(t_i) = \bar{x}$ be given.

With reference to such system consider also the functional:

$$\max \sum_{j=t_i+1}^{T} (f_j, x(j)) \tag{2.2}$$

and the constraints

$$W(t)x(t) \leq M(t) \quad t_i < t \leq T. \tag{2.3}$$

For each t, $W(t)$ is a $m \times n$ matrix and $M(t)$ an m-dimensional vector. The polyhedron defined by each of these inequalities is denoted by D_t.

We define the following feasibility (F) and optimality (O) problems.

Definition 1 *Problem F: Given the system (2.1) the feasibility problem consists in ascertaining whether there exists a control $\bar{u}$ defined on the interval $[t_i, T)$, such that the corresponding solution satisfies (2.1) and (2.3), and, if these is the case, in computing one such control, called feasible or admissible control.*

The problem F is said to be feasible if the solution exists.

Definition 2 *Problem O: the optimality problem consists in ascertaining whether there exists a control u^o defined on the interval $[t_i, T)$, such that the corresponding solution satisfies (2.1), (2.3) and maximizes the functional (2.2) on the set of all admissible inputs, and, if this is the case, in computing one such control, called optimal control.*

To solve these problems we shall at times introduce further problems referred to initial and final times t_i' and T', with $t_i \leq t_i' \leq T' \leq T$.

In the following sections we make systematic use of a duality result (see [4]) which will be recalled now for reader's convenience.

Let Q be a $s \times n$ matrix and let v be a vector in R^n. Denote by $\mathcal{R}(G)$ the range of the matrix G and by P the nonnegative orthant of R^s (the same symbol will be used in the sequel to denote the nonnegative orthant of any Euclidean space, leaving to the context the determination of the space itself).

The convex cone $\mathcal{R}(G)^\perp \cap P$ is polyhedral and pointed. A set of vectors formed taking a nonzero vector from each of its extreme rays is called a set of generators of the cone.

At this point we can state the following:

Theorem 1 *The convex polyhedron $\{x : Gx \leq v,\ G \in R^{m \times n}\}$ is nonvoid if and only if*

$$Qv \geq 0 \tag{2.4}$$

where $\{rows\ of\ Q\} = \{generators\ of\ the\ cone : \mathcal{R}(G)^\perp \cap P\}$.

3 Feasibility Problem

We start introducing a sequence of sets of admissible states, which can be computed recursively backward in time. Let us call S_t the set of all initial states, for which the problem is feasible, assuming initial time $t_i' = t < T$ and constraints $W(s)x(s) \leq M(s)$, $t < s \leq T$. There is a backward recursion involving the sets S_t. In fact S_t can be obtained as the set of all states for which the feasibility problem defined by the following constraints for system (2.1)

$$t'_i = t$$
$$t'_f = t + 1$$
$$x(t+1) \in E_{t+1} = S_{t+1} \cap D_{t+1}$$

is feasible. Of course the initial condition for this recursion is given by $S_T = D_T$.

At this point we can state the following theorem, which solves the feasibility problem and gives the explicit expression of the sets S_t and E_t.

Theorem 2 *(i) The sets S_t, $t_i \le t < T$, and E_t, $t_i < t < T$, are polyhedra and have the following recursive expressions:*

$$S_t = \{x : Q_{t+1}\hat{W}(t+1)A(t)x \le Q_{t+1}\hat{M}(t+1)\} \tag{3.1}$$

$$E_t = \{x : \hat{W}(t)x \le \hat{M}(t)\} \tag{3.2}$$

where

$$\hat{W}(t) = \begin{pmatrix} Q_{t+1}\hat{W}(t+1)A(t) \\ W(t) \end{pmatrix} \tag{3.3}$$

$$\hat{M}(t) = \begin{pmatrix} Q_{t+1}\hat{M}(t+1) \\ M(t) \end{pmatrix} \tag{3.4}$$

with initial conditions $\hat{W}(T) = W(T)$ and $\hat{M}(T) = M(T)$ and where:

$$\{rows\ of\ Q_{t+1}\} = \{generators\ of\ \mathcal{R}(W(t+1)B(t))^{\perp} \cap P\}. \tag{3.5}$$

