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Frequency Domain Criteria for Hurwitz Stability of Generalized Disc Polynomials*

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Abstract

This paper derives a frequency domain criterion for Hurwitz stability of polynomials with complex coefficients in L_p domains for a fixed real $p \in [1, \infty]$ (generalized disc polynomials). The frequency domain criterion only requires one frequency domain plot to check the robustness of generalized disc polynomials for all real $p \in [1, \infty]$. Furthermore the largest allowable perturbation bounds for all real $p \in [1, \infty]$ can be graphically estimated from the same frequency domain plot.

1 Introduction

The stability of polynomial sets with complex coefficients in L_p domains for a fixed real $p \in [1, \infty]$ has been satisfactorily tackled for the special case of $p = \infty$ and $p = 2$. Kharitonov [5] has shown that a polynomial set with complex coefficients in L_∞ domains or interval polynomials is Hurwitz stable if and only if eight specific polynomials are Hurwitz stable. Recently, Chapellat et al. [2] have shown that a polynomial set with complex coefficients in L_2 domains or disc polynomials is Hurwitz stable if and only if the nominal polynomial is Hurwitz stable and the H_∞-norms of two specific stable rational functions are less than one. However the criteria of Chapellat et al. [2] for $p = 2$ appears unrelated to the well-known result of Kharitonov [5] for $p = \infty$.

This paper derives a frequency domain criterion for Hurwitz stability of polynomial sets with complex coefficients in L_p domains for a fixed real $p \in [1, \infty]$ (generalized disc polynomials). The frequency domain criterion only requires one frequency domain plot to check the robustness of generalized disc polynomials for all real $p \in [1, \infty]$. Furthermore, the largest allowable perturbation bounds for all real $p \in [1, \infty]$ can be graphically estimated

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from the same frequency domain plot. The results are useful in several engineering applications where it is required to ensure that the zeros of a polynomial set with complex coefficients belong to the open left half plane. Such requirements are necessary, for example, in the theory of whirling shafts [3], in the design of asymmetric bandpass and bandrejection filters from a complex coefficient low-pass design [1], and models of vibrational systems and stable control of such systems [4].

The paper is organized as follows. In section 2, the notation to be used is defined. The required supporting results are derived in section 3. In section 4, the frequency domain criterion for Hurwitz stability of generalized disc polynomials is derived. In section 5, we show that the frequency domain criterion is also applicable to handle a wider class of polynomial sets. The use of the frequency domain criterion for Hurwitz stability of generalized disc polynomials is illustrated in section 6.

2 Notation

Consider the complex polynomial set for a fixed real $p \in [1, \infty]$

$$P(s,p,r) = t_n s^n + t_{n-1} s^{n-1} + \ldots + t_0 \ ; \ t_n \neq 0 \tag{2.1}$$

where $t_i = \dot{t}_i + \Delta t_i$. The coefficients $\dot{t}_i \ ; \ i = 0, \ldots, n$ are fixed complex coefficients and $\Delta t_i \in D_i(p)$ where $D_i(p)$ is a L_p domain in the complex plane of radius $\alpha_i r \geq 0$, centred at the origin and defined by

$$D_i(p) = \{z = x + jy : [|x|^p + |y|^p]^{1/p} \leq \alpha_i r\} \tag{2.2}$$

where $\alpha_i \geq 0$ for $i = 1, \ldots, n,\ \alpha_0 > 0$ and $r > 0$. Let

$$S(w) = \sum_{i=0}^{n} \alpha_i |w|^i \tag{2.3}$$

and

$$P_0(s) = \sum_{i=0}^{n} \dot{t}_i s^i \ . \tag{2.4}$$

Note that $S(w) > 0$ for all real w.

Define

$$P(w) = \frac{P_0(s = jw)}{S(w)} \tag{2.5}$$

and $L_p(r)$ to be a L_p domain in the complex plane centred at the origin of radius $r > 0$ and defined by

$$L_p(r) = \{z = x + jy : [|x|^p + |y|^p]^{1/p} \leq r\} \ . \tag{2.6}$$

3 Supporting Results

In this section, we develop the geometrical tools necessary for attaining the ultimate objective.

