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Receding Horizon Control for the Stabilization of Nonlinear Uncertain Systems Described by Differential Inclusions*

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Abstract

In this paper the stabilization problem of nonlinear uncertain systems described by differential inclusions is considered. The nominal system is assumed to be affine in the control variable. For the stabilization of the nominal system, a variant of the receding horizon control method is proposed which is based on the solution of a certain Bolza problem. The results are illustrated by an example.

Key words: feedback stabilization; nonlinear systems; uncertain systems; receding horizon control

1 Introduction

In the past one and a half decades, a great deal of interest has been devoted to the design of stabilizing controllers for uncertain deterministic systems; see [2,5-7,9-10] and the references therein. In this context, the uncertainty is meant in a deterministic sense: it arises as a result of approximation, imprecision, or imperfect knowledge introduced during the modelling procedure. Moreover, realistic processes are frequently subject to extraneous disturbances with unknown statistical characterization but with known structural properties and with known bounds. They lead also to some uncertainty in the model of the system. In the present paper, similarly to [9], the uncertain system is defined by a differential inclusion, the right-hand side of which is a known multifunction $(t, x, u) \mapsto \mathcal{F}(t, x, u)$,

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but in contrast to [9], the nominal system (i.e., the system in the absence of uncertainties) is not assumed to be linear; instead, it is allowed to be nonlinear and affine in the control. The literature contains two main approaches for the stabilization of control-affine nonlinear uncertain systems, the min-max controller discussed by Gutman and Palmor [10] and the Corless-Leitmann approach [7]. Both of these approaches assume that the nominal free system (i.e., the system with $u(t) \equiv 0$) is uniformly asymptotically stable or, at least practically stable, with known Lyapunov function V the gradient of which is used in constructing the stabilizing feedback. There exist relatively few methods for designing stabilizing feedback controllers for nonlinear systems. This paper proposes a variant of the receding horizon control method which may serve as a starting point for the design of the controller of uncertain systems.

The notion of receding horizon control is not new; it goes back to an early publication of Kleinman [12]. The method has been since revisited by other authors, e.g. [14], [13], [18] for linear time-varying systems. Recently, Mayne and Michalska [15], [16] have established the stability properties of nonlinear systems with receding horizon control. In these works, a fixed-time Lagrange-type optimal control problem is solved at every instant of time under the terminal constraint $x(t_1) = 0$. This constraint can be interpreted in the terminology of [13] as the case of infinite final weighting. The present paper applies a finite but not zero final weighting solving a Bolza problem at every instant of time for finding a stabilizing controller. This approach can be considered as the generalization for nonlinear systems of the method proposed in [13]. Our assumptions, similarly to that of the work [15], are necessarily very strong, since they ensure the continuous differentiability of the value function. At the same time, these conditions—except Assumption A3—can be verified relatively simply in advance.

Standard notation is adapted. In particular, the Euclidean scalar product and the induced norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. The transpose of a vector or a matrix X is denoted by X'. $\mathcal{B}_Y$ denotes the closed unit ball of the space Y. For a compact set $K \neq \emptyset$, $\xi(K) := \max\{\|v\| : v \in K\}$ and $\xi(\emptyset) = 0$. Finally, for $x \in \mathbf{R}^n$ and $S \subset \mathbf{R}^n$, $\ll x, S \gg := \{\langle x, s\rangle : s \in S\} \subset \mathbf{R}$.

2 Problem and Assumptions

Consider a system described by

$$\dot{x}(t) \in \mathcal{F}(t, x(t), u(t)) \tag{2.1}$$

where $\mathcal{F} : \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^m \rightsquigarrow \mathbf{R}^n$ is a known multifunction with nonempty values. For a given control function $u : I \to \mathbf{R}^m$, with $I \subset \mathbf{R}$, $x : I \to \mathbf{R}^n$

is a trajectory of (2.1) if it is absolutely continuous and satisfies (2.1) almost everywhere on I. The fundamental problem to be studied is that of stabilization (2.1) by feedback. To make the problem statement more precise, we have to impose certain assumptions concerning the multifunction $\mathcal{F}$ and the admissible class of feedbacks. We will require the following assumptions.

A1. There exist (known) functions $f : \mathbf{R}^n \to \mathbf{R}^n$, $B : \mathbf{R}^n \to \mathbf{R}^{n\times m}$ and a multifunction $\mathcal{G} : \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^m \rightsquigarrow \mathbf{R}^n$, $\mathcal{G}(t,x,u) = \mathcal{F}(t,x,u) - f(x) - B(x)u$ with the following properties:

- A1.1. f and B are locally Lipschitz continuous, differentiable functions and there exists a constant $M > 0$ such that $\|f(x)\| + \|B(x)\| \le M(1 + \|x\|)$. Moreover, $f(0) = 0$.
- A1.2. There exist an upper semicontinuous multifunction $\mathcal{G}_m : \mathbf{R} \times \mathbf{R}^n \rightsquigarrow \mathbf{R}^m$ with nonempty convex and compact values and a continuous function $\beta : \mathbf{R} \to [0, \kappa_0]$, $\kappa_0 < 1$, such that $\mathcal{G}(t,x,u) = B(x)[\mathcal{G}_m(t,x) + \beta(t)\mathcal{G}_c(u)]$ for all (t,x,u), where $\mathcal{G}_c : \mathbf{R}^m \rightsquigarrow \mathbf{R}^m$, $\mathcal{G}_c(u) = \|u\|\mathcal{B}_{\mathbf{R}^m}$.