(ii) The feasibility problem F has solution if and only if

$$[Q_{t_i+1}\hat{W}(t_i+1)A(t_i)]\bar{x} \le [Q_{t_i+1}\hat{M}(t_i+1)] \tag{3.6}$$

(iii) The condition (ii) is necessary and sufficient for the existence of the following closed loop (or feedback) controller

$$\hat{W}(t+1)B(t)u(t) \le \hat{M}(t+1) - \hat{W}(t+1)A(t)x(t)\ t_i \le t < T. \tag{3.7}$$

These inequalities in fact, coupled with the state equation and solved recursively forward in time, define the set of all the feasible control function and corresponding feasible state functions.

Proof: (i) The state constraint at T is:

$$W(T)x(T) \le M(T).$$

Substituting for $x(T)$:

$$W(T)(A(T-1)x(T-1)+B(T-1)u(T-1)) \leq M(T)$$

or

$$W(T)B(T-1)u(T-1) \leq M(T) - W(T)A(T-1)x(T-1). \tag{3.8}$$

In view of the dual nonvoidness condition (Theorem 1), the set of all bounds that make the latter equation feasible is given by:

$$Q_T[M(T) - W(T)A(T-1)x(T-1)] \geq 0$$

where Q_T is the matrix, whose rows are the generators of the pointed polyhedral cone $\mathcal{R}[W(T)B(T-1)]^{\perp} \cap P$. Therefore:

$$Q_T W(T)A(T-1)x(T-1) \leq Q_T M(T). \tag{3.9}$$

The latter inequality defines the set of admissible states S_{T-1}, showing, at the same time, that S_{T-1} is a polyhedron.

The set E_{T-1} is obtained intersecting S_{T-1} with the constraining set $D_{T-!}$ and hence it is given by the polyhedron:

$$E_{T-1} = \{x : \hat{W}(T-1)x \leq \hat{M}(T-1)\} \tag{3.10}$$

where

$$\hat{W}(T-1) = \begin{pmatrix} Q_T W(T)A(T-1) \\ W(T-1) \end{pmatrix}$$

$$\hat{M}(T-1) = \begin{pmatrix} Q_T M(T) \\ M(T-1) \end{pmatrix}.$$

At this point it is easy to see that generalizing the above formulas for the generic instant of time t, the desired expression of S_t and E_t are obtained.

(ii) The feasibility problem has solution if and only if the initial state $\bar{x}$ belongs to the set S_{t_i}. From (3.1), with $t = t_i$, we have that this is true if and only if

$$[Q_{t_i+1}\hat{W}(t_i+1)A(t_i)]\bar{x} \leq [Q_{t_i+1}\hat{M}(t_i+1)]$$

as we wanted to prove.

(iii) The inequality defining the closed loop controller is obtained simply generalizing (3.8) to arbitrary t.□

It may be useful, to better clarify the present result, to illustrate its application in operative terms. First of all we have to compute the sequence of the $\hat{W}(t)$'s, $\hat{M}(t)$'s and Q_t's. This allows us to verify the feasibility condition. If the problem is feasible, then, by means of such sequences, the closed loop controller is completely defined. We start from time t_i and substitute in the equation of the closed loop controller the initial state $\bar{x}$ and solve for the corresponding $u(t_i)$. Once this latter is known it is substituted in the state equation to compute the feasible state $x(t_i+1)$, which in turns feeds the feedback controller yielding the feasible control $u(t_i+1)$, to be fed to the state equation, and so on. Thus we have obtained a closed loop solution, which makes it possible to compute automatically and recursively any feasible solution.

