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Parameterized Linear Systems in the Behavioral Approach*

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Abstract

In the behavioral approach a dynamical system is essentially determined by a set of trajectories $\mathcal{B}$, which is called behavior. There exist various ways for representing behaviors that are linear and shift-invariant: kernel representations, image representations and latent variable representations. In this paper we deal with families of parametrized linear shift-invariant behaviors and with the problem of representing such families in an efficient way. The representation of parametrized families of behaviors we propose is based on the algebraic properties of a class of rings that are called Jacobson rings. Also in this case parametrized kernel representations, parametrized image representations, and parametrized latent variable representations play an essential role. Finally, algorithms for passing from one representation to another are proposed. This also solves the parametrized latent variable elimination problem.

Key words: parametrized systems, behavioral approach, parametrized kernel and image representations, Jacobson rings, linear systems over rings, controllable systems, latent variable representations

AMS Subject Classifications: 93B25, 93C25, 93C55

1 Introduction

In the behavioral approach to systems theory [15, 16, 17], a *dynamical system* is defined as a triple

$$\Sigma = (T, W, \mathcal{B}),$$

where T is the *time set* (in general $\mathbb{R}$ or $\mathbb{Z}$), W is the set of *signal variables* and $\mathcal{B} \subseteq W^T$, called *behavior*, is a set of time trajectories which describes

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the way the system can interact with the external environment. Typically, $\mathcal{B}$ is represented as the solution set of a system of difference or differential equations in the signal variables. In this case $\mathcal{B}$ can be described as the kernel of the difference or differential operator associated to the difference or differential equation representing $\mathcal{B}$. Therefore this way of representing the behavior $\mathcal{B}$ is called *kernel representation*.

However, in the modelling procedure it may come naturally, and sometimes even unavoidably, to introduce a set L of auxiliary variables, and to describe the behavior through a system of difference or differential equations involving both the variables in W and in L. Variables in L are usually called *latent* and the corresponding descriptions, *latent variable representations* [17, 20]. A special latent variable representation of independent interest is the *image representation* which corresponds to the situation when $\mathcal{B}$ is described as the image of a difference or differential operator acting on l-valued time functions. This type of representation provides an efficient way to parameterize the trajectories of a system and is, therefore, a useful instrument for simulation purpose. It can be seen, moreover, that image representations are more suitable for solving control problems [18, 19]. On the other hand, kernel representations are obviously very useful if we want to check whether an observed trajectory is in the behavior $\mathcal{B}$ and so in any fault detection process. It can be seen, moreover, that also in filtering kernel representations, latent variable representation and their relations play an essential role.

A basic problem, of evident theoretical and practical importance, is to study which systems admit such representations and how these different descriptions of a system are related to each other. When we restrict our attention to representations that are based on linear and constant coefficient difference or differential operators, this problem is completely understood. More precisely, let k be any field and let σ denote the backward shift operator on any of the spaces of bi-infinite sequences $(k^r)^{\mathbb{Z}}$. Every polynomial matrix $\phi \in k[u, u^{-1}]^{g\times q}$

$$\phi = \sum_{i=m}^{n} \phi_i u^i, \quad \phi_i \in k^{g\times q}$$

naturally induces an operator

$$\phi(\sigma, \sigma^{-1}) : (k^q)^{\mathbb{Z}} \to (k^g)^{\mathbb{Z}}, \tag{1.1}$$

$$\phi(\sigma, \sigma^{-1})w := \sum_{i=m}^{n} \phi_i(\sigma^i w).$$

The operators defined in this way are linear shift-invariant (namely, $\phi \circ \sigma = \sigma \circ \phi$) and are called *shift operators*.

One of the main results proved in [16] is that the class of systems admitting kernel representations through shift operators coincides with the class of systems having $T = \mathbb{Z}$, $W = k^q$ and the behavior $\mathcal{B}$ which is a linear shift-invariant and closed (with respect to the pointwise convergence topology: see Section 2) subspace of W^T. In other words, given a system $\Sigma = (\mathbb{Z}, k^q, \mathcal{B})$, where the behavior $\mathcal{B}$ is linear shift-invariant and closed, there exists a polynomial matrix $\phi \in k[u, u^{-1}]^{g \times q}$ such that

$$\mathcal{B} = \ker \phi(\sigma, \sigma^{-1}).$$

These systems are called *linear autoregressive* (AR) *systems*. Among all possible kernel representations of a given behavior $\mathcal{B}$, we call minimal the ones which attain the minimum number of rows g in ϕ (and this happens if and only if the corresponding shift operator is onto).

It can be shown that the class of systems admitting image representations is strictly smaller than the class of systems admitting kernel representations. Indeed, [15, 16] on one hand, it can be seen that images of shift operators can always be expressed as kernels. On the other hand, the behavior $\mathcal{B}$ of an AR system can be expressed as the image of a shift operator

$$\mathcal{B} = \text{im } \psi(\sigma, \sigma^{-1})$$

if and only if it is *controllable* (see Section 2). Moreover, in this case the shift operator can be chosen to be injective: such image representations are called *observable* and are of evident importance.

The latent variable representation generalizes both the concepts of kernel and image representations presented above. In this case a latent variable representation is defined as follows: suppose we have polynomial matrices $\eta \in k[u, u^{-1}]^{r \times q}$ and $\psi \in k[u, u^{-1}]^{r \times p}$ and consider the difference equation

$$\eta(\sigma, \sigma^{-1})w = \psi(\sigma, \sigma^{-1})l. \tag{1.2}$$

This difference equation describes an AR system with signal space $k^q \times k^p$. However, we can also consider the system $\Sigma = (\mathbb{Z}, k^q, \mathcal{B})$, where

$$\mathcal{B} := \{w \in (k^q)^{\mathbb{Z}} \mid \exists l \in (k^p)^{\mathbb{Z}}, (w, l) \text{ satisfies } (1.2)\}.$$

The difference equation (1.2) is thus a latent variable representation of $\mathcal{B}$. The most relevant example of latent variable representation is the state space representation. A basic result in [15, 16] is that Σ defined in this way is an AR system. Namely, there exists $\phi \in k[u, u^{-1}]^{g \times q}$ such that $\mathcal{B} = \ker \phi(\sigma, \sigma^{-1})$. The problem of obtaining a kernel representation from a latent variable representation is called *latent variable elimination*.

Another important problem solved in [15, 16] is how to obtain concretely one representation from another one. More specifically, given a kernel representation of $\mathcal{B}$, $\mathcal{B} = \ker \phi(\sigma, \sigma^{-1})$, there exists an efficient algorithm, based on the Smith canonical form, that allows us to check whether $\mathcal{B}$ is controllable and, if this is the case, allows us to obtain an observable image representation of $\mathcal{B}$. Conversely, given an image representation $\mathcal{B} = \text{im } \phi(\sigma, \sigma^{-1})$, still computing the Smith canonical form of ϕ, a minimal kernel representation of $\mathcal{B}$ can be obtained. Observe that the problem of obtaining kernel representations from image representations is a particular case of the latent variable elimination. However, once we know how to pass from image representations to kernel representations, the general latent variable elimination problem can be solved easily. Actually, given a latent variable representation (1.2) of $\mathcal{B}$, if ϕ is such that im $\psi(\sigma, \sigma^{-1}) = \ker \phi(\sigma, \sigma^{-1})$, then it follows easily that $\mathcal{B} = \ker(\phi\eta)(\sigma, \sigma^{-1})$.

The problems we address in this paper are the natural extension of the above ones for parameterized families of linear systems which we now introduce. For the sake of simplicity of notation, we will drop the more precise system notation as a triple, and we will simply consider behaviors. Behaviors coming from AR systems will be simply called *linear* or k-behaviors.

Let X be a set and let k be a field. Consider a subalgebra R of the algebra k^X of all the maps from X to k. Let $\phi \in R[u, u^{-1}]^{g \times q}$. Given $x \in X$, denote by $\phi_x \in k[u, u^{-1}]^{g \times q}$ the polynomial matrix obtained by evaluating at x all the coefficients of ϕ. For every $x \in X$ we can consider the linear behavior

$$\ker \phi_x(\sigma, \sigma^{-1}). \tag{1.3}$$

This will be called an *R-family of k-kernels*. Similarly, we can consider

$$\text{im } \phi_x(\sigma, \sigma^{-1}) \tag{1.4}$$

which instead will be called an *R-family of k-images*. Generally, (1.3) and (1.4) will be called *R-families of k-behaviors*.

The first question we address is to find the conditions under which an R-family of k-kernels is also representable as an R-family of k-images and vice versa. If $R = k^X$, the problem is trivial: it follows from previous considerations that a family of k-images is always a family of k-kernels while the converse holds if and only if each kernel is controllable. The interesting situation is when R is a proper subalgebra, since this gives restrictions on the way the polynomial matrices must depend on the parameters. The case on which we will focus our attention is when X is an affine k-variety and R is the algebra of polynomial k-valued maps on X. As we will see in this setting, new obstructions to this change of representation can come out. In case of obstructions, we will characterize subsets $X_0 \subseteq X$ for which the change of representation can be achieved. Another interesting issue in this setting is constituted by minimality and observability: as we will see, even if a family of k-kernels can be represented as a family of k-images, it may not admit image representations in the class R which are observable at every point $x \in X$. Similar problems can occur regarding minimality of kernel representations.

Notice that, given an R-family of latent variable representations

$$\eta_x(\sigma, \sigma^{-1})w = \psi_x(\sigma, \sigma^{-1})l, \tag{1.5}$$

where $\eta \in R[u, u^{-1}]^{r \times q}$ and $\psi \in R[u, u^{-1}]^{r \times p}$, then it can be seen that also the parametrized latent variable elimination problem can be solved by obtaining an R-family of k-kernels from an R-family of k-images. More precisely, if there exists $\phi \in R[u, u^{-1}]^{g \times q}$ such that

$$\text{im } \psi_x(\sigma, \sigma^{-1}) = \ker \phi_x(\sigma, \sigma^{-1}), \quad \forall x \in X_0 \subseteq X,$$

then, for all $x \in X_0$, we have that the behavior represented by the parametrized latent variable representation (1.5) is represented by the parametrized kernel representation $\ker(\phi_x \eta_x)(\sigma, \sigma^{-1})$. For this reason on we will concentrate our attention only on kernel and image representations.

We will approach these problems by studying more general behaviors over rings. Notice indeed that $\phi \in R[u, u^{-1}]^{g \times q}$ induces, exactly as in (1.1), a shift operator

$$\phi(\sigma, \sigma^{-1}) : (R^q)^{\mathbb{Z}} \to (R^g)^{\mathbb{Z}}.$$

In analogy with the linear case, we can thus consider objects like

$$\ker \phi(\sigma, \sigma^{-1}), \quad \text{im } \phi(\sigma, \sigma^{-1}), \tag{1.6}$$

which can be thought as behaviors over the signal variable set, respectively, R^q and R^g. Let us see how they are related to the families of k-behaviors introduced above. Let $x \in X$ and consider $\eta_x : R \to k$, the evaluation map at the point x (i.e. $\eta_x(r) := r(x)$). The map η_x admits a trivial extension to vectors which will be denoted in the same way while η_x^∞ will denote the extension of η_x to sequences. Put $\mathcal{B} := \ker \phi(\sigma, \sigma^{-1})$ and consider $\mathcal{B}_x := \eta_x^\infty(\mathcal{B}) \subseteq (k^q)^{\mathbb{Z}}$. Clearly,

$$\mathcal{B}_x \subseteq \ker \phi_x(\sigma, \sigma^{-1}). \tag{1.7}$$

If we have equality in (1.7), this means that, in a certain sense, all the information about the family $\ker \phi_x(\sigma, \sigma^{-1})$ is contained inside $\ker \phi(\sigma, \sigma^{-1})$. More precisely, if equality holds in (1.7), given $v \in \ker \phi_x(\sigma, \sigma^{-1})$, we can find $\tilde{v} \in (R^q)^{\mathbb{Z}}$ such that

$$\eta_x^\infty \tilde{v} = v,$$

$$\eta_y^\infty \tilde{v} \in \ker \phi_y(\sigma, \sigma^{-1}) \quad \forall y \in X.$$

This shows that also the single trajectories admit a sort of parametrization in the class R. Conditions under which equality in (1.7) holds will be studied in Section 2.

As far as families of images are concerned, it is clear that, if η_x is onto (very mild assumption which is true in the algebraic case), we always have that

$$\eta_x^\infty(\text{im } \phi(\sigma, \sigma^{-1})) = \text{im } \phi_x(\sigma, \sigma^{-1}).$$

Sections 3 and 4 will be devoted to studying the relation between behaviors in kernel and image representation, as (1.6), and generalizing many results known in the field case. Beside the importance for parameterized systems, such extensions have independent interest also considering the recent extensions of the theory of convolutional codes to group and ring setting [5]. Finally, in Section 5, we will come back to parameterized families of linear behaviors and show how the abstract results of Sections 3 and 4 can be specifically used in this setting. A number of illustrative examples will also be presented. A few of them are briefly introduced now.

Example 1: Let $k = \mathbb{R}$, $X = \mathbb{R}^2$ and $R = \mathbb{R}[z_1, z_2]$. Consider

$$\phi := \begin{pmatrix} z_1 u & z_2 & z_2 + 1 \\ 0 & 1 & 1 + z_1 u \end{pmatrix} \in R[u, u^{-1}]^{2\times 3}.$$

The R-family of kernels $\ker \phi_{(x,y)}(\sigma, \sigma^{-1})$ admits the following observable image representation which can be obtained with the techniques of section 3.2

$$\ker \phi_{(x,y)}(\sigma, \sigma^{-1}) = \text{im } \begin{pmatrix} xy\sigma \Leftrightarrow 1 \\ \Leftrightarrow x\sigma(1 + x\sigma) \\ x\sigma \end{pmatrix} \quad \forall (x, y) \in \mathbb{R}^2.$$

Example 2: Let $k = \mathbb{R}$, $X = S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ and R be the algebra of polynomial k-valued maps on X. It is known that R is isomorphic to $\mathbb{R}[z_1, z_2]/(1 \Leftrightarrow z_1^2 \Leftrightarrow z_2^2)$. Consider

$$\phi := \begin{pmatrix} z_1 u^2 + z_2 & z_2 \end{pmatrix} \in R[u, u^{-1}]^{1\times 2}.$$

The R-family of kernels $\ker \phi_{(x,y)}(\sigma, \sigma^{-1})$ admits the following observable parametrized image representation

$$\ker \phi_{(x,y)}(\sigma, \sigma^{-1}) = \text{im } \begin{pmatrix} \Leftrightarrow y \\ x\sigma^{-2} \end{pmatrix} \quad \forall (x, y) \in S^1.$$

Example 3: Let $k = \mathbb{R}$, $X = \mathbb{R}$ and $R := \mathbb{R}[z]$. Consider

$$\phi := \begin{pmatrix} zu & z \end{pmatrix}.$$

We will show later that, in this case, the R-family of kernels $\ker \phi_{(x,y)}(\sigma, \sigma^{-1})$ does not admit a parametrized image representation.