Remark System $\dot{x}(t) = f(x(t)) + B(x(t))u(t)$ is called the *nominal system*. The uncertainty of the system is represented by the multifunction $\mathcal{G}$. According to A1.2, it is assumed to belong to the range of the input matrix. Such uncertainties are called to be *matched* (see e.g. [7]). Now, a feedback strategy $(t,x) \to \mathcal{H}(t,x)$ has to be determined so that by substitution of $u(t) = \mathcal{H}(t, x(t))$ into (2.1), the resulted closed loop system

$$\dot{x}(t) \in \mathcal{F}_{\mathcal{H}}(t, x(t)) \,, \tag{2.2}$$

$$\mathcal{F}_{\mathcal{H}}(t,x) = f(x) + B(x)\mathcal{H}(t,x) + B(x)\left[\mathcal{G}_m(t,x) + \beta(t)(\mathcal{G}_c \circ \mathcal{H})(t,x)\right] \tag{2.3}$$

is globally uniformly asymptotically stable about the origin (for a definition, see e.g. [9]).

To assure the existence of solutions of (2.2) at least on small intervals, the feedback $\mathcal{H}$ is assumed to be an upper semicontinuous multifunction $\mathcal{H} : \mathbf{R} \times \mathbf{R}^n \rightsquigarrow \mathbf{R}^m$ with nonempty, convex, compact values. In keeping with the usage of Goodall and Ryan [9], such multifunctions will be referred to as *generalized feedbacks*. The existence of local solutions can be stated on the basis of the following proposition.

Proposition 1 [1, p.98] *If $\mathcal{F}_{\mathcal{H}}$ in (2.2) is upper semicontinuous with nonempty, convex, compact images, then, for each (t_0, x_0), there exists a local solution $x : [t_0, \tau) \to \mathbf{R}^n$ with $x(t_0) = x_0$.*

Since the sum and the composition of upper semicontinuous multifunctions is again upper semicontinuous (see e.g. [1, p 41]), and the same is true if an upper semicontinuous multifunction is multiplied by a continuous single-valued function, the existence of local solutions follows from proposition 1, if $\mathcal{H}$ is upper semicontinuous and chosen so that $\mathcal{F}_{\mathcal{H}}$ is convex, compact valued.

3 Receding Horizon Control for the Stabilization of the Nominal System

Consider the nominal system

$$\dot{x}(t) = f(x(t)) + B(x(t))u(t) \tag{3.1}$$

and let it be subject to the cost function

$$J(t_0, t_1, u, x_0) = g(x^u(t_1, x_0, t_0)) + \int_{t_0}^{t_1} [Q(x^u(t; x_0, t_0)) + R(u(t)))] \, \mathrm{d}t, \tag{3.2}$$

where $x^u(\cdot, x_0, t_0)$ denotes the solution of (3.1) due to control u and with initial condition $x^u(t_0; x_0, t_0) = x_0$. The receding horizon control is defined as follows. Fix a time length $T > 0$ and consider the optimization problem

$$P(t,x) : \inf_{u \in \mathbf{L}_\infty^m[t,t+T]} J(t, t+T, u, x) \, ,$$

where $\mathbf{L}_\infty^m[t, t+T]$ denotes the class of all Lebesgue measurable and essentially bounded functions $v : [a,b] \to \mathbf{R}^m$.

Let the minimizing solution to problem $P(t,x)$ be denoted by $\hat{u}(\cdot; x, t)$ and the corresponding value of the cost function by $V(t,x)$, i.e.

$$V(t,x) = J(t, t+T, \hat{u}(\cdot; x, t), x) \, .$$

To determine the receding horizon control at every instant of time t, we shall solve the problem $P(t, x(t))$ and we shall apply the control $u(t) = \hat{u}(t; x(t), t)$.

We observe that because of the time invariance of (3.1) and (3.2),

$$V(t,x) = V(0,x) \quad \text{and} \quad \hat{u}(\tau; x, t) = \hat{u}(\tau - t; x, 0)$$

for all t and all $\tau \geq t$; therefore it is enough to consider problem $P(0,x)$. If problem $P(0,x)$ is solvable for any x, it is possible to define the mapping $x \mapsto \hat{u}(0; x, 0)$. The receding horizon feedback $h : \mathbf{R}^n \to \mathbf{R}^m$ is defined then by

$$h(x) = \hat{u}(0; x, 0) \quad \text{for all} \quad x \in \mathbf{R}^n \, .$$

As a result, we obtain the closed-loop nominal system

$$\dot{x}(t) = f(x(t)) + B(x(t))h(x(t)). \tag{3.3}$$

The aim of this section is to show that, under some additional conditions for problem $P(0,x)$, system (3.3) is globally asymptotically stable.