Finally, it is very important to stress that essentially all the computational effort to solve the problem is connected to the determination of the Q_t's. Moreover, neither the sequence of the $\hat{W}(t)$'s nor that of the Q_t's depend on the initial state or on the sequence of the $\hat{M}(t)$'s, and the computation of this latter is absolutely trivial. Thus once these two sequences are computed (a computation that can be carried out off-line) we can obtain the solutions for all feasible problem corresponding to any initial state and any bounds' sequence at marginal additional numeric effort, possibly even in real time.

4 Two Interesting Special Cases

There are some interesting special cases, in which the above theory greatly simplify. They arise comparing, at each time, S_t and D_t. Incidentally, these special cases play a crucial role in the theory developed in [7].

For simplicity we refer to stationary system and constraints, that is, $A(t)=A$, $B(t)=B$, $D_t=D$, $W(t)=W$ and $M(t)=M$. Thus, in the present case we can set for simplicity $t_i=0$. Again we let $x(0)=\bar{x}$.

If we suppose that $S_{T-1}\supset D$ then, clearly, E_{T-1} is D. Therefore S_{T-2} is equal to S_{T-1}, so that E_{T-2} is again D, because the system is time-invariant, and so on. Clearly, at the end of the recursion, we will find $S_0=S_{T-1}$. Thus the backward recursion radically simplifies, reducing to a single step.

In this case:

$$\hat{W}(t)=W \quad \hat{M}(t)=M \quad t=1,\ldots,T. \tag{4.1}$$

So that, by Theorem 2 it follows:

$$S_0=\{x: QWAx\le QM\} \tag{4.2}$$

where the rows of Q are the generators of $\mathcal{R}(WB)^\perp\cap P$.

Moreover, the problem is feasible if and only if

$$QWA\bar{x}\le QM. \tag{4.3}$$

Finally, the closed loop controller becomes

$$WBu(t) \leq M - WAx(t) \quad 0 \leq t < T. \tag{4.4}$$

Before passing on to the second particular case in point, it is useful to state the following general remark.

The behavior of the sequence $\{E_t : t = T-1, \ldots, 0\}$ is monotone with respect to the constraining set sequence $\{D_t : t = 1, \ldots, T\}$. In fact, if we relax the constraints, the terms of the corresponding new sequence $\{E'_t : t = T-1, \ldots, 0\}$ contain those of the original sequence, and, conversely, if we strengthen the constraints, they are contained.

Consider now the case in which the reverse of the preceding inclusion occurs, that is, $S_{T-1} \subset D$. Bearing in mind the above monotonicity principle and the stationarity of the problem, it is clear that $S_{T-2} \subset D$. By recursion $S_t \subset D$. Thus we can solve the problem removing all the state constraints except that at the final instant T.

The formulas in Theorem 2 remain unchanged except (3.3) and (3.4). The new simplified expressions of the coefficients $\hat{W}(t)$ and $\hat{M}(t)$ are given by

$$\hat{W}(t) = Q\hat{W}(t+1)A \tag{4.4}$$

$$\hat{M}(t) = Q\hat{M}(t+1) \tag{4.5}$$

with initial conditions $\hat{W}(T) = W$ and $\hat{M}(T) = M$, and where:

$$\{rows\ of\ Q_{t+1}\} = \{generators\ of\ \mathcal{R}(\hat{W}(t+1)B)^{\perp} \cap P\}$$

Once it is known that no constraint but the last is active, the trivial non-recursive solution simplifies too. For example, compute the set S_0, using again the conical machinery. We can express the state $x(T)$ in the form:

$$x(T) = \mathcal{C}(T)h(T) + A^T\bar{x} \tag{4.6}$$

where:

$$\mathcal{C}(T) = (A^{T-1}B \ldots AB\ B) \tag{4.7}$$

$$h(T) = col(u(0), \ldots, u(T-1)). \tag{4.8}$$

Substituting in the given constraint at time T we obtain the polyhedron of admissible input functions relative to the initial state $\bar{x}$:

$$W\mathcal{C}(T)h(T) \leq M - WA^T\bar{x}. \tag{4.9}$$

As usual, the dualization of this inequality yields the inequality defining the polyhedron of the admissible initial states:

$$S_0 = \{x : \mathbf{Q}(T)WA^Tx \leq \mathbf{Q}(T)M\} \tag{4.10}$$

where:

$$\{rows\ of\ \mathbf{Q}(T)\} = \{generators\ of\ \mathcal{R}(W\mathcal{C}(T))^{\perp} \cap P\}.$$

