Journal of Mathematical Systems, Estimation, and Control
Vol. 6, No. 4, 1996, pp. 1–12
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Robust Stability Analysis of Characteristic Polynomials Whose Coefficients are Polynomials of Interval Parameters*

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Abstract

In this paper, stability of the characteristic polynomial $F(s)$ of which parameters appear nonlinearly in coefficients is studied. If real and imaginary part of $F(jw)$ are monotone of parameters in frequency domain, their maximum and minimum values can be calculated from the endpoints of parameters. Using this monotonicity, we will extend the mapping theorem, to the case of polynomial coefficient of parameters. And sufficient conditions of the monotonicity are derived for the case of one parameter and multi-parameters.

Key words: interval polynomial, Sturm's theorem, mapping theorem, Mikhailov's theorem

AMS Subject Classifications: 93D09, 93B25

1 Introduction

Since the appearance of the Kharitonov's theorem [1], the simple stability criterions of interval polynomials have been studied by many researchers ([2],[3],[4],[5],[6],[7]). On the other hand, there is the mapping theorem for the analysis of multilinear coefficient of polynomials ([8, p.476], [2, pp. 147-152], [9], [10]). The mapping theorem is useful to analyze the stability of open loop and closed loop system in the frequency domain. In these studies, coefficients of interval polynomial are supposed to be linear or multilinear in system parameters. And little attention has been given to the nonlinear case which is more common in practice.

In this paper, the mapping theorem is extended to the general nonlinear case by means of the monotonicity of functions on the intervals of

*Received December 10, 1994; received in final form September 12, 1995. Summary appeared in Volume 6, No. 4, 1996.

parameters. If our conditions on monotonicity of $Re\ F(jw)$ and $Im\ F(jw)$ are satisfied, then all polynomials $F(jw)$ belong to a rectangle $F_h(w)$ in the frequency domain, whose vertices are calculated from the endpoints of parameter intervals. Using this property, we can check the stability of the interval polynomial. If the coefficient is a polynomial of the only one parameter, its monotonicity can be studied by applying Sturm's theorem ([11], [12, pp. 81-85]) to its derivatives with respect to parameter. In order to obtain the monotonicity test for multiparameter case, it is sufficient for us to apply the monotonicity test for one-parameter recursively. An example is worked out to illustrate our result.

2 Description of the System

We study the characteristic polynomial

$$\begin{aligned} F(s) \quad = \quad & s^n + c_1(\mathbf{p})s^{n-1} + c_2(\mathbf{p})s^{n-2} + \cdots + c_{n-1}(\mathbf{p})s + c_n(\mathbf{p}) \\ & \text{where } c_j(\mathbf{p})\text{'s are in polynomials of } \mathbf{p} = (p_1, \ldots, p_m) \\ & p_i \in [\ \underline{p}_i,\ \overline{p}_i\ ],\ p_i \in \Re,\ i = 1, 2, \ldots, m \end{aligned} \tag{2.1}$$

It is clear that for each $c_j(\mathbf{p})$ there exists an interval $[\ \underline{c}_j,\ \overline{c}_j\ ]$ to which the value of $c_j(\mathbf{p})$ belongs under the restrictions $p_i \in [\ \underline{p}_i,\ \overline{p}_i\ ]$ $(i = 1, \ldots, m;\ j = 1, \ldots, n)$. And the relations between the interval $[\ \underline{c}_j,\ \overline{c}_j\ ]$ and the interval $[\ \underline{p}_i,\ \overline{p}_i\ ]$ are complicated in general. For example, the system with structural uncertainties:

$$\dot{x}(t) = \mathbf{A}x(t) \tag{2.2}$$

$$\mathbf{A} = \mathbf{A}_0 + p_1\mathbf{A}_1 + p_2\mathbf{A}_2 + \cdots + p_m\mathbf{A}_m$$

$$\text{where} \quad p_i \in [\ \underline{p}_i, \overline{p}_i\ ],\ p_i \in \Re;\ \mathbf{A}, \mathbf{A}_i \in \Re^{n\times n},\ i = 1, 2, \ldots, m$$

has the characteristic polynomial in the form of equation (2.1). It is known that if c_j's are multilinear in p_i's, the interval $[\ \underline{c}_j, \overline{c}_j\ ]$ is directly determined by $\underline{p}_i$'s and $\overline{p}_i$'s. In this case, the mapping theorem can be used to check the stability of the system. In the following, we will study a more general case where coefficients c_j 's are polynomials.