Lemma 3.1 *Suppose p is a fixed real number which lies in the range* $[1, \infty]$. *Let U_i be a L_p domain in the complex plane of radius $R_i \geq 0$, centred at the origin and defined by $L_p(R_i)$ in (2.6).*

Then the sum of the L_p domains $\sum_{i=0}^{n} U_i$ is also a L_p domain in the complex plane of radius $S_r \geq 0$, centred at the origin and defined by $L_p(S_r)$ in (2.6) where

$$S_r = \sum_{i=0}^{n} R_i \ .$$

Proof: First note that the result is geometrically obvious for

$$\sum_{i=0}^{1} U_i = U_0 + U_1 \ .$$

By induction, this implies that the general result is also correct since

$$\sum_{i=0}^{i+1} U_i = \sum_{i=0}^{i} U_i + U_{i+1} \ .$$

Lemma 3.2 *Suppose p is a fixed real number which lies in the range* $[1, \infty]$.

Let

$$\Delta P(w) = \sum_{i=0}^{n} \Delta t_i (jw)^i$$

where $\Delta t_i \in D_i(p)$; $i = 0, \ldots, n$.

Then

$$\Delta P(w) = S(w) L_p(r) \ .$$

Proof: First note that $\Delta t_i (jw)^i$ is a L_p domain in the complex plane of radius $\alpha_i r |w|^i \geq 0$, centred at the origin and defined by $L_p(\alpha_i r |w|^i)$ in (2.6).

Using Lemma 3.1, $\Delta P(w)$ is also a L_p domain in the complex plane of radius

$$(\sum_{i=0}^{n} \alpha_i |w|^i) r = S(w) r$$

centred at the origin and defined by $L_p(S(w)r)$ in (2.6). Since

$$L_p(S(w)r) = S(w) L_p(r) \ ,$$

it also follows that

$$\Delta P(w) = S(w)L_p(r) \ .$$

Lemma 3.3 *Let F be a simply connected region in the complex plane and F_c be the complex plane not including the open region F. Define the common boundary of F and F_c to be δF. Let B(s) be a connected set of degree q polynomials. Then every polynomial $P(s) \in B(s)$ has $m(0 \leq m \leq q)$ zeros in the open region F and q-m zeros in the open region F_c if and only if*

1. *There exists at least one polynomial $P_0(s) \in B(s)$ which has the specified distribution of zeros;*
2. *For every $s \in \delta F$, the image of B(s) does not include the origin of the complex plane.*

Proof: The necessity of conditions (1) and (2) is obvious. To prove sufficiency, we proceed by contradiction. Suppose conditions (1) and (2) hold but there exists $P_1(s) \in B(s)$ which does not have the specified distribution of zeros. We need to show that there exists some $P(s) \in B(s)$ such that $P(s_0) = 0$ where $s_0 \in \delta F$. By connectedness of B(s), we can construct a continuous path Γ in B(s) connecting $P_0(s)$ and $P_1(s)$. Then, Γ induces at least one continuous path in the complex plane connecting a zero of $P_0(s)$ in the open region F (or open region F_c) with a zero of $P_1(s)$ in F_c(or F). This guarantees the existence of some $P(s) \in \Gamma$ with $P(s) = 0$ for at least one value of $s \in \delta F$.

4 Frequency Domain Criterion

We now derive the frequency domain criterion for robust stability of generalized disc polynomials for a fixed real $p \in [1, \infty]$.

Theorem 4.1 *Suppose p is a fixed real number which lies in the range $[1, \infty]$. Then every polynomial in the polynomial set P(s,p,r) in (2.1) has $m(0 \leq m \leq n)$ zeros in the open left half plane and n-m zeros in the open right half plane if and only if*

1. *$P_0(s)$ has the same distribution of zeros;*
2. *The polar plot $P(w)$ in (2.5) does not intersect the domain $L_p(r)$ defined in (2.6) for all real w.*

Proof: First note that the frequency domain image of P(s,p,r) in (2.1) is given by

$$P(s = jw, p, r) = P_0(s = jw) + \Delta P(w)$$

where

$$\Delta P(w) = \sum_{i=0}^{n} \Delta t_i (jw)^i$$

and $\Delta t_i \in D_i(p)$; $i = 0, \ldots, n$. Using Lemma 3.2,

$$P(s = jw, p, r) = P_0(s = jw) + S(w)L_p(r) \ .$$

Scaling the frequency domain image of $P(s = jw, p, r)$ above by $1/S(w)$ gives

$$\frac{P(s = jw, p, r)}{S(w)} = \frac{P_0(w)}{S(w)} + L_p(r) = P(w) + L_p(r) \ .$$