2 Behaviors over Rings and Their Representations

2.1 Behaviors and shift operators

We now introduce some basic concepts and notations that will be needed in the sequel. If A is a commutative ring, denote by A^* the multiplicative group of units of A and by $A^{g\times q}$ the A-module of $g \times q$ matrices with entries in A. If $\phi \in A^{g\times q}$, $\ker\phi$ and im ϕ denote the kernel and the image of ϕ thought as a homomorphism from A^q to A^g. If A is an integral domain, rk (ϕ) denotes the rank of ϕ in the field of fractions of A. The symbol $\simeq_A$ denotes isomorphism in the category of A-modules. Subscript A will be dropped whenever it is clear from the context. Finally $\mathrm{Max}(A)$ denotes the set of all maximal ideals of A.

In the sequel, R will always denote a Noetherian commutative ring with identity. Moreover, we will always assume that R is Jacobson [2], namely, that every prime ideal can be obtained as intersection (possibly infinite) of maximal ideals. Rings of polynomial functions on an affine variety as well the ring $\mathbb{Z}$ of integers are examples of Jacobson rings. Denote by $R[u,u^{-1}]$ the ring of Laurent polynomials with coefficients in R. If V is any R-module, then the symbol $V^{\mathbb{Z}}$ will denote the R-module of bi-infinite sequences over V. On $V^{\mathbb{Z}}$ we can introduce a module structure over the ring $R[u,u^{-1}]$ by defining for $v \in V^{\mathbb{Z}}$

$$(u \cdot v)(t) := (\sigma v)(t) := v(t+1) \quad \forall t \in \mathbb{Z}. \tag{2.1}$$

The operator σ is called the *backward shift.* From now on we will always assume that V is a finitely generated R-module equipped with the discrete topology and $V^{\mathbb{Z}}$ with the corresponding product topology. It is clear that the topology just introduced on $V^{\mathbb{Z}}$ is metrizable and a sequence $v_n \in V^{\mathbb{Z}}$ converges to $v \in V^{\mathbb{Z}}$ if and only if for every $t \in \mathbb{Z}$, $v_n(t) = v(t)$, for n sufficiently large. For this reason this topology is called pointwise convergence topology. If V is a finitely generated R-module, then an R-*behavior* on V is any closed $R[u,u^{-1}]$-submodule $\mathcal{B} \subseteq V^{\mathbb{Z}}$. In particular the R-behavior $V^{\mathbb{Z}}$ is called the *full* R-behavior over V.

If $I \subseteq \mathbb{Z}$, denote by $\mathcal{B}_{|I}$ the R-module of restrictions to I of the bi-infinite sequences in $\mathcal{B}$. We recall now the concept of memory [15, 16]. An R-behavior $\mathcal{B}$ is said to have *memory* $n \in \mathbb{N}$ if

$$v \in V^{\mathbb{Z}} \text{ and } v_{|[t,t+n]} \in \mathcal{B}_{|[t,t+n]} \; \forall t \in \mathbb{Z} \Rightarrow v \in \mathcal{B}.$$

$\mathcal{B}$ is said to have *finite memory* (or to be of *finite type*), if it has memory n for some $n \in \mathbb{N}$. Let $\mathcal{B} \subseteq V^{\mathbb{Z}}$ be an R-behavior. $\mathcal{B}$ is said to be *controllable* [15, 16] if for all $v_1, v_2 \in \mathcal{B}$, there exists $n \in \mathbb{N}$ and $v \in \mathcal{B}$ with

$$v(t) = v_1(t) \;\; \forall t < 0, \quad (\sigma^n v)(t) = v_2(t) \;\; \forall t \geq 0. \tag{2.2}$$

It is shown in [21] that in our context every controllable R-behavior $\mathcal{B}$ satisfies a stronger notion of controllability in the sense that the number $n \in \mathbb{N}$ in (2.2) can be chosen a priori for all pairs $v_1, v_2 \in \mathcal{B}$. Moreover, if $\mathcal{B}$ is a finite memory R-behavior, then there exists the largest controllable R-behavior contained in $\mathcal{B}$ which is denoted by $\mathcal{B}_c$ and which is called the controllable part of $\mathcal{B}$. In this case it can be shown that $\mathcal{B}_c = \overline{\mathcal{B}_f}$, where $\mathcal{B}_f$ is the submodule of $\mathcal{B}$ constituted by the finite supported trajectories in $\mathcal{B}$ and $\overline{\cdot}$ means closure.

It follows from (2.1) that any polynomial $p \in R[u, u^{-1}]$ can be interpreted as a continuous $R[u, u^{-1}]$-homomorphism

$$p(\sigma, \sigma^{-1}) : R^{\mathbb{Z}} \to R^{\mathbb{Z}}.$$

In an analogous way, if V, W are R-modules, every $\phi \in \mathrm{Hom}_R(V, W)[u, u^{-1}]$ naturally induces a continuous $R[u, u^{-1}]$-homomorphism

$$\phi(\sigma, \sigma^{-1}) : V^{\mathbb{Z}} \to W^{\mathbb{Z}}.$$

The homomorphisms defined in this way are called *shift operators*. In case $V = R^q$ and $W = R^l$, we identify $\mathrm{Hom}_R(V, W)[u, u^{-1}]$ with $R[u, u^{-1}]^{g \times q}$. In this case, corresponding shift operators are also called *matrix shift operators*. It can be shown that all continuous $R[u, u^{-1}]$-homomorphisms between full R-behaviors are given by shift operators [21].

In this paper we will focus our attention on R-behaviors which are kernels or images of matrix shift operators. It can be easily shown that if an R-behavior is a kernel of a matrix shift operator (called *matrix kernel representation*), it has finite memory. The converse is in general not true. Finite memory easily yields the existence of a kernel representation but not necessarily of matrix type. On the other hand, it has been shown in [21] that an R-behavior $\mathcal{B} \subseteq (R^q)^{\mathbb{Z}}$ is the image of a matrix shift operator (called *matrix image representation*) if and only if it is controllable. As we will see in the sequel, sometimes we will be forced to consider more general image representations through shift operators induced by objects of the type $\phi \in \mathrm{Hom}(V, R^q)[u, u^{-1}]$, where V is usually a projective finitely generated R-module. An image representation is said to be *observable* if it is one to one.

When $R = k$ is a field, all the previous properties become stronger. Every k-behavior has finite memory, which in this setting is equivalent to possessing matrix kernel representations: therefore linear behaviors considered in the introduction always admit a matrix kernel representation. If $\mathcal{B}$ is a k-behavior, $\mathcal{B}_f$ is then a free finite-dimensional $k[u, u^{-1}]$-module. Define the *rank* of $\mathcal{B}$ (denoted rk $(\mathcal{B})$) as the dimension of $\mathcal{B}_f$. If $\mathcal{B} = \ker \phi(\sigma, \sigma^{-1})$, $\phi \in k[u, u^{-1}]^{g \times q}$, is a kernel representation of $\mathcal{B}$ and $\mathcal{B}_c = \mathrm{im}\ \psi(\sigma, \sigma^{-1})$, $\psi \in k[u, u^{-1}]^{q \times r}$, is an image representation of the controllable part $\mathcal{B}_c$ of

$\mathcal{B}$, we have that

$$q \Leftrightarrow \text{rk }(\phi) = \text{rk }(\psi) = \text{rk }(\mathcal{B}). \tag{2.3}$$

Moreover, $\ker \phi(\sigma, \sigma^{-1})$ is a minimal kernel representation, as we have defined in the previous section, if and only if rk $(\phi) = g$, while im $\psi(\sigma, \sigma^{-1})$ is an observable image representation if and only if rk $(\psi) = r$. Finally, if $\mathcal{B}_1 \subseteq \mathcal{B}_2$ are two k-behaviors and rk $(\mathcal{B}_1) = \text{rk}(\mathcal{B}_2)$, then $(\mathcal{B}_1)_c = (\mathcal{B}_2)_c$. This will be used in the sequel.

2.2 Projection of behaviors over quotient rings

Let V be a finitely generated R-module and let L be an ideal in R. Consider the quotient module V/LV. As we will show in the sequel, it is interesting, particularly when dealing with parametrized systems, to investigate how a property of an R-behavior $\mathcal{B} \subseteq V^{\mathbb{Z}}$ is connected with the analogous property of the projection of $\mathcal{B}$ on the signal space $(V/LV)^{\mathbb{Z}}$.

To this aim consider the quotient projection $p : V \to V/LV$. As mentioned above we can obtain from p the projection

$$p^{\infty} : V^{\mathbb{Z}} \Leftrightarrow\!\!\rightarrow (V/LV)^{\mathbb{Z}}.$$

Define

$$\mathcal{B}_L := p^{\infty}(\mathcal{B}).$$

Clearly, $\mathcal{B}_L$ is an $(R/L)[u, u^{-1}]$-submodule of $(V/LV)^{\mathbb{Z}}$, but it is not a priori clear that it is closed and so that it is an (R/L)-behavior. As we will see, this turns out to be true for a particular class of R-behaviors over R^q admitting matrix kernel representations that will be called regular.

Definition An R-behavior $\mathcal{B} \subseteq (R^q)^{\mathbb{Z}}$ is said to be *regular* if there exist $\phi_1 \in R[u, u^{-1}]^{q_1 \times q}, \phi_2 \in R[u, u^{-1}]^{q_2 \times q_1}, \dots, \phi_n \in R[u, u^{-1}]^{q_n \times q_{n-1}}$ such that the following sequence is exact

$$0 \Leftrightarrow\!\!\rightarrow \mathcal{B} \overset{i}{\hookrightarrow} (R^q)^{\mathbb{Z}} \overset{\phi_1(\sigma,\sigma^{-1})}{\Leftrightarrow\!\!\rightarrow} (R^{q_1})^{\mathbb{Z}} \overset{\phi_2(\sigma,\sigma^{-1})}{\Leftrightarrow\!\!\rightarrow} \cdots \overset{\phi_n(\sigma,\sigma^{-1})}{\Leftrightarrow\!\!\twoheadrightarrow} (R^{q_n})^{\mathbb{Z}} \Leftrightarrow\!\!\rightarrow 0, \tag{2.4}$$

where i is the injection map. We say that an R-behavior $\mathcal{B} \subseteq (R^q)^{\mathbb{Z}}$ is *regular of order* n if it is regular and if the exact sequence of minimum length associated to $\mathcal{B}$ has length n. We say that the full R-behavior $(R^q)^{\mathbb{Z}}$ is regular of order 0. $\mathcal{B} = \ker \phi_1(\sigma, \sigma^{-1})$ as in (2.4) is called a *regular* kernel representation of $\mathcal{B}$.

If R is a principal ideal domains (PID), it follows from standard factorization results [7, 4] that if an R-behavior admits a matrix kernel representation, then it admits one that is onto, so it is regular of order 1. Moreover, we will show in the sequel that when $R[u, u^{-1}]$ is Hermite, then a regular behavior has always order less than or equal to 1 (see Proposition

6). Regular behaviors of order strictly greater than 1, however, do exist (see Section 4).

For regular R-behaviors the following important technical result holds true.

Lemma 1 *Assume that $\mathcal{B} \subseteq (R^q)^{\mathbb{Z}}$ is a regular R-behavior and let L be an ideal in R. Then*

$$\mathcal{B} \cap (L^q)^{\mathbb{Z}} = L\mathcal{B}. \tag{2.5}$$

Proof: It is trivial to verify that $L\mathcal{B} \subseteq \mathcal{B} \cap (L^q)^{\mathbb{Z}}$. In order to prove the converse we need to verify something slightly stronger, that is the following: If $v_1, \dots, v_m \in \mathcal{B}$ and if there exist $g_{ij} \in R$ such that

$$v_i = \sum_{j=1}^{n} g_{ij} w_j,$$

where $w_1, \dots, w_n \in (R^q)^{\mathbb{Z}}$, then there exist $\bar{w}_1, \dots, \bar{w}_n \in \mathcal{B}$ such that

$$v_i = \sum_{j=1}^{n} g_{ij} \bar{w}_j.$$

We will show the assertion by induction on the regularity order of $\mathcal{B}$. If $\mathcal{B}$ has order 0, then the assertion is trivial. Suppose that the assertion holds for R-behaviors of order $n-1$ and suppose that $\mathcal{B}$ is an R-behavior of order n. Consider the exact sequence of length n associated to $\mathcal{B}$

$$0 \longrightarrow \mathcal{B} \stackrel{i}{\hookrightarrow} (R^q)^{\mathbb{Z}} \stackrel{\phi_1(\sigma,\sigma^{-1})}{\longrightarrow} (R^{q_1})^{\mathbb{Z}} \stackrel{\phi_2(\sigma,\sigma^{-1})}{\longrightarrow} \cdots \stackrel{\phi_n(\sigma,\sigma^{-1})}{\twoheadrightarrow} (R^{q_n})^{\mathbb{Z}} \longrightarrow 0. \tag{2.6}$$

Then, $\mathcal{B} = \ker \phi_1(\sigma, \sigma^{-1})$. Let $v_1, \dots, v_m \in \mathcal{B}$ and let

$$v_i = \sum_{j=1}^{n} g_{ij} w_j,$$

where $w_1, \dots, w_n \in (R^q)^{\mathbb{Z}}$ and $g_{ij} \in R$. Then,

$$0 = \phi_1(\sigma, \sigma^{-1}) v_i = \sum_{j=1}^{n} g_{ij} \phi_1(\sigma, \sigma^{-1}) w_j.$$