Remark In [15], [16], another version of receding horizon control is investigated: function g in (3.2) is taken to be identically zero and the final state $x(T;x,0)$ is restricted to be zero.

The Hamilton function belonging to problem $P(0,x)$ is defined by

$$H(x,p) = \sup_u\{\langle p, f(x)\rangle - Q(x) + \langle p, B(x)u\rangle - R(u)\}. \qquad (3.4)$$

Concerning problem $P(0,x)$, the following assumptions will be imposed.

A2.1. Q is differentiable and for any $r > 0$ there is a constant $L_{Q,r}$ such that $|Q(x_1) - Q(x_2)| \leq L_{Q,r}\|x_1 - x_2\|$ for all $x_1, x_2 \in r\mathcal{B}_{\mathbf{R}^n}$. Moreover, $Q(0) = 0$ and $Q(x) > 0$ if $x \neq 0$.

A2.2. Function R is convex, and there exists a constant $c > 0$ such that $R(u) \geq c\|u\|^2$. Moreover $\bigtriangledown R$ exists and it is a diffeomorphism from $\mathbf{R}^m$ to itself.

A2.3. $g \in C^1(\mathbf{R}^n)$, $g(0) = 0$, $g(x) > 0$ if $x \neq 0$ and $\lim_{\|x\|\to\infty} g(x) = \infty$.

A2.4. g satisfies the inequality $H(x, -\bigtriangledown g(x)) \geq 0$ for all $x \in \mathbf{R}^n$, where H is given by (3.4).

Remark 1 Under the Assumption A2.2, there exists a unique u for which the supremum is achieved, namely,

$$u = u(x,p) = (\bigtriangledown R)^{-1}(B(x)'p)\,.$$

Remark 2 Assumption A2.4 is very restrictive: on the one hand, it implies that a continuous stabilizing controller may directly be constructed by means of function g, and it is known that many nonlinear systems cannot be stabilized with continuous feedback. On the other hand, it is not always easy to determine a function satisfying A2.4. Moreover, the controller proposed here needs an additional optimization process. Nevertheless, it may be useful, among the others, when the magnitude of the control is important.

The value function for problem $P(0,x)$ can be defined as $V^0 : [0,T] \times \mathbf{R}^n \to \mathbf{R}$,

$$V^0(s,x) = \inf J(s,T,u,x)\,, \quad (s,x) \in [0,T] \times \mathbf{R}^n\,.$$

Clearly, $V^0(0,x) = V(0,x)$.

Let us now make the following additional assumption.

A3. For all $(t_0, x_0) \in [0,T] \times \mathbf{R}^n$, there exists a unique $\overline{x}(\cdot)$ trajectory of (3.1) for which

$$V^0(t_0, x_0) = g(\overline{x}^{\overline{u}}(T; x_0, t_0)) + \int_{t_0}^{T} \left[Q(\overline{x}^{\overline{u}}(s; x_0, t_0)) + R(\overline{u}(s)) \right] \mathrm{d}s \, ,$$

where $\overline{u}(\cdot)$ is a corresponding control.

In [3] it has been shown, that under the Assumptions A1 and A2.1-A2.3, the statement of A3 is equivalent to the continuous differentiability of V^0 on $[0,T] \times \mathbf{R}^n$. It is well-known (see e.g. [17]), that at every point (s,x), where V^0 is differentiable, it satisfies the Hamilton-Jacobi-Bellman equation:

$$-\frac{\partial V^0}{\partial s}(s,x) + H(x, -\frac{\partial V^0}{\partial x}(s,x)) = 0 \, , \tag{3.5}$$

with the final condition

$$V^0(T,x) = g(x) \, . \tag{3.6}$$

The optimal solution to problem $P(0, x(t))$ can be given by

$$\hat{u}(s; x(t), 0) = (\bigtriangledown R)^{-1} \left(-B(x^{\hat{u}}(s; x(t), 0)) \frac{\partial V^0}{\partial x} \left(s, x^{\hat{u}}(s; x(t), 0) \right) \right) . \tag{3.7}$$

Since all of the functions on the right-hand side of (3.7) are continuous in s, $\hat{u}(\cdot; x(t), 0)$ is continuous.

The Hamilton-Jacobi-Bellman equation (3.5)-(3.6) can be investigated by the method of characteristics, i.e., by means of the following system of ordinary differential equations

$$\frac{dx}{dt} = \frac{\partial H}{\partial p}(x,p) \quad , \; x(T) = x^T \, , \tag{3.8}$$

$$-\frac{dp}{dt} = \frac{\partial H}{\partial x}(x,p) \quad , \; p(T) = -\bigtriangledown g(x^T) \, , \tag{3.9}$$

$$\frac{dp_{n+1}}{dt} = 0 \quad , \; p_{n+1}(T) = -H(x^T, -\bigtriangledown g(x^T)) \, , \tag{3.10}$$

$$\frac{dV}{dt} = \langle -p, \frac{\partial H}{\partial p}(x,p) \rangle - p_{n+1} \; , \; V(T) = g(x^T) \, . \tag{3.11}$$