3 Extension of the Mapping Theorem

First, we need the definition of monotone functions.

Definition 1 *(Monotone function of* $\mathbf{p}$*) If a function* $f(\mathbf{p}) = f(p_1, \ldots, p_m)$ *is a polynomial of* p_i, $i = 1, \ldots, m$ *and satisfies the condition*

$$\frac{\partial f(\mathbf{p})}{\partial p_i} \leq 0 \tag{3.1}$$

or

$$\frac{\partial f(\mathbf{p})}{\partial p_i} \geq 0 \tag{3.2}$$

for all $p_i \in (\ \underline{p}_i,\ \overline{p}_i\ ), p_k \in [\ \underline{p}_k,\ \overline{p}_k\ ],\ k \neq i;\ i, k = 1, 2, \cdots, m,$

it is called a monotone function of $\mathbf{p}$. □

Using this definition, we have the following theorem.

Theorem 1 *If* $Re\ F(jw)$ *and* $Im\ F(jw)$ *are monotone functions of* $\mathbf{p}$, $F(jw)$ *belongs to the rectangle* $F_h(w)$ *in the complex plane.*

$$\begin{aligned} F_h(w) &= \{x + jy : \alpha_m \leq x \leq \alpha_M,\ \beta_m \leq y \leq \beta_M\} \qquad (3.3)\\ \alpha_m &= \min_{\substack{p_i = \underline{p}_i,\ \overline{p}_i \\ i=1,2,\ldots,m}} Re\ F(jw),\ \alpha_M = \max_{\substack{p_i = \underline{p}_i,\ \overline{p}_i \\ i=1,2,\ldots,m}} Re\ F(jw) \\ \beta_m &= \min_{\substack{p_i = \underline{p}_i,\ \overline{p}_i \\ i=1,2,\ldots,m}} Im\ F(jw),\ \beta_M = \max_{\substack{p_i = \underline{p}_i,\ \overline{p}_i \\ i=1,2,\ldots,m}} Im\ F(jw). \end{aligned}$$

□

Proof: Let us consider the monotonicity at an arbitrary frequency w. If only a parameter p_i is varied on $[\ \underline{p}_i,\ \overline{p}_i\ ]$ with the other parameters fixed, it is shown that $Re\ F(jw)$ have the relative maximal value at $\underline{p}_i$ or $\overline{p}_i$ due to the monotonicity with p_i. The relative minimum value of $Re\ F(jw)$ is determined in the same way. Then, another parameter p_k can be varied and so on. Consequently, $Re\ F(jw)$ has the maximum value and the minimum value at the endpoints of $p_i;\ i = 1, 2, \ldots, m$. The same argument is applied to Im $F(jw)$. Therefore, $F(jw)$ belongs to the convex hull F_h. □

Using this theorem, we can check the stability of $F(s)$. $F(jw)$ belongs to the rectangle $F_h(w)$.

Theorem 2 *If the plot of* $F_h(w)$ *satisfies the following two conditions, the system is stable.*
1) $F_h(0)$ *is on the positive part of real axis.*
2) As w *increases, the plot of* $F_h(w)$ *encircles the origin in a counterclockwise direction and its vertex's phase goes to* $n\frac{\pi}{2}$ *for* $w \to \infty$. □

Thus all rectangles $F_h(w)$ do not include the origin.

Proof: According to Mikhailov's Theorem [2, p. 113], the above conditions are derived. □

This stability analysis is similar to the mapping theorem [2, pp. 147-150]. And the monotonicity of a function is very important in our analysis method. Next, we study conditions of the monotonicity.

4 Criterion of Monotonicity

In this section we will discuss how to check the monotonicity. First we study one parameter case using both endpoints of interval parameters.