Let $u_j := \phi_1(\sigma, \sigma^{-1}) w_j$. Then, $u_j \in \operatorname{im} \phi_1(\sigma, \sigma^{-1}) \subseteq (R^{q_1})^{\mathbb{Z}}$, that is regular of order $n-1$. Moreover,

$$\sum_{j=1}^{n} g_{ij} u_j = 0 \quad i = 1, 2, \dots, m. \tag{2.7}$$

The vectors u_j can be expressed as follows

$$u_j = \begin{bmatrix} u_{1j} \\ \vdots \\ u_{q_1 j} \end{bmatrix},$$

where $u_{ij} \in R^{\mathbb{Z}}$. Define the R-module

$$M := \{[a_1, \ldots, a_n] \in R^{1\times n} : \sum_{j=1}^{n} a_j g_{ij} = 0,\ i = 1, 2, \ldots, m\}.$$

Since M is finitely generated, let $f_1, \ldots, f_l \in R^{1\times n}$ be its generators. Notice that, if $f_k = [f_{k1}, \ldots, f_{kn}]$, then

$$\sum_{j=1}^{n} f_{kj} g_{ij} = 0. \tag{2.8}$$

Moreover, by (2.7), we have that $[u_{i1}(t), \ldots, u_{in}(t)] \in M$ for all $t \in \mathbb{Z}$ and so $[u_{i1}, \ldots, u_{in}] = \sum_{k=1}^{l} \alpha_{ik} f_k$, where $\alpha_{ij} \in R^{\mathbb{Z}}$. Define

$$\alpha_k := \begin{bmatrix} \alpha_{1k} \\ \vdots \\ \alpha_{q_1 k} \end{bmatrix} \in (R^{q_1})^{\mathbb{Z}}.$$

It is easy to verify that

$$u_j = \sum_{k=1}^{l} f_{kj} \alpha_k \in \mathrm{im}\ \phi_1(\sigma, \sigma^{-1}).$$

By induction, there exist $\bar{\alpha}_1, \ldots, \bar{\alpha}_l \in \mathrm{im}\ \phi_1(\sigma, \sigma^{-1})$ such that

$$u_j = \sum_{k=1}^{l} f_{kj} \bar{\alpha}_k. \tag{2.9}$$

Therefore $\bar{\alpha}_k = \phi_1(\sigma, \sigma^{-1}) \bar{u}_k$. Put finally

$$\bar{w}_j = w_j \Leftrightarrow \sum_{k=1}^{l} f_{kj} \bar{u}_k. \tag{2.10}$$

It is easy to verify both that $\bar{w}_j \in \ker \phi_1(\sigma, \sigma^{-1})$ and that

$$\sum_{j=1}^{n} g_{ij} \bar{w}_j = v_i.$$

■

Consider a polynomial matrix $\phi \in R[u, u^{-1}]^{g\times q}$ and let L be an ideal in R. Using the projection p mentioned above, we can construct the polynomial matrix $\phi_L \in (R/L)[u, u^{-1}]^{g\times q}$ in the obvious way. It can be easily seen that if an R-behavior $\mathcal{B}$ admits an image representation $\mathcal{B} = \text{im } \phi(\sigma, \sigma^{-1})$, $\phi \in R[u, u^{-1}]^{g\times q}$, then $\mathcal{B}_L = \text{im } \phi_L(\sigma, \sigma^{-1})$ and consequently if $\mathcal{B}$ is controllable, then $\mathcal{B}_L$ is controllable. It is possible to prove an analogous result for kernel representations only for regular R-behaviors as stated in the following theorem.

Theorem 2 *Let $\mathcal{B} \subseteq (R^q)^{\mathbb{Z}}$ be a regular R-behavior and suppose that $\mathcal{B}$ admits the regular kernel representation $\mathcal{B} = \ker \phi(\sigma, \sigma^{-1})$, where $\phi \in R[u, u^{-1}]^{g\times q}$. Then, for each ideal L in R*

$$\mathcal{B}_L = \ker \phi_L(\sigma, \sigma^{-1}).$$

Proof: Suppose that $l_1, \ldots, l_m$ is a set of generators for L. It is immediate to see that $\mathcal{B}_L \subseteq \ker \phi_L(\sigma, \sigma^{-1})$. Suppose conversely that $\bar{v} \in \ker \phi_L(\sigma, \sigma^{-1})$ and let $v \in (R^q)^{\mathbb{Z}}$ be any representative of $\bar{v}$. Then, $w := \phi(\sigma, \sigma^{-1})v \in (L^q)^{\mathbb{Z}} \cap \text{im } \phi(\sigma, \sigma^{-1})$. Since im $\phi(\sigma, \sigma^{-1})$ is regular, then

$$\text{im } \phi(\sigma, \sigma^{-1}) \cap (L^q)^{\mathbb{Z}} = L\text{im } \phi(\sigma, \sigma^{-1})$$

and so there exist $u_1, \ldots, u_n$ such that

$$w = \sum_{i=1}^{m} l_i \phi(\sigma, \sigma^{-1}) u_i.$$

We can argue that

$$\tilde{v} := v \Leftrightarrow \sum_{i=1}^{m} l_i u_i \in \mathcal{B}$$

and so $\bar{v} \in \mathcal{B}_L$. ■

As mentioned above, given an R-behavior $\mathcal{B} \subseteq (R^q)^{\mathbb{Z}}$, it is not always true that $\mathcal{B}_L$ is an R/L-behavior since, in general, it is not clear if it is closed. The direct consequence of the previous result is that if $\mathcal{B}$ is a regular R-behavior, then $\mathcal{B}_L$ admits a kernel representation, and so, in particular, it is an (R/L)-behavior.

3 From Kernel to Image Representations

In this section we will consider the problem of verifying whether a certain finite memory R-behavior $\mathcal{B}$, given through a kernel representation, is controllable and, therefore, whether it can be expressed by an image representation. Moreover, we will also develop some techniques that allow to obtain concretely an image representation of $\mathcal{B}$, starting from its kernel representation. Finally, we will consider some important extensions to non-controllable cases.

3.1 Controllability of behaviors admitting kernel representations

We start by analyzing the first part of the problem mentioned above, that is equivalent to find techniques that allow to check whether a certain R-behavior $\mathcal{B}$ is controllable starting from its kernel representation. Given a polynomial matrix $\phi \in R[u, u^{-1}]^{g \times q}$, we introduce the following subset of $\mathrm{Max}(R)$

$$U_c(\phi) := \{m \in \mathrm{Max}(R) : \ker \phi_m(\sigma, \sigma^{-1}) \text{ is controllable}\},$$

$$U_o(\phi) := \{m \in \mathrm{Max}(R) : \phi_m(\sigma, \sigma^{-1}) \text{ is onto}\},$$

$$U_{co}(\phi) := U_c(\phi) \cap U_o(\phi).$$

Moreover, we associate to ϕ two ideals of $R[u, u^{-1}]$: the ideal J_ϕ generated by all the $g \times g$ minors of ϕ (we put $J_\phi = (0)$ if $q < g$), and the ideal

$$I_\phi := \{p \in R[u, u^{-1}] \mid \exists \psi \in R[u, u^{-1}]^{q \times g} \;\; : \phi\psi = pI\},$$

where I is the identity matrix. It is easy to check the following inclusions

$$J_\phi \subseteq I_\phi \subseteq \sqrt{J_\phi}, \tag{3.1}$$

where $\sqrt{\cdot}$ means the radical ideal. Standard linear theory shows that

$$m \in U_{co}(\phi) \Leftrightarrow I_{\phi_m} = (R/m)[u, u^{-1}] \Leftrightarrow J_{\phi_m} = (R/m)[u, u^{-1}].$$

Theorem 3 *Let $\phi \in R[u, u^{-1}]^{g \times q}$. Then, the following facts are equivalent*

1. $\mathcal{B} := \ker \phi(\sigma, \sigma^{-1})$ *is controllable and* $\phi(\sigma, \sigma^{-1})$ *is onto.*
2. $J_\phi = I_\phi = R[u, u^{-1}]$.
3. $U_{co}(\phi) = \mathrm{Max}(R)$.

Proof: 1.$\Rightarrow$3. Fix an $m \in \mathrm{Max}(R)$. The fact that $\phi_m(\sigma, \sigma^{-1})$ is onto is obvious. On the other hand, it is clear that the controllability of $\mathcal{B}$ implies the controllability of $\mathcal{B}_m$ and so, since $\mathcal{B}_m = \ker \phi_m(\sigma, \sigma^{-1})$ by Proposition 2, then $\ker \phi_m(\sigma, \sigma^{-1})$ is controllable.

3.⇒2. Suppose that $I_\phi \neq R[u,u^{-1}]$. Then, there exists a maximal ideal $\tilde{m} \in \mathrm{Max}(R[u,u^{-1}])$ such that $I_\phi \subseteq \tilde{m}$. Let $m = \tilde{m} \cap R$. Since R is Jacobson, [2] it follows that $m \in \mathrm{Max}(R)$. Since $mR[u,u^{-1}] \subseteq \tilde{m}$, then

$$I_\phi + mR[u,u^{-1}] \subseteq \tilde{m} \neq R[u,u^{-1}].$$

This implies that $I_{\phi_m} \neq (R/m)[u,u^{-1}]$.

2.⇒1. The fact that $\phi(\sigma,\sigma^{-1})$ is onto is obvious. Let $\psi \in R[u,u^{-1}]^{q\times g}$ such that $\phi\psi = I$. It is immediate to see that

$$\mathcal{B} = \mathrm{im}\ (I - \psi(\sigma,\sigma^{-1})\phi(\sigma,\sigma^{-1})). \tag{3.2}$$

■

Notice that Theorem 3 provides an almost direct way to construct an image representation, as shown in (3.2). The problem of constructing an image representation reduces to the problem of finding ψ such that $\phi\psi = I$. Notice that, if $r_1, \ldots, r_s \in R[u,u^{-1}]$ are the $g \times g$ minors of ϕ, then the problem of finding ψ reduces to determining $h_1, \ldots, h_s \in R[u,u^{-1}]$ such that

$$\sum_{i=1}^{s} h_i r_i = 1. \tag{3.3}$$

Actually, suppose that S_i is the selection matrix (i.e., a matrix in $R^{q\times g}$ with only zeros and ones) such that $r_i = \det(\phi S_i)$. Then,

$$I = \sum_{i=1}^{s} h_i r_i I = \sum_{i=1}^{s} h_i (\det \phi S_i) I = \phi(\sum_{i=1}^{s} h_i S_i \mathrm{Adj}\ (\phi S_i)).$$

It is clear that we can let $\psi := \sum_{i=1}^{s} h_i S_i \mathrm{Adj}\ (\phi S_i)$. Note that, for a large class of Noetherian rings R, problems like checking if an ideal I generated by $r_1, \ldots, r_s \in R[u,u^{-1}]$ contains 1 and like determining $h_1, \ldots, h_s \in R[u,u^{-1}]$ such that (3.3) holds can be solved efficiently by Gröbner basis techniques [3, 11, 14].

3.2 Observable image representations

The drawback of the image representation (3.2) is that, in general, it is not observable. As we will see, for some rings, more refined constructions will yield observable image representations. The first important case is given when the ring $R[u,u^{-1}]$ is Hermite. We recall that a ring A is called Hermite if given any onto $\phi \in A^{g\times q}$, there exists $\phi' \in A^{(q-g)\times q}$ such that

$$\begin{bmatrix} \phi \\ \phi' \end{bmatrix}$$

is invertible in the ring A. Equivalently, A is Hermite if every finitely generated R-module V which is stably free (i.e. $V \oplus A^g \simeq_A A^q$ for some g, q) is then free. The connection between the two definitions is given by considering $V = \ker \phi$ [8]. It is difficult in general to determine if a certain ring is Hermite. There are, however, important examples: as a consequence of the Quillen-Suslin result [8] (Corollary 4.12) we have that, if $R = A[z_1, \ldots, z_n]$, where A is a PID, then both R and $R[u, u^{-1}]$ are Hermite.

We have the following.

Proposition 4 *Suppose that $R[u, u^{-1}]$ is Hermite. Let $\phi \in R[u, u^{-1}]^{g \times q}$ and let $\mathcal{B} := \ker \phi(\sigma, \sigma^{-1})$. Assume that any of the equivalent conditions in Theorem 3 holds. Then, there exists $\psi \in R[u, u^{-1}]^{q \times (q-g)}$ such that*

$$\mathcal{B} = \text{im } \psi(\sigma, \sigma^{-1}),$$

where $\psi(\sigma, \sigma^{-1})$ is one–to–one and admits a continuous inverse on the image.

Proof: The assumptions imply that there exists $\phi' \in R[u, u^{-1}]^{(q-g) \times q}$ such that

$$\begin{bmatrix} \phi \\ \phi' \end{bmatrix} \tag{3.4}$$

is invertible in $R[u, u^{-1}]$. Then, there exists $\psi' \in R[u, u^{-1}]^{q \times g}$ and $\psi \in R[u, u^{-1}]^{q \times (q-g)}$ such that

$$\begin{bmatrix} \phi \\ \phi' \end{bmatrix} [\psi' \quad \psi] = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

It is easy to verify that

$$\mathcal{B} = \text{im } \psi(\sigma, \sigma^{-1})$$

and that the matrix shift operator $\psi(\sigma, \sigma^{-1})$ is one to one. Now observe that the inverse of $\psi(\sigma, \sigma^{-1})$ on the image $\mathcal{B}$ is given by $\phi'(\sigma, \sigma^{-1})_{|\mathcal{B}}$ which is continuous. ■

The assumption that $R[u, u^{-1}]$ is Hermite in Proposition 4 is crucial as we will see shortly. First notice that we can easily prove that if $R[u, u^{-1}]$ is Hermite, than also R must be Hermite. Actually assume that R is not a Hermite ring. Then, there exists an onto R-homomorphism

$$\eta : R^q \to R^g$$

such that $V = \ker \eta$ is not free. Clearly $I_\eta = R[u, u^{-1}]$. Since V is not free, it easily follows that $V^{\mathbb{Z}} = \ker \eta(\sigma, \sigma^{-1})$ can not have observable image representation. There are plenty of Jacobson Noetherian rings which are not Hermite. An example is the ring of real polynomial functions on the 2-sphere: it possesses a stably free R submodule of R^3 or rank equal to 2 which is not free [8]. This example will be taken up again in Section 5.