Lemma 1 *Suppose that the Assumptions A1–A3 are valid, $\bigtriangledown H$ is locally Lipschitz continuous and system (3.8)-(3.9) is complete. Then*

$$\frac{\partial V^0}{\partial s}(0,x) \geq 0 \quad \textit{for all} \quad x \in \mathbf{R}^n \, . \tag{3.12}$$

Proof: Similarly to [3], we can introduce the set

$$M_t = \{(x(t), p(t)) : \exists x^T \in \mathbf{R}^n \quad \text{such that} \quad (x(\cdot), p(\cdot)) \quad \text{is} \qquad (3.13)$$
$$\text{the solution of} \ \ (3.8) - (3.9)\}.$$

In [3] it has been proved that, under the Assumption A1, A2.1–A2.3, Assumption A3 is equivalent to the existence of a mapping $\Pi(t, \cdot) : \mathbf{R}^n \to \mathbf{R}^n$ such that $M_t = \text{graph}\,(-\Pi(t, \cdot))$. This means that the mapping $x^T \mapsto x(t)$, where $(x(\cdot), p(\cdot))$ is a solution of (3.8)-(3.9) is one to one. Thus, using the method of characteristics, for any x, an x^T can be shown such that

$$p_{n+1}(0) = -\frac{\partial V^0}{\partial t}(0, x) = -H(x^T, -\bigtriangledown g(x^T)).$$

Then (3.12) follows by Assumption A2.4.

Let us consider now the function

$$\hat{V}(x) = V^0(0, x) = V(0, x)\,. \qquad (3.14)$$

We want to show that $\hat{V}$ can be used as a Lyapunov function to system (3.3).

Lemma 2 *Suppose that the Assumptions A1–A3 are valid, $\bigtriangledown H$ is locally Lipschitz continuous and system (3.8)-(3.9) is complete. Then $\hat{V}(0) = 0$, $\hat{V}(x) > 0$ if $x \neq 0$ and*

$$\frac{d}{dt}\hat{V}(\hat{x}(t)) \leq -[Q(\hat{x}(t)) + R(h(\hat{x}(t)))]\,, \qquad (3.15)$$

where $\hat{x}(\cdot)$ is a trajectory of system (3.3).

Proof: Because of Assumptions A2, $J(0, T, x, u) \geq 0$; therefore $\hat{V}(x) \geq 0$. If $x = 0$, then $J(0, T, 0, 0) = 0$, thus $\hat{V}(0) = 0$. If $x \neq 0$, then there is an interval $[0, \tau)$ of positive length such that $x^{\hat{u}}(s; x, 0) \neq 0$ for $s \in [0, \tau)$. Since $V(0, x) = J(0, T, x, \hat{u}) \geq \int_0^\tau Q(x^{\hat{u}}(s; x, 0))ds$, $\hat{V}(x) > 0$ follows immediately from A2.1.

Let us estimate now the time derivative of $\hat{V}$ along the trajectories of system (3.3). To be short, we introduce the notation

$$F(x) := f(x) + B(x)h(x). \qquad (3.16)$$

Since $\hat{V}$ is continuously differentiable, it is locally Lipschitz. Therefore,

$$\begin{aligned}\frac{d}{dt}\hat{V}(\hat{x}(t)) &= \lim_{\tau \to 0+} \frac{1}{\tau}\left[\hat{V}(\hat{x}(t) + \tau F(\hat{x}(t))) - \hat{V}(\hat{x}(t)))\right] = \\ &= \lim_{\tau \to 0+} \frac{1}{\tau}\left[V^0(\tau, x^{\hat{u}}(\tau; \hat{x}(t), 0)) - V^0(0, \hat{x}(t))\right] +\end{aligned}$$

$$
\begin{aligned}
&+ \quad \lim_{\tau\to 0_+} \frac{1}{\tau}\left[V^0(0,x^{\hat u}(\tau;\hat x(t),0)) - V^0(\tau,x^{\hat u}(\tau;\hat x(t),0))\right] + \\
&+ \quad \lim_{\tau\to 0} \frac{1}{\tau}\left[\hat V(\hat x(t)+\tau F(\hat x(t))) - V^0(0,x^{\hat u}(\tau;\hat x(t),0)\right] . \qquad (3.17)
\end{aligned}
$$

We know that V^0 is differentiable and it satisfies the Hamilton-Jacobi-Bellman equation (3.5)-(3.6); thus

$$
\begin{aligned}
&\lim_{\tau\to 0_+} \frac{1}{\tau}\left[V^0(\tau,x^{\hat u}(\tau;\hat x(t),0)) - V^0(0,\hat x(t))\right] = \\
&= \quad \frac{\partial V^0}{\partial s}(0,\hat x(t)) + \frac{\partial V^0}{\partial x}(0,\hat x(t)) \cdot \left.\frac{dx^{\hat u}(s;\hat x(t),0)}{ds}\right|_{s=0} = \\
&= \quad \frac{\partial V^0}{\partial s}(0,\hat x(t)) + \frac{\partial V^0}{\partial x}(0,\hat x(t))\left[f(\hat x(t)) + B(\hat x(t))\hat u(0;\hat x(t),0)\right] = \\
&= \quad -Q(\hat x(t)) - R(h(\hat x(t))) . \qquad (3.18)
\end{aligned}
$$