Since $S(w) > 0$ for all real w, the scaled frequency domain image above does not include the origin of the complex plane if and only if the original frequency domain image does not include the origin of the complex plane. Furthermore

$$-L_p(r) = L_p(r) \ ,$$

which implies that

$$\frac{P(s = jw, p, r)}{S(w)} \neq 0$$

is equivalent to

$$P(w) \neq L_p(r) \ .$$

It also follows that the frequency domain image of $P(s = jw, p, r)$ does not include the origin of the complex plane for all real w if and only if condition (2) of Theorem 4.1 holds. The proof is completed by noting that every polynomial in a connected set of degree n polynomials has a specified number of zeros in the open left half plane and the other zeros in the open right half plane if and only if the frequency domain image of the connected set of degree n polynomials does not include the origin of the complex plane for all real w and at least one polynomial in the connected set of polynomials has the specified distribution of zeros (Lemma 3.3).

Remark 4.1 First note that the polar plot $P(w) = P_0(s = jw)/S(w)$ is independent of p. Therefore only one polar plot is required to check the robustness of generalized disc polynomials for all real $p \in [1, \infty]$. The largest disc polynomial set $P(s, p, r)$ for any fixed real $p \in [1, \infty]$, say $P(s, p, r_m)$, which has the same distribution of zeros as $P_0(s)$, can be graphically estimated by finding the radius r_m of the largest L_p domain centred at the origin which can be inscribed within the polar plot $P(w) = P_0(s = jw)/S(w)$. This is the geometrical implication of Theorem 4.1.

Remark 4.2 Consider the contour of the left half plane specified by

$$s_1 = jw \ ; \ -\infty < w < \infty$$

and

$$s_2 = \lim_{R \to \infty} Rexp(j\phi) \ ; \ \pi/2 \le \phi \le 3\pi/2 \ .$$

Since

$$ARG(P_0(s = jw)) = ARG(P_0(s = jw)/S(w)) \ ,$$

the number of encirclements of the origin of the complex plane by the polar plot $P_0(s = jw)$ as w increases from $-\infty$ to $+\infty$ is the same as the number of encirclements, say k, of the origin of the complex plane by the polar plot $P(w) = P_0(s = jw)/S(w)$ as w increases from $-\infty$ to $+\infty$. Using the Argument Principle, a necessary and sufficient condition for $P_0(s)$ to have m zeros in the open left half plane and $n - m$ zeros in the open right half plane is the condition that $k + n/2 = m$ and $P_0(s = jw)$ (or $P(w) = P_0(s = jw)/S(w)$) does not touch the origin of the complex plane for all real w. This is because the number of encirclements of the origin of the complex plane by $P_0(s)$ as s traverses the segment s_2 once in the counter-clockwise direction is $n/2$ and $P_0(s = jw)/S(w) \neq 0$ for all real w implies that $P_0(s)$ has no zeros on the imaginary axis. Hence condition (1) of Theorem 4.1 can be replaced with the condition that $k + n/2 = m$ since condition (2) of Theorem 4.1 guarantees that $P_0(s)$ has no zeros on the imaginary axis.

Remark 4.3 Consider

$$P_0(s = jw) = R(w) + jI(w)$$

where $R(w)$ and $I(w)$ are real polynomials specified by

$$R(w) = \sum_{i=0}^{n_1} a_i w^i \ ; \ a_{n_1} \neq 0$$

and

$$I(w) = \sum_{i=0}^{n_2} b_i w^i \ ; \ b_{n_2} \neq 0 \ .$$

Marden [6] has shown that all the zeros of $R(w)$ lie within the circle centred at the origin of radius r_1 given by

$$r_1 = 1 + \max_{i=0,\ldots,n_1-1} \{|a_i/a_n|\} \ .$$

It also follows that all the real zeros of $R(w)$ lie on the real segment $(-r_1, r_1)$. Similarly, all the real zeros of $I(w)$ lie on the real segment $(-r_2, r_2)$ where

