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Variational Stability Analysis of Optimal Control Problems for Systems Governed by Nonlinear Second Order Evolution Equations*

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Abstract

The variational stability of optimal control problems governed by second order nonlinear evolution abstract equations is studied. First we prove an existence theorem for optimal solutions. Then admitting perturbations in all the data of the control problem, we show the results on the asymptotic behavior of optimal solutions to control problems as well as on the convergence of minimal values and reachable sets. The notions of τ-convergence of functionals and the Kuratowski-Mosco convergence of sets are employed. Finally an example of nonlinear hyperbolic control problem demonstrates the applicability of the results.

Key words: control problem, τ-convergence, Kuratowski-Mosco convergence, monotone operator, compact embedding, hyperbolic system

AMS Subject Classifications: 34G20, 49J27, 49K40, 49J20

1 Introduction and Notation

In this paper we investigate the optimal control problems governed by abstract second order evolution equations. We consider a sequence of such problems indexed by the parameter $n \in \mathbb{N}$ which appears in all the data including the nonlinear operator of the state equation, the integrand of cost functional and the control constraint set.

The goal is to identify a "limit problem" which is obtained from the perturbed control problems as n tends to infinity. First we deal with a nonlinear second order evolution equation for which we show the results on existence and uniqueness of solutions and on the continuous dependence

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of the solutions on the data. Next we deliver a theorem on the existence of optimal solutions for control problems under consideration. Then we find conditions which are sufficient for the convergence of optimal solutions of perturbed control problems to an optimal solution of the limit problem. Moreover, we will show the convergence of the corresponding minimal values and we provide the stability result for the reachable sets. Finally, we apply our results to a sequence of Bolza-type, nonlinear hyperbolic, distributed parameter optimal control problems.

The problem of existence of optimal controls, being an important question in optimal control theory, was extensively investigated in the past; see Lions [16] and Ahmed and Teo [2] and the references therein for control of distributed-parameter linear systems and Cesari [9] for nonlinear infinite dimensional control systems. The sensitivity of optimal control problems for partial differential equations to changes in the parameter has been studied by Zolezzi [30], Buttazzo and Dal Maso [7], Migórski [18] who considered elliptic systems, by Bennati [6], Carja [8], Denkowski and Migórski [11], Papageorgiou [26], Denkowski and Mortola [12] who examined parabolic equations and by Migórski in [19], where the hyperbolic state equations were dealt with. However, in all these works the differential equations are linear in the state and often the objective functionals are of particular forms. Recently, the asymptotic limits of control problems have been considered in Papageorgiou [25], [27] and Migórski [20] for systems described by nonlinear parabolic evolution equations. To our knowledge, the problem of variational stability of control problems for second order equations has not been treated in the literature. In this paper we study this problem by considering control problems for a class of systems governed by abstract monotone equations admitting nonlinearities.

We conduct the sensitivity analysis using the notions of τ-convergence of functions and the Kuratowski-Mosco convergence of sets. We underline that the τ-convergence of functions is a particular case of Γ-convergence introduced by De Giorgi (see, for instance, [10]) and it can be expressed as the Kuratowski-Mosco convergence of their epigraphs [3].

Throughout the paper, we make use of the definitions and facts listed below. Let H denote a separable Hilbert space and V a subspace of H having the structure of a reflexive Banach space which is continuously and densely embedded in H. Identifying H with its dual H', we have the Gelfand triple $V \subset H \subset V'$, where V' is the dual of V. We will suppose that these embeddings are compact. Let $\langle \cdot, \cdot \rangle$ be the duality of V and V' as well as the inner product on H, let $||.||$, $|.|$ and $||.||_{V'}$ denote the norms in V, H and V', respectively. Given a fixed real number $T > 0$ and $2 \leq p < +\infty$, we introduce the following spaces $\mathcal{V} = L^p(0,T;V)$, $\mathcal{H} = L^p(0,T;H)$, $\mathcal{H}' = L^q(0,T;H)$, $\mathcal{V}' = L^q(0,T;V')$, $(1/p + 1/q = 1)$ and $\mathcal{W} = \{w \in \mathcal{V} \,|\, w' \in \mathcal{V}'\}$, where the derivative is understood in the sense of vector valued distributions. Clearly $\mathcal{W} \subset \mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$. The pairing of $\mathcal{V}$ and $\mathcal{V}'$ and the duality between $\mathcal{H}$ and $\mathcal{H}'$ are denoted by

$$\langle\langle f, v\rangle\rangle = \int_0^T \langle f(s), v(s)\rangle \, ds.$$

It is well known (see [17], [15], [2], [5]) that every $u \in \mathcal{W}$ is, after eventual modification on a set of measure zero, continuous from $[0,T]$ in H and the embedding $\mathcal{W} \subset C(0,T;H)$ is continuous. Furthermore, if $V \subset H$ compactly then also $\mathcal{W} \subset \mathcal{H}$ compactly; see Lions ([15], Theorem 5.1, p. 57) and Simon [29].

Given a Banach space X, the symbols $w-X$, $s-X$ are used to indicate the space X equipped with the weak and the strong (norm) topology, respectively. Let (Ω, Σ, μ) be a measure space. A multifunction F defined on Ω with values in the space 2^X of all nonempty subsets of X is called measurable if $F^-(E) := \{\omega \in \Omega : F(\omega) \cap E \neq \emptyset\} \in \Sigma$ for every closed set $E \subset X$. We denote by S_F^r $(1 \leq r \leq \infty)$ the set of all selectors of F that belong to $L^r(\Omega;X)$; i.e. $S_F^r = \{f \in L^r(\Omega;X) : f(\omega) \in F(\omega)\ \mu \text{ a.e.}\}$. It is known that $S_F^r \neq \emptyset$ if and only if $\omega \mapsto \inf\{||x|| : x \in F(\omega)\} \in L_+^r$. For $A \subset 2^X$ we also put $|A| = \sup\{|a| : a \in A\}$.

Given $\{S_n, S\}_{n\in\mathbb{N}} \subseteq 2^X$, we recall (see [14]) that S_n converge to S in the Kuratowski-Mosco sense (denoted by $S_n \xrightarrow{K-M} S$), if $w-\limsup_n S_n \subseteq S \subseteq s-\liminf_n S_n$, where the sequential Kuratowski upper and lower limits are defined respectively by $s-\liminf_n S_n = \{x \in X : \exists\, x_n \in S_n, x_n \to x \text{ in } X, \text{ as } n \to +\infty\} = \{x \in X : \lim_n d(x,S_n) = 0\}$ and $w-\limsup_n S_n = \{x \in X : \exists\,\{n_\nu\}, x_{n_\nu} \in S_{n_\nu}, x_{n_\nu} \to x \text{ in } w-X, \text{ as } \nu \to +\infty\}$.

We shall denote by $\Gamma_0(X)$ the set of all functions $f\colon X \to \mathbb{R} \cup \{\infty\}$ which are convex, lower semicontinuous and not identically equal to $+\infty$. A function $f\colon \Omega \times X \to \mathbb{R} \cup \{+\infty\}$ is called a normal convex integrand (cf. [13] and the references therein) if $f(\cdot,\cdot)$ is jointly measurable and $f(\omega,\cdot) \in \Gamma_0(X)$ for each $\omega \in \Omega$. For a sequence $f_n, f\colon X \to \mathbb{R} \cup \{+\infty\}$, $n \in \mathbb{N}$ of functions which are not identically $+\infty$, we say (following [3]) that f_n τ-converges to f, as $n \to +\infty$ if and only if $\text{epi} f_n \xrightarrow{K-M} \text{epi} f$, where the epigraph of a function g is defined as follows $\text{epi} g = \{(x,\lambda) \in X \times \mathbb{R} : g(x) \leq \lambda\}$. Furthermore, by [22], we know that $f_n \xrightarrow{\tau} f$ is equivalent to the following two conditions

(i) if $x_n \to x$ in $w-X$, then $f(x) \leq \liminf_n f_n(x_n)$;

(ii) for every $x \in X$, there exists $x_n \to x$ in $s-X$ such that $\lim_n f_n(x_n) = f(x)$.

This convergence is also equivalent to M_s^w-convergence studied by Salvadori in [28] and it is a particular case of De Giorgi's Γ-convergence (see [10], [3]). In the sequel we also need the notion of continuous convergence of functions. Namely, we say that f_n is sequentially continuously convergent to f in $\sigma - X$ (and we write $f_n \stackrel{c}{\longrightarrow} f$) if for every $x_n \to x$ in $\sigma - X$, we have $\lim\limits_n f_n(x_n) = f(x)$, where σ is a given topology on X.

Finally, by $\mathcal{L}(V, H)$ we will denote the space of linear continuous operators from Y into H endowed with the uniform operator topology.

Remark 1.1 It was proved by Nagy in [23] (and exploited, for instance, in [27], [1]) that if V is a Hilbert space and $V \subset H$ compactly, then the embedding $\mathcal{W} \subset C(0,T;H)$ is compact. However, in [21], the present author has delivered an example which shows that the above embedding can not be compact and therefore the result of Nagy is false.