Using again the continuous differentiability of V^0, the mean value theorem gives

$$
\begin{aligned}
&\lim_{\tau\to 0_+} \frac{1}{\tau}\left[V^0(0,x^{\hat u}(\tau;\hat x(t),0)) - V^0(\tau,x^{\hat u}(\tau;\hat x(t),0))\right] = \\
&= \quad \lim_{\tau\to 0} -\frac{\partial V^0}{\partial s}(\theta\tau,x^{\hat u}(\tau;\hat x(t),0)) = -\frac{\partial V^0}{\partial s}(0,\hat x(t)) \le 0 , \qquad (3.19)
\end{aligned}
$$

where the last inequality follows from Lemma 1. Since V^0 is locally Lipschitz continuous, we have

$$
\begin{aligned}
&\lim_{\tau\to 0_+} \frac{1}{\tau}\left[\hat V(\hat x(t)+\tau F(\hat x(t))) - V^0(0,x^{\hat u}(\tau;\hat x(t),0)\right] \le \\
&\le \quad \lim_{\tau\to 0} \frac{L_{V^0}}{\tau}\|x^{\hat u}(\tau;\hat x(t),0) - \hat x(t) - \tau F(\hat x(t))\| \le \\
&\le \quad \lim_{\tau\to 0} \frac{L_{V^0}}{\tau}\int_0^\tau \|[f(x^{\hat u}(s;\hat x(t),0)) + B(x^{\hat u}(s;\hat x(t),0)\hat u(s;\hat x(t),0)] - \\
&\qquad -[f(\hat x(t)) + B(\hat x(t))\hat u(0;\hat x(t),0)]\|ds = 0 , \qquad (3.20)
\end{aligned}
$$

for the integrand is continuous in $s = 0$. From (3.17)-(3.20), (3.15) follows immediately.

Theorem 1 *Suppose that Assumptions A1–A3 are valid, ∇H is Lipschitz continuous and system (3.8)-(3.9) is complete. Then system (3.3) is locally asymptotically stable about the origin.*

Proof: We take as a Lyapunov function the optimal value function $\hat V$. The assertion follows by Lemma 2 and by Lyapunov's second theorem [8, p. 240].

To show global asymptotic stability we need an additional assumption on the rate of growth of the right hand side of (3.1).

A4. There exist constants $K > 0$ and $\rho > 0$ such that

$$\|f(x) + B(x)u\| \leq K(Q(x) + R(u)) \quad \text{for every} \quad (x, u) \notin \rho\mathcal{B}_{\mathbf{R}^n \times \mathbf{R}^m}.$$

Lemma 3 *Suppose that Assumptions A1-A4 are satisfied. Then $\hat{V}$ is radially unbounded, i.e. $\hat{V}(x) \to \infty$ if $\|x\| \to \infty$.*

Proof: By Assumption A3, for any x_0 there exists a unique optimal trajectory $\overline{x}(\cdot)$ for the problem $P(0; x_0)$ with the optimal control $\overline{u}$. Hence the following sets are well-defined:

$$\begin{aligned} N_1 &= \{x_0 \in \mathbf{R}^n : \|\overline{x}(T)\| \geq \frac{1}{2}\|x_0\|\} \\ N_2 &= \{x_0 \in \mathbf{R}^n : \|\overline{x}(T)\| < \frac{1}{2}\|x_0\|\} . \end{aligned}$$

If $x_0 \in N_2$, then

$$\frac{1}{2}\|x_0\| < \|x(T) - x_0\| = \| \int_0^T (f(\overline{x}(s)) + B(\overline{x}(s))\overline{u}(s))ds\| ;$$

therefore it can be proved in the same way as in [16] that

$$\hat{V}(x_0) \geq K^{-1}\left(\frac{1}{2}\|x_0\| - M\mu(I_{rB})\right) ,$$

where I_{rB} is the subset of $[0, T]$ such that $(\overline{x}(t), \overline{u}(t)) \in \rho\mathcal{B}_{\mathbf{R}^n \times \mathbf{R}^m}$ and $M = \max_{(x,u) \in \rho\mathcal{B}_{\mathbf{R}^n \times \mathbf{R}^m}} \|f(x) + B(x)u\|$.