This last example suggests the fact that we might look for more general image representations of shift operators which are not necessarily of matrix type. We are thinking of shift operators induced by elements in $\mathrm{Hom}(V, R^q)[u, u^{-1}]$, where V is a finitely generated, usually stably free, R-module. We start with a lemma which will also be used later.

Lemma 5 *Let $\phi \in \mathrm{Hom}(V, R^q)[u, u^{-1}]$, where V is a stably free finitely generated R-module, and assume that $\mathcal{B} := \mathrm{im}\ \phi(\sigma, \sigma^{-1}) \subseteq (R^q)^{\mathbb{Z}}$ is a regular R-behavior. The following conditions are equivalent:*

1. *$\phi(\sigma, \sigma^{-1})$ is one–to–one;*
2. *$\phi_m(\sigma, \sigma^{-1})$ is one to one for all $m \in \mathrm{Max}(R)$.*

Proof: 1.⇒2. If $\bar{v} \in (V/mV)^{\mathbb{Z}}$ is such that $\phi_m(\sigma, \sigma^{-1})\bar{v} = 0$ and $v \in V^{\mathbb{Z}}$ is a representative of $\bar{v}$, then $w := \phi(\sigma, \sigma^{-1})v \in m(R^q)^{\mathbb{Z}}$. Applying Lemma 1 we have that $w \in m\mathcal{B}$. Therefore, if $a_1, \ldots, a_n \in R$ generate m, then there exist $v_1, \ldots, v_n \in V^{\mathbb{Z}}$ such that

$$w = \phi(\sigma, \sigma^{-1})v = \sum_{i=1}^{n} a_i \phi(\sigma, \sigma^{-1})v_i.$$

Since $\phi(\sigma, \sigma^{-1})$ is one to one, then

$$v = \sum_{i=1}^{n} a_i v_i \in mV^{\mathbb{Z}}$$

and so $\bar{v} = 0$.

2.⇒1. Denote $\bar{\mathcal{B}} := \ker \phi(\sigma, \sigma^{-1})$. If $v \in \bar{\mathcal{B}}$, it follows from the assumption 2. that $v \in (mV)^{\mathbb{Z}}$ for all $m \in \mathrm{Max}(R)$. Denote by K the Jacobson radical of R. Since V is stably free, it easily follows that

$$\bar{\mathcal{B}} \subseteq (KV)^{\mathbb{Z}}. \tag{3.5}$$

On the other hand, since im $\phi(\sigma, \sigma^{-1})$ is regular, it easily follows that also $\bar{\mathcal{B}}$ is regular. Therefore, it follows from Lemma 1 that

$$\bar{\mathcal{B}} \cap (KV)^{\mathbb{Z}} = K\bar{\mathcal{B}}$$

and, because of (3.5), we thus obtain

$$\bar{\mathcal{B}} = K\bar{\mathcal{B}}.$$

In particular,

$$\bar{\mathcal{B}}_{|[0,n]} = (K\bar{\mathcal{B}})_{|[0,n]} = K(\bar{\mathcal{B}}_{|[0,n]}) \quad \forall n \in \mathbb{N}.$$

Since $\bar{\mathcal{B}}_{|[0,n]}$ is finitely generated, then it follows from Nakayama's lemma that

$$\bar{\mathcal{B}}_{|[0,n]} = 0 \quad \forall n \in \mathbb{N}.$$

This clearly yields 1. ■

We can now state and prove the following result that generalize Proposition 4 to rings that are not Hermite.

Proposition 6 *Let $\phi \in R[u,u^{-1}]^{g\times q}$ and let $\mathcal{B} := \ker \phi(\sigma,\sigma^{-1})$. Assume that any of the equivalent conditions in Theorem 3 holds and moreover that $\mathcal{B}_f$ is R-extended (i.e., there exists an R-module W such that $\mathcal{B}_f \simeq W[u,u^{-1}]$). Then, there exist a stably free R-module V and a $\psi \in \mathrm{Hom}(V,R^q)[u,u^{-1}]$ such that*

$$\mathcal{B} = \mathrm{im}\ \psi(\sigma,\sigma^{-1}),$$

where $\psi(\sigma,\sigma^{-1})$ is one–to–one and admits a continuous inverse on the image. Moreover, we have that $V \simeq W$.

Proof: Assume that $I_\phi = R[u,u^{-1}]$ and that $\mathcal{B}_f \simeq W[u,u^{-1}]$ for some finitely generated R-module W. We have an exact sequence

$$0 \to W[u,u^{-1}] \overset{j}{\hookrightarrow} R[u,u^{-1}]^q \overset{\phi}{\longrightarrow} R[u,u^{-1}]^g \to 0.$$

Consider the splitting map $\phi' := j^{-1}(I \Leftrightarrow \eta\phi)$, where $\eta \in R[u,u^{-1}]^{q\times g}$ is such that $\phi\eta = I$. It is easy to see that

$$\begin{bmatrix} \phi \\ \phi' \end{bmatrix} \in \mathrm{Hom}(R^q, R^g \oplus W)[u,u^{-1}]$$

is invertible. Write the inverse as $[\psi',\psi]$. It is immediate to check that ψ has all the required properties and that W is stably free.

We want to prove now that $\psi(\sigma,\sigma^{-1})$ has a continuous inverse on the image. More precisely, we will prove that there exists $\xi \in \mathrm{Hom}(R^q,V)[u,u^{-1}]$ such that $\xi\psi = I$. It follows from Lemma 5 and the remark following it that $\psi_m(\sigma,\sigma^{-1})$ is one to one for every $m \in \mathrm{Max}(R)$. Hence, by standard linear theory, it follows that $(\psi^*)_m$ is onto for every m, where $\psi^* \in \mathrm{Hom}(V^*,R^q)$ denotes the dual homomorphism [6]. A standard argument using Jacobson property then shows that for every $\tilde{m} \in \mathrm{Max}(R[u,u^{-1}])$, the quotient homomorphism

$$(\psi^*)_{\tilde{m}} : V^*[u,u^{-1}]/\tilde{m}V^*[u,u^{-1}] \longrightarrow R^q[u,u^{-1}]/\tilde{m}R^q[u,u^{-1}]$$

is onto. Hence [2, II.3.3, Prop. 11] ψ^* is onto. Since $V^*[u,u^{-1}]$ is projective, there exists $\eta \in \mathrm{Hom}(V^*, R^q)[u,u^{-1}]$ such that $\psi^*\eta = I$. It is now sufficient to take $\xi = \eta^*$. Notice that ψ yields an isomorphism $\mathcal{B}_f \simeq V[u,u^{-1}]$.

■

There are interesting examples [8] in which hypotheses of Proposition 6 may be satisfied without R being Hermite. We will present a couple of examples in Section 5.

Proposition 6 has another interesting consequence. Assume that R is such that there exists a stably free $R[u,u^{-1}]$-module M which is not R-extended. Consider $\phi \in R[u,u^{-1}]^{g\times q}$ and $\psi \in R[u,u^{-1}]^{q\times g}$ such that $\ker\phi \simeq M$ and $\phi\psi = I$. Define $\mathcal{B} := \ker\phi(\sigma,\sigma^{-1})$. Notice that $\mathcal{B} = \mathrm{im}\ (I \Leftrightarrow \psi(\sigma,\sigma^{-1})\phi(\sigma,\sigma^{-1}))$. It is immediate to check that $\mathcal{B}_f \simeq M$. Proposition 6 implies that $\mathcal{B}$ does not possess observable image representations even of general type as discussed above. It remains the problem if such rings do exist and the answer is positive. It can be proven, but it is highly non-trivial, that if R is the ring of the complex polynomials on the 4-sphere, then $R[u,u^{-1}]^3$ possesses a stably free submodule of rank 2, which is not R-extended.

3.3 A remark on regularity in the Hermite case

An interesting consequence of Theorem 3 is the fact that, when $R[u,u^{-1}]$ is Hermite, then all regular behaviors have order less than or equal to 1. Surprisingly enough, also the inverse holds true.

Proposition 7 *$R[u,u^{-1}]$ is Hermite if and only if every regular R-behavior $\mathcal{B} \subseteq (R^q)^{\mathbb{Z}}$ has at most order 1.*

Proof: Suppose that $R[u,u^{-1}]$ is Hermite and let $\mathcal{B} \subseteq (R^q)^{\mathbb{Z}}$ be a regular R-behavior. Suppose that

$$0 \Leftrightarrow\!\rightarrow \mathcal{B} \stackrel{i}{\hookrightarrow} (R^q)^{\mathbb{Z}} \stackrel{\phi_1(\sigma,\sigma^{-1})}{\Leftrightarrow\!\rightarrow} (R^{q_1})^{\mathbb{Z}} \stackrel{\phi_2(\sigma,\sigma^{-1})}{\Leftrightarrow\!\rightarrow} \cdots \stackrel{\phi_n(\sigma,\sigma^{-1})}{\Leftrightarrow\!\twoheadrightarrow} (R^{q_n})^{\mathbb{Z}} \Leftrightarrow\!\rightarrow 0 \tag{3.6}$$

is the exact sequence associated to $\mathcal{B}$. Denote moreover

$$\mathcal{B}_i := \ker\phi_{i+1}(\sigma,\sigma^{-1}) = \mathrm{im}\ \phi_i(\sigma,\sigma^{-1}).$$

Note that $\mathcal{B}_{n-1} = \ker\phi_n(\sigma,\sigma^{-1})$ is controllable and that $\phi_n(\sigma,\sigma^{-1})$ is onto. Therefore, by Theorem 3, there exists a polynomial matrix $\bar{\psi}_n$ such that $\phi_n\bar{\psi}_n = I$. Since $R[u,u^{-1}]$ is Hermite, this implies that there exists a polynomial matrix ϕ'_n of suitable dimensions such that

$$\begin{bmatrix} \phi_n \\ \phi'_n \end{bmatrix}$$

is a square unimodular polynomial matrix. Let $[\psi_n \ \psi_n']$ be its inverse partitioned in a conform way and let $\bar{\phi}_{n-1} := \phi_n' \phi_{n-1}$.

First note that

$$\mathcal{B}_{n-2} = \ker \bar{\phi}_{n-1}(\sigma, \sigma^{-1}). \tag{3.7}$$

"$\subseteq$" is evident. On the other hand, note that $\phi_n \phi_{n-1} = 0$ and that

$$\psi_n \phi_n + \psi_n' \phi_n' = I. \tag{3.8}$$

Hence

$$\phi_{n-1} = \psi_n \phi_n \phi_{n-1} + \psi_n' \phi_n' \phi_{n-1} = \psi_n' \bar{\phi}_{n-1}.$$

From this (3.7) immediately follows.

Moreover, note that

$$\ker \phi_n(\sigma, \sigma^{-1}) = \text{im } \psi_n'(\sigma, \sigma^{-1}).$$

The inclusion "$\subseteq$" simply follows from $\phi_n \psi_n' = 0$. On the other hand, let $w \in \ker \phi_n(\sigma, \sigma^{-1})$. Then, applying (3.8), we have that

$$\begin{aligned} w &= \psi_n(\sigma, \sigma^{-1}) \phi_n(\sigma, \sigma^{-1}) w + \psi_n'(\sigma, \sigma^{-1}) \phi_n'(\sigma, \sigma^{-1}) w = \\ &= \psi_n'(\sigma, \sigma^{-1}) \phi_n'(\sigma, \sigma^{-1}) w \end{aligned}$$

that implies that $w \in \text{im } \psi_n'(\sigma, \sigma^{-1})$.

Therefore

$$\begin{aligned} \text{im } \bar{\phi}_{n-1}(\sigma, \sigma^{-1}) &= \phi_n'(\sigma, \sigma^{-1}) \text{im } \phi_{n-1}(\sigma, \sigma^{-1}) = \\ &= \phi_n'(\sigma, \sigma^{-1}) \ker \phi_n(\sigma, \sigma^{-1}) = \\ &= \phi_n'(\sigma, \sigma^{-1}) \text{im } \psi_n'(\sigma, \sigma^{-1}) = (R^{q_{n-1} - q_n})^{\mathbb{Z}}. \end{aligned}$$

Therefore it is possible to associate to $\mathcal{B}$ the exact sequence

$$0 \to \mathcal{B} \stackrel{i}{\hookrightarrow} (R^q)^{\mathbb{Z}} \stackrel{\phi_1(\sigma, \sigma^{-1})}{\Leftrightarrow\!\!\to} (R^{q_1})^{\mathbb{Z}} \cdots (R^{q_{n-2}})^{\mathbb{Z}} \stackrel{\bar{\phi}_{n-1}(\sigma, \sigma^{-1})}{\Leftrightarrow\!\!\twoheadrightarrow} (R^{q_{n-1} - q_n})^{\mathbb{Z}} \to 0,$$

that is one step shorter than the exact sequence (3.6). By induction, this proves that it is always possible to associate to $\mathcal{B}$ a sequence of length less than or equal to 1.