2 Setting of the Problem and Preliminary Result

In this section, we formulate the control problems and we present an auxiliary result on compactness in the space $L^p(0,T;X)$.
We study the following sequence of control problems

$$\inf J_n(x,u) = m_n \tag{P$_n$}$$

subject to the state and control constraints

$$\begin{cases} \ddot{x}(t) + A_n(t, \dot{x}(t)) + B_n x(t) = f_n(t)u(t) \quad \text{a.e.}, \\ x(0) = x_0^n, \quad \dot{x}(0) = x_1^n, \\ u(t) \in U_n(t) \quad \text{a.e.}, \quad u \in L^1(0,T;Y), \end{cases} \tag{2.1$_n$}$$

where the cost functionals are given by

$$J_n(x,u) = l_n(x(T), \dot{x}(T)) + \int_0^T L_n(t, x(t), \dot{x}(t), u(t))\, dt. \tag{2.2$_n$}$$

We also consider the following unperturbed problem

$$\inf J(x,u) = m \tag{P}$$

such that

$$\begin{cases} \ddot{x}(t) + A(t, \dot{x}(t)) + Bx(t) = f(t)u(t) \quad \text{a.e.}, \\ x(0) = x_0, \quad \dot{x}(0) = x_1, \\ u(t) \in U(t) \quad \text{a.e.}, \quad u \in L^1(0,T;Y) \end{cases} \tag{2.1}$$

and

$$J(x,u) = l(x(T),\dot{x}(T)) + \int_0^T L(t,x(t),\dot{x}(t),u(t))\,dt. \tag{2.2}$$

Our aim is to provide conditions under which the optimal solutions to $(\mathrm{P})_n$, (P) exist and we have their convergence as well as the convergence of minimal values. By an optimal solution we mean a trajectory-control pair (x,u) (sometimes denoted also by $((x,\dot{x}),u)$) on which the infimum is attained.

For the problems $(\mathrm{P})_n$ and (P), we need the following hypotheses. The control space is modelled by Y which is assumed to be separable and reflexive Banach space.

$\underline{H(A)}$: $A_n\colon [0,T]\times V \to V'$, $n \in \mathbb{N}$, $A\colon [0,T]\times V \to V'$ are maps such that

(1) $t \to A_n(t,v)$, $t \to A(t,v)$ are measurable,

(2) $v \to A_n(t,v)$, $v \to A(t,v)$ are monotone and hemicontinuous,

(3) $\langle A_n(t,v),v\rangle \geq c||v||^p - d|v|^2$ a.e. and $\langle A(t,v),v\rangle \geq c||v||^p - d|v|^2$ a.e. with $c>0$ and $d \geq 0$,

(4) $||A_n(t,v)||_{V'} \leq \alpha_n(t) + b||v||^{p-1}$ a.e. and $||A(t,v)||_{V'} \leq \alpha(t) + b||v||^{p-1}$ a.e., with $\alpha_n, \alpha \in L^q_+(0,T)$, $\sup\{||\alpha_n||_{L^q} : n \in \mathbb{N}\} < +\infty$ and $b>0$.

$\underline{H(B)}$: $B_n \in \mathcal{L}(V,V')$, $n \in \mathbb{N}$ and $B \in \mathcal{L}(V,V')$ are symmetric (i.e. $\langle B_n v, w\rangle = \langle v, B_n w\rangle$ for all $v,w \in V$) and coercive (i.e. $\langle B_n v, v\rangle \geq c'||v||^2$ for all $v \in V$ with $c'>0$).

$\underline{H(f)}$: $f_n \in L^\infty(0,T;\mathcal{L}(Y,H))$, $n \in \mathbb{N}$ and $f \in L^\infty(0,T;\mathcal{L}(Y,H))$.

$\underline{H(U)}$: $U_n\colon [0,T] \to 2^Y$, $n \in \mathbb{N}$ and $U\colon [0,T] \to 2^Y$ are measurable multifunctions with closed, convex values such that $|U_n(t)| \leq \gamma(t)$, $|U(t)| \leq \gamma(t)$ a.e. with $\gamma \in L^q_+(0,T)$.

$\underline{(H_0)}$: $x_0^n, x_0 \in V$ and $x_1^n, x_1 \in H$ for $n \in \mathbb{N}$.

$\underline{H(L)}$: $L_n\colon [0,T]\times V\times H\times Y \to \mathbb{R}\cup\{+\infty\}$, $n \in \mathbb{N}$ and $L\colon [0,T]\times V\times H\times Y \to \mathbb{R}\cup\{+\infty\}$ are normal convex integrands such that $\varphi_1(t) + \varphi_2(t)\left(||x||^2 + |y|^2\right) + \varphi_3(t)||u||_Y^2 \leq L_n(t,x,y,u), L(t,x,y,u) \leq \psi_1(t) + \psi_2(t)\left(||x||^2+|y|^2\right) + \psi_3(t)||u||_Y^2$ a.e., where $\varphi_1, \psi_1 \in L^2(0,T)$, $\varphi_2,\varphi_3,\psi_2,\psi_3 \in L^\infty(0,T)$.

$\underline{H(l)}$: $l_n : V \times H \to \mathbb{R} \cup \{+\infty\}$, $n \in \mathbb{N}$ and $l : V \times H \to \mathbb{R} \cup \{+\infty\}$ are lower semicontinuous on $(w - V) \times (s - H)$.

In the following we prove a compactness result that will be useful in the sequel. We use the space of scalarly continuous functions defined by Lions and Magenes in [17], as follows $C_s(0,T;V) := \{f \in L^\infty(0,T;V) \,|\, t \to \langle h, f(t)\rangle_{V \times V'}$ belongs to $C([0,T])$ for any fixed $h \in V'\}$. We have

Lemma 2.1 *Let $V \subset H$ compactly and let $\{z_n\}$ be a sequence which satisfies*

$$z_n \to z \ \text{ in } \ w - * - L^\infty(0,T;V), \tag{2.3}$$

$$\dot{z}_n \to \dot{z} \ \text{ in } \ w - L^\delta(0,T;H), \tag{2.4}$$

where $\delta > 1$. Then

$$z_n \to z \ \text{ in } \ s - C_s(0,T;V). \tag{2.5}$$

Proof: First, applying Simon (Corollary 4, p.85 in [29]), from (2.3) and (2.4), we obtain

$$z_n \to z \ \text{ in } \ s - C(0,T;H). \tag{2.6}$$

Next, due to the inclusion $L^\infty(0,T;V) \cap C(0,T;H) \subset C_s(0,T;V)$ (see Lions and Magenes in [17], Chapter 3, Lemma 8.1), we have that z_n belongs to $C_s(0,T;V)$. We will show that (2.3) and (2.6) imply (2.5). To this end, it is sufficient to prove that for any $h \in V'$ the sequence $\{\langle h, z_n(t)\rangle\}_n$ is Cauchy in $C([0,T])$. Let $h \in V'$, $\eta \in H' \simeq H$. For $m, n \in \mathbb{N}$, we have

$$|\langle h, z_n(t)\rangle_{V \times V'} - \langle h, z_m(t)\rangle_{V \times V'}|$$

$$\leq |\langle h - \eta, z_n(t) - z_m(t)\rangle_{V \times V'} + \langle \eta, z_n(t) - z_m(t)\rangle_{H \times H}|$$

$$\leq \left(||z_n||_{L^\infty(0,T;V)} + ||z_m||_{L^\infty(0,T;V)}\right) ||h - \eta||_{V'} + ||z_n - z_m||_{C(0,T;H)} ||\eta||_H.$$

By virtue of (2.6) and the density of H in V', we immediately get that $\{\langle h, z_n(t)\rangle\}$ is a Cauchy sequence in $C([0,T])$. □

Remark 2.1 The convergence (2.5) implies in particular that $z_n(t) \to z(t)$ weakly in V for every fixed $t \in [0,T]$.

3 Results on Evolution Problems

In this section we investigate the evolution problems $(2.1)_n$ and (2.1). We address the questions of existence and uniqueness of solutions to these problems and of the dependence of the solutions on the data. We recall (see [1]) that by a solution to $(2.1)_n$ we understand a pair $(x_n, \dot{x}_n) \in C(0,T;V) \times \mathcal{W}$ such that $(2.1)_n$ is satisfied. Similarly for the problem (2.1).

In the proofs of the next two lemmas, we follow the methods used in [1].

Lemma 3.1 *Under hypotheses $H(A)$, $H(B)$, $H(f)$, $H(U)$ and (H_0), for each $n \in \mathbb{N}$, the problems $(2.1)_n$ admit unique solutions which satisfy*

(i) $x_n \in L^\infty(0,T;V)$,

(ii) $\dot{x}_n \in \mathcal{V} \cap L^\infty(0,T;H)$,

(iii) $\ddot{x}_n \in \mathcal{V}'$,

(iv) $A_n(\cdot, \dot{x}_n(\cdot)) \in \mathcal{V}'$.

An analogous statement holds for the problem (2.1).

The proof of Lemma 3.1 follows from standard application of the Galerkin method (cf. e.g. [16], [1]) and the *a priori* estimates given in Lemma 3.2. We only remark that by Lemma 2.1, (i) and (ii) of Lemma 3.1 imply $x_n \in C(0,T;V)$ while from (ii) and (iii) we have $\dot{x}_n \in \mathcal{W}$.

Lemma 3.2 *Let $n \in \mathbb{N}$ be fixed. Under the hypotheses of Lemma 3.1, if $(x_n, \dot{x}_n)$ is a solution to $(2.1)_n$, then*

$$||x_n(t)||^2 + |\dot{x}_n(t)|^2 + ||\dot{x}_n||^2_{\mathcal{W}} \leq C\left(1 + ||x_0^n||^2 + |x_1^n|^2 + ||B_n||^2_{\mathcal{L}(V,V')}\right) \tag{3.1}$$

for any $t \in [0,T]$ with $C > 0$ independent of n. A similar conclusion is valid for the problem (2.1).