$$r_2 = 1 + \max_{i=0,\ldots,n_2-1} \{|b_i/b_{n_2}|\} \ .$$

Let

$$r_3 = max\{r_1, r_2\} \ .$$

Then the polar plot of $P_0(s = jw)$ (or $P(w) = P_0(s = jw)/S(w)$) only intersects the real and imaginary axis for finite values of w in the range $(-r_3, r_3)$. This implies that the polar plot $P(w)$ in condition (2) of Theorem 4.1 is not necessarily to be plotted for all real w and should first be plotted from $w = -r_3$ to $w = r_3$ and adjusted accordingly. Furthermore, since the number of encirclements of the origin of the complex plane by the polar plot $P(w)$ as w increases from $-\infty$ to $+\infty$ is the same as the number of encirclements of the origin of the complex plane, say k, by the straight line segment connecting the asymptotic value of $\lim_{w\to-\infty} P(w)$ with $P(w = -r_3)$, polar plot $P(w)$ from $w = -r_3$ to $w = r_3$ and the straight line segment connecting $P(w = r_3)$ with the asymptotic value of $\lim_{w\to\infty} P(w)$, condition (1) of Theorem 4.1 can be replaced with the condition $k + n/2 = m$.

5 Extensions

We now extend the frequency domain criterion to be applicable to handle a linear combination of generalized disc polynomial sets for a fixed real $p \in [1, \infty]$ described by

$$H(s,p,r) = \sum_{k=1}^{d} Q_k(s) P_k(s,p,r) \tag{5.1}$$

where

$$P_k(s,p,r) = \sum_{i=0}^{n(k)} t_{ki} s^i \tag{5.2}$$

and $Q_k(s)$; $k = 1, \ldots, d$ are fixed complex polynomials such that

$$Q_k(s = jw) = Re(Q_k(s = jw))$$

for all real w or

$$Q_k(s = jw) = Im(Q_k(s = jw))$$

for all real w. The coefficients $t_{ki} = \dot{t_{ki}} + \Delta t_{ki}$; $i = 0, \ldots, n(k)$ where $\dot{t_{ki}}$; $i = 0, \ldots, n(k)$ are fixed complex coefficients and $\Delta t_{ki} \in D_{ki}(p)$ where $D_{ki}(p)$ is a L_p domain in the complex plane of radius $\alpha_{ki} r \geq 0$, centred at the origin and defined by

$$D_{ki}(p) = \{z = x + jy : [|x|^p + |y|^p]^{1/p} \leq \alpha_{ki} r\} \tag{5.3}$$

where $\alpha_{ki} \geq 0$ for $i = 0, \ldots, n(k)$ and $r > 0$.

Let

$$G(w) = \sum_{k=1}^{d} |Q_k(s = jw)| (\sum_{i=0}^{n(k)} \alpha_{ki} |w|^i) \tag{5.4}$$

and

$$H_0(s) = \sum_{k=1}^{d} Q_k(s) P_{0k}(s) \tag{5.5}$$

where

$$P_{0k}(s) = \sum_{i=0}^{n(k)} \dot{t_{ki}} s^i \ .$$

Note that $G(w) \geq 0$ for all real w.

Define the frequency domain plot

$$H(w) = \begin{cases} H_0(s = jw)/G(w) & ; \quad G(w) > 0 \\ H_c(w) & ; \quad G(w) = 0 \end{cases} \tag{5.6}$$

where

$$H_c(w) = \begin{cases} (\sqrt{2}r + 1) H_0(s = jw)/(|H_0(s = jw)|) & ; \quad H_0(s = jw) \neq 0 \\ 0 & ; \quad H_0(s = jw) = 0 \end{cases} \tag{5.7}$$

We now derive the frequency domain criterion to check the robustness of a linear combination of generalized disc polynomial sets.

Theorem 5.1 *Suppose p is a fixed real number which lies in the range* $[1, \infty]$ *and H(s,p,r) in (5.1) is a polynomial set of degree q. Then every polynomial in the polynomial set H(s,p,r) in (5.1) has* $m (0 \leq m \leq q)$ *zeros in the open left half plane and q-m zeros in the open right half plane if and only if*

1. $H_0(s)$ *has the same distribution of zeros;*
2. *The polar plot* $H(w)$ *defined in (5.6) does not intersect the* L_p *domain* $L_p(r)$ *defined in (2.6) for all real* w*.*