4.1 One-parameter function

We consider a polynomial $f(p)$ with a parameter $p \in [\ \underline{p},\ \overline{p}\ ]$. It is clear that if $\frac{df(p)}{dp} \neq 0$ on the interval $[\ \underline{p},\ \overline{p}\ ]$, $f(p)$ is monotone. Therefore, it is necessary to know how many roots of the equation $\frac{df(p)}{dp} = 0$ are in the interval $[\ \underline{p},\ \overline{p}\ ]$. According to the Euclidean Algorithm, the following procedure is derived.

$$
\begin{aligned}
r_1 &= \frac{df(p)}{dp}, \quad r_2 = \frac{d^2 f(p)}{dp^2} && (4.1)\\
r_1 &= q_1 r_2 - r_3 && (4.2)\\
r_2 &= q_2 r_3 - r_4 && (4.3)\\
&\cdots\cdots\cdots \\
r_{w-2} &= q_{w-2} r_{w-1} - r_w, && (4.4)
\end{aligned}
$$

where r_w is the greatest common factor of r_1 and r_2.

Next, we evaluate the values $r_1(\underline{p})$ $r_2(\underline{p})$, $\ldots$, $r_w(\underline{p})$, and count how many times $r_1(\underline{p})$, $r_2(\underline{p})$, $\ldots$, $r_w(\underline{p})$ change its signs, which is denoted by $V(\underline{p})$. $V(\overline{p})$ is calculated in the same way.

Theorem 3 *In the above system, if the following condition 1) or 2) holds, $f(p)$ is a monotone function of p on the interval $[\ \underline{p},\ \overline{p}\ ]$.*

$$
\begin{aligned}
&1) \qquad V(\underline{p}) - V(\overline{p}) = 0 && (4.5)\\
&2) \qquad V(\underline{p}) - V(\overline{p}) = 1 \text{and sign } (r_1(\underline{p}) r_1(\overline{p})) > 0 && (4.6)
\end{aligned}
$$

□

Remark 1 [12, pp. 81-85]. If r_1 has α-tuple roots at $p = \underline{p}$ and/or β-tuple roots at $p = \overline{p}$, then all functions (4.1), (4.2), (4.3) and (4.4) become 0. In this case, $V(\mathbf{p})$ can not be calculated in the neighborhood of $\underline{p}$ and/or $\overline{p}$. So we must eliminate $(p - \underline{p})^\alpha$ or/and $(p - \overline{p})^\beta$ in r_1 and r_2. For example,

$$
r_1 = \frac{d}{dp} \frac{f(p)}{(p - \underline{p})^\alpha}, \quad r_2 = \frac{d^2}{dp^2} \frac{f(p)}{(p - \underline{p})^\alpha}.
$$

We can calculate $V(\underline{p}) - V(\overline{p})$ using the above algorithm. □

Proof: According to the Sturm's theorem, the number of roots of $\frac{df(p)}{dp} = 0$ on the interval $(\ \underline{p},\ \overline{p}\ ]$ is given by $V(\underline{p}) - V(\overline{p})$. If $V(\underline{p}) - V(\overline{p}) = 0$,

$\frac{df(p)}{dp} = 0$ has no roots on the interval ($\underline{p}$, $\overline{p}$], and if $V(\underline{p}) - V(\overline{p}) = 1$, $\frac{df(p)}{dp} = 0$ has one root on the interval ($\underline{p}$, $\overline{p}$]. Since $r_1(\underline{p})$ and $r_1(\overline{p})$ have the same sign, $f(p)$ is a monotone function with one point of inflection. □

4.2 Multi-parameter function

For multiparameter functions, we examine the monotonicity.

Definition 2 *For parameters p_1, p_2, ..., p_m, we define the set of parameters p_i $(i = 1, 2, \ldots, m)$ as $p(1_{j_1} 2_{j_2} 3_{j_3} \cdots m_{j_m})$ where each $j_1, j_2, \cdots, j_m$ takes 0 or 1 respectively. In this form, we denote $j_i = 0$ if $p_i = \underline{p}_i$ and also $j_i = 1$ if $p_i = \overline{p}_i$. For example, $p(i_0 j_1)$ means $p_i = \underline{p}_i$ and $p_j = \overline{p}_j$.* □

Then we suppose a function $f(\mathbf{p}) = f(p_1, \ldots, p_m)$ to be a polynomial of p_i, $i = 1, \ldots, m$, where $p_i \in [\ \underline{p}_i,\ \overline{p}_i\ ]$ and $f(\mathbf{p}) \in [\ \underline{f},\ \overline{f}\ ]$. If $\frac{\partial f(\mathbf{p})}{\partial p_i} \neq 0$ on $[\ \underline{p}_i,\ \overline{p}_i\ ]$ for $i = 1, \ldots, m$, then the function $f(\mathbf{p})$ is monotone by definition. In the same way as one parameter case, we examine the existence of roots of $\frac{\partial f(\mathbf{p})}{\partial p_i} = 0$ on $[\ \underline{p}_i,\ \overline{p}_i\ ]$ for all i.