Proof: The proof will be given for $(2.1)_n$ and it holds also for (2.1). Let n be fixed and $(x_n, \dot{x}_n)$ be solution to $(2.1)_n$. Then we have

$$\int_0^t \langle \ddot{x}_n(s), \dot{x}_n(s)\rangle\, ds + \int_0^t \langle A_n(s, \dot{x}_n(s)), \dot{x}_n(s)\rangle\, ds$$

$$+ \int_0^t \langle B_n x_n(s), \dot{x}_n(s)\rangle\, ds = \int_0^t \langle f_n(s) u_n(s), \dot{x}_n(s)\rangle\, ds$$

for every $t \in [0,T]$ with $u_n(t) \in U_n(t)$ a.e. Using the integration by parts formula ([5]), Schwarz inequality and $H(A)(3)$, we have

$$|\dot{x}_n(t)|^2 - |\dot{x}_n(0)|^2 + 2c\int_0^t ||\dot{x}_n(s)||^p\, ds - 2d\int_0^t |\dot{x}_n(s)|^2\, ds$$

$$+2\int_0^t \langle B_n x_n(s), \dot{x}_n(s)\rangle\, ds \leq 2\int_0^t |f_n(s) u_n(s)||\dot{x}_n(s)|\, ds.$$

On the other hand, the symmetry of the operator B_n gives

$$\langle B_n x_n(s), \dot{x}_n(s)\rangle = (1/2)(d/dt)\langle B_n x_n(s), x_n(s)\rangle \quad \text{a.e.}$$

Hence and from the fact that $H(f)$, $H(U)$ imply

$$|f_n(s)u_n(s)|^q \le ||f_n(s)||^q_{\mathcal{L}(Y,H)} ||u_n(s)||^q_Y \le C_1(\gamma(s))^q \quad \text{a.e.},$$

using the coerciveness of B_n and applying Cauchy's inequality, we obtain

$$|\dot{x}_n(t)|^2 + 2c\int_0^t ||\dot{x}_n(s)||^p\,ds - 2d\int_0^t |\dot{x}_n(s)|^2\,ds + c'||x_n(t)||^2$$

$$\le |x_1^n|^2 + ||B_n||_{\mathcal{L}(V,V')}||x_0^n||^2 + \frac{2\varepsilon^p}{p}\int_0^t |\dot{x}_n(s)|^p ds + \frac{2C_1\varepsilon^{-q}}{q}\int_0^t (\gamma(s))^q\,ds$$

for every $\varepsilon > 0$. Using the fact that $|\cdot| \le \beta||\cdot||$ with $\beta > 0$ (recall $V \subset H$ continuously) and then taking $\varepsilon := (1/\beta)(pc/2)^{1/p}$, we have

$$|\dot{x}_n(t)|^2 + c\int_0^t ||\dot{x}_n(s)||^p\,ds + c'||x_n(t)||^2 \tag{3.2}$$

$$\le |x_1^n|^2 + ||B_n||_{\mathcal{L}(V,V')}||x_0^n||^2 + 2d\int_0^t |\dot{x}_n(s)|^2\,ds + \frac{(2\beta)^q C_1}{q(pc)^{q-1}}||\gamma||^q_{L^q}.$$

Invoking Gronwall's inequality, it follows from (3.2) that there exists a positive constant C_2 independent of n such that

$$|\dot{x}_n(t)|^2 \le C_2\left(|x_1^n|^2 + ||B_n||\,||x_0^n||^2 + ||\gamma||^q_{L^q}\right), \quad \forall\, t \in [0,T]. \tag{3.3}$$

Now, again from (3.2) and (3.3), we get

$$c||\dot{x}_n||^p_{\mathcal{V}} + c'||x_n(t)||^2 \le C_3\left(1 + |x_1^n|^2 + ||B_n||\,||x_0^n||^2\right) \tag{3.4}$$

for $t \in [0,T]$. Let us take $\zeta \in \mathcal{V}$. Multiplying our equation in $(2.1)_n$ in duality by ζ, we have

$$|\langle\langle \ddot{x}_n, \zeta\rangle\rangle| \le \left(||\widehat{A}_n(\dot{x}_n)||_{\mathcal{V}'} + ||B_n x_n||_{\mathcal{V}'} + ||\widehat{f}_n u_n||_{\mathcal{V}'}\right)||\zeta||_{\mathcal{V}}$$

$$\le \left(2^{1/p}\left(||\alpha_n||_{L^q} + b||\dot{x}_n||^{p/q}_{\mathcal{V}}\right) + C_4||B_n||\,||x_n||_{L^q(0,T;V)} + C_5||\gamma||_{L^q}\right)||\zeta||_{\mathcal{V}}$$

$$\le C_6\left(\sup_n\{||\alpha_n||\} + ||\dot{x}_n||^{p/q}_{\mathcal{V}} + ||B_n||\,||x_n||_{\mathcal{V}} + ||\gamma||_{L^q}\right)||\zeta||_{\mathcal{V}},$$

where by $\widehat{A}_n\colon \mathcal{V} \to \mathcal{V}'$ we denote the Nemitsky operator corresponding to A_n (i.e. $(\widehat{A}_n v)(t) = A_n(t, v(t))$ for a.e. t) and $(\widehat{f}v)(t) = f(t)v(t)$ a.e. Since ζ is arbitrary, we deduce that

$$||\ddot{x}_n||_{\mathcal{V}'} \le C_7\left(1 + ||\dot{x}_n||_{\mathcal{V}} + ||B_n||\,||x_n||_{\mathcal{V}}\right). \tag{3.5}$$

Now, the inequality (3.1) follows immediately from (3.3), (3.4) and (3.5). This finishes the proof. □

Next we present a result on the continuous dependence of solutions to $(2.1)_n$ with respect to perturbations in the data. We will need the following hypotheses.

$\underline{H(A)_1}$: $A_n, A: [0,T] \times V \to V'$ are maps satisfying $H(A)$ and $A_n(\cdot, w(\cdot)) \to A(\cdot, w(\cdot))$ in $s - \mathcal{V}'$ for all $w \in \mathcal{V} \cap L^\infty(0,T;H)$.

$\underline{H(B)_1}$: $B_n \in \mathcal{L}(V,V')$, $n \in \mathbb{N}$ are symmetric, coercive and $B_n \to B$ in $\mathcal{L}(V,V')$.

$\underline{H(f)_1}$: $f_n \in L^\infty(0,T;\mathcal{L}(Y,H))$, $n \in \mathbb{N}$ and $f_n \to f$ in $L^\infty(0,T;\mathcal{L}(Y,H))$.

$\underline{H(U)_1}$: $U_n: [0,T] \to 2^Y$, $n \in \mathbb{N}$ are measurable multifunctions with closed, convex values such that $|U_n(t)| \leq \gamma(t)$ a.e. with $\gamma \in L^q_+(0,T)$ and $U_n(t) \stackrel{K-M}{\longrightarrow} U(t)$ a.e.

$\underline{(H_0)_1}$: the hypothesis (H_0) holds and $x_0^n \to x_0$ in $s - V$, $x_1^n \to x_1$ in $s - H$.

Remark 3.1 It can be shown (see [25]) that under hypothesis $H(U)_1$, the multifunction U is measurable with nonempty, convex and weakly compact values and $|U(t)| \leq \gamma(t)$ a.e.

Theorem 3.1 *If hypotheses $H(A)_1$, $H(B)_1$, $H(f)_1$, $H(U)_1$ and $(H_0)_1$ hold, then the sequence $\{(x_n, \dot{x}_n)\}$ of solutions to $(2.1)_n$ converges to $(x, \dot{x})$ in $C(0,T;V \times H)$, where $(x,\dot{x})$ is a solution to the problem (2.1).*

Proof: Let $(x_n, \dot{x}_n)$ be solution to $(2.1)_n$. From $H(U)_1$ we have $||u_n(t)||_Y \leq \gamma(t)$ a.e., where $u_n \in S^q_{U_n}$. Since $L^q(0,T;Y)$ is reflexive (recall that $1 < q \leq 2$), by Alaoglu's theorem we know that $\{u_n\}$ is sequentially weakly compact in this space. Therefore, by passing to a subsequence if necessary, we may assume that

$$u_n \to u \quad \text{in} \quad w - L^q(0,T;Y). \tag{3.6}$$

From $H(U)_1$ and Papageorgiou ([24], Theorem 4.4), it follows that $S^q_{U_n} \stackrel{K-M}{\longrightarrow} S^q_U$. Hence we deduce that $u \in S^q_U$.

Let $(x, \dot{x})$ be the (unique) solution to (2.1) corresponding to the control $u \in S^q_U$. From (2.1) and $(2.1)_n$, we have

$$\langle \ddot{x}_n(s) - \ddot{x}(s), \dot{x}_n(s) - \dot{x}(s) \rangle + \langle A_n(s, \dot{x}_n(s)) - A(s, \dot{x}(s)), \dot{x}_n(s) - \dot{x}(s) \rangle$$

$$+ \langle B_n x_n(s) - Bx(s), \dot{x}_n(s) - \dot{x}(s) \rangle = \langle f_n(s)u_n(s) - f(s)u(s), \dot{x}_n(s) - \dot{x}(s) \rangle$$

a.e., for every $n \in \mathbb{N}$. Integrating this equality, using the monotonicity of $A_n(s, \cdot)$ and the fact that

$$(d/dt)\langle Bv(t), v(t) \rangle = 2\langle B\dot{v}(t), v(t) \rangle \quad \text{a.e.,} \tag{3.7}$$

we get

$$|\dot{x}_n(t) - \dot{x}(t)|^2 - |\dot{x}_n(0) - \dot{x}(0)|^2 + 2\int_0^t \langle A_n(s, \dot{x}(s)) - A(s, \dot{x}(s)), \dot{x}_n(s) - \dot{x}(s) \rangle \, ds$$

$$+2\int_0^t \langle B_n x_n(s) - Bx_n(s), \dot{x}_n(s) - \dot{x}(s)\rangle\, ds + \langle Bx_n(t) - Bx(t), x_n(t) - x(t)\rangle$$

$$-\langle Bx_n(0) - Bx(0), x_n(0) - x(0)\rangle \leq 2\int_0^t \langle (f_n(s) - f(s))u_n(s), \dot{x}_n(s) - \dot{x}(s)\rangle\, ds$$

$$+2\int_0^t \langle f(s)\,(u_n(s) - u(s))\,, \dot{x}_n(s) - \dot{x}(s)\rangle\, ds,$$