Suppose, conversely, that every regular R-behavior $\mathcal{B}$ has at most order 1 and let $\phi \in R[u, u^{-1}]^{g \times q}, \psi \in R[u, u^{-1}]^{q \times g}$ be polynomial matrices such that $\phi \psi = I$. Define the R-behavior $\mathcal{B} := \text{im } \psi(\sigma, \sigma^{-1})$. Then it is easy to see that

$$\mathcal{B} = \ker(I \Leftrightarrow \psi(\sigma, \sigma^{-1}) \phi(\sigma, \sigma^{-1})).$$

Let $\bar{\phi} := I \Leftrightarrow \psi \phi$. Then, it is easy to see that the following sequence is exact

$$0 \Leftrightarrow\!\!\to \mathcal{B} \stackrel{i}{\hookrightarrow} (R^q)^{\mathbb{Z}} \stackrel{\bar{\phi}(\sigma, \sigma^{-1})}{\Leftrightarrow\!\!\to} (R^q)^{\mathbb{Z}} \stackrel{\phi(\sigma, \sigma^{-1})}{\Leftrightarrow\!\!\twoheadrightarrow} (R^l)^{\mathbb{Z}} \Leftrightarrow\!\!\to 0.$$

Then, $\mathcal{B}$ is regular and so it is regular of order less than or equal to 1. Suppose that $\phi' \in R[u,u^{-1}]^{l' \times q}$ such that $\mathcal{B} = \ker \phi'(\sigma, \sigma^{-1})$ and $\phi'(\sigma,\sigma^{-1})$ is onto. If we look at ψ and ϕ' as polynomial matrices with entries in the field of fractions of $R[u,u^{-1}]$, then it is easy to verify that the range of ψ and the kernel of ϕ' coincide and so $l' = \text{rank}\ \phi' = q \Leftrightarrow \text{rank}\ \psi = q \Leftrightarrow l$. Moreover, since $\mathcal{B}$ is controllable, then by Theorem 3, there exists $\psi' \in R[u,u^{-1}]^{q \times g'}$ such that $\phi'\psi' = I$. Moreover $\phi'\psi = 0$. Therefore we have that

$$\begin{bmatrix} \phi \\ \phi' \end{bmatrix} [\psi \quad \psi'] = \begin{bmatrix} I & * \\ 0 & I \end{bmatrix}.$$

This implies that

$$\begin{bmatrix} \phi \\ \phi' \end{bmatrix}$$

is square and unimodular. ■

3.4 Loss of controllability and partial image representations

From the point–of–view of parametrized systems, the situation expressed in Theorem 3 corresponds to systems that remain controllable for all values of the parameters. In the following, we will deal with a more general situation in which we may lose controllability for some values of the parameters. We need some preliminary notations. If A is a ring we will consider on $\text{Max}(A)$ the Zarisky topology whose closed sets are the subsets of $\text{Max}(A)$ of the following kind

$$V(I) := \{m \in \text{Max}(R) \mid I \subseteq m\},$$

where I is an ideal of A.

First we need to develop some new concepts. Since R is a Jacobson ring, the inclusion

$$j : R \hookrightarrow R[u,u^{-1}]$$

induces [1, pag. 13] a continuous mapping

$$j^* : \text{Max}(R[u,u^{-1}]) \to \text{Max}(R)$$

where $j^*(\tilde{m}) := \tilde{m} \cap R$. Notice first of all that j^* is onto: indeed, if $m \in \text{Max}(R)$, consider an $\tilde{m} \in \text{Max}(R[u,u^{-1}])$ such that $m[u,u^{-1}] \subseteq \tilde{m}$. Clearly, $j^*(\tilde{m}) = m$. Moreover, we have a canonical homeomorphism

$$(j^*)^{-1}(m) \simeq \text{Max}((R/m)[u,u^{-1}]) \tag{3.9}$$

obtained by associating to $\tilde{m} \in (j^*)^{-1}(m)$ its projection into $(R/m)[u,u^{-1}]$.

Let $\phi \in R[u,u^{-1}]^{g \times q}$. Let us remind that $U_{co}(\phi) = U_c(\phi) \cap U_o(\phi)$ is the subset of $\text{Max}(R)$ consisting of the maximal ideals m yielding $I_{\phi_m} = (R/m)[u,u^{-1}]$. We have the following

Lemma 8 *The following relations hold true:*

1. $U_{co}(\phi) = j^*(V(I_\phi))^c$;
2. $(U_{co}(\phi))^o = V(I_\phi \cap R)^c$, *where* $(\ \cdot\ )^o$ *means the interior;*
3. $U_o(\phi) = j^*(V(I_\phi)^c)$.

Moreover, if I_ϕ *contains a bimonic polynomial* ($p = \sum_{i=n}^N p_i u^i$ *with* $p_n, p_N \in R^*$), *then*

4. $U_{co}(\phi) = V(I_\phi \cap R)^c$.

Proof: 1. Let $\tilde{m} \in \mathrm{Max}(R[u,u^{-1}])$ and put $m = \tilde{m} \cap R$. Consider the commutative diagram

$$\begin{array}{ccc} (R/m)[u,u^{-1}]^q & \overset{\phi_m}{\Longleftrightarrow} & (R/m)[u,u^{-1}]^p \\ \downarrow & & \downarrow \\ (R[u,u^{-1}]/\tilde{m})^q & \overset{\phi_{\tilde{m}}}{\Longleftrightarrow} & (R[u,u^{-1}]/\tilde{m})^p. \end{array} \tag{3.10}$$

Now, if $\tilde{m} \in j^*(V(I_\phi))$, we have that $J_\phi \subseteq \tilde{m}$ and, consequently, $\phi_{\tilde{m}}$ is not onto. This then shows (3.10) that also ϕ_m can not be onto and this yields $m \notin U_{co}(\phi)$. On the other hand, if $m \notin U_{co}(\phi)$, ϕ_m is not onto and this implies that there exists $\bar{m} \in \mathrm{Max}((R/m)[u,u^{-1}])$ such that the quotient of ϕ_m by $\bar{m}$ is not onto. Let $\tilde{m} \in (j^*)^{-1}(m)$ be the maximal ideal corresponding to $\bar{m}$ through the identification (3.9). It is clear that $\phi_{\tilde{m}}$ is not onto and this implies that $\tilde{m} \in V(I_\phi)$. This proves 1.

2. We will prove the equivalent fact that

$$\overline{U_{co}(\phi)^c} = V(I_\phi \cap R). \tag{3.11}$$

Clearly,

$$\overline{U_{co}(\phi)^c} = \overline{j^*(V(I_\phi))} = V(I),$$

for some ideal I in R. We have that

$$\sqrt{I} = \bigcap_{m \supseteq I} m \subseteq \bigcap_{\tilde{m} \supseteq I_\phi} (\tilde{m} \cap R) = \sqrt{I_\phi} \cap R \subseteq \sqrt{I_\phi \cap R}.$$

Hence

$$\overline{U_{co}(\phi)^c} \supseteq V(I_\phi \cap R).$$

On the other hand, it follows from 1 that

$$\overline{U_{co}(\phi)^c} \subseteq V(I_\phi \cap R)$$

and this proves (3.11) and hence 2.

3. Arguing as in the proof of 1, we can easily see that $m \in U_o(\phi)$ if and only if there exists $\tilde{m} \in (j^*)^{-1}(m)$ such that $\phi_{\tilde{m}}$ is onto. From this 3 easily follows.

4. The assumption made easily implies that $R[u, u^{-1}]/I_\phi$ is an integral extension of $R/(I_\phi \cap R)$. This implies [13] that

$$j^*_{|V(I_\phi)} : V(I_\phi) \to V(I_\phi \cap R)$$

is surjective. ∎

An example where $U_{co}(\phi)$ is not open is not hard to find and will be given in Section 5.

Given a behavior $\mathcal{B}$, we would like to find an image representation whose behavior coincides with the the given one 'in as many as possible maximal ideals'. More precisely, we would like to find $\psi \in R[u, u^{-1}]^{q \times g}$ such that

$$\mathcal{B}_m = \text{im } \psi_m(\sigma, \sigma^{-1})$$

for all $m \in U_c(\phi)$. Of course, this is the best we can hope, but, as it will be shown in Section 5, this is in general not possible. It will be proven in the sequel that it is possible if we restrict instead to $U_{co}(\phi)$. For the moment, however, we will limit ourselves to something less ambitious considering maximal ideals in $V(I_\phi \cap R)^c$. It follows from Lemma 8 that this constitutes a consistent part of $U_{co}(\phi)$.

Let $a_1, \ldots, a_n$ be a family of generators for the ideal $I_\phi \cap R$ and let $\psi_i \in R[u, u^{-1}]^{q \times g}$ be such that $\phi \psi_i = a_i I$. It is easy to see that

$$\text{im } (a_i I \Leftrightarrow \psi_i(\sigma, \sigma^{-1}) \phi(\sigma, \sigma^{-1})) \subseteq \ker \phi(\sigma, \sigma^{-1}),$$

$$(V(I_\phi \cap R))^c = \bigcup_{i=1}^{n} V((a_i))^c,$$

where (a_i) is the ideal generated by a_i. If $m \in (V(I_\phi \cap R))^c$, then $m \in V((a_i))^c$ for some i and it easily follows that

$$\ker \phi_m(\sigma, \sigma^{-1}) = \text{im } (a_i I \Leftrightarrow \psi_i \phi)_m(\sigma, \sigma^{-1}).$$

The disadvantage of this type of representation is that it has to be changed according to the open set in which we are considering the maximal ideals. If R is a PID, however, this will indeed provide a global image representation over all $(V(I_\phi \cap R))^c$. Moreover, this is a possible way to find observable image representations. Indeed, consider the ring

$$R_i := \{ \frac{r}{a_i^k} \mid r \in R,\ k \in \mathbb{N} \}.$$

The polynomial matrix ϕ, as a matrix in $R_i[u, u^{-1}]^{q \times g}$, will be denoted by ϕ_i. Put $\mathcal{B}_i := \ker \phi_i(\sigma, \sigma^{-1})$. It follows that $I_{\phi_i} = R_i[u, u^{-1}]$. Since R_i is still a Jacobson ring [2], it follows that Theorem 3 holds true for ϕ_i and $\mathcal{B}_i$. If $m \in V((a_i))^c$ and we denote by m_i the maximal ideal in R_i obtained by extending m, we have that

$$(\mathcal{B}_i)_{m_i} = \mathcal{B}_m \quad (\phi_i)_{m_i} = \phi_m$$

under the canonical identification $R_i/m_i = R/m$. This shows that if we can find an observable image representation for $\mathcal{B}_i$, we are done. If $R_i[u, u^{-1}]$ is Hermite, then everything goes through. Notice that if R is a PID, also R_i is a PID and therefore $R_i[u, u^{-1}]$ is Hermite. However in general $R_i[u, u^{-1}]$ may well not be Hermite even if $R[u, u^{-1}]$ was such. More specific considerations will be taken up in Section 5.

3.5 Alternative methods for finding image representations

We now present an alternative method that has general validity and which permits to find more general image representations. First we have the following lemma.

Lemma 9 *Let $\phi \in R[u, u^{-1}]^{q\times g}$ and consider $\mathcal{B} := \ker \phi(\sigma, \sigma^{-1})$. Let $\psi \in R[u, u^{-1}]^{q\times r}$ be any polynomial matrix whose columns generate the $R[u, u^{-1}]$-submodule*

$$\ker \phi = \{x \in R[u, u^{-1}]^q \ : \ \phi x = 0\}. \tag{3.12}$$

Then the controllable part of $\mathcal{B}$ is given by

$$\mathcal{B}_c = \text{im } \psi(\sigma, \sigma^{-1}).$$

Proof: Put $\mathcal{F} := ((R^q)^{\mathbb{Z}})_f$. Then $\mathcal{B}_f = \psi(\sigma, \sigma^{-1})(\mathcal{F})$. This yields

$$\mathcal{B}_c = \overline{\mathcal{B}_f} = \overline{\psi(\sigma, \sigma^{-1})(\mathcal{F})} \supseteq \psi(\sigma, \sigma^{-1})(\overline{\mathcal{F}}) = \text{im } \psi(\sigma, \sigma^{-1}).$$

On the other hand, since $\mathcal{B}_c$ is controllable, there exists [21] $\hat{\psi} \in R[u, u^{-1}]^{q\times s}$ such that

$$\mathcal{B}_c = \text{im } \hat{\psi}(\sigma, \sigma^{-1}).$$

We have that

$$\hat{\psi}(\sigma, \sigma^{-1})(\mathcal{F}) \subseteq \mathcal{B}_f = \psi(\sigma, \sigma^{-1})(\mathcal{F})$$

and so there exists a polynomial matrix Y of suitable dimensions such that $\hat{\psi} = \psi Y$ and so

$$\mathcal{B} = \text{im } \hat{\psi}(\sigma, \sigma^{-1}) \subseteq \text{im } \psi(\sigma, \sigma^{-1}).$$

The proof is thus complete. ■

Notice that Gröbner basis techniques can be of some use also for finding $\psi \in R[u, u^{-1}]^{g\times q}$ whose columns generate the submodule (3.12) [10].

This is our more general result regarding image representations.

Theorem 10 *Let* $\phi \in R[u,u^{-1}]^{g\times q}$ *and let* $\mathcal{B} = \ker\phi(\sigma,\sigma^{-1})$. *If* $\psi \in R[u,u^{-1}]^{q\times r}$ *is such that* $\mathcal{B}_c = \text{im }\psi(\sigma,\sigma^{-1})$, *then for all* $m \in U_o(\phi)$

$$(\mathcal{B}_m)_c = (\ker\phi_m(\sigma,\sigma^{-1}))_c = \text{im }\psi_m(\sigma,\sigma^{-1}).$$

Proof: Consider the exact sequence

$$0 \rightarrow \mathcal{B}_f \hookrightarrow R[u,u^{-1}]^q \stackrel{\phi}{\Leftrightarrow\!\rightarrow} R[u,u^{-1}]^g.$$

Localization at $\tilde{m} \in V(I_\phi)^c$ yields another exact sequence

$$0 \rightarrow (\mathcal{B}_f)_{(\tilde{m})} \hookrightarrow (R[u,u^{-1}]_{(\tilde{m})})^q \stackrel{\phi}{\Leftrightarrow\!\rightarrow} (R[u,u^{-1}]_{(\tilde{m})})^g \rightarrow 0, \tag{3.13}$$

which implies that $(\mathcal{B}_f)_{(\tilde{m})}$ is a projective $R[u,u^{-1}]_{(\tilde{m})}$-module, hence free, of dimension exactly $q \Leftrightarrow g$. But,

$$\dim_{R[u,u^{-1}]_{(\tilde{m})}}(\mathcal{B}_f)_{(\tilde{m})} = \dim_{R[u,u^{-1}]/\tilde{m}}\left[(\mathcal{B}_f)_{(\tilde{m})}/\tilde{m}(\mathcal{B}_f)_{(\tilde{m})}\right] = \tag{3.14}$$

$$= \dim_{R[u,u^{-1}]/\tilde{m}}\left[\mathcal{B}_f/\tilde{m}\mathcal{B}_f\right]. \tag{3.15}$$

It follows from (3.13) that

$$\tilde{m}(\mathcal{B}_f)_{(\tilde{m})} = (\mathcal{B}_f)_{(\tilde{m})} \cap \tilde{m}(R[u,u^{-1}]_{(\tilde{m})})^q. \tag{3.16}$$

This implies that

$$\tilde{m}\mathcal{B}_f = \mathcal{B}_f \cap \tilde{m}R[u,u^{-1}]^q. \tag{3.17}$$

Indeed, if $v \in \mathcal{B}_f \cap \tilde{m}R[u,u^{-1}]^q$, it follows from (3.16) that there exists $a \in R[u,u^{-1}] \setminus \tilde{m}$ such that $av \in \tilde{m}\mathcal{B}_f$. Since $\mathcal{B}_f/\tilde{m}\mathcal{B}_f$ is an $R[u,u^{-1}]/\tilde{m}$-vector space, it follows that $v \in \tilde{m}\mathcal{B}_f$. It follows from (3.14) and (3.17) that

$$\dim_{R[u,u^{-1}]/\tilde{m}}\left[\mathcal{B}_f/(\mathcal{B}_f \cap \tilde{m}R[u,u^{-1}]^q)\right] = q \Leftrightarrow g. \tag{3.18}$$

Let now $m = \tilde{m} \cap R$. It easily follows from (3.18) that $(\mathcal{B}_f)_m$ is a free $(R/m)[u,u^{-1}]$-module of dimension $q \Leftrightarrow g$. Hence, $(\mathcal{B}_c)_m$ has rank $q \Leftrightarrow g$. Consider now $m \in U_o(\phi)$ and let $\tilde{m} \in V(I_\phi)^c$ such that $m = \tilde{m} \cap R$. We have the inclusions

$$(\mathcal{B}_c)_m \subseteq (\mathcal{B}_m)_c \subseteq (\ker\phi_m(\sigma,\sigma^{-1}))_c.$$

By previous considerations they all have the same rank, hence they are equal. This clearly proves the result. ■

4 From Image to Kernel Representations

In this section we will study conditions under which a controllable R-behavior $\mathcal{B}$, given by an image representation, admits a kernel representation. Moreover, we will see how to develop a procedure that provides constructively a kernel representation for $\mathcal{B}$ starting from its image representation. Many of the results are symmetric to the ones presented in Section 4. Many proofs are omitted or only sketched.