Noticing that $\mathcal{R}(W\mathcal{C}(T)) = W\mathcal{R}(\mathcal{C}(T))$ and that for $T \geq n$ $\mathcal{R}(\mathcal{C}(T+1)) = \mathcal{R}(\mathcal{C}(T)) = \mathcal{R}(\mathcal{C}(n))$, it follows that for any $T \geq n$ the polyhedron S_0 is given by

$$S_0 = \{x : \mathbf{Q}(n)WA^T x \leq \mathbf{Q}(n)M\}. \tag{4.11}$$

5 The Optimality Problem

In this section we extend the results developed for the feasibility problem to the optimality problem. To this purpose we begin incorporating the functional in the dynamics of the system. Then we will express the optimality problem as a parameterized feasibility problem. In this way the application of feasibility theory will be straightforward.

To carry out the first step of this program, we reformulate the problem introducing a new state variable v:

$$v(t) = \sum_{j=t_i}^{t-1} (f_j, x(j)) \tag{5.1}$$

for which the following dynamic equation can be given:

$$v(t+1) = v(t) + (f_t, x(t)) \quad v(t_i) = 0 \quad f_{t_i} = (0\,0\ldots 0)^*.$$

The new state vector is defined by:

$$\tilde{x}(t) = \begin{pmatrix} x(t) \\ v(t) \end{pmatrix}. \tag{5.2}$$

The dynamics of the augmented state system is described by:

$$\tilde{x}(t+1) = \tilde{A}(t)\tilde{x}(t) + \tilde{B}(t)u(t) \tag{5.3}$$

where:

$$\tilde{A}(t) = \begin{pmatrix} A(t) & 0 \\ f_t & 1 \end{pmatrix} \quad \tilde{B}(t) = \begin{pmatrix} B(t) \\ 0 \end{pmatrix}$$

with $\tilde{x}(t) \in R^{n+1}$ and $u(t) \in R^p$. The initial state is now given by $\tilde{x}(t_i) = \begin{pmatrix} \bar{x} \\ 0 \end{pmatrix}$.

With these positions we can state an equivalent optimization problem as follows:

$$\max(f, \tilde{x}(T)) \quad f = (f_T \;\; 1) \tag{5.4}$$

subject to:

$$\tilde{x}(t+1) = \tilde{A}(t)\tilde{x}(t) + \tilde{B}(t)u(t) \quad t_i \leq t < T$$
$$\bar{W}(t)\tilde{x}(t) \leq M(t) \quad t_i < t \leq T$$

where $\bar{W}(t) = (W(t)\ 0),\ t_i < t \leq T$.

Finally, and according to a usual technique, the optimization problem is transformed into a parameterized feasibility problem, introducing, in lieu of the functional, the additional constraint:

$$(-f, \tilde{x}(T)) \leq -h \tag{5.5}$$

which depends on the scalar parameter h. In this way, solving the optimization problem becomes equivalent to finding the maximum h such that the resulting feasibility problem is feasible. Before giving a detailed formulation let's state the final form of such feasibility problem, incorporating the functional into the constraints. Consider the constrained dynamical system:

$$\tilde{x}(t+1) = \tilde{A}(t)\tilde{x}(t) + \tilde{B}(t)u(t) \tag{5.6}$$

subject to:

$$\tilde{W}(t)\tilde{x}(t) \leq \tilde{M}(t) \quad t_i < t \leq T \tag{5.7}$$
$$\tilde{x}(t_i) = \begin{pmatrix} \bar{x} \\ 0 \end{pmatrix}$$

with:

$$\tilde{W}(t) = (W(t) \;\; 0) \quad t_i < t < T$$
$$\tilde{W}(T) = \begin{pmatrix} W(T) & 0 \\ -f & 0 \end{pmatrix}$$
$$\tilde{M}(t) = M(t) \quad t_i < t < T$$
$$\tilde{M}(T)(h) = \begin{pmatrix} M(T) \\ -h \end{pmatrix}.$$