for every $t \in [0,T]$. Hence applying Hölder inequality and using $H(B)_1$, we obtain

$$|\dot{x}_n(t) - \dot{x}(t)|^2 + c'||x_n(t) - x(t)||^2 \leq ||B||\,||x_0^n - x_0|| + |x_1^n - x_1|^2 \tag{3.8}$$

$$+2||\widehat{A}_n(\dot{x}) - \widehat{A}(\dot{x})||_{\mathcal{V}'}||\dot{x}_n - \dot{x}||_{\mathcal{V}} + C||B_n - B||\,||x_n||_{\mathcal{V}}\,||\dot{x}_n - \dot{x}||_{\mathcal{V}}$$

$$+2\int_0^t ||f_n(s) - f(s)||_{\mathcal{L}(Y,H)}||u_n(s)||_Y|\dot{x}_n(s) - \dot{x}(s)|\, ds$$

$$+2\int_0^t \langle u_n(s) - u(s), f^*(s)(\dot{x}_n(s) - \dot{x}(s))\rangle_{Y\times Y'}\, ds, \quad \forall\, t \in [0,T],$$

where C is a positive constant independent of n. From $H(B)_1$ and $(H_0)_1$, by Lemma 3.2, after possible passing to subsequences, we have

$$\dot{x}_n \to z \quad \text{in} \quad w - \mathcal{W} \quad \text{and} \quad s - \mathcal{H}, \quad \text{as} \quad n \to +\infty \tag{3.9}$$

with $z \in \mathcal{W}$ (due to the compactness of the embedding $\mathcal{W} \subset \mathcal{H}$). Since $\mathcal{W} \subset C(0,T;H)$ continuously, we know that

$$|\dot{x}_n(s) - \dot{x}(s)| \leq \overline{c}||\dot{x}_n - \dot{x}||_{\mathcal{W}} \tag{3.10}$$

for all $s \in [0,T]$. On the other hand, from $H(f)_1$ and (3.9), we infer that

$$f^*(\cdot)\dot{x}_n(\cdot) \to f^*(\cdot)z(\cdot) \quad \text{in} \quad s - L^p(0,T;Y'). \tag{3.11}$$

Using (3.6), (3.9), (3.10), (3.11) and our hypotheses, from (3.8) we get $(x_n(t), \dot{x}_n(t)) \to (x(t), \dot{x}(t))$ in $V \times H$, for every $t \in [0,T]$, as $n \to +\infty$. Since the solution to (2.1) is unique, we deduce that the whole sequence $\{(x_n, \dot{x}_n)\}$ converges to $(x, \dot{x})$ in $C(0,T;V \times H)$. The proof is completed. $\square$

4 Existence Theorem for Control Problems

The aim of this section is to demonstrate the existence of optimal pairs for problems $(\mathrm{P})_n$ and (P).

Theorem 4.1 *If hypotheses $H(A)$, $H(B)$, $H(f)$, $H(U)$, (H_0), $H(L)$ and $H(l)$ hold, then the control problems (P) and $(P)_n$, for any n fixed, admit an optimal solution.*

Proof: It will be carried out for problem (P) and holds also for problems $(\mathrm{P})_n$. Let $\{(x_k, u_k)\}_{k\geq 1}$ be a minimizing sequence for (P); i.e. the pairs (x_k, u_k) are admissible for (P) and $\lim_k J(x_k, u_k) = m$. From Lemma 3.2, we know that $\{(x_k, \dot{x}_k)\}_{k\geq 1}$ belongs to a bounded subset of $(L^\infty(0,T;V)\cap W^{1,\infty}(0,T;H)) \times \mathcal{W}$. Extracting a subsequence one has

$$x_k \to x \ \text{ in } \ w - * - L^\infty(0,T;V), \tag{4.1}$$

$$\dot{x}_k \to \dot{x} \ \text{ in } \ w - * - L^\infty(0,T;H) \ \text{ and in } \ w - \mathcal{W}, \tag{4.2}$$

as $k \to +\infty$, where $x \in L^\infty(0,T;V)$ is such that $\dot{x} \in \mathcal{V} \cap L^\infty(0,T;H)$, $\ddot{x} \in \mathcal{V}'$. In view of Lemma 2.1, Remark 2.1 and recalling that $\mathcal{W} \subset \mathcal{H}$ compactly, from (4.1) and (4.2), we deduce that

$$x_k \to x \ \text{ in } \ C(0,T; w - V), \tag{4.3}$$

$$\dot{x}_k \to \dot{x} \ \text{ in } \ s - \mathcal{H}, \ \text{ as } \ k \to +\infty. \tag{4.4}$$

From (4.3) and the fact that $x_k(0) = x_0$, it follows immediately that

$$x(0) = x_0. \tag{4.5}$$

Note that by the continuity of the embedding $\mathcal{W} \subset C(0,T;H)$ and Mazur's lemma, the convex set $\mathcal{W}_{x_1} := \{w \in \mathcal{W} \,|\, w(0) = x_1\}$ is a weakly closed subset of $\mathcal{W}$. Since $\dot{x}_k \in \mathcal{W}_{x_1}$, from (4.2), we have

$$\dot{x}(0) = x_1. \tag{4.6}$$

We claim that

$$\dot{x}_k \to \dot{x} \ \text{ in } \ C(0,T;H), \ \text{ as } \ k \to +\infty. \tag{4.7}$$

(We emphasize that (4.7) does not follow from (4.2) merely, since the embedding $\mathcal{W} \subset C(0,T;H)$ is not compact, cf. Remark 1.1). To prove (4.7), we will first show that

$$\limsup_k \langle\langle \widehat{A}(\dot{x}_k), \dot{x}_k - \dot{x} \rangle\rangle \leq 0. \tag{4.8}$$

Multiplying the equation $\ddot{x}_k + \widehat{A}(\dot{x}_k) + Bx_k = \widehat{f}u_k$ in duality with $\dot{x}_k - \dot{x}$ and integrating by parts, we obtain

$$(1/2)|\dot{x}_k(T) - \dot{x}(T)|^2 - (1/2)|\dot{x}_k(0) - \dot{x}(0)|^2 - \langle\langle \ddot{x}, \dot{x}_k - \dot{x} \rangle\rangle \tag{4.9}$$

$$+\langle\langle \widehat{A}(\dot{x}_k), \dot{x}_k - \dot{x}\rangle\rangle + \langle\langle Bx_k, \dot{x}_k - \dot{x}\rangle\rangle = \langle\langle \widehat{f}u_k, \dot{x}_k - \dot{x}\rangle\rangle.$$

Next, because of $H(U)$, we know that S^q_U is sequentially weakly compact. Hence, we may suppose that

$$u_k \to u \quad \text{in} \quad w - L^q(0,T;Y) \tag{4.10}$$

and $u \in S^q_U$. Since $\widehat{f} \in \mathcal{L}(L^q(0,T;Y), \mathcal{H}')$, we readily deduce that

$$\widehat{f}u_k \to \widehat{f}u \quad \text{in} \quad w - \mathcal{H}', \quad \text{as} \quad k \to +\infty. \tag{4.11}$$

Using (4.2), (4.4), (4.6), (4.11), from (4.9) after dropping the positive term, we obtain

$$\limsup_k \Big(\langle\langle \widehat{A}(\dot{x}_k), \dot{x}_k - \dot{x}\rangle\rangle + \langle\langle Bx_k, \dot{x}_k - \dot{x}\rangle\rangle \Big) \tag{4.12}$$

$$\leq \limsup_k \Big(\langle\langle \widehat{f}_k u_k, \dot{x}_k - \dot{x}\rangle\rangle + \langle\langle \ddot{x}, \dot{x}_k - \dot{x}\rangle\rangle + (1/2)|x_1 - \dot{x}(0)|^2 \Big) = 0.$$

Utilizing the equality (3.7) and the coerciveness of B, we get

$$c'||x_k(T) - x(T)||^2 + 2\langle\langle Bx, \dot{x}_k - \dot{x}\rangle\rangle + \langle B(x_k(0) - x(0)), x_k(0) - x(0)\rangle$$

$$\leq 2\langle\langle Bx_k, \dot{x}_k - \dot{x}\rangle\rangle.$$

From the above inequality, (4.2), (4.3) and (4.5), it follows that

$$0 \leq \liminf_k \langle\langle Bx_k, \dot{x}_k - \dot{x}\rangle\rangle. \tag{4.13}$$

Combining (4.12) and (4.13), we easily get (4.8).

Let

$$\rho_k(t) = \langle A(t, \dot{x}_k(t)) - A(t, \dot{x}(t)), \dot{x}_k(t) - \dot{x}(t)\rangle$$

for a.e. $t \in (0,T)$. Applying (4.2) and (4.8), Fatou's lemma ensures that

$$0 \leq \int_0^T \liminf_k \rho_k(s)\, ds \leq \liminf_k \int_0^T \rho_k(s)\, ds \leq \limsup_k \int_0^T \rho_k(s)\, ds$$

$$\leq \limsup_k \langle\langle \widehat{A}(\dot{x}_k), \dot{x}_k - \dot{x}\rangle\rangle - \lim_k \langle\langle \widehat{A}(\dot{x}), \dot{x}_k - \dot{x}\rangle\rangle \leq 0.$$

From the above inequalities, we deduce that $\lim_k \int_0^T \rho_k(s)\, ds = 0$, which implies, thanks to the property $\rho_k(t) \geq 0$ a.e., that $\rho_k \to 0$ strongly in $L^1(0,T)$. Thus, we may assume, passing to next subsequence if necessary, that

$$\rho_k(t) \to 0 \quad \text{a.e.} \quad t \in (0,T). \tag{4.14}$$

Using hypothesis $H(A)(3)(4)$, for a.e. $t \in (0,T)$, we have

$$\rho_k(t) \geq c||\dot{x}_k(t)||^p - d|\dot{x}_k(t)|^2 - a(t)||\dot{x}(t)|| - b||\dot{x}(t)||\,||\dot{x}_k(t)||^{p-1}$$
$$- \left(a(t) + b||\dot{x}(t)||^{p-1}\right)||\dot{x}_k(t)|| + c||\dot{x}(t)||^p - d|\dot{x}(t)|^2.$$

Combining (4.14) and the last inequality, it follows that $\{||\dot{x}_k(t)||\}$ is bounded for a.e. $t \in (0,T)$ and k sufficiently large. So we have shown that the sequence $\{\dot{x}_k\}$ belongs to a bounded set of $L^\infty(0,T;V)$. Moreover, since $\{\ddot{x}_k\}$ lies in a bounded subset of $\mathcal{V}'$ (cf. Lemma 3.2) and $V \subset H$ compactly, we infer from the version of the Arzelá-Ascoli theorem (see [29], Corollary 4, p. 85) that (4.7) holds. This finishes the proof of the claim.