Proof: First note that from the proof of Theorem 4.1, the frequency domain image of $P_k(s, p, r)$ in (5.2) is given by

$$P_k(s = jw, p, r) = P_{0k}(s = jw) + \Delta P_k(w)$$

where

$$\Delta P_k(w) = (\sum_{i=0}^{n(k)} \alpha_{ki} |w|^i) L_p(r) \ .$$

First note that

$$L_p(r) = -L_p(r) = jL_p(r) = -jL_p(r) \ .$$

Since

$$Q_k(s = jw) = Re(Q_k(s = jw)) \ ,$$

for all real w or

$$Q_k(s = jw) = Im(Q_k(s = jw))$$

for all real w, we have

$$Q_k(s = jw)\Delta P_k(w) = |Q_k(s = jw)|(\sum_{i=0}^{n(k)} \alpha_{ki}|w|^i)L_p(r) \ ,$$

which implies that the frequency domain image of H(s,p,r) in (5.1) is given by (see Lemma 3.1 and the proof of Theorem 4.1)

$$H(s = jw, p, r) = H_0(s = jw) + G(w)L_p(r) \ .$$

The proof then follows similar arguments as used in the proof of Theorem 4.1 for the case of $G(w) > 0$. For the case of $G(w) = 0$,$H(s = jw, p, r) = H_0(s = jw)$. We now recall that the intersection of the polar plot $H(w)$ with the region $L_p(r)$ in (2.6) is equivalent to $H(s = jw, p, r)$ containing the origin of the complex plane. Since $H(s = jw, p, r) = H_0(s = jw)$ for this special case, we only need to consider the case of the polar plot

$$H(w) = H_c(w) = (\sqrt{2}r+1)H_0(s = jw)/(|H_0(s = jw)|) \ ; \ H_0(s = jw) \neq 0 \ .$$

Note that $|H_c(w)| = \sqrt{2}r + 1$ implies that $H_c(w)$ lies outside the region $L_p(r)$ in (2.6) for any fixed real $p \in [1, \infty]$. This is because $L_p(r) \subset L_\infty(r)$ for any real $p \in [1, \infty)$ and the points $(\pm r, \pm r)$ in $L_\infty(r)$ with euclidean distance of $\sqrt{2}r$ are the furthest from the origin. Hence $H_c(w) \notin L_\infty(r)$ which is equivalent to $H_0(s = jw) \neq 0$.

Remark 5.1 Remarks 4.1 and 4.3 are also applicable.

Remark 5.2 For the special case of $p = \infty$, Soh [7] has shown that $H(s, \infty, r)$ in (5.1) is Hurwitz stable if and only if 2×4^d specific polynomials are Hurwitz stable which is a generalization of the result of Kharitonov [5].

We now generalize Theorem 4.1 to take into consideration the case where the nonnegative weights α_i's are dependent on each other.

Theorem 5.2 *Consider again the polynomial set P(s,p,r) in (2.1). Suppose p is a fixed real number which lies in the range* $[1, \infty]$ *and the nonnegative weights* α_i*'s satisfy the following constraints*

$$\alpha_i = 0 \; ; \; i = b, \ldots, n$$

$$[\sum_{i=0}^{b-1} |\alpha_i/\beta_i|^a]^{1/a} \leq \tau \; ; \; b > 1$$

where $\tau > 0$ *and* $\beta_i > 0$ *for all* $i = 0, \ldots, b-1$ *and* a *is a fixed real number which lies in the range* $(0, \infty]$. *Then every polynomial in the polynomial set P(s,p,r) in (2.1) has* $m (0 \leq m \leq n)$ *zeros in the open left half plane and n-m zeros in the open right half plane if and only if*

1. $P_0(s)$ *has the same distribution of zeros;*
2. *The polar plot*
$$P(w) = \frac{P_0(s = jw)}{S(w, a)}$$
where
$$S(w,a) = \begin{cases} \tau(\sum_{i=0}^{b-1}(\beta_i|w|^i)^f)^{1/f} & ; \quad a > 1 \\ \tau \max_{i=0,\ldots,b-1}(\beta_i|w|^i) & ; \quad 0 < a \leq 1 \end{cases}$$
where f *is a real value satisfying*
$$1/f + 1/a = 1$$
does not intersect the domain $L_p(r)$ *defined in (2.6) for all real* w.

Proof: First note that

$$S(w, a) = max\{\sum_{i=0}^{b-1} \alpha_i |w|^i\}$$

subject to the constraints on the nonnegative weights α_i's in Theorem 5.2 [8]. Since $\tau > 0$, $S(w, a)$ is real and greater than zero. Furthermore note that $S(w, a)r$ is the radius of the L_p domain that encloses the frequency domain images $P(s = jw, p, r)$ in Theorem 4.1 for all fixed real $a \in (0, \infty]$. Therefore the polar plot in condition (2) of Theorem 4.1 that is most likely to intersect the domain $L_p(r)$ is given by the polar plot in condition (2) of Theorem 5.2.