The polyhedra defined by (5.7) will be denoted by $\tilde{D}_t$. Formally we have obtained a problem which has the same structure as those examined in the preceding Section 3, except that the final bound vector depends on the scalar parameter h. Thus, at any t, the polyhedron of admissible states at t depends on h, and will be denoted by $\tilde{S}_t(h)$. Similarly $\tilde{E}_t(h) = \tilde{S}_t(h) \cap \tilde{D}_t$. To solve the optimization problem, we have to find an $\bar{h}$ such that

$$if\ h\ \leq \bar{h}\ then\ \tilde{x}(t_i) \in \tilde{S}_{t_i}(h)$$

$$if\ h > h\ then\ \tilde{x}(t_i) \notin \tilde{S}_{t_i}(h).$$

Such an $\bar{h}$, if it exists, is the maximum of the problem, which therefore would be feasible bounded. The controller computed in correspondence to the maximum $\bar{h}$, according to the theory developed in the preceding sections, is the optimal closed loop controller. Clearly such controller exists if and only if $\bar{h}$ exists. If, for any h, $\tilde{x}(t_i) \in \tilde{S}_{t_i}(h)$ then the problem is feasible unbounded. In this case we can find feasible but not optimal solutions. To this purpose it suffices to fix an arbitrary value of h and compute the solution with the already developed formulas.

Finally, if $\tilde{x}(t_i) \notin \tilde{S}_{t_i}(h)$ for any h, then the problem is infeasible.

Dealing with this problem with the conical method we are lead to solving linear programming problems of the following simple structure:

$$\max h, \quad subject\ to\ Fh \leq K$$

where F and K are two column vectors and F is non negative. It is straightforward to demonstrate ([4] and [11]) the following

Lemma 1 *Define the set:*

$$J = \{i : F_i = 0\ and\ K_i < 0\}$$

then there are only three mutually exclusive and exhaustive cases:

- *a) $J \neq \emptyset$. In this case problem O is not feasible.*
- *b) $J = \emptyset$ and $\forall i,\ F_i = 0$. In this case problem O is feasible and unbounded.*
- *c) $J = \emptyset$ and $\exists i :\ F_i > 0$. In this case problem O is feasible bounded. The maximum $\bar{h}$ of the functional is given by:*

$$\bar{h} = \min\left\{\frac{K_i}{F_i} : F_i > 0\right\}$$

At this point we can state the following theorem:

Theorem 3 *(i) Considering the constrained system (5.6), (5.7), the sets $\tilde{S}_{t_i}(h)$, $t_i \leq t < T$, have the following expression:*

$$\tilde{S}_{t_i}(h) = \{x : \mathcal{Q}_{t+1}\mathcal{W}(t+1)\tilde{A}(t)\tilde{x} \leq \mathcal{Q}_{t+1}\mathcal{M}(t+1)(h)\} \tag{5.8}$$

where

$$\mathcal{W}(t) = \begin{pmatrix} \mathcal{Q}_{t+1}\mathcal{W}(t+1)A(t) \\ \tilde{W}(t) \end{pmatrix} \tag{5.9}$$

$$\mathcal{M}(t)(h) = \begin{pmatrix} \mathcal{Q}_{t+1}\mathcal{M}(t+1)(h) \\ \tilde{M}(t) \end{pmatrix} \tag{5.10}$$

with terminal conditions $\mathcal{W}(T) = \tilde{W}(T)$ and $\mathcal{M}(T) = \tilde{M}(T)$ and where:

$$\{rows\ of\ \mathcal{Q}_{t+1}\} = \{generators\ of\ \mathcal{R}(\mathcal{W}(t+1)\tilde{B}(t))^{\perp} \cap P\}.$$