The convergences (4.3), (4.4), (4.10) and $H(L)$ allow us to deduce, by applying Theorem 2.1 of Balder [4], that

$$\int_0^T L(t,x(t),\dot{x}(t),u(t))\,dt \leq \liminf_k \int_0^T L(t,x_k(t),\dot{x}_k(t),u_k(t))\,dt. \quad (4.15)$$

From (4.3), (4.7) and hypothesis $H(l)$, we have

$$l(x(T),\dot{x}(T)) \leq \liminf_k l(x_k(T),\dot{x}_k(T)),$$

which together with (4.15) implies

$$J(x,u) \leq \liminf_k J(x_k,u_k) = m.$$

In order to show that (x,u) is the optimal "state-control" pair, it is now sufficient to prove that (x,u) is admissible for (P). To this end, we observe in view of hypothesis $H(A)(4)$ that $\{\widehat{A}\dot{x}_k\}$ is bounded in $\mathcal{V}'$. So we may suppose that

$$\widehat{A}(\dot{x}_k) \to \chi \;\text{ in }\; w - \mathcal{V}' \quad (4.16)$$

with $\chi \in \mathcal{V}'$. From $H(A)(2)$, it follows (see Proposition 2.5 in Chapter 2 of [15]) that $\widehat{A}$ has the generalized pseudomonotone property. Therefore, (4.2), (4.8) and (4.16) yield $\chi = \widehat{A}(\dot{x})$. As a consequence of (4.2), (4.3), (4.11) and (4.16), we pass to the limit in the equation

$$\langle\langle \ddot{x}_k, \eta\rangle\rangle + \langle\langle \widehat{A}(\dot{x}_k), \eta\rangle\rangle + \langle\langle Bx_k, \eta\rangle\rangle = \langle\langle \widehat{f}u_k, \eta\rangle\rangle, \quad \forall\, \eta \in \mathcal{V}$$

and we get $\langle\langle \ddot{x}, \eta\rangle\rangle + \langle\langle \widehat{A}(\dot{x}), \eta\rangle\rangle + \langle\langle Bx, \eta\rangle\rangle = \langle\langle \widehat{f}u, \eta\rangle\rangle$, for every $\eta \in \mathcal{V}$. Thus $\ddot{x}(t) + A(t,\dot{x}(t)) + Bx(t) = f(t)u(t)$ a.e., which together with (4.5), (4.6) and $u \in S_U^q$ means that the pair (x,u) is admissible for (P). Hence we conclude that $J(x,u) = m$. The proof of Theorem 4.1 is completed. □

Remark 4.1 For a slightly different class of second order state equations (more general right hand sides with L^∞ controls), a result on existence of optimal solutions was proved in Theorem 3.1 of [1].

5 Sensitivity of Control Problems

This section is devoted to state and prove the results on the asymptotic behavior of the sequence of control problems $(\mathrm{P})_n$, as n tends to infinity. We admit the perturbations appear in all the data of the control problem i.e. in the operators and initial conditions of the state equations, in the objective functionals, and in the control constraint sets. We provide the stability theorems for the sets of optimal solutions and for the reachable sets. We also show the convergence of the minimal values of $(\mathrm{P})_n$ to a minimal value of (P).

We introduce the following notation. For every $n \in \mathbb{N}$ fixed, let us consider the map $p_n : L^q(0,T;Y) \to C(0,T;V \times H) \times L^q(0,T;Y) \subseteq L^1(0,T;V \times H \times Y)$ given by $p_n(u) = (x_n, \dot{x}_n, u)$, where $(x_n, \dot{x}_n)$ is the solution of $(2.1)_n$ corresponding to the control u. Analogously, we define map p which is associated with problem (2.1). Then, let $F_n, F : L^q(0,T;Y) \to \mathbb{R} \cup \{+\infty\}$ be functionals respectively given by

$$F_n(u) = \widetilde{J}_n(p_n(u)) + \chi_{C(0,T;V \times H) \times S^q_{U_n}}(p_n(u)),$$

$$F(u) = \widetilde{J}(p(u)) + \chi_{C(0,T;V \times H) \times S^q_U}(p(u)),$$

where we write $\widetilde{J}_n(x_n, \dot{x}_n, u) = J_n(x_n, u)$, $\widetilde{J}(x, \dot{x}, u) = J(x, u)$ and χ_E denotes the indicator function of a set E; i.e. $\chi_E(e) = 0$ if $e \in E$ and $\chi_E(e) = +\infty$, otherwise. Under these notations, we have $m_n = \inf\{F_n(u) : u \in L^q(0,T;Y)\}$ and $m = \inf\{F(u) : u \in L^q(0,T;Y)\}$. Moreover, we introduce $G_n, G : L^1(0,T;V \times H \times Y) \to \mathbb{R} \cup \{+\infty\}$ defined as follows:

$$G_n(x,y,u) = \int_0^T L_n(t, x(t), y(t), u(t))\, dt$$

and

$$G(x,y,u) = \int_0^T L(t, x(t), y(t), u(t))\, dt.$$

In the first theorem we will study the τ-convergence of functionals F_n. To this end, we need the following assumptions.

$\underline{H(L)_1}$: $L_n : [0,T] \times V \times H \times Y \to \mathbb{R} \cup \{+\infty\}$, $n \in \mathbb{N}$ satisfy $H(L)$ and $L_n(t,\cdot,\cdot,\cdot) \xrightarrow{\tau} L(t,\cdot,\cdot,\cdot)$ a.e.

$\underline{H(l)_1}$: the hypothesis $H(l)$ holds and $l_n \xrightarrow{c} l$ in $s-(V \times H)$.

Theorem 5.1 *If hypotheses $H(A)_1$, $H(B)_1$, $H(f)_1$, $H(U)_1$, $(H_0)_1$, $H(L)_1$ and $H(l)_1$ hold, then $F_n \xrightarrow{\tau} F$, as $n \to +\infty$.*

Proof: It is enough to show that

(i) if $u \in S_U^q$, $u_n \in S_{U_n}^q$ and $u_n \to u$ in $w - L^q(0,T;Y)$, then $F(u) \leq \liminf_n F_n(u_n)$;

(ii) if $u \in S_U^q$, then there exists $u_n \in S_{U_n}^q$ such that $u_n \to u$ in $s - L^q(0,T;Y)$ and $F(u) = \lim_n F_n(u_n)$.

We first prove (i). Let $u \in S_U^q$, $u_n \in S_{U_n}^q$ and

$$u_n \to u \text{ in } w - L^q(0,T;Y), \text{ as } n \to +\infty. \tag{5.1}$$

Denote by $(x_n, \dot{x}_n)$ (and $(x, \dot{x})$ respectively) the unique solutions to $(2.1)_n$ (and (2.1) respectively) corresponding to control u_n (and u respectively). Using Theorem 3.1, we know that

$$(x_n, \dot{x}_n) \to (x, \dot{x}) \text{ in } C(0,T;V \times H). \tag{5.2}$$

From $H(l)_1$ and (5.2), we easily get

$$l(x(T), \dot{x}(T)) = \lim_n l_n(x_n(T), \dot{x}_n(T)). \tag{5.3}$$

Because of $H(L)_1$ and Theorem 3.1 of Salvadori [28], we have $G_n \xrightarrow{\tau} G$, as $n \to +\infty$. Thus, by the definition of τ-convergence, using (5.1) and (5.2), we have

$$G(x, \dot{x}, u) \leq \liminf_n G_n(x_n, \dot{x}_n, u_n). \tag{5.4}$$

From (5.3) and (5.4) we deduce that $F(u) \leq \liminf_n F_n(u_n)$.