$$r_{i\,1} = \frac{\partial f(\mathbf{p})}{\partial p_i}, \quad r_{i\,2} = \frac{\partial^2 f(\mathbf{p})}{\partial p_i^2} \tag{4.7}$$

$$r_{i\,1} = q_{i\,1} r_{i\,2} - r_{i\,3} \tag{4.8}$$

$$r_{i\,2} = q_{i\,2} r_{i\,3} - r_{i\,4} \tag{4.9}$$

$$\cdots\cdots\cdots$$

$$r_{i\,w_i-2} = q_{i\,w_i-2} r_{i\,w_i-1} - r_{i\,w_i}, \tag{4.10}$$

where $r_{i\,1}$, $r_{i\,2}$, ..., $r_{i\,w_i}$ are polynomials of p_1, p_2, ..., p_m. And $r_{i\,w_i}$ is the greatest common factor of r_1 and r_2.

As the one-parameter case, $V(p(1_0 2_0 \cdots (i-1)_0 (i+1)_0 \cdots m_0))(\underline{p}_i)$ is calculated by counting the sign changes of the sequence

$$r_{i\,1}(p(1_0 2_0 \cdots (i-1)_0 (i+1)_0 \cdots m_0))(\underline{p}_i), \tag{4.11}$$

$$r_{i\,2}(p(1_0 2_0 \cdots (i-1)_0 (i+1)_0 \cdots m_0))(\underline{p}_i), \tag{4.12}$$

$$\cdots\cdots\cdots,$$

$$r_{i\,w_i}(p(1_0 2_0 \cdots (i-1)_0 (i+1)_0 \cdots m_0))(\underline{p}_i). \tag{4.13}$$

And $V(p(1_0 2_0 \cdots (i-1)_0 (i+1)_0 \cdots m_0))(\overline{p}_i)$ is calculated in the same way.

Using these expressions, we have the following theorem.

Theorem 4 *If the following conditions 1), and 2) or 3) hold, $f(\mathbf{p})$ is a monotone function of $\mathbf{p}$ on $[\ \underline{p}_1,\ \overline{p}_1\ ] \otimes [\ \underline{p}_2,\ \overline{p}_2\ ] \otimes \cdots \otimes [\ \underline{p}_w,\ \overline{p}_w\ ]$.*
1) Let $r_{i\,1}$ be regarded as a polynomial of p_i. All coefficients of p_i in $r_{i\,1}$

are monotone of p_k *(* $k \neq i, k = 1, 2, \ldots, m$ *).*
2) For all pairs $\underline{p}_k$ *and* $\overline{p}_k$ $(k \neq i)$*,*

$$\begin{array}{rl} & V(p(1_{j_1} 2_{j_2} \cdots (i-1)_{j_{i-1}} (i+1)_{j_{i+1}} \cdots m_{j_m}))(\underline{p}_i) \\ - & V(p(1_{j_1} 2_{j_2} \cdots (i-1)_{j_{i-1}} (i+1)_{j_{i+1}} \cdots m_{j_m}))(\overline{p}_i) = 0 \end{array} \quad (4.14)$$

where $j_1, j_2, \ldots, j_{i-1}, j_{i+1}, \ldots, j_m = 0, 1;\ i = 1, 2, \ldots, m$