(ii) To solve problem O consider the following optimization problem O_e:

$$\max h \quad h \in R \tag{5.11}$$

$$subject\ to\ F(T)h \leq K(T)$$

where

$$F(T) = \mathcal{Q}'_{t_i+1}\mathcal{Q}'_{t_i+2}\ldots\mathcal{Q}'_{T-1}\mathcal{Q}'_T$$

$$K(T) = \sum_{j=1}^{T-t_i} \mathcal{Z}(t_i+j)M(t_i+j) - \mathcal{Q}_{t_i+1}\mathcal{W}(t_i+1)\tilde{A}(t_i)\tilde{x}(t_i)$$

$$\mathcal{Z}(t_i+1) = \mathcal{Q}''_{t_i+1}$$
$$\mathcal{Z}(t_i+j) = \mathcal{Q}'_{t_i+1}\mathcal{Q}'_{t_i+2}\ldots\mathcal{Q}'_{t_i+j-1}\mathcal{Q}''_{t_i+j} \quad 1 < j < T - t_i$$
$$\mathcal{Z}(T) = \mathcal{Q}'_{t_i+1}\mathcal{Q}'_{t_i+2}\ldots\mathcal{Q}'_T$$

and the partitions $\mathcal{Q}_t = [\mathcal{Q}'_t|\mathcal{Q}''_t]$ are defined by the block structures of $\mathcal{M}(t)(h)$ on the row side.

Then the problems O and O_e are equivalent in the sense that O is infeasible or feasible unbounded or feasible bounded if and only if the same is true for O_e, a circumstance that can be verified by means of the lemma. In this latter case the maximum is the same for both problems and hence it is given by:

$$\bar{h} = \min\left\{\frac{K(T)_i}{F(T)_i} : F(T)_i > 0\right\}. \tag{5.12}$$

(iii) An optimal feedback controller exists if and only the problem O is feasible bounded, and it is defined by:

$$\mathcal{W}(t+1)\tilde{B}(t)u^o(t) \leq \mathcal{M}(t+1)(\bar{h}) - \mathcal{W}(t+1)\tilde{A}(t)x(t) \quad t_i \leq t < T \tag{5.13}$$

These inequalities in fact, coupled with the state equation and solved recursively forward in time, define the set of all the optimal control functions u^o and corresponding optimal state functions.

Proof: (i): apply statement (i) of Theorem 1 to the system (5.6 - 5.7). (ii): the set $\tilde{S}_{t_i}(h)$ has the expression

$$\{x : \mathcal{Q}_{t_i+1}\mathcal{W}(t+1)\tilde{A}(t_i)x \leq \mathcal{Q}_{t_i+1}\mathcal{M}(t_i+1)(h)\}.$$

Let us examine more in detail the structure of $\mathcal{M}(t_i+1)(h)$. From (5.9) we have

$$\mathcal{M}(t_i+1)(h) = \begin{pmatrix} \mathcal{Q}_{t_i+2}\mathcal{M}(t_i+2)(h) \\ \tilde{M}(t_i+1) \end{pmatrix}$$

$$\mathcal{M}(t_i+2)(h) = \begin{pmatrix} \mathcal{Q}_{t_i+3}\mathcal{M}(t_i+3)(h) \\ \tilde{M}(t_i+2) \end{pmatrix}$$

$$\mathcal{M}(t_i+3)(h) = \begin{pmatrix} \mathcal{Q}_{t_i+4}\mathcal{M}(t_i+4)(h) \\ \tilde{M}(t_i+3) \end{pmatrix}$$

and so on, hence

$$\mathcal{M}(t_i+1)(h) = \begin{pmatrix} \mathcal{M}_1(t_i+1)(h) \\ \tilde{M}(t_i+1) \end{pmatrix}$$