Let us consider now the case when $x_0 \in N_1$. For this we introduce the functions $\alpha(\cdot)$, $\beta(\cdot)$ and the set valued maps Λ_1, Λ_2 by the following definitions

$$\begin{aligned} &\max_{\|x\| \leq \frac{1}{2}\|y\|} g(x) = \alpha(\|y\|) , \\ &\Lambda_1(r) = \{x \in \mathbf{R}^n : g(x) \leq \alpha(r)\} , \quad r > 0, \\ &\Lambda_2(r) = (\Lambda_1 \cup r\mathcal{B}_{\mathbf{R}^n}) \backslash \frac{1}{2} r\mathcal{B}^0_{\mathbf{R}^n} , \quad r > 0, \\ &\min_{x \in \Lambda_2(r)} g(x) = \beta(r) . \quad r > 0. \end{aligned}$$

Since g is continuous, α is well defined and $\Lambda_1(r) \supset \frac{1}{2} r\mathcal{B}_{\mathbf{R}^n}$. Furthermore, $g(x) \to \infty$ if $\|x\| \to \infty$; therefore $\Lambda_1(r)$ is a nonempty compact and the same is true also for $\Lambda_2(r)$. Thus β is also well defined. Clearly, $\alpha(r) \to \infty$

and $\beta(r) \to \infty$. Now, for $x_0 \in N_1$, two cases are possible. If $\overline{x}(T) \notin \Lambda_1(\|x_0\|)$, then $g(\overline{x}(T)) > \alpha(\|x_0\|)$. On the other hand, if $\overline{x}(T) \in \Lambda_1(\|x_0\|)$, then necessarily $\overline{x}(T) \in \Lambda_2(\|x_0\|)$; therefore $g(\overline{x}(T)) \geq \beta(\|x_0\|)$. Since $\hat{V}(x_0) \geq g(\overline{x}(T))$, for every $x^0 \in \mathbf{R}_1^n$, we obtained a lower bound for $\hat{V}$ tending to infinity with $\|x_0\| \to \infty$.

Theorem 2 *Suppose that Assumptions A1-A4 hold true, ∇H is locally Lipschitz continuous and system (3.8)-(3.9) is complete. Then system (3.3) is globally asymptotically stable about the origin.*

Proof: The proof is an immediate consequence of Lemma 2, Lemma 3 and the Barbashin-Krasowsky theorem [8, p. 248].

Remark The receding horizon control redefines the performance criterion at every instant of time; thus it does not optimize any overall criterion. Nevertheless, a cost function determined by Q and R can be accepted as a measure for the performance of the process. Under the conditions of Theorem 1, this cost has the following bounds:

$$\int_{\overline{t}}^{\infty} [Q(\hat{x}(t)) \, + \, R(h(\hat{x}(t)))] \, \mathrm{d}t \, \leq \, V(0, \hat{x}(\overline{t})) \, \leq \, g(\hat{x}(\overline{t}))\,.$$

In fact, it has been proved in Lemma 2 that

$$\frac{d}{dt}\hat{V}(\hat{x}(t)) \, \leq \, - \, [Q(\hat{x}(t)) \, + \, R(h(\hat{x}(t)))] \; .$$

Therefore,

$$\int_{\overline{t}}^{t_f} [Q(\hat{x}(t)) \, + \, R(h(\hat{x}(t)))] \, \mathrm{d}t \, \leq \, \hat{V}(\hat{x}(\overline{t})) \, - \, \hat{V}(\hat{x}(t_f))\,.$$

We know that $\hat{x}(t_f) \to 0$ and $\hat{V}(\hat{x}(t_f)) \to 0$ if $t_f \to \infty$; thus the integral on the left hand side of the above inequality is convergent and has the upper bound

$$\hat{V}(\hat{x}(\overline{t})) \, = \, \hat{V}(\hat{x}(t_f))\,.$$

On the other hand, the assertion of Lemma 1 is valid not only for $(0, x)$, $x \in \mathbf{R}^n$, but for any $(t, x) \in [0, T] \times \mathbf{R}^n$, i.e.,

$$\frac{\partial V^0}{\partial t}(t, x) \, \geq \, 0 \qquad \text{for all} \quad t \in [0, T], \, x \in \mathbf{R}^n \,.$$

It follows that, for any fixed $x \in \mathbf{R}^n$

$$V(0, x) \, = \, V^0(0, x) \, \leq \, V^0(T, x) \, = \, g(x)\,.$$

The estimation above generalizes the result of [13].

4 Generalized Feedback by Receding Horizon

In this part, we shall give the generalized feedback $(t,x) \mapsto \mathcal{H}(t,x)$ for which system (2.2)-(2.3) is globally uniformly asymptotically stable about the origin.

Consider the receding horizon feedback law $x \mapsto h(x)$ defined in the previous section and let $(t,x) \mapsto \rho(t,x)$ be any continuous function satisfying the inequality

$$\rho(t,x) \geq \rho_0(t,x) := \frac{1}{1-\beta(t)}\{\beta(t)\|h(x)\| + \xi(\mathcal{G}_m(t,x))\} , \tag{4.1}$$

where β and $\mathcal{G}_m$ are given in A1. Define the multifunction $\mathcal{D} : \mathbf{R}^m \rightsquigarrow \mathbf{R}^m$ by

$$\mathcal{D}(u) = \begin{cases} u/\|u\|, & \text{if } u \neq 0, \\ \mathcal{B}_{\mathbf{R}^m}, & \text{if } u = 0 \end{cases} \tag{4.2}$$

which is clearly upper semicontinuous. Let $\hat{V}$ be the receding horizon value function given by (3.14). We know that, under the Assumptions A1-A3, $\hat{V}$ is continuously differentiable. The proposed feedback can now be defined as

$$(t,x) \mapsto \mathcal{H}(t,x) := h(x) + \mathcal{K}(t,x) , \tag{4.3}$$

where

$$\mathcal{K}(t,x) := -\rho(t,x)\mathcal{D}(B'(x) \bigtriangledown \hat{V}(x)) . \tag{4.4}$$

As a result, (2.3) will take now the form

$$\mathcal{F}_{\mathcal{H}}(t,x) = F(x) + B(x)\mathcal{K}(t,x) + B(x)(\mathcal{G}_m(t,x) + \beta(t)(\mathcal{G}_c \circ \mathcal{H})(t,x)) , \tag{4.5}$$

where F is given by (3.16).