4.1 Regularity of behaviors admitting image representations

First we set some more notation. Given $\phi \in R[u, u^{-1}]^{q\times g}$, we define the ideal $\hat{J}_\phi$ generated by all the $g \times g$ minors of ϕ (we put $\hat{J}_\phi = 0$ if $q < g$) and the ideal

$$\hat{I}_\phi := \{p \in R[u, u^{-1}] \mid \exists \psi \in R[u, u^{-1}]^{g\times q} \; : \psi\phi = pI\}.$$

Clearly, $\hat{J}_\phi = J_{\phi^t}$ and $\hat{I}_\phi = I_{\phi^t}$ as previously defined. Define

$$U_i(\phi) := \{m \in \mathrm{Max}(R) \mid \; \phi_m(\sigma, \sigma^{-1}) \text{ is one–to–one}\}.$$

Standard linear theory shows again that

$$m \in U_i(\phi) \Leftrightarrow \hat{I}_{\phi_m} = (R/m)[u, u^{-1}] \Leftrightarrow \hat{J}_{\phi_m} = (R/m)[u, u^{-1}].$$

Theorem 11 *Let $\phi \in R[u, u^{-1}]^{q\times g}$ and suppose that $\mathcal{B} := \mathrm{im}\ \phi(\sigma, \sigma^{-1})$. Then, the following facts are equivalent*

1. *$\mathcal{B}$ is regular and $\phi(\sigma, \sigma^{-1})$ is on–t–one;*
2. *$\mathcal{B}$ is regular of order less than or equal to 2 and $\phi(\sigma, \sigma^{-1})$ is on–to–one;*
3. *$\hat{I}_\phi = \hat{J}_\phi = R[u, u^{-1}]$;*
4. *$U_i(\phi) = \mathrm{Max}(R)$.*

Proof: 2.⇒1. Obvious.

1.⇒4. Direct consequence of Lemma 5.

4.⇒3. follows from Theorem 3 since $U_i(\phi) = U_{co}(\phi^t)$.

3.⇒2. Let $\psi \in R[u, u^{-1}]^{g\times q}$ be such that $\psi\phi = I$. Then, it is clear that $\phi(\sigma, \sigma^{-1})$ is one to one. We have that

$$\mathrm{im}\ \phi(\sigma, \sigma^{-1}) = \ker(I \Leftrightarrow \phi(\sigma, \sigma^{-1})\psi(\sigma, \sigma^{-1}))$$

and moreover that

$$\mathrm{im}\ (I \Leftrightarrow \phi(\sigma, \sigma^{-1}) \ker(\sigma, \sigma^{-1})) = \mathrm{im}\ \psi(\sigma, \sigma^{-1}).$$

Therefore the following sequence is exact

$$0 \Leftrightarrow\!\rightarrow \mathcal{B} \overset{i}{\hookrightarrow} (R^q)^{\mathbb{Z}} \overset{\bar{\phi}(\sigma,\sigma^{-1})}{\Leftrightarrow\!\rightarrow} (R^q)^{\mathbb{Z}} \overset{\psi(\sigma,\sigma^{-1})}{\Leftrightarrow\!\twoheadrightarrow} (R^l)^{\mathbb{Z}} \Leftrightarrow\!\rightarrow 0,$$

where $\bar{\phi} := (I \Leftrightarrow \phi\psi)$. This shows that $\mathcal{B}$ is regular of order less than or equal to 2. ■

Assume that $\phi \in R[u, u^{-1}]^{q\times g}$ is such that $\hat{I}_\phi = R[u, u^{-1}]$ and that $\mathcal{B} = \mathrm{im}\ \phi(\sigma, \sigma^{-1})$. Then there exists $\psi \in R[u, u^{-1}]^{g\times q}$ such that $\psi\phi = I$. It is immediate to see that

$$\mathcal{B} = \ker(I \Leftrightarrow \phi(\sigma, \sigma^{-1})\psi(\sigma, \sigma^{-1})). \tag{4.1}$$

The problem of finding ψ has been already discussed in Section 3.

4.2 Minimal kernel representations

As it could be expected in analogy with Section 3, if $R[u, u^{-1}]$ is Hermite, it is possible to construct matrix kernel representations which are onto, as illustrated in the following result which can be proven as Proposition 4.

Proposition 12 *Suppose that $R[u, u^{-1}]$ is Hermite. Let $\phi \in R[u, u^{-1}]^{q \times g}$ and let $\mathcal{B} := \mathrm{im}\ \phi(\sigma, \sigma^{-1})$. Assume that any of the equivalent conditions in Theorem 11 holds. Then there exists $\psi \in R[u, u^{-1}]^{(q-g) \times q}$ such that*

$$\mathcal{B} = \ker \psi(\sigma, \sigma^{-1})$$

where $\psi(\sigma, \sigma^{-1})$ is onto and open.

It follows from the proof of Proposition 7 that if $R[u, u^{-1}]$ is not Hermite, there exist cases of R-behaviors $\mathcal{B} = \mathrm{im}\ \phi(\sigma, \sigma^{-1})$, where $\phi(\sigma, \sigma^{-1})$ is one-to-one and $\mathcal{B}$ is regular of order strictly greater than 1. This shows that the assumption that $R[u, u^{-1}]$ is Hermite in Proposition 12 can not be weakened.

In analogy to Section 4 we now show how a more general result can be obtained if we allow kernel representations with final space which is no longer free. We have an analogue of Proposition 6 that can be proved along the same lines.

Proposition 13 *Let $\phi \in R[u, u^{-1}]^{q \times g}$ and let $\mathcal{B} := \mathrm{im}\ \phi(\sigma, \sigma^{-1})$. Assume that any of the equivalent conditions in Theorem 11 holds and that $R[u, u^{-1}]^q / \mathcal{B}_f$ is R-extended. Then, there exist a stably free R-module V and a $\psi \in \mathrm{Hom}(R^q, V)[u, u^{-1}]$ such that*

$$\mathcal{B} = \ker \psi(\sigma, \sigma^{-1}),$$

where psi is such that $\psi\eta = I$for some $\eta \in \mathrm{Hom}(V, R^q)[u, u^{-1}]$. Moreover, we have that $R[u, u^{-1}]^q / \mathcal{B}_f \simeq V[u, u^{-1}]$.

4.3 Loss of observability and partial kernel representations

From the point–of–view of parametrized systems, the situation expressed in Theorem 11 corresponds to families of k-behaviors which admit image representations that remain observable for all values of the parameters. In the following, we will deal with a more general situation in which we may loose observability for some values of the parameters.

Notice that, given $\phi \in R[u, u^{-1}]^{q \times g}$, by Lemma 8 we have that $U_i(\phi)^o = V(\hat{I}_\phi \cap R)^c$. Analogously to what we did in section 3.4, we show how to find kernel representations for maximal ideals in $V(\hat{I}_\phi \cap R)^c$. Let $a_1, \ldots, a_n$ be a family of generators for the ideal $\hat{I}_\phi \cap R$ and let $\psi_i \in R[u, u^{-1}]^{q\times g}$ such that $\psi_i \phi = a_i I$. It is easy to see that

$$\text{im } \phi(\sigma, \sigma^{-1}) \subseteq \ker(a_i I \Leftrightarrow \phi(\sigma, \sigma^{-1})\psi_i(\sigma, \sigma^{-1})).$$

Write

$$V(\hat{I}_\phi \cap R)^c = \bigcup_{i=1}^{n} V((a_i))^c.$$

If $m \in (V(\hat{I}_\phi \cap R))^c$, then $m \in V((a_i))^c$ for some i and it easily follows that

$$\text{im } \phi(\sigma, \sigma^{-1})_m = \ker(a_i I \Leftrightarrow \phi\psi_i)_m(\sigma, \sigma^{-1}).$$

We now present an alternative method to find image representation which is, in a certain sense, the dual of the one discussed in section 3.5. If $\mathcal{B} \subseteq (R^q)^{\mathbb{Z}}$ is an R-behavior, consider

$$\mathcal{B}^\perp := \{r \in R[u, u^{-1}]^{1\times q} \mid r(\sigma, \sigma^{-1})\mathcal{B} \equiv 0\}$$

which is clearly a finitely generated $R[u, u^{-1}]$-module. Conversely, given an $R[u, u^{-1}]$-submodule M of $R[u, u^{-1}]^{1\times q}$, we can consider

$$M^\perp := \{v \in (R^q)^{\mathbb{Z}} \mid r(\sigma, \sigma^{-1})v = 0, \ \forall r \in M\}.$$

Since M is finitely generated, $M^\perp$ is a finite memory R-behavior and indeed a kernel representation of it can be obtained in the following obvious way. Fix a set of generators $r_1, \ldots, r_g$ of M and put

$$\psi = \begin{bmatrix} r_1 \\ \vdots \\ r_g \end{bmatrix}.$$

Then,

$$M^\perp = \ker \psi(\sigma, \sigma^{-1}).$$

Clearly, given an R-behavior $\mathcal{B} \subseteq (R^q)^{\mathbb{Z}}$, we have that $\mathcal{B} \subseteq \mathcal{B}^{\perp\perp}$ and we have equality if and only if $\mathcal{B}$ itself admits a matrix kernel representation.

The following is the most general result regarding kernel representations we will present.

Theorem 14 *Assume that R is a domain. Let $\phi \in R[u, u^{-1}]^{q\times g}$ and let $\mathcal{B} = \text{im } \phi(\sigma, \sigma^{-1})$. If $\psi \in R[u, u^{-1}]^{r\times q}$ is a polynomial matrix whose rows generate $\mathcal{B}^\perp$, then for all $m \in U_i(\phi)$,*

$$\text{im } \phi_m(\sigma, \sigma^{-1}) = \mathcal{B}_m = \ker \psi_m(\sigma, \sigma^{-1}).$$

Proof: We only need to prove that, if $m \in U_i(\phi)$, then

$$\ker \psi_m(\sigma, \sigma^{-1}) \subseteq \mathcal{B}_m.$$

This will be proven by showing that

1. $\ker \psi_m(\sigma, \sigma^{-1})$ is controllable;
2. $\text{rk } \mathcal{B}_m = \text{rk } \ker \psi_m(\sigma, \sigma^{-1})$.

Let us start with 1. It will follow from the following stronger result

$$\text{rk } \psi_{\tilde{m}} = q \Leftrightarrow g \quad \forall \tilde{m} \notin V(\hat{J}_\phi).$$

To see this consider the exact sequence

$$0 \to R[u, u^{-1}]^g \overset{\phi}{\hookrightarrow} R[u, u^{-1}]^q \to R[u, u^{-1}]^q / \text{im } \phi \to 0.$$

Localization at $\tilde{m} \notin V(\hat{J}_\phi)$ gives

$$0 \to (R[u, u^{-1}]_{(\tilde{m})})^g \overset{\phi_{(\tilde{m})}}{\hookrightarrow} (R[u, u^{-1}]_{(\tilde{m})})^q \to (R[u, u^{-1}]_{(\tilde{m})})^q / \text{im } \phi_{(\tilde{m})} \to 0.$$

By the way we have chosen $\tilde{m}$, $\phi_{(\tilde{m})}$ splits. Hence

$$(R[u, u^{-1}]_{(\tilde{m})})^q / \text{im } \phi_{(\tilde{m})} \simeq (R[u, u^{-1}]_{(\tilde{m})})^{q-g}.$$

This implies that

$$((\text{im } \phi)^\perp)_{(\tilde{m})} \simeq ((\text{im } \phi)_{(\tilde{m})})^\perp) = (\text{im } \phi_{(\tilde{m})})^\perp \simeq (R[u, u^{-1}]_{(\tilde{m})})^{q-g}.$$

Since $\mathcal{B} = \overline{\text{im } \phi}$, we thus obtain

$$\text{rk } \psi_{(\tilde{m})} = q \Leftrightarrow g,$$

which also yields

$$\text{rk } \psi_{\tilde{m}} = q \Leftrightarrow g.$$

This proves 1. We now come to point 2. Since by previous considerations

$$\text{rk } \psi_m = q \Leftrightarrow g$$

and $\phi_m(\sigma, \sigma^{-1})$ is injective, it follows that

$$\text{rk } \ker \psi_m(\sigma, \sigma^{-1}) = g = \text{rk } \mathcal{B}_m.$$

This proves 2. ■

5 Parameterized Linear Systems

Let $X \subseteq k^n$ be an affine variety, namely the common zeroes of a set of polynomials in $k[z_1, \dots z_n]$. Denote by $I(X)$ the ideal of all the polynomials which are zero on X and let $R = k[X] := k[z_1, \dots z_n]/I(X)$ be the k-algebra of polynomial functions on X. On X we will consider the Zarisky topology whose closed sets are exactly the affine varieties inside X. They can be represented in the following way: let I be an ideal in R. Then define

$$V(I) := \{x \in X \mid p(x) = 0 \ \forall p \in I\}.$$

Notice now that if $x \in X$, then

$$m_x := \{p \in R \mid p(x) = 0\} \in \mathrm{Max}(R). \tag{5.1}$$

If $\phi \in R[u, u^{-1}]^{g\times q}$, we recall that $\phi_x \in k[u, u^{-1}]^{g\times q}$ is obtained from ϕ by evaluating in x all the coefficients. It is easy to see that under the canonical identification of R/m_x and k, we have that $\phi_x = \phi_{m_x}$. This clearly shows how the representation problems for families of linear behaviors are connected to the problems considered in the previous sections.