We demonstrate (ii). Let $u \in S_U^q$. We define $u_n \in S_{U_n}^q$ such that $d(u, S_{U_n}^q) = ||u - u_n||_{L^q(0,T;Y)}$. Such u_n exists, since $S_{U_n}^q$ is compact in $w - L^q(0,T;Y)$ and $L^q(0,T;Y)$ with $1 < q \leq 2$ is reflexive. From $H(U)_1$ and Theorem 4.4 of Papageorgiou [24], we know that $S_{U_n}^q \xrightarrow{K-M} S_U^q$. Hence and from the fact that $u_n \in S_{U_n}^q$, we have $d(u, S_{U_n}^q) \to 0$, which implies that

$$u_n \to u \text{ in } s - L^q(0,T;Y), \text{ as } n \to +\infty. \tag{5.5}$$

As in part (i) of the proof, by Theorem 3.1, we obtain that the convergence (5.2) of solutions of $(2.1)_n$ corresponding to u_n holds and (5.3) is satisfied. Moreover, since $G_n \xrightarrow{\tau} G$, we find a sequence $\{(\widetilde{x}_n, \widetilde{y}_n, \widetilde{u}_n)\}_{n \geq 1} \subseteq L^1(0,T;V \times H \times Y)$ such that

$$(\widetilde{x}_n, \widetilde{y}_n, \widetilde{u}_n) \to (x, \dot{x}, u) \text{ in } s - L^1(0,T;V \times H \times Y) \tag{5.6}$$

and $\lim_n G_n(\widetilde{x}_n, \widetilde{y}_n, \widetilde{u}_n) = G(x, \dot{x}, u)$. Due to $H(L)_1$, $G_n(\cdot,\cdot,\cdot)$ are locally equilipschitzean and therefore

$$|G_n(x_n, \dot{x}_n, u_n) - G_n(\widetilde{x}_n, \widetilde{y}_n, \widetilde{u}_n)|$$

$$\leq M\left(||x_n - \widetilde{x}_n||_{L^1(0,T;V)} + ||\dot{x}_n - \widetilde{y}_n||_{L^1(0,T;H)} + ||u_n - \widetilde{u}_n||_{L^1(0,T;Y)}\right).$$

In view of (5.5), (5.2), (5.6), we obtain $\lim_n G_n(x_n, \dot{x}_n, u_n) = G(x, \dot{x}, u)$. Hence and from (5.3), we immediately have $\lim_n F_n(u_n) = F(u)$. This finishes the proof of (ii) and completes the proof of the theorem. □

Remark 5.1 We observe that it is not true in general that the τ-limit of a sum of two τ-convergent sequences of functionals is the sum of their τ-limits. This property holds if we require more than τ-convergence for one of these sequences, for instance, the continuous convergence. Such a situation is met in Theorem 5.1; compare the convergences in $H(L)_1$ and $H(l)_1$. For conditions implying hypothesis $H(L)_1$, we refer to Attouch [3].

Theorem 5.2 *If the hypotheses of Theorem 5.1 hold, then* $m_n \to m$, *as* $n \to +\infty$.

Proof: Let $u_n \in S^q_{U_n}$ be such that $m_n = F_n(u_n)$. From $H(U)_1$, we may assume that $u_n \to u$ weakly in $L^q(0,T;Y)$. The condition (i) of the proof of Theorem 5.1 yields

$$m \leq F(u) \leq \liminf_n F_n(u_n) = \liminf_n m_n. \tag{5.7}$$

Next, let $u \in S^q_U$ be such that $m = F(u)$. From the step (ii) of the proof of Theorem 5.1, we know that there exists $u_n \in S^q_{U_n}$ such that $u_n \to u$ in $s - L^q(0,T;Y)$ and

$$m = \lim_n F_n(u_n) \geq \lim_n \{\inf F_n(u) : u \in L^q(0,T;Y)\} = \lim_n m_n. \tag{5.8}$$

The thesis of the theorem follows from (5.7) and (5.8). □

We show a stability result for the set of optimal solutions. We introduce the following sets:

$$\mathcal{O}_n = \{(x, \dot{x}, u) \,|\, ((x, \dot{x}), u) \text{ is an optimal pair for } (\mathrm{P})_n\}, \quad n \in \mathbb{N},$$

$$\mathcal{O} = \{(x, \dot{x}, u) \,|\, ((x, \dot{x}), u) \text{ is an optimal pair for } (\mathrm{P})\}.$$

Under the hypotheses of Theorem 5.1, the sets $\mathcal{O}_n$, $\mathcal{O}$ are nonempty subsets of $C(0,T;V \times H) \times L^q(0,T;Y)$ (see Theorem 4.1).

Theorem 5.3 *Under the same hypotheses as in Theorem 5.1, we have* $\limsup_n \mathcal{O}_n \subseteq \mathcal{O}$, *where* $C(0,T;V \times H)$ *is endowed with its norm topology and in* $L^q(0,T;Y)$ *we use the weak topology.*

Proof: Let $(x, \dot{x}, u) \in \limsup_n \mathcal{O}_n$. By the definition we find $(x_{n_k}, \dot{x}_{n_k}, u_{n_k}) \in \mathcal{O}_{n_k}$ such that $(x_{n_k}, \dot{x}_{n_k}) \to (x, \dot{x})$ in $C(0,T; V \times H)$ and $u_{n_k} \to u$ weakly in $L^q(0,T;Y)$. So $(x_{n_k}, \dot{x}_{n_k}, u_{n_k})$ is admissible for $(\mathrm{P})_{n_k}$ and $m_{n_k} = F_{n_k}(u_{n_k})$. From Theorem 5.2, we get $m = \lim_k m_{n_k} = \lim_k F_{n_k}(u_{n_k})$. Since $S^1_{U_n} \xrightarrow{K-M} S^1_U$, $u_{n_k} \in S^q_{U_{n_k}}$ and u_{n_k} converges weakly, we have $u \in S^q_U$. As in Theorem 3.1, we obtain that $(x, \dot{x})$ is a solution to (2.1) corresponding to the control u. Then, Theorem 5.1 gives $m \leq F(u) \leq \liminf_k F_{n_k}(u_{n_k})$. Hence $m = F(u)$ which implies $(x, \dot{x}, u) \in \mathcal{O}$. □

From Lemma 3.2, Theorems 3.1 and 5.3, we have the following

Corollary 5.1 *Under the hypotheses of Theorem 5.1, every sequence* $\{((x_n, \dot{x}_n), u_n)\}$ *of optimal solutions to* $(\mathrm{P})_n$ *possesses a subsequence which is convergent in* $C(0,T; V \times H) \times (w - L^q(0,T;Y))$ *topology to an optimal solution of the limit problem* (P). *If the limit problem admits the unique solution, then the whole sequence* $\{((x_n, \dot{x}_n), u_n)\}$ *converges to this solution. Moreover, the minimal values* $\min(\mathrm{P})_n$ *converge to the minimal value* $\min(\mathrm{P})$.

In order to state a theorem on the stability of the reachable sets, for every $t \in [0,T]$, we introduce

$$\begin{aligned} R_n(t) = \{v \in V \times H : v = (x_n(t), \dot{x}_n(t)),\ &\text{where } (x_n, \dot{x}_n) \text{ is a solution} \\ &\text{to } (2.1)_n \text{ corresponding to some } u \in S^q_{U_n}\}, \\ R(t) = \{v \in V \times H : v = (x(t), \dot{x}(t)),\ &\text{where } (x, \dot{x}) \text{ is a solution} \\ &\text{to (2.1) corresponding to some } u \in S^q_U\}. \end{aligned}$$

Theorem 5.4 *If the hypotheses of Theorem 5.1 hold, then for every* $t \in [0,T]$, *we have* $R_n(t) \xrightarrow{K-M} R(t)$, *as* $n \to +\infty$.

Proof: We first prove that $w - \limsup_n R_n(t) \subseteq R(t)$ for all $t \in [0,T]$. To this end, let $t \in [0,T]$ and $v \in w - \limsup_n R_n(t)$. So we find $v_{n_k} \in R_{n_k}(t)$ such that $v_{n_k} \to v$ weakly in $V \times H$. Moreover, $v_{n_k} = (x_{n_k}(t), \dot{x}_{n_k}(t))$, where $(x_{n_k}, \dot{x}_{n_k})$ is the solution to $(2.1)_{n_k}$ for some $u_{n_k} \in S^q_{U_{n_k}}$. By passing to a further subsequence if necessary, we suppose that $u_{n_k} \to u$ weakly in $L^q(0,T;Y)$, where $u \in L^q(0,T;Y)$. Due to the fact that $S^1_{U_n} \xrightarrow{K-M} S^1_U$, we have $u \in S^q_U$. From Theorem 3.1, we get $(x_{n_k}(t), \dot{x}_{n_k}(t)) \to (x(t), \dot{x}(t))$ strongly in $V \times H$ for every $t \in [0,T]$, where $(x, \dot{x})$ is the solution to (2.1) corresponding to control u. We obtain $v = (x(t), \dot{x}(t))$ i.e. $v \in R(t)$.

We now show that $R(t) \subseteq s - \liminf_n R_n(t)$ for every $t \in [0,T]$. Let $t \in [0,T]$, $v \in R(t)$. Then $v = (x(t), \dot{x}(t))$, where $(x, \dot{x})$ solves (2.1) with some $u \in S^q_U$. We can find $u_n \in S^q_{U_n}$, defined as in the proof of

Theorem 5.1 such that $u_n \to u$ weakly in $L^q(0,T;Y)$. Again by Theorem 3.1, we know that the sequence of solutions $(x_n, \dot{x}_n)$ corresponding to u_n converges in $C(0,T;V \times H)$ to $(x,\dot{x})$. We have $\lim\limits_{n\to+\infty} d(v, R_n(t)) \leq \lim\limits_{n\to+\infty} ||(x_n(t), \dot{x}_n(t)) - (x(t), \dot{x}(t))||_{V\times H} = 0$. Hence we infer that $v \in s - \liminf\limits_n R_n(t)$. □