Being clearly upper semicontinuous, $(t,x) \mapsto \mathcal{H}(t,x)$ is a generalized feedback. Since for any (t,x), its value is either a singleton or a closed ball, it is a convex, compact valued multifunction. Using the definition of $\mathcal{G}_c$, we can see that $\beta(t)(\mathcal{G}_c \circ \mathcal{H})(t,x)$ is a closed ball (with radius $\beta(t)\xi(h(x) - \rho(t,x)\mathcal{B}_{\mathbf{R}^m})$ or $\beta(t)\|h(x) - \rho(t,x)B'(x) \bigtriangledown \hat{V}(x)/\|B'(x) \bigtriangledown \hat{V}(x)\|\|$, respectively). Thus the multifunction $\mathcal{F}_{\mathcal{H}}$ in (2.2), (4.5) is the sum of convex and compact valued multifunctions; hence it has itself these properties. The existence of local solutions for (2.2), (4.5) follows now by proposition 1.

Theorem 3 *Suppose that Assumptions A1-A4 are valid, $\bigtriangledown H$ is locally Lipschitz continuous and system (3.8)-(3.9) is complete. Then system (2.2)-(2.3) with (4.1)-(4.4) is globally asymptotically stable about the origin.*

Proof: We shall use again $\hat{V}$ as a Lyapunov function. Since it is continuously differentiable, we have to investigate

$$\frac{d}{dt}\hat{V}(x(t)) = (\bigtriangledown \hat{V}(x(t)))'\dot{x}(t)$$

along the solutions of (2.2), (4.5). We have

$$\begin{aligned}\frac{d}{dt}\hat{V}(x(t)) \in \ll \bigtriangledown\hat{V}(x(t)), F(x(t)) + B(x(t))\mathcal{K}(t,x(t)) + \\ +B(x(t))[\mathcal{G}_m(t,x(t)) + \beta(t)(\mathcal{G}_c\circ\mathcal{H})(t,x(t))] \gg \ .\end{aligned} \tag{4.6}$$

In the proof of Lemma 2, we have seen that

$$\langle\bigtriangledown\hat{V}(x(t)), F(x(t))\rangle \leq -[Q(x(t)) + R(x(t)))]\ . \tag{4.7}$$

By the definition of $\mathcal{K}$, we obtain that

$$\begin{aligned}&\ll B'(x)\bigtriangledown\hat{V}(x), \mathcal{K}(t,x) \gg = \\ &\begin{cases}\{-\rho(t,x)\|B'(x)\bigtriangledown\hat{V}(x)\|\}, & \text{if } B'(x)\bigtriangledown\hat{V}(x)\neq 0,\\ \{0\}, & \text{if } B'(x)\bigtriangledown\hat{V}(x) = 0.\end{cases}\end{aligned} \tag{4.8}$$

On the other hand,

$$\begin{aligned}&\beta(t)(\mathcal{G}_c\circ\mathcal{H})(t,x) = \\ &\begin{cases}\beta(t)\|h(x)-\rho(t,x)\frac{B'(x)\bigtriangledown\hat{V}(x)}{\|B'(x)\bigtriangledown\hat{V}(x)\|}\|\mathcal{B}_{\mathbf{R}^m}, & \text{if } B'(x)\bigtriangledown\hat{V}(x)\neq 0\,,\\ \beta(t)\|h(x)\|\mathcal{B}_{\mathbf{R}^m}, & \text{if } B'(x)\bigtriangledown\hat{V}(x) = 0.\end{cases}\end{aligned}$$

Therefore, in the case of $B'(x)\bigtriangledown\hat{V}(x)\neq 0$,

$$\begin{aligned}&\ll B'(x)\bigtriangledown\hat{V}(x), \mathcal{G}_m(t,x) + \beta(t)(\mathcal{G}_c\circ\mathcal{H})(t,x) \gg\subset \\ &\subset [(\beta(t)(\|h(x)\| + \rho(t,x)) + \xi(\mathcal{G}_m(t,x))]\|B'(x)\bigtriangledown\hat{V}(x)\|\cdot[-1,1]\end{aligned} \tag{4.9}$$

while in the case of $B'(x)\bigtriangledown\hat{V}(x) = 0$,

$$\ll B'(x)\bigtriangledown\hat{V}(x), \mathcal{G}_m(t,x) + \beta(t)(\mathcal{G}_c\circ\mathcal{H})(t,x) \gg = \{0\}\ . \tag{4.10}$$

Comparing (4.6)-(4.10), the result can be summarized in the inequality

$$\frac{d}{dt}\hat{V}(x(t)) \leq -[Q(x(t)) + R(h(x(t))]\ .$$

Thus the Barbashin-Krasowsky theorem can again be applied to prove the statement of the theorem.