3) For all pairs $\underline{p}_k$ *and* $\overline{p}_k$ $(k \neq i)$*,*

$$\begin{array}{rl} & V(p(1_{j_1} 2_{j_2} \cdots (i-1)_{j_{i-1}} (i+1)_{j_{i+1}} \cdots m_{j_m}))(\underline{p}_i) \\ - & V(p(1_{j_1} 2_{j_2} \cdots (i-1)_{j_{i-1}} (i+1)_{j_{i+1}} \cdots m_{j_m}))(\overline{p}_i) = 1 \end{array} \quad (4.15)$$

and

$$\begin{array}{l} sign(r_{i\,1}(p(1_{j_1} 2_{j_2} \cdots (i-1)_{j_{i-1}} (i+1)_{j_{i+1}} \cdots m_{j_m})(\underline{p}_i)) \\ r_{i\,1}(p(1_{j_1} 2_{j_2} \cdots (i-1)_{j_{i-1}} (i+1)_{j_{i+1}} \cdots m_{j_m})(\overline{p}_i))) > 0 \end{array} \quad (4.16)$$

where $j_1, j_2, \ldots, j_{i-1}, j_{i+1}, \ldots, j_m = 0, 1;\ i = 1, 2, \ldots, m.$

□

Remark 2 As the same as Remark 1, if $r_{i\,1}$ has α-tuple roots at $p_i = \underline{p}_i$ and/or β-tuple roots at $p_i = \overline{p}_i$, we must eliminate $(p_i - \underline{p}_i)^\alpha$ and/or $(p_i - \overline{p}_i)^\beta$ in $r_{i\,1}$ and $r_{i\,2}$. We can calculate $V(p(1_{j_1} 2_{j_2} \cdots (i-1)_{j_{i-1}} (i+1)_{j_{i+1}} \cdots m_{j_m}))(\underline{p}_i) - V(p(1_{j_1}\ 2_{j_2} \cdots (i-1)_{j_{i-1}} (i+1)_{j_{i+1}} \cdots m_{j_m}))(\overline{p}_i)$ using the above algorithm as the one-parameter function case. □

Remark 3 In this theorem, we have to check whether $r_{i\,1}$ is monotone of p_k ($k \neq i, k = 1, 2, \ldots, m$) or not. If $r_{i\,1}$ can be expressed by the polynomial of p_i, the coefficients of the powers of p_i are expressed by polynomials of p_k $(k \neq i)$. Applying this procedure repeatedly to the coefficients, we finally obtain coefficient functions with only one parameter. Then we can check its monotonicity by means of Theorem 3 and calculate the maximum value and minimum value of these coefficient functions. Using these maximum and minimum values, we obtain interval polynomials whose monotonicity can be checked, and we calculate its maximum and minimum values, which yields another interval polynomials and so on. This backward procedure terminates with finite steps and we can check the monotonicity of $r_{i\,1}(\mathbf{p})$. □

Proof: In this theorem, $V(p(1_{j_1} 2_{j_2} \cdots (i-1)_{j_{i-1}} (i+1)_{j_{i+1}} \cdots m_{j_m}))(p_i)$ is equal to the number of sign changes of the sequence $r_{i\,j}$, $j = 1, 2, \ldots, w_i$. $r_{i\,j}$ is a polynomial of p_i and coefficients of powers of p_i are polynomials of p_k ($k \neq i, k = 1, 2, \ldots, m$). If condition 1) is satisfied, $r_{i\,1}$ is a monotone function of p_k and $r_{i\,1}$ belongs to the convex hull spanned by functions whose coefficients are derived from all combinations of coefficients of $r_{i\,1}$ where p_k takes $\underline{p}_k$ or $\overline{p}_k$. According to Sturm's theorem, if the equation (4.14) holds, $r_{i\,1} = 0$ has no roots on the interval $(\ \underline{p}_i,\ \overline{p}_i\ ]$. This means directly $f(p_i)$ is a monotone function with respect to p_i. And also if the equation (4.15) holds, $r_{i\,1} = 0$ has one root on the interval $(\ \underline{p}_i,\ \overline{p}_i\ ]$. As the one parameter case, if (4.15) and (4.16) hold, $f(p_i)$ is a monotone function with respect to p_i. Thus, if condition 1), and 2) or 3) hold for all p_i, $f(\mathbf{p})$ is monotone. □

Remark 4 In the applications of our test to $ReF(jw)$ and $ImF(jw)$, we have to examine the monotonicity in parameters for all w belonging to the infinite interval $[0, \infty)$. In the coefficients of higher powers of w, parameters appear in lower powers. Hence, if we differentiate the polynomial with respect to parameters, the terms with higher powers of w often disappear. $ReF(jw)$ consists of only the even powers of w and $ImF(jw)$ odd powers. Thus, if we put $\xi = w^2$, it is clear that we can check the monotonicity of examining the lower order polynomial in ξ. □