Remark 5.3 Remarks 4.1,4.2 and 4.3 are also applicable.

Remark 5.4 Suppose the nonnegative weights α_i's are subject to d constraints

$$f_i(\alpha_n, \alpha_{n-1}, \ldots, \alpha_0) \ ; \ i = 1, \ldots, d\,. \tag{5.8}$$

Let the maximization of $(\sum_{i=0}^{b-1} \alpha_i |w|^i)$ subject to the d constraints in 5.8 be denoted by $S(\alpha_n, \alpha_{n-1}, \ldots, \alpha_0, w)$. Suppose $S(\alpha_n, \alpha_{n-1}, \ldots, \alpha_0, w) > 0$. Then condition (1) of Theorem 5.2 and the condition that the polar plot

$$P(w) = \frac{P_0(s = jw)}{S(\alpha_n, \alpha_{n-1}, \ldots, \alpha_0, w)}$$

does not intersect the domain $L_p(r)$ defined in (2.6) for all real w are necessary and sufficient conditions for every polynomial in $P(s,p,r)$ in (2.1) for a fixed real $p \in [1, \infty]$ and with nonnegative weights α_i's satisfying the d constraints in 5.8to have the same distribution of zeros as $P_0(s)$.

We now generalize Theorem 5.1 to take into consideration the case of the nonnegative weights α_{ki}'s being dependent on each other.

Theorem 5.3 *Consider again the polynomial set H(s,p,r) in (5.1) with the nonnegative weights α_{ki}'s for each $k \in [1, d]$ satisfying d_k constraints $f_{ki}(\alpha_k)$; $i = 1, \ldots, d_k$ where $\alpha_k^T = [\alpha_{kn(k)}, \ldots, \alpha_{k0}]$. Let the maximization of $(\sum_{i=0}^{n(k)} \alpha_{ki} |w|^i)$ for each $k \in [1, d]$ subject to the d_k constraints $f_{ki}(\alpha_k)$; $i = 1, \ldots, d_k$ be denoted by $E_k(w)$. Suppose p is a fixed real number which lies in the range $[1, \infty]$ and the polynomial set H(s,p,r) is a polynomial set of degree q. Then every polynomial in the polynomial set H(s,p,r) has $m (0 \le m \le q)$ zeros in the open left half plane and q-m zeros in the open right half plane if and only if*

1. *$H_0(s)$ has the same distribution of zeros,*
2. *The polar plot*

$$H_1(w) = \begin{cases} H_0(s = jw)/G_1(w) & ; \quad G_1(w) > 0 \\ H_c(w) & ; \quad G_1(w) = 0 \end{cases}$$

where

$$G_1(w) = \sum_{k=1}^{d} |Q_k(s = jw)| E_k(w)$$

does not intersect the L_p domain $L_p(r)$ defined in (2.6) for all real w. $H_c(w)$ and $H_0(s)$ are defined in (5.7) and (5.5) respectively.

Proof: Using Theorem 5.1, the proof is similar to the proof of Theorem 5.2.

Remark 5.5 Remarks 4.1,4.2 and 4.3 are also applicable.

Remark 5.6 Suppose D is a quarterplane or half-plane of the complex plane and D_c is the complex plane not including the open region D. Define δD to be the common boundary of D and D_c. It is easily verified that the image of $H(s,p,r)$ in (5.1) (or $P(s,p,r)$ in (2.1)) for every $s \in \delta D$ is also L_p domain centred on $H_0(s)$ (centred on $P_0(s)$). This also implies that all the results obtained can be generalized to be applicable to constrain zeros in a specified manner to lie in the open region D and the rest of the zeros to lie in the open region D_c.

6 Illustrative Examples

6.1 Example 1

Consider the polynomial set $P(s,p,r)$ in (2.1) with

$$P_0(s) = s^4 + (2 + j1.1)s^3 + (3.4 + j2.1)s^2 + (2.4 + j1.7)s + (1 + j1.5)$$

being Hurwitz stable and $\alpha_4 = 0$ and $\alpha_3 = \alpha_2 = \alpha_1 = \alpha_0 = 1$.