6 An Example

In this section we present an example which illustrates the application of the abstract framework and of results of the theory developed in the previous sections.
Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\Gamma = \partial\Omega$, $Q = (0,T) \times \Omega$, $\Sigma = (0,T) \times \Gamma$. We consider the following sequence of nonlinear hyperbolic optimal control problems:

$$\begin{gathered} J_n(\varphi, u) = l_n(\varphi(T), \varphi_t(T)) \\ + \int_Q L_n(t, x, \varphi(t,x), \varphi_t(t,x), u(t,x))\, dtdx \to \inf = \widetilde{m}_n, \end{gathered} \tag{6.1$_n$}$$

where

$$\begin{cases} \dfrac{\partial^2 \varphi}{\partial t^2} - \operatorname{div}\left(a^n(t,x,D\varphi_t)\right) - \displaystyle\sum_{i,j=1}^N D_i\left(b_{ij}^n(x) D_j \varphi\right) = \\ \qquad\qquad = (g_n(t,x), u(t,x)) \ \text{ a.e. in } Q \\ \varphi|_\Sigma = 0, \quad \varphi(0,x) = \varphi_0^n(x), \quad \varphi_t(0,x) = \varphi_1^n(x), \\ ||u(t,\cdot)||_{L^2(\Omega;\mathbb{R}^k)} \leq r_n(t) \ \text{ a.e.} \end{cases} \tag{6.2$_n$}$$

and the limit problem

$$\begin{gathered} J(\varphi, u) = l(\varphi(T), \varphi_t(T)) \\ + \int_Q L(t, x, \varphi(t,x), \varphi_t(t,x), u(t,x))\, dtdx \to \inf = \widetilde{m} \end{gathered} \tag{6.1}$$

such that

$$\begin{cases} \dfrac{\partial^2 \varphi}{\partial t^2} - \operatorname{div}\left(a(t,x,D\varphi_t)\right) - \displaystyle\sum_{i,j=1}^N D_i\left(b_{ij}(x) D_j \varphi\right) = \\ \qquad\qquad = (g(t,x), u(t,x)) \ \text{ a.e. in } Q \\ \varphi|_\Sigma = 0, \quad \varphi(0,x) = \varphi_0(x), \quad \varphi_t(0,x) = \varphi_1(x), \\ ||u(t,\cdot)||_{L^2(\Omega;\mathbb{R}^k)} \leq r(t) \ \text{ a.e.} \end{cases} \tag{6.2}$$

The parentheses on the right hand sides of the state equations denote the inner product in $\mathbb{R}^k$, $D_i = \partial/\partial x_i$, $i = 1, \ldots, N$, $D\varphi = (D_1\varphi, \ldots, D_N\varphi)$ is the gradient of φ. Let $V = H_0^1(\Omega)$, $H = L^2(\Omega)$ and $V' = H^{-1}(\Omega)$. We know that (V, H, V') is an evolution triple with compact embeddings. Let $Y = L^2(\Omega; \mathbb{R}^k)$ and $p = 2$. Given positive real constant c_1 and functions $c_2 \in L_+^2(Q)$, $c_3 \in L_+^\infty(\Omega)$, we define the class $\mathbb{M} = \mathbb{M}(c_1, c_2, c_3)$ of maps $a: Q \times \mathbb{R}^N \to \mathbb{R}^N$ which satisfy

(j) $a(\cdot, \cdot, \xi)$ is measurable for every $\xi \in \mathbb{R}^N$,

(jj) $a(t, x, \cdot)$ is continuous a.e. in Q,

(jjj) $(a(t, x, \xi_1) - a(t, x, \xi_2), \xi_1 - \xi_2)_{\mathbb{R}^N} \geq 0$ a.e. in Q, $\forall\, \xi_1, \xi_2 \in \mathbb{R}^N$,

(jv) $(a(t, x, \xi), \xi)_{\mathbb{R}^N} \geq c_1 |\xi|^2$ a.e. in Q, $\forall\, \xi \in \mathbb{R}^N$,

(v) $|a(t, x, \xi)| \leq c_2(t, x) + c_3(x)|\xi|$ a.e. in Q, $\forall\, \xi \in \mathbb{R}^N$.

Given $\lambda > 0$, we denote by $E(\lambda)$ the class of operators $B: H_0^1(\Omega) \to H^{-1}(\Omega)$ such that $B = -\sum_{i,j=1}^N D_i\,(b_{ij}(x) D_j)$ for some $b_{ij} \in L^\infty(\Omega)$ and $b_{ij} = b_{ji}$, $\lambda |\xi|^2 \leq \sum_{i,j=1}^N b_{ij}(x)\xi_i\xi_j$ for $\xi \in \mathbb{R}^N$.

We introduce the following hypotheses on the data of problems under consideration.

$\underline{H(A)_2}$: $a, a^n: Q \times \mathbb{R}^N \to \mathbb{R}^N$, $n \in \mathbb{N}$, are such that

(1) $a \in \mathbb{M}(c_1, c_2, c_3)$, $a^n \in \mathbb{M}(c_1, c_2^n, c_3)$ with $c_1 > 0$, $\{c_2^n, c_2\} \subset L_+^2(Q)$, $\sup\limits_n \|c_2^n\|_{L^2} < +\infty$ and $c_3 \in L_+^\infty(\Omega)$,

(2) $a^n(t, \cdot, Dv(\cdot)) \to a(t, \cdot, Dv(\cdot))$ a.e. for every $v \in V$.

$\underline{H(B)_2}$: $B, B_n \in E(\lambda)$, $n \in \mathbb{N}$ with some $\lambda > 0$ and the coefficients b_{ij}^n of B_n satisfy $b_{ij}^n \to b_{ij}$ in $L^\infty(\Omega)$ for all $i, j = 1, \ldots, n$, where b_{ij} are the coefficients of B.

$\underline{H(g)}$: $g, g_n \in L^\infty(0, T; Y))$, $n \in \mathbb{N}$ satisfy $g_n(t, \cdot) \to g(t, \cdot)$ in $s - Y$ uniformly with respect to t.

$\underline{H(r)}$: $r, r_n \in L_+^q(0, T)$, $n \in \mathbb{N}$ are such that $r_n(t) \leq \eta(t)$ a.e. with an $\eta \in L_+^q(0, T)$, $r_n(t) \to r(t)$ a.e.

$\underline{(H_0)_2}$: $\varphi_0^n, \varphi_0 \in V$, $\varphi_0^n \to \varphi_0$ in $s - V$, $\varphi_1^n, \varphi_1 \in H$, $\varphi_1^n \to \varphi_1$ in $s - H$.

$\underline{H(L)_2}$: $L, L_n: Q \times \mathbb{R}^{k+2} \to \mathbb{R} \cup \{+\infty\}$ are normal convex (in (w, z, u)) integrands such that

$$\begin{aligned} \phi_1(t,x) \;&+\; \phi_2(t,x)\left(|w|^2 + |z|^2\right) + \phi_3(t,x)\|u\|^2 \\ &\leq\; L(t,x,w,z,u), L_n(t,x,w,z,u) \\ &\leq\; \psi_1(t,x) + \psi_2(t,x)\left(|w|^2 + |z|^2\right) \\ &+\; \psi_3(t,x)\|u\|^2 \end{aligned}$$

with $\phi_1, \psi_1 \in L^2(Q)$, $\phi_2, \phi_3, \psi_2, \psi_3 \in L^\infty(Q)$ and $L_n(t,x,\cdot,\cdot,\cdot) \xrightarrow{\tau} L(t,x,\cdot,\cdot,\cdot)$ a.e. on Q.

We consider the following Dirichlet forms associated to the nonlinear operators appearing in state equations $(6.2)_n$ and (6.2):

$$d_n(t,u,v) = \int_\Omega (a^n(t,x,Du), Dv)\, dx, \quad n \in \mathbb{N}, \quad u,v \in V,$$

$$d(t,u,v) = \int_\Omega (a(t,x,Du), Dv)\, dx, \quad u,v \in V.$$

Using $H(A)_2$ and Hölder inequality, we have

$$|d_n(t,u,v)| \le \left(\int_\Omega |a^n(t,x,Du)|^2\, dx \right)^{1/2} \left(\int_\Omega |Dv|^2\, dx \right)^{1/2} \le$$

$$\le \left(2 \left(\int_\Omega |c_2^n(t,x)|^2\, dx + \int_\Omega c_3^2(x)|Du|^2 \right) \right)^{1/2} ||v|| \le \left(\widetilde{\alpha}_n(t) + \widetilde{b} \right) ||u||\, ||v||,$$

where $\widetilde{\alpha}_n(t) = \sqrt{2}||c_2^n(t,\cdot)||_{L^2} \in L^2_+(0,T)$ and $\widetilde{b} = \sqrt{2}||c_3||_{L^\infty}$. Similarly $|d(t,u,v)| \le (\widetilde{\alpha}(t) + \widetilde{b}||u||)||v||$ with $\widetilde{\alpha}(t) = \sqrt{2}||c_2(t,\cdot)||_{L^2} \in L^2_+(0,T)$. Therefore there exist operators A_n and A from $[0,T] \times V$ to V' such that $d_n(t,u,v) = \langle A_n(t,u), v\rangle$, $d(t,u,v) = \langle A(t,u), v\rangle$ and moreover they satisfy condition $H(A)(4)$.

Next, making use of separability of V', from the Pettis measurability theorem, we get that $A_n(\cdot,v), A(\cdot,v)$ are measurable. The continuity of $A(t,\cdot)$ follows from the relation

$$||A(t,v_k) - A(t,v)||_{V'} = \sup_{||z|| \le 1} |\langle A(t,v_k) - A(t,v), z\rangle| \le$$

$$\le \sup_{||z|| \le 1} \int_\Omega |a(t,x,Dv_k) - a(t,x,Dv)||Dz|\, dx \to 0$$

which is true for every $v_k \to v$ in $s - V$. Analogously one proves the continuity of $A_n(t,\cdot)$. The monotonicity of $A_n(t,\cdot)$ and $A(t,\cdot)$ readily follows from (jjj) of the definition of the class $\mathbb{M}$, while (jv) implies that both A_n and A satisfy $H(A)(3)$ with $c = c_1$ and $d = 0$.