where $\mathcal{M}_1(t_i+1)(h) = \mathcal{Q}'_{t_i+2}\mathcal{Q}_{t_i+3}\mathcal{M}(t_i+3)(h) + \mathcal{Q}''_{t_i+2}\tilde{M}(t_i+2) = \mathcal{Q}'_{t_i+2}\mathcal{Q}'_{t_i+3}\mathcal{Q}_{t_i+4}\mathcal{M}(t_i+4)(h) + \mathcal{Q}'_{t_i+2}\mathcal{Q}''_{t_i+3}\tilde{M}(t_i+3) + \mathcal{Q}''_{t_i+2}\tilde{M}(t_i+2)$.

By finite recursion (and bearing in mind that $\mathcal{M}(T) = \tilde{M}(T)$), we obtain

$$\begin{aligned} &\mathcal{M}_1(t_i+1)(h) = \\ &\quad \mathcal{Q}'_{t_i+2}\mathcal{Q}'_{t_i+3}\ldots\mathcal{Q}_T\tilde{M}(T)(h) + \mathcal{Q}'_{t_i+2}\ldots\mathcal{Q}'_{T-2}\mathcal{Q}''_{T-1}\tilde{M}(T-1) \\ &\quad + \ldots + \mathcal{Q}''_{t_i+2}\tilde{M}(t_i+2). \end{aligned}$$

Substituting for $\tilde{M}(T) = \begin{pmatrix} M(T) \\ -h \end{pmatrix}$ we arrive at the following expression: $\mathcal{M}_1(t_i+1)(h) = \mathcal{Q}'_{t_i+2}\mathcal{Q}'_{t_i+3}\ldots\mathcal{Q}'_T M(T) - \mathcal{Q}'_{t_i+2}\mathcal{Q}'_{t_i+3}\ldots\mathcal{Q}''_T h + \mathcal{Q}'_{t_i+2}\ldots\mathcal{Q}'_{T-2}\mathcal{Q}''_{T-1}\tilde{M}(T-1) + \ldots + \mathcal{Q}''_{t_i+2}\tilde{M}(t_i+2)$.

Finally, partitioning the matrix $\mathcal{Q}_{t+1}$ and bearing in mind $\tilde{M}(t) = M(t)$, $t_i < t < T$, we can write for the bound appearing in the expression (5.13):

$$\mathcal{Q}_{t_i+1}\mathcal{M}(t_i+1)(h) = -(\mathcal{Q}'_{t_i+1}\mathcal{Q}'_{t_i+2}\ldots\mathcal{Q}''_T)h + \sum_{j=1}^{T-t_i} \mathcal{Z}(t_i+j)M(t_i+j)$$

with $\mathcal{Z}(t_i+j)$ as in (5.10).

Thus $\tilde{x}(t_i) \in \tilde{S}_{t_i}(h)$ if and only if:

$$\mathcal{Q}_{t_i+1}\mathcal{W}(t+1)\tilde{A}(t_i)\tilde{x}(t_i) \leq -(\mathcal{Q}'_{t_i+1}\mathcal{Q}'_{t_i+2}\ldots\mathcal{Q}''_T)h + \sum_{j=1}^{T-t_i} \mathcal{Z}(t_i+j)M(t_i+j)$$

or, equivalently, if and only if

$$F(T)h \leq K(T)$$

with $F(T)$ and $K(T)$ defined as in (5.10).

(iii): Apply statement (iii) of Theorem 1.$\square$

From a computational point of view the sequences $\{\mathcal{Q}_t\}$ and $\{\mathcal{W}_t\}$ can be computed once and forever. This requires a good deal of numerical effort, but the computation can be carried out off line. Then for each determination of the initial state and of the bound sequence, it is immediate to solve the optimization problem O_e and obtain the maximum $\bar{h}$. At this point it is easy to implement the automatic feedback optimal controller (which works in a manner analogous to that illustrated for the feasibility case) to compute recursively any optimal solution.

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