Denote by $\bar{k}$ the algebraic closure of k and by $\bar{X}$ the closure of X inside $\bar{k}^n$. Denote moreover $\bar{R} := \bar{k}[\bar{X}] = R \otimes_k \bar{k}$. The advantage of working in the algebraically closed situation is that, in this case, by the Hilbert's Nullstellensatz all the maximal ideals of $\bar{R}$ are of the type m_x as above with $x \in \bar{X}$. This permits an identification of $\overline{X}$ with $\mathrm{Max}(\bar{R})$ which is a homeomorphism if both are equipped with the Zarisky topologies. We will assume this identification in the sequel.

5.1 From families of kernels to families of images

Let $\phi \in R[u, u^{-1}]^{g\times q}$. Analogously to what we did in Section 3, we introduce some important sets connected to ϕ:

$$U_c(\phi) := \{x \in X \mid \ker \phi_x(\sigma, \sigma^{-1}) \text{ is controllable}\},$$

$$U_o(\phi) := \{x \in X \mid \phi_x(\sigma, \sigma^{-1}) \text{ is onto}\},$$

$$U_{co}(\phi) := U_c(\phi) \cap U_o(\phi).$$

We start with the following result.

Theorem 15 *Let X be a k-affine variety and let $R = k[X]$. Let $\phi \in R[u, u^{-1}]^{g\times q}$ be such that $U_{co}(\bar{\phi}) = \bar{X}$. Then*

1.
$$\ker \phi_x(\sigma, \sigma^{-1}) = \mathcal{B}_x, \quad \forall x \in X,$$
where $\mathcal{B} := \ker \phi(\sigma, \sigma^{-1})$.

2. *There exists* $\psi \in R[u,u^{-1}]^{q\times r}$ *such that*

$$\ker \phi_x(\sigma,\sigma^{-1}) = \text{im } \psi_x(\sigma,\sigma^{-1}), \quad \forall x \in X.$$

Proof: Since $\bar{R}$ is a Jacobson Noetherian ring [2], everything immediately follows from the above considerations and Theorems 2 and 3. ■

In the case the field k is not algebraically closed, the result expressed by Theorem 15 is not very satisfactory since it can not be applied if we have only that $U_{co}(\phi) = X$. This drawback will be overcome in a little while.

Next example, on the other hand, shows that the assumption $U_c(\bar{\phi}) = \bar{X}$ is not sufficient to obtain an image representation.

Example 3 (cont.): Let $k = \mathbb{R}$, $X = \mathbb{R}$ and $R := \mathbb{R}[z]$. Consider

$$\phi := \begin{pmatrix} zu & z \end{pmatrix}.$$

Clearly, $U_c(\bar{\phi}) = \mathbb{C}$ and $U_{co}(\bar{\phi}) = \mathbb{C} \setminus \{0\}$. Assume by contradiction that there exists $\psi \in R[u,u^{-1}]^{q\times r}$ such that

$$\ker \phi_x(\sigma,\sigma^{-1}) = \text{im } \psi_x(\sigma,\sigma^{-1}), \quad \forall x \in \mathbb{R}. \tag{5.2}$$

It then follows from (2.3) that

$$\text{rk } (\psi_0) = \text{rk } (\ker \phi_0(\sigma,\sigma^{-1})) = 2.$$

Hence rk $(\psi_x) \geq 2$ for x in an open set containing 0. However, this is not possible since by (5.2) and (2.3) we have that rk $(\psi_x) =$ rk $(\ker \phi_x(\sigma,\sigma^{-1})) =$ 1, if $x \in \mathbb{R} \setminus \{0\}$.

We discuss now the observability. Observable image representations are guaranteed in certain cases.

Theorem 16 *Assume that* $X = k^n$. *In the same assumptions of Theorem 15, there exists* $\psi \in R[u,u^{-1}]^{q\times(q-g)}$ *such that*

$$\ker \phi_x(\sigma,\sigma^{-1}) = \text{im } \psi_x(\sigma,\sigma^{-1}), \quad \forall x \in X,$$

where $\psi_x(\sigma,\sigma^{-1})$ *is one–to–one for all* $x \in X$.

Proof: It follows from the assumptions and from Theorem 3 that $J_{\bar{\phi}} = \bar{R}[u,u^{-1}]$. It is easy to see that this implies $J_\phi = R[u,u^{-1}]$. The result then easily follows from Proposition 4 since $R = k[X] = k[z_1,\ldots,z_n]$ is Hermite. ■

Notice that the construction of the image representation suggested in Proposition 4 is computable when the Hermite ring is $k[z_1,\ldots,z_n]$ [9].

In certain cases more general results can be obtained using Proposition 6, if we allow more general image representations induced by a $\psi \in \mathrm{Hom}(V, R^q)[u, u^{-1}]$, where V is a stably free R-module. To such a ψ we can indeed associate a sort of generalized family of k-images by considering

$$\mathrm{im}\ \psi_{m_x}(\sigma, \sigma^{-1}),$$

where m_x is the maximal ideal (5.1) associated to the point $x \in X$. We can give a nice geometric interpretation of this generalized family of images. In fact, the module V, being stably free, can be interpreted as the R-module of global polynomial sections of a certain algebraic vector bundle [12]

$$\pi : E \to X.$$

Namely,

$$V = k[X, E] := \{s : X \to E \text{ polynomial with } \pi \circ s = i_{|E}\},$$

where i is the identity map on X. E can be constructed in a canonical way, so that the fibers E_x, with $x \in X$, are identified with V/m_xV. Let us write $\psi = \sum \psi_i u^i$, with $\psi_i \in \mathrm{Hom}(V, R^q)$. Each ψ_i gives rise to a bundle morphism

$$\psi_i^* : E \to k^q$$

given, if $v \in E_x$, by

$$\psi_i^*(v) = (\psi_i)_{m_x}(v).$$

Consider $\psi^* := \sum \psi_i^* u^i$. Then

$$\psi^*_{|E_x} = \sum (\psi_i)_{m_x} u^i = \psi_{m_x}.$$

$\{\psi_{m_x}(\sigma, \sigma^{-1})\}$ can be thus interpreted as a family of linear shift operators polynomially parameterized by x, but whose initial space is not a fix vector space k^g but the fiber E_x of a vector bundle E on X.

Example 1 (cont.): Consider the R-family of kernels introduced in Example 1 in the introduction. It is immediate to check that $J_{\bar{\phi}} = \mathbb{C}[u, u^{-1}]$ so that, by Theorem 3, $U_{co}(\bar{\phi}) = \mathbb{C}^2$. According to Theorem 16, the R-family of kernels $\ker \phi_{(x,y)}(\sigma, \sigma^{-1})$ admits an observable image representation which can easily be found with the techniques of Section 3.2. First complete ϕ to a square polynomial matrix with determinant equal to 1:

$$\tilde{\phi} := \begin{pmatrix} z_1 u & z_2 & z_2 + 1 \\ 0 & 1 & 1 + z_1 u \\ \Leftrightarrow 1 & 0 & z_2 \end{pmatrix}.$$

The last column of the inverse of $\tilde{\phi}$ is

$$\psi := \begin{pmatrix} z_1 z_2 u \Leftrightarrow 1 \\ \Leftrightarrow z_1 u(1 + z_1 u) \\ z_1 u \end{pmatrix}.$$

Hence,

$$\ker \phi_{(x,y)}(\sigma, \sigma^{-1}) = \operatorname{im} \begin{pmatrix} xy\sigma \Leftrightarrow 1 \\ \Leftrightarrow x\sigma(1 + x\sigma) \\ x\sigma \end{pmatrix} \quad \forall (x,y) \in \mathbb{R}^2.$$

Example 2 (cont.): Consider the R-family of kernels introduced in Example 2 in the introduction. It is immediate to check that $U_{co}(\bar{\phi}) = \bar{X}$. By Theorem 15, the R-family of kernels $\ker \phi_{(x,y)}(\sigma, \sigma^{-1})$ admits a matrix image representation. Actually, it admits an observable one for the following general argument which is worth illustrating. Put $\mathcal{B} = \ker \phi(\sigma, \sigma^{-1})$. Since R is a Dedekind domain, it follows from [8] (Corollary 4.12 chapter 2) that $\mathcal{B}_f$ is the R-extension of a stably free R-module M of rank 1. Since M is stably free of rank 1, it must be free [8] (Theorem 4.11 chapter 1). This shows, by Proposition 6, that our R-family of kernels, admits an observable matrix image representation which can be obtained as in Example 3 completing the matrix ϕ:

$$\tilde{\phi} := \begin{pmatrix} z_1 u^2 + z_2 & z_2 \\ \Leftrightarrow z_2 + z_1 u^{-2} & z_1 u^{-2} \end{pmatrix}$$

and considering the second column of the inverse:

$$\psi := \begin{pmatrix} \Leftrightarrow z_2 \\ z_1 u^{-2} \end{pmatrix}.$$

Example 4: Let $k = \mathbb{R}$, $X = S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$ and $R = \mathbb{R}[S^2] = \mathbb{R}[z_1, z_2, z_3]/(1 \Leftrightarrow z_1^2 \Leftrightarrow z_2^2 \Leftrightarrow z_3^2)$. Consider

$$\phi := \begin{pmatrix} z_1 u + z_2 & z_2 & z_3 \end{pmatrix}.$$

Again, $U_{co}(\bar{\phi}) = \bar{X}$. Put $\mathcal{B} = \ker \phi(\sigma, \sigma^{-1})$ and consider the R-module

$$M = \ker \begin{pmatrix} z_1 & z_2 & z_3 \end{pmatrix}.$$

We have a natural $R[u, u^{-1}]$-isomorphism

$$g : M \otimes_R R[u, u^{-1}] \to \mathcal{B}_f$$

given, if $w = (w_1, w_2, w_3)^T \in M$, by

$$g(w \otimes u^j) := \begin{pmatrix} u^{-1} & 0 & 0 \\ \Leftrightarrow u^{-1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w_1 u^j \\ w_2 u^j \\ w_3 u^j \end{pmatrix}.$$

It is easy to realize that M is the R-module of sections of the tangent bundle of S^2 which is not trivial (not even topologically). Hence M is not free [8]. Consequently, it follows from Proposition 6 that we can not find in this case an observable matrix image representation of the R-family of kernels. At this point there are two alternatives. If we want a matrix image representation in the class R valid for all points of the sphere, we have to drop observability and, indeed, we can find one with initial space $\mathbb{R}^3$ as follows. Put

$$\psi := \begin{pmatrix} u^{-1} & 0 & 0 \\ \Leftrightarrow u^{-1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_2 & z_3 & 0 \\ \Leftrightarrow z_1 & 0 & z_3 \\ 0 & \Leftrightarrow z_1 & \Leftrightarrow z_2 \end{pmatrix}. \tag{5.3}$$

Then, it is straightforward to verify that

$$\ker \phi_{(x_1,x_2,x_3)}(\sigma, \sigma^{-1}) = \text{im } \psi_{(x_1,x_2,x_3)}(\sigma, \sigma^{-1}), \quad \forall (x_1, x_2, x_3) \in S^2.$$

Note that, by using condition (3.2), we would find a different ψ. The image representation (5.3) can easily be found using Lemma 9, the map g and observing that the image of the matrix at right in (5.3) is exactly M. On the other hand, if we do not want to lose observability we are forced to consider an image representation which has as initial space not a fixed vector space but the linear fiber of dimension 2 of the tangent bundle TS^2 of S^2. Since TS^2 can be trivialized on S^2 deprived of a point, it follows that we can find observable matrix image representations which hold on the all sphere minus one point. Using for instance the stereographic projection from the point $(0,0,1)$, a straightforward computation shows that, if $(x_1, x_2, x_3) \neq (0,0,1)$, $TS^2_{(x_1,x_2,x_3)}$ can be represented as the image of the matrix

$$\begin{pmatrix} [(1 \Leftrightarrow x_3)^2 \Leftrightarrow x_1^2 + x_2^2] & \Leftrightarrow 2x_1x_2 \\ \Leftrightarrow 2x_1x_2 & [(1 \Leftrightarrow x_3)^2 + x_1^2 \Leftrightarrow x_2^2] \\ 2(1 \Leftrightarrow x_3)x_1 & 2(1 \Leftrightarrow x_3)x_2 \end{pmatrix}.$$

Hence, if we consider

$$\psi := \begin{pmatrix} u^{-1} & 0 & 0 \\ \Leftrightarrow u^{-1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} [(1 \Leftrightarrow z_3)^2 \Leftrightarrow z_1^2 + z_2^2] & \Leftrightarrow 2z_1z_2 \\ \Leftrightarrow 2z_1z_2 & [(1 \Leftrightarrow z_3)^2 + z_1^2 \Leftrightarrow z_2^2] \\ 2(1 \Leftrightarrow z_3)z_1 & 2(1 \Leftrightarrow z_3)z_2 \end{pmatrix}$$

we have that

$$\ker \phi_{(x_1,x_2,x_3)}(\sigma,\sigma^{-1}) = \text{im}\ \psi_{(x_1,x_2,x_3)}(\sigma,\sigma^{-1}), \forall (x_1,x_2,x_3) \in S^2 \backslash \{(0,0,1)\}.$$

Something similar can be done outside of $(0,0,-1)$ so that we can represent our R-family of kernels with two observable image representations.