Using Theorem 4.1, the frequency domain plot of

$$P_1(w) = \frac{P_0(s = jw)}{(\sum_{i=0}^{3} |w|^i)}$$

from $w = -4$ to $w = 4$ is plotted in Figure 1.

From Figure 1, the largest L_1 domain and L_2 domain centred at the origin of the complex plane that can be inscribed within the polar plot $P_1(w)$ is estimated to have a radius of 0.253 and 0.246 respectively. Using Theorem 4.1, the largest polynomial set $P(s,1,r_1)$ which is Hurwitz stable for $p = 1$ is estimated to be $r_1 = 0.253$. Similarly, the largest polynomial set $P(s,2,r_2)$ which is Hurwitz stable for $p = 2$ is estimated to be $r_2 = 0.246$.

6.2 Example 2

Consider the polynomial set $P(s,p,r)$ in (2.1) with the same $P_0(s)$ as in example 1, $\alpha_4 = 0$ and

$$\sum_{i=0}^{3} |\alpha_i| \leq 1 \ .$$

Using Theorem 5.2, the frequency domain plot of

$$P_2(w) = \frac{P_0(s = jw)}{S(w,1)}$$

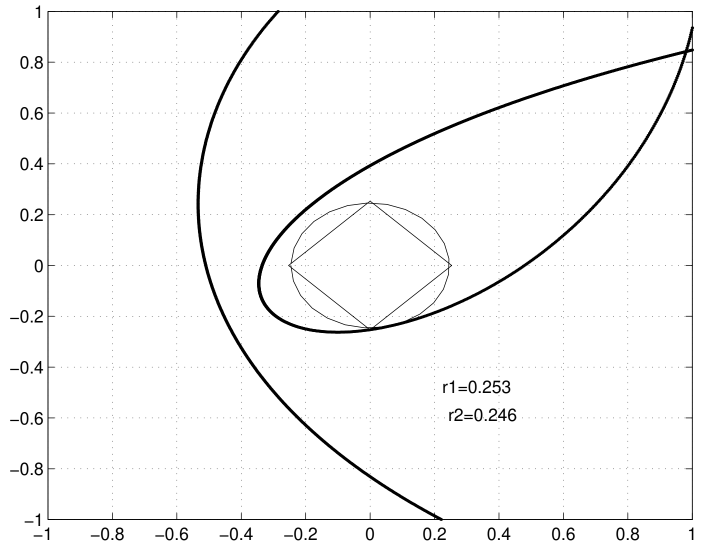

Figure 1: Polar Plot for Example 1

where

$$S(w,1) = \begin{cases} 1 & ; \quad |w| \le 1 \\ |w|^3 & ; \quad |w| > 1 \end{cases}$$

is plotted in Figure 2.

From Figure 2, the largest L_1 domain and L_2 domain centred at the origin of the complex plane that can be inscribed within the polar plot $P_2(w)$ is estimated to have a radius of 0.691 and 0.583 respectively. Using Theorem 5.2, the largest polynomial set $P(s,1,r_1)$ which is Hurwitz stable for $p=1$ is estimated to be $r_1 = 0.691$. Similarly, the largest polynomial set $P(s,2,r_2)$ which is Hurwitz stable for $p=2$ is estimated to be $r_2 = 0.583$.

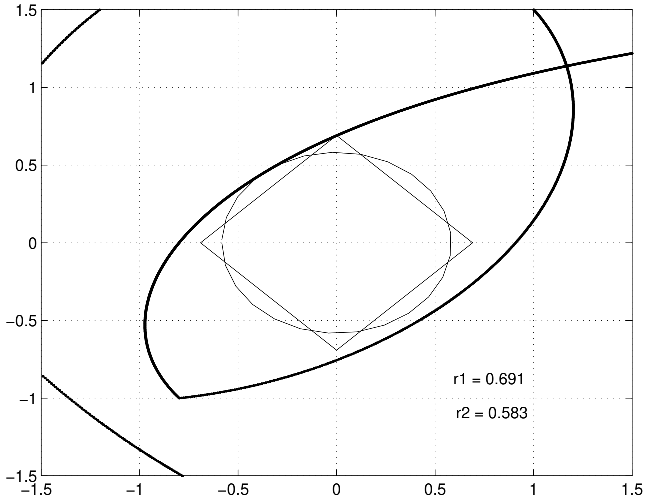

Figure 2: Polar Plot for Example 2

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