Let $w \in V$. In view of $H(A)_2(2)$ we obtain

$$||A_n(t,w) - A(t,w)||_{V'} \le \sup_{||z|| \le 1} \int_\Omega |a^n(t,x,Dw) - a(t,x,Dw)||Dz|\, dx \to 0,$$

for a.e. t. Applying the dominated convergence theorem, we infer that for any $v \in \mathcal{V} \cap L^\infty(0,T;H)$, we have $\lim_{n\to+\infty} ||A_n(\cdot,v(\cdot)) - A(\cdot,v(\cdot))||_{\mathcal{V}'} = 0$. So

we have verified hypothesis $H(A)_1$ for the operators A_n and A. We also observe that hypothesis $H(B)_1$ holds as a consequence of $H(B)_2$.
Subsequently, we define $f(t), f_n(t): Y \to H$ by $(f(t)u)(\cdot) = (g(t,\cdot), u(\cdot))$ and $(f_n(t)u)(\cdot) = (g_n(t,\cdot), u(\cdot))$, respectively. It is clear that $f, f_n \in L^\infty(0,T;\mathcal{L}(Y,H))$. Since

$$||f_n(t) - f(t)||_{\mathcal{L}(Y,H)} = \sup_{||z||_Y \le 1} |(f_n(t) - f(t))z|_H \le$$

$$\le \sup_{||z||_Y \le 1} ||g_n(t,\cdot) - g(t,\cdot)||_Y ||z||_Y \le ||g_n(t,\cdot) - g(t,\cdot)||_Y \to 0$$

uniformly in t, by hypothesis $H(g)$, we deduce that $H(f)_1$ holds.
Let us define $U_n(t) = \{u \in Y : ||u||_Y \le r_n(t)\}$ and $U(t) = \{u \in Y : ||u||_Y \le r(t)\}$. Due to hypothesis $H(r)$, it is easy to check that $H(U)_1$ is satisfied.

Finally, we define $\widetilde{L}_n$ and $\widetilde{L}$ by putting

$$\widetilde{L}_n(t,v,w,u) = \int_\Omega L_n(t,x,v(x),w(x),u(x))\,dx, \quad n \in \mathbb{N},$$

$$\widetilde{L}(t,v,w,u) = \int_\Omega L(t,x,v(x),w(x),u(x))\,dx,$$

where $(v,w,u) \in V \times H \times Y$. From Theorem 3.1 of Salvadori [28] and τ-convergence of L_n to L (see $H(L)_2$), it follows that $\widetilde{L}_n(t,\cdot,\cdot,\cdot) \xrightarrow{\tau} \widetilde{L}(t,\cdot,\cdot,\cdot)$ a.e. Moreover the growth conditions on $\widetilde{L}_n, \widetilde{L}$ follow from $H(L)_2$. Thus $\widetilde{L}_n$ and $\widetilde{L}$ satisfy $H(L)_1$.

We observe that the problems $(6.1)_n$ and (6.1) can be formulated by using the above notation in the abstract forms $(\mathrm{P})_n$ and (P), respectively. From Theorems 4.1, 5.4 and Corollary 5.1, we conclude

Corollary 6.1 *If hypotheses $H(A)_2$, $H(B)_2$, $H(g)$, $H(r)$, $(H_0)_2$, $H(L)_2$ and $H(l)_1$ hold, then*

(1) the control problems $(6.1)_n$, for every $n \in \mathbb{N}$ and (6.1) admit optimal solutions;

(2) every sequence of optimal solutions to $(6.1)_n$ has a subsequence which is convergent, as $n \to +\infty$, to an optimal solution of the problem (6.1) in $C(0,T;V \times H) \times (w - L^q(0,T;Y))$ topology;

(3) $\widetilde{m}_n \to \widetilde{m}$, as $n \to +\infty$;

(4) the reachable sets $R_n(t)$ and $R(t)$ for $(6.1)_n$ and (6.1), respectively, satisfy $R_n(t) \xrightarrow{K-M} R(t)$ for every $t \in [0,T]$, as $n \to +\infty$.

Remark 6.1 It is easy to see that in the above example, hypothesis $H(l)_1$ is satisfied if, for instance, $l, l_n \colon V \times H \to \mathbb{R}$ are defined as follows

$$l_n(v,w) = |\mathcal{D}_n v - z_d^n|^2_{\mathcal{Y}_1} + |\mathcal{E}_n w - \overline{z}_d^n|^2_{\mathcal{Y}_2}, n \in \mathbb{N}$$

and

$$l(v,w) = |\mathcal{D} v - z_d|^2_{\mathcal{Y}_1} + |\mathcal{E} w - \overline{z}_d|^2_{\mathcal{Y}_2},$$

where $\mathcal{Y}_1$, $\mathcal{Y}_2$ are Hilbert spaces, $\mathcal{D}, \mathcal{D}_n \in \mathcal{L}(V, \mathcal{Y}_1)$, $\mathcal{D}_n \to \mathcal{D}$ in $\mathcal{L}(V, \mathcal{Y}_1)$, $\mathcal{E}, \mathcal{E}_n \in \mathcal{L}(H, \mathcal{Y}_2)$, $\mathcal{E}_n \to \mathcal{E}$ in $\mathcal{L}(H, \mathcal{Y}_2)$ and $\{z_d^n\} \subset \mathcal{Y}_1$, $\{\overline{z}_d^n\} \subset \mathcal{Y}_2$ are two sequences which converge strongly in $\mathcal{Y}_1$ and $\mathcal{Y}_2$ to z_d and $\overline{z}_d$, respectively.

References

[1] N. U. Ahmed and S. Kerbal. Optimal control of nonlinear second order evolution equations, *J. Appl. Math. and Stoch. Anal.,* **6** (1993), 123-136.

[2] N. U. Ahmed and K. L. Teo. *Optimal Control of Distributed Parameter Systems.* New York: North Holland, 1981.

[3] H. Attouch. *Variational Convergence for Functions and Operators.* Boston: Pitman, 1984.

[4] E. Balder. Necessary and sufficient conditions for L^1-strong-weak lower semicontinuity of integral functionals, *Nonlin. Anal. Th. Meth. Appl.,* **11** (1987), 1399-1404.

[5] V. Barbu. *Nonlinear Semigroups and Differential Equations in Banach Spaces.* Leyden, The Netherlands: Noordhoff International Publishing, 1976.

[6] M. L. Bennati. On the convergence of dual variables in optimal control problems, *J. Optim. Th. Appl.,* **34** (1981), 263-277.

[7] G. Buttazzo and G. Dal Maso. Γ-convergence and optimal control problems, *J. Optim. Th. Appl.,* **38** (1982), 385-407.

[8] O. Carja. On variational perturbations of control problems: minimum-time and minimum-effort problem, *J. Optim. Th. Appl.,* **44** (1984), 407-433.

[9] L. Cesari. Existence of solutions and existence of optimal solutions, in *Mathematical Theories of Optimization,* (J.P. Cecconi and T. Zolezzi, eds.), 88-107, 1983.

[10] E. De Giorgi. Convergence problems for functionals and operators, *Proc. Int. Meeting on Recent Methods in Nonlinear Anal.,* (E. De Giorgi, E. Magenes and U. Mosco, eds.). Bologna: Pitagora, 1979, pp. 131-188.

[11] Z. Denkowski and S. Migórski. Control problems for parabolic and hyperbolic equations via the theory of G- and Γ-convergence, *Ann. Mat. Pura Appl.,* **149** (1987), 23-39.

[12] Z. Denkowski and S. Mortola. Asymptotic behavior of optimal solutions to control problems for systems described by differential inclusions corresponding to partial differential equations, *J. Optim. Th. Appl.,* **78** (1993), 365-391.

[13] I. Ekeland and R. Temam. *Convex Analysis and Variational Problems.* Amsterdam: North-Holland, 1976.

[14] K. Kuratowski. *Topology,* Vol. I. New York: Academic Press and Warszawa: PWN-Polish Scientific Publishers, 1966.

[15] J. L. Lions. *Quelques Méthodes de Résolution des Problémes aux Limites Non Linéaires.* Paris: Dunod, 1969.

[16] J. L. Lions. *Optimal Control of Systems Governed by Partial Differential Equations.* New York: Springer-Verlag, 1971.

[17] J. L. Lions and E. Magenes. *Problémes aux Limites Non Homogénes et Applications,* Vol. 1. Paris: Dunod, 1968.

[18] S. Migórski. Asymptotic behavior of optimal solutions in control problems for elliptic equations, *Riv. Mat. Pura Appl.,* **11** (1992), 7-28.

[19] S. Migórski. On asymptotic limits of control problems with parabolic and hyperbolic equations, *Riv. Mat. Pura Appl.,* **12** (1992), 33-50.

[20] S. Migórski. Sensitivity analysis of distributed parameter optimal control problems for nonlinear parabolic equations, *J. Optim. Th. Appl.,* **87** (1995), 595–613.

[21] S. Migórski. A counterexample to a compact embedding theorem for functions with values in a Hilbert space, *Proc. Amer. Math. Soc.,* **123** (8) (1995), 2447–2450.

[22] U. Mosco. On the continuity of the Young-Fenchel transform, *J. Math. Anal. Appl.,* **35** (1971), 518-535.

[23] E. V. Nagy. A theorem on compact embedding for functions with values in an infinite dimensional Hilbert space, *Ann. Univ. Sci. Budapest., Budapest, Eötvös Sectio Mathem.,* **22-23** (1979-80), 243-245.

[24] N. S. Papageorgiou. Convergence theorems for Banach space valued integrable multifunctions, *Int. J. Math. Math. Sci.,* **10** (1987), 433-442.

[25] N. S. Papageorgiou. Variational stability of infinite dimensional optimal control problems, *Int. J. Systems Sci.,* **21** (1990), 1473-1488.

[26] N. S. Papageorgiou. A convergence result for a sequence of distributed parameter optimal control problems, *J. Optim. Th. Appl.,* **68** (1991), 305-320.

[27] N. S. Papageorgiou. Nonlinear Volterra integrodifferential evolution inclusions and optimal control, *Kodai Math. J.,* **14** (1991), 254-280.

[28] A. Salvadori. On the M-convergence for integral functionals on L^p_X, *Atti Sem. Mat., Univ. Modena,* **33** (1984), 137-154.

[29] J. Simon. Compact sets in the space $L^p(0,T;B)$, *Ann. Mat. Pura Appl.,* **146** (1987), 65-96.

[30] T. Zolezzi. Characterization of some variational perturbations of the abstract linear-quadratic problem, *SIAM J. Control and Optim.,* **16** (1978), 106-121.

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