Example 5: Let $k = \mathbb{R}$, $X = S^2$ and $R = \mathbb{R}[S^2]$. Consider

$$\phi := \begin{pmatrix} u+1 & z_1 & z_2 & z_3 \end{pmatrix}.$$

Again, $U_{co}(\bar{\phi}) = \bar{X}$. Put $\mathcal{B} = \ker \phi(\sigma,\sigma^{-1})$. It is easy to verify that

$$\mathcal{B}_f \simeq (R \oplus M) \otimes_R R[u,u^{-1}],$$

where M is defined as in Example 4. On the other hand, it follows from the definition of M that $R \oplus M \simeq R^3$. Hence, there exists in this case an observable matrix image representation which can be found with the usual techniques. Notice indeed that we can explicitly complete ϕ to a square invertible matrix

$$\phi := \begin{pmatrix} u+1 & z_1 & z_2 & z_3 \\ -z_3 & z_2 & -z_1 & 0 \\ -z_1 & 0 & z_3 & -z_2 \\ -z_2 & z_3 & 0 & -z_1 \end{pmatrix}.$$

From this, by passing to the algebraic adjoint, we can again find the desired image representation. We omit the long but straightforward calculations.

We now pass to examine more general cases when controllability may be lost for certain values of the parameters. We have the following nice result.

Theorem 17 *Assume that $X = k$ and let $R = k[z]$. Let $\phi \in R[u,u^{-1}]^{g\times q}$ and assume that $I_\phi \cap R \neq \{0\}$. Then*

1. *$V(I_\phi \cap R)^c = U_{co}(\phi)$;*
2. *There exists $\psi \in R[u,u^{-1}]^{q\times(q-g)}$ such that*

$$\ker \phi_x(\sigma,\sigma^{-1}) = \text{im}\ \psi_x(\sigma,\sigma^{-1}) \quad \forall x \in U_{co}(\phi), \tag{5.4}$$

where $\psi_x(\sigma,\sigma^{-1})$ is one–to–one for all $x \in U_{co}(\phi)$.

Proof:

1. Since $V(I_{\bar{\phi}} \cap \bar{R})^c$ is a non-empty open set, it is equal to $\bar{k}$ deprived of a finite number of points. Since, by Lemma 8, $V(I_{\bar{\phi}} \cap \bar{R})^c$ is the interior of $U_{co}(\bar{\phi})$, it immediately follows that they must be equal. From this, 1 easily follows.

2. Let f be a generator of $I_\phi \cap R$ and let $R_f := \{r/f^i : r \in R,\ i \in \mathbb{N}\}$. It follows from 1 and the considerations in section 3.4 that there exists $\psi \in R_f[u, u^{-1}]^{q\times(q-g)}$ satisfying the requirements of the proposition. Multiplying ψ by a suitable power of f we obtain the result. ■

Remark: Notice that, in general, the inclusion

$$V(I_\phi \cap R)^c \subseteq U_{co}(\phi) \tag{5.5}$$

may be proper. In fact, $U_{co}(\phi)$ may well not be open as shown in the following example. Let $X = \bar{X} = \mathbb{C}^2$ and $R = \mathbb{C}[z_1, z_2]$. Take $\phi = (z_1, 1 + z_2 u)$. It is easy to see that

$$U_{co}(\phi)^c = \{(x_1, x_2) \in X \mid x_1 = 0\ x_2 \neq 0\},$$

while

$$V(I_\phi \cap R) = \{(x_1, x_2) \in X \mid x_1 = 0\}.$$

A straightforward application of the results in Section 3 gives the following general result.

Theorem 18 *Let X be a k-affine variety and let $R = k[X]$. Let $\phi \in R[u, u^{-1}]^{g\times q}$. Then*

1. $$(\ker \phi_x(\sigma, \sigma^{-1}))_c = (\mathcal{B}_x)_c, \quad \forall x \in U_o(\phi),$$

   where $\mathcal{B} := \ker \phi(\sigma, \sigma^{-1})$.

2. *There exists $\psi \in R[u, u^{-1}]^{q\times r}$ such that*

   $$(\ker \phi_x(\sigma, \sigma^{-1}))_c = \text{im}\ \psi_x(\sigma, \sigma^{-1}), \quad \forall x \in U_o(\phi).$$

This has an important consequence which permits to overcome the drawback of Theorem 15.

Corollary 19 *Let X be a k-affine variety and let $R = k[X]$. Let $\phi \in R[u, u^{-1}]^{g\times q}$ be such that $U_{co}(\phi) = X$. Then*

1. $$\ker \phi_x(\sigma, \sigma^{-1}) = \mathcal{B}_x \quad \forall x \in X,$$

   where $\mathcal{B} := \ker \phi(\sigma, \sigma^{-1})$.

2. *There exists $\psi \in R[u, u^{-1}]^{q\times r}$ such that*

   $$\ker \phi_x(\sigma, \sigma^{-1}) = \text{im}\ \psi_x(\sigma, \sigma^{-1}), \quad \forall x \in X.$$

Example 6 Let $k = \mathbb{R}$, $X = \mathbb{R}$ and $R = \mathbb{R}[z]$. Consider

$$\phi := \begin{pmatrix} (1 - z)u & 1 & 0 \\ 0 & zu & z(z - 1) \end{pmatrix}.$$

It is easy to check that $U_{co}(\phi) = \mathbb{R} \setminus \{0, 1\}$. Working in the ring $\tilde{R} := \{p/(z(z - 1))^i : p \in R[z],\ i \in \mathbb{N}\} = R[z, z^{-1}, (1 - z)^{-1}]$, we complete ϕ to an invertible matrix

$$\tilde{\phi} := \begin{pmatrix} (1 - z)u & 1 & 0 \\ 0 & zu & z(z - 1) \\ z^{-1}(1 - z)^{-1} & 0 & 0 \end{pmatrix}.$$

Taking the last column of the inverse and simplifying, we obtain that

$$\ker \phi_x(\sigma, \sigma^{-1}) = \mathrm{im}\ \psi_x(\sigma, \sigma^{-1}), \quad \forall x \in \mathbb{R} \setminus \{0, 1\},$$

where

$$\psi := \begin{pmatrix} 1 \\ -(1 - z)u \\ u \end{pmatrix}.$$

Example 7 Let $k = \mathbb{R}$, $X = \mathbb{R}$ and $R = \mathbb{R}[z]$. Consider

$$\phi := \begin{pmatrix} z + u & zu - 1 \end{pmatrix}.$$

It is immediate to verify that $U_{co}(\phi) = \mathbb{R}$, while $U_{co}(\bar{\phi}) = \mathbb{C} \setminus \{i, -i\}$. An observable matrix image representation for $x \in \mathbb{R}$ can be obtained in this case, by considering

$$\psi := \begin{pmatrix} 1 - zu \\ z + u \end{pmatrix}.$$

Indeed, it is clear that, if $x \in \mathbb{R}$,

$$\mathrm{im}\ \psi_x(\sigma, \sigma^{-1}) \subseteq \ker \phi_x(\sigma, \sigma^{-1})$$

and equality holds by standard rank considerations.

Example 8 Let $k = \mathbb{R}$, $X = \mathbb{R}^2$ and $R = \mathbb{R}[z_1, z_2]$. Consider

$$\phi := \begin{pmatrix} z_1 u^2 + u - 2 & z_1 u + z_2 \end{pmatrix}.$$

It is easy to see that $U_o(\phi) = \mathbb{R}^2$. We want to characterize $U_{co}(\phi)$. It is clear that $(x, y) \in U_{co}(\phi)$ if and only if $xu^2 + u - 2$ and $xu + y$ do not have non-trivial common factors and this can be easily checked by using elimination theory. Indeed, consider the resultant of the two polynomials

$$, := \det \begin{pmatrix} -2 & 1 & z_1 \\ z_2 & z_1 & 0 \\ 0 & z_2 & z_1 \end{pmatrix} = -z_1[-2z_1 + z_2(1 - z_2)].$$

Hence

$$U_{co}(\phi) := ((\{0\} \oplus \mathbb{R}) \cup \{(x,y) \mid 2x = y(1 \Leftrightarrow y)\})^c.$$

Consider now

$$\psi := \begin{pmatrix} \Leftrightarrow z_1 u \Leftrightarrow z_2 \\ z_1 u^2 + u \Leftrightarrow 2 \end{pmatrix}.$$

Arguing like in Example 7, it follows that

$$\text{im } \psi_{(x,y)}(\sigma, \sigma^{-1}) = \ker \phi_{(x,y)}(\sigma, \sigma^{-1}), \quad \forall (x,y) \in U_{co}(\phi).$$

Example 9 Let $k = \mathbb{R}$, $X = \mathbb{R}^3$ and $R = \mathbb{R}[z_1, z_2, z_3]$. Consider

$$\phi := \begin{pmatrix} z_1 u + z_2 & z_2 & z_3 \end{pmatrix}.$$

Formally, it is the same polynomial matrix than in Example 4. We have that $U_{co}(\phi) = \mathbb{R}^3 \setminus \{(0,0,0)\}$. A matrix image representation out of $(0,0,0)$ is exactly given by (5.3) and it is clear, by the arguments developed in Example 4 that there are not observable matrix image representations.

5.2 From families of images to families of kernels

An analogous application of the results in Section 4 provides a number of results on the representation of a family of images as a family of kernels. We here present them without further comments.

If $\phi \in R[u, u^{-1}]^{q \times g}$, define

$$U_i(\phi) := \{x \in X \mid \phi_x(\sigma, \sigma^{-1}) \text{ is one to one}\}.$$

The following theorems are immediate consequences of Theorem 11 and Propositions 12 and 13.

Theorem 20 *Let X be a k-affine variety and let $R = k[X]$. Let $\phi \in R[u, u^{-1}]^{q \times g}$ be such that $U_i(\bar{\phi}) = \bar{X}$. Then there exists $\psi \in R[u, u^{-1}]^{r \times q}$ such that*

$$\text{im } \phi_x(\sigma, \sigma^{-1}) = \ker \psi_x(\sigma, \sigma^{-1}), \quad \forall x \in X.$$

Theorem 21 *Let $X = k^n$. In the same assumptions of Theorem 20, there exist $\psi \in R[u, u^{-1}]^{(q-g) \times q}$ such that such that*

$$\text{im } \phi_x(\sigma, \sigma^{-1}) = \ker \psi_x(\sigma, \sigma^{-1}), \quad \forall x \in X,$$

where $\psi_x(\sigma, \sigma^{-1})$ is onto for all $x \in X$.

If $U_i(\bar{\phi}) \neq \bar{X}$ we have partial kernel representations.

Theorem 22 *Assume that $X = k$ and let $R = k[z]$. Let $\phi \in R[u, u^{-1}]^{q \times g}$ and assume that $\hat{I}_\phi \cap R \neq \{0\}$. Then,*

1. *$V(\hat{I}_\phi \cap R)^c = U_i(\phi)$.*

2. *There exists $\psi \in R[u, u^{-1}]^{(q-g)\times q}$ such that*

$$\text{im } \phi_x(\sigma, \sigma^{-1}) = \ker \psi_x(\sigma, \sigma^{-1}), \quad \forall x \in U_i(\phi),$$

where $\phi_x(\sigma, \sigma^{-1})$ is onto for all $x \in U_i(\phi)$.

Theorem 23 *Let X be an irreducible k-affine variety and let $R = k[X]$. Let $\phi \in R[u, u^{-1}]^{q\times g}$. Then, there exists $\psi \in R[u, u^{-1}]^{r\times q}$ such that*

$$\text{im } \phi_x(\sigma, \sigma^{-1}) = \ker \psi_x(\sigma, \sigma^{-1}), \quad \forall x \in U_i(\phi).$$

This allows us to improve Theorem 20.

Corollary 24 *Let X be an irreducible k-affine variety and let $R = k[X]$. Let $\phi \in R[u, u^{-1}]^{q\times g}$ be such that $U_i(\phi) = X$. Then, there exists $\psi \in R[u, u^{-1}]^{r\times q}$ such that*

$$\text{im } \phi_x(\sigma, \sigma^{-1}) = \ker \psi_x(\sigma, \sigma^{-1}), \quad \forall x \in X.$$

Completely analogous examples to the ones developed in previous subsection can be easily presented in this context. Here instead we illustrate a simple application to a more classical system theoretic setting.

Example 10: Consider a k-affine variety X and put $R = k[X]$. Consider then matrices $A \in R^{n\times n}$, $B \in R^{n\times q_1}$, $C \in R^{q_2\times n}$, $D \in R^{q_2\times q_1}$ to which we can associate, as x varies in X, the family of classical input/state/output system

$$\begin{cases} \sigma l = A_x l + B_x u \\ \\ y = C_x l + D_x u. \end{cases} \tag{5.6}$$

We wonder if it is possible to 'eliminate' the variable l and to get difference equations involving u and y only, whose coefficients are still polynomials in $x \in X$. This can be interpreted in the behavioral context as follows. Consider

$$\mathcal{B}_x := \{(u, y) \in (k^{q_1+q_2})^{\mathbb{Z}} \mid \exists l \in (R^n)^{\mathbb{Z}} : (u, l, y) \text{ satisfies } (5.6)\}.$$

Clearly, (5.6) is a latent variable representation of $\mathcal{B}_x$. To use the results of this section, we first rewrite (5.6):

$$\begin{pmatrix} B_x & 0 \\ D_x & \Leftrightarrow I \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} \sigma I \Leftrightarrow A_x \\ \Leftrightarrow C_x \end{pmatrix} l.$$

Consider now

$$\phi := \begin{pmatrix} uI \Leftrightarrow A \\ \Leftrightarrow C \end{pmatrix} \in R[u, u^{-1}]^{(n+q_2)\times n}.$$

If $U_i(\bar{\phi}) = \bar{X}$, then there exists $\psi \in R[u, u^{-1}]^{r \times (n+q_2)}$ such that

$$\text{im } \phi_x(\sigma, \sigma^{-1}) = \ker \psi_x(\sigma, \sigma^{-1}), \quad \forall x \in X.$$

Then,

$$\mathcal{B}_x = \ker \psi_x(\sigma, \sigma^{-1}) \begin{pmatrix} B_x & 0 \\ D_x & \Leftrightarrow I \end{pmatrix}, \tag{5.7}$$

which is the desired difference equation representation of $\mathcal{B}_x$. Note that the condition that $U_i(\bar{\phi}) = \bar{X}$ is equivalent to the fact that the pair (A_x, C_x) is observable for every $x \in \bar{X}$.

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