5 Example

Let us consider the characteristic polynomial with interval parameters.

$$\begin{aligned} F(s) &= s^3 + (2 - p_1 - p_2)s^2 \\ &+ (p_1^2 + p_1 + 2p_1p_2 - 2p_2^2 + p_2 + 3)s \\ &+ 3p_1^3 + 4p_1p_2 + p_2^3 + 1, \qquad (5.1) \\ \text{where} &\quad p_1 \in [0, 0.5],\ p_2 \in [0, 0.25] \end{aligned}$$

In order to study the stability of (5.1), we have to check the monotonicity of *Re* $F(jw)$ and *Im* $F(jw)$.

$$\begin{aligned} F(jw) &= (p_1 + p_2 - 2)w^2 + 3p_1^3 + 4p_1p_2 + p_2^3 + 1 \\ &+ j\{-w^3 + (p_1^2 + p_1 + 2p_1p_2 - 2p_2^2 + p_2 + 3)w\} \qquad (5.2) \\ &= \alpha + j\beta \qquad (5.3) \\ &\quad \text{where } p_1 \in [0, 0.5],\ p_2 \in [0, 0.25] \end{aligned}$$

1) Check the monotonicity of α.
i) Regard α as a polynomial of p_1.

$$\begin{aligned} \alpha &= (p_1 + p_2 - 2)w^2 + 3p_1^3 + 4p_1p_2 + p_2^3 + 1 \\ \frac{\partial \alpha}{\partial p_1} &= w^2 + 9p_1^2 + 4p_2. \qquad (5.4) \end{aligned}$$

It is obvious the equation (5.4) takes positive value on intervals of p_1 and p_2 and $\frac{\partial \alpha}{\partial p_1}$ is monotone with respect to p_2. In this case, we will check the monotonicity according to theorem 3.

$$r_{11} = \frac{\partial \alpha}{\partial p_1} = w^2 + 9p_1^2 + 4p_2, \ r_{12} = \frac{\partial^2 \alpha}{\partial p_1^2} = 18p_1, \ r_{13} = 4p_2 + w^2 \quad (5.5)$$

case 1: $p_2 = \underline{p}_2 = 0$

$$r_{11} = w^2 + 9p_1^2, \ r_{12} = 18p_1, \ r_{13} = w^2 \quad (5.6)$$

$V(\underline{p}_2)(\underline{p}_1) = 0$, $V(\underline{p}_2)(\overline{p}_1) = 0$. Thus, $V(\underline{p}_2)(\underline{p}_1) - V(\underline{p}_2)(\overline{p}_1) = 0$.

case 2: $p_2 = \overline{p}_2 = 0.25$

$$r_{11} = w^2 + 9p_1^2 + 1, \ r_{12} = 18p_1, \ r_{13} = w^2 + 1 \quad (5.7)$$

$V(\overline{p}_2)(\underline{p}_1) = 0$, $V(\overline{p}_2)(\overline{p}_1) = 0$, $V(\overline{p}_2)(\underline{p}_1) - V(\overline{p}_2)(\overline{p}_1) = 0$.
Then α is a monotone function of p_1.
ii) Regard α as a polynomial of p_2.
α is a monotone function of p_2, because $\frac{\partial \alpha}{\partial p_2} = w^2 + 3p_2^2 + 4p_1 > 0$ holds on interval p_1 and p_2.

2) Check the monotonicity of β.
i) Regard β as a polynomial of p_1.

$$\begin{aligned} \beta &= -w^3 + (p_1^2 + p_1 + 2p_1p_2 - 2p_2^2 + p_2 + 3)w \\ \frac{\partial \beta}{\partial p_1} &= (2p_1 + (1 + 2p_2))w \end{aligned} \quad (5.8)$$

All coefficients of (5.8) are monotone on the interval. Thus we can check the monotonicity at the endpoints of p_2.
case 1: $p_2 = 0$

$$\frac{\partial \beta}{\partial p_1} = (2p_1 + 1)w > 0; \ p_1 \in [0, 0.5] \quad (5.9)$$

case 2: $p_2 = 0.25$

$$\frac{\partial \beta}{\partial p_1} = (2p_1 + 1.5)w > 0; \ p_1 \in [0, 0.5] \quad (5.10)$$