5 Examples

The stabilizing property of the controllers discussed in Sections 3 and 4 can be illustrated by the following examples.

Example 1 Let us consider the nominal system

$$\begin{pmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{pmatrix} = \begin{pmatrix}-x_2(t)\\ x_1(t)\end{pmatrix} + \begin{pmatrix}x_1(t) & -x_2(t)\\ x_2(t) & x_1(t)\end{pmatrix}\begin{pmatrix}u_1(t)\\ u_2(t)\end{pmatrix}\ ,$$

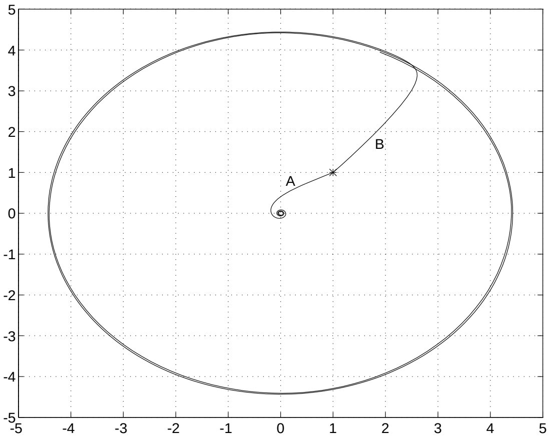

Fig. 1

and let us choose the functions $x \to Q(x) = 2a^2(x_1^2 + x_2^2)^2$, $u \to R(u) = \frac{1}{2}u'u$, $x \to g(x) = b(x_1^2 + x_2^2)$. It can easily be verified that these functions satisfy Assumption A2, if $0 < a \leq b$. Simulation results are obtained with parameters $a = 0.5$, $b = 8$. The horizon length was $T = 0.5$. We notice that the maximal magnitude of the receding horizon controller for the initial state $(5, 4)$ is 41.14, while that of the controller defined by means of the function g is 328.

Let the uncertainty for this system be given by $\mathcal{G}_m(t, x) = 20\mathcal{B}_{\mathbf{R}^2}$ and $\beta(t) \equiv 0$. A realization of this uncertainty was simulated as the function

$$20(\cos(0.01t(|x_1| + |x_2|)), \sin(0.01t(|x_1| + |x_2|)))'.$$

In Fig. 1, curve B is the trajectory obtained by the use of the receding horizon control law only, while curve A resulted by the controller of Section 4.

Example 2 Consider now the nominal system

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} x_1^3(t) - x_2^3(t) \\ x_1^3(t) + x_2^3(t) \end{pmatrix} + \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} u(t)$$

which has a highly unstable free part. We choose $x \to Q(x) = x_1^4 + x_2^4$, $u \to R(u) = \frac{1}{2}u^2$, $x \to g(x) = c(x_1^2 + x_2^2)$. If $c \geq 1.5$, then Assumption A2 holds true. Simulation results are obtained with parameter $c = 5$ and horizon length $T = 0.5$. If we compare the maximal control magnitudes again for the initial point $(5, 4)$, we find the values 71.64 and 410 for the receding horizon controller and for the controller defined by function g, respectively.

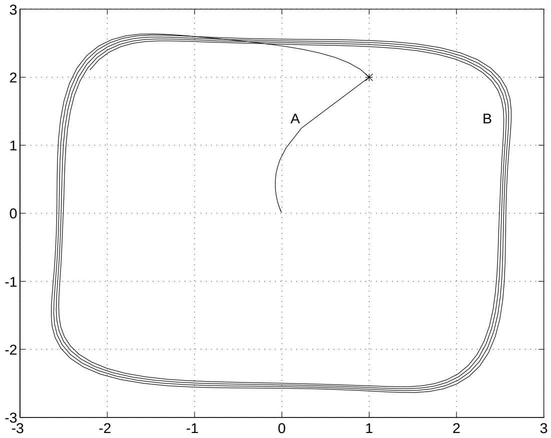

Fig. 2

The uncertainty for this system was taken to be $\mathcal{G}_m(t,x) = 12\mathcal{B}_{\mathbf{R}^2}$, $\beta(t) \equiv 0$. A realization of this uncertainty was simulated as the function

$$12\exp(-0.005t(|x_1| + |x_2|)) .$$

Fig. 2 shows the trajectories resulted from the receding horizon strategy (trajectory B) and from the control method of Section 4 (trajectory A).

We note that the robustness of the receding horizon controller is indicated by the fact that the function

$$(t,x) \to 10\sin(1 + t(|x_1| + |x_2|)) \in \mathcal{G}_m(t,x)$$

leaves the system controlled by the receding horizon method to be asymptotically stable.

For the solution of the two-point boundary value problem necessary in the calculation of the controllers, the computer program described in [11] was used.

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