Then β is monotone function of p_1.
ii) Regard β as a polynomial of p_2. We also check the monotonicity.

$$\frac{\partial \beta}{\partial p_2} = (-4p_2 + (1 + 2p_1))w \quad (5.11)$$

All coefficients of (5.11) are monotone with respect to p_1 on the interval $[\,\underline{p}_1, \ \overline{p}_1\,]$.
case 1: $p_1 = 0$

$$\frac{\partial \beta}{\partial p_2} = (-4p_2 + 1)w \geq 0; \ p_2 \in [0, 0.25] \quad (5.12)$$

case 2: $p_1 = 0.5$

$$\frac{\partial \beta}{\partial p_2} = (-4p_1 + 2)w > 0;\ p_2 \in [0, 0.25] \tag{5.13}$$

β is a monotone function of p_2. Thus, β is a monotone function of p_1, p_2. Consequently, $Re\ F(jw)$ and $Im\ F(jw)$ are monotone with respect to parameters p_1, p_2. Thus maximum and minimum values of $Re\ F(jw)$ and $Im\ F(jw)$ are expressed by endpoints of p_1 and p_2. Drawing the rectangles $F_h(w)$ for this example (Fig. 1), we know that F(s) is stable. □

In some cases as shown in our example, it is possible to check the monotonicity even if w varies from 0 to ∞. But in more complicated cases, we have to divide the interval $[0, \infty)$ into subintervals where the monotonicity holds.

6 Concluding Remarks

In this paper, we present a new criterion of stability of interval polynomials. Defining the monotonicity of multivariable function, the mapping theorem is extended to the general case, where parameters appear nonlinearly. Using our extended mapping theory, stability of characteristic polynomials with polynomial coefficients is checked by means of endpoints of parameters.

We introduce the rectangle $F_h(w)$ in the complex plane and show the condition that the characteristic polynomial with interval polynomials lies in the rectangle $F_h(w)$ for an arbitrary frequency w. The sufficient condition for stability of $F(s)$ are derived by means of $F_h(w)$ by our extended mapping theorem.

For one parameter functions and multi-parameter functions, we show how to check the monotonicity. For one parameter coefficients, we apply Sturm's theorem and for multi-parameter coefficients, we can check the monotonicity by our multiparameter test.

Our result may be applied to the stability problem for the characteristic polynomial, of which coefficients are polynomials of parameters, multilinear and linear (affine) with respect to the parameters. And there is room for argument on the conservativeness of our sufficient conditions. It needs further investigation for nonlinear functions with interval parameters.

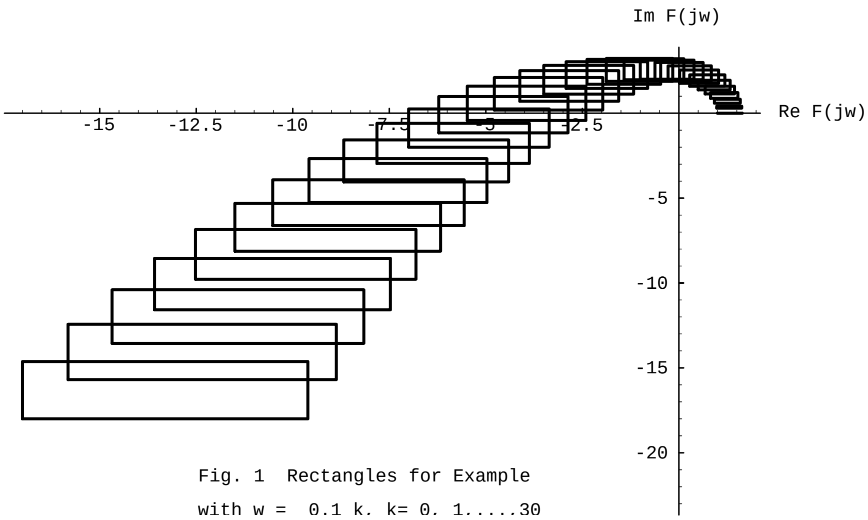
Im F(jw)
Re F(jw)
-15
-12.5
-10
-7.5
-5
-2.5
-5
-10
-15
-20
Fig. 1 Rectangles for Example
with w = 0.1 k. k= 0. 1.....30

Figure 1:

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Communicated by Clyde